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ABSTRACT

Fixed-term employment contracts have been introduced in number of European countries as a way to provide flexibility to economies with high employment protection levels. We introduce these contracts into the equilibrium search model in Alvarez and Veracierto (1999), a version of the Lucas and Prescott island model, adapted to have undirected search and variable labor force participation. We model a contract of length J as a tax on separations of workers with tenure higher than J. We show a version of the welfare theorems, and characterize the efficient allocations. This requires solving a control problem, whose solution is characterized by two dimensional inaction sets. For J=1 these contracts are equivalent to the case of firing taxes, and for large J they are equivalent to the laissez-faire case. In a calibrated verion of the model, we find that temporary contracts with J equivalent to three years length close about half of the gap between those two extremes.

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1 Introduction

This paper constructs a general equilibrium search model to analyze the effects of fixed-term employment contracts (or *temporary* contracts). These contracts were introduced in economies with high employment protection levels in Europe and Latin America, as a way of giving firms some type of flexibility in the process of hiring and firing workers. Fixed-term contracts stipulate a period of time, typically between one and three years, during which workers can be dismissed at very low or zero separation costs. If workers are retained beyond this period, standard separation costs apply.

Since the introduction of fixed-term contracts during the 1980s, the fraction of workers hired under this modality has expanded steadily in Europe, reaching more than 13 percent in 2000. However, there are large cross-country differences in the scope and duration of fixed-term contracts. For instance, some countries restrict these contracts to certain occupations and type of workers, while others give them broad applicability. In this paper we focus on the case of Spain, because in 1984 Spain substantially liberalized the applicability of temporary contracts at a time when the country had one of the highest employment protection levels in Europe (see Cabrales and Hopenhayn, 1997, and Heckman and Pages-Serra, 2000). From 1984 to 1991, the fraction of workers with fixed-term contracts in Spain went from 11 percent to more than 30 percent, and almost all the hiring in the economy was done under this form of contract (see Hopenhayn and Garcia-Fontes, 1996).[1]

Figure 1, which is taken from Hopenhayn and Cabrales (1997), displays estimates for the one-quarter transition probabilities from employment to unemployment during the six years before and after the 1984 reform, as a function of the length of the employment spells. It shows that the firing rates increased significantly after the reform and that a spike formed at an employment duration of three years, which (not surprisingly) corresponds to the maximum fixed-term contract length allowed by the reform. Thus, the introduction of fixed-term contracts appears to have significant effects on worker reallocation. In fact, there is considerable agreement in the literature

[1] These reforms were partially undone during the 1990s, when the maximum length of the fixed-term contracts was reduced from three years to one year, and the severance payments for ordinary indefinite-length contracts were substantially reduced. However, even after this partial reversal, the fraction of workers under fixed-term contracts stabilized at about 33 percent.

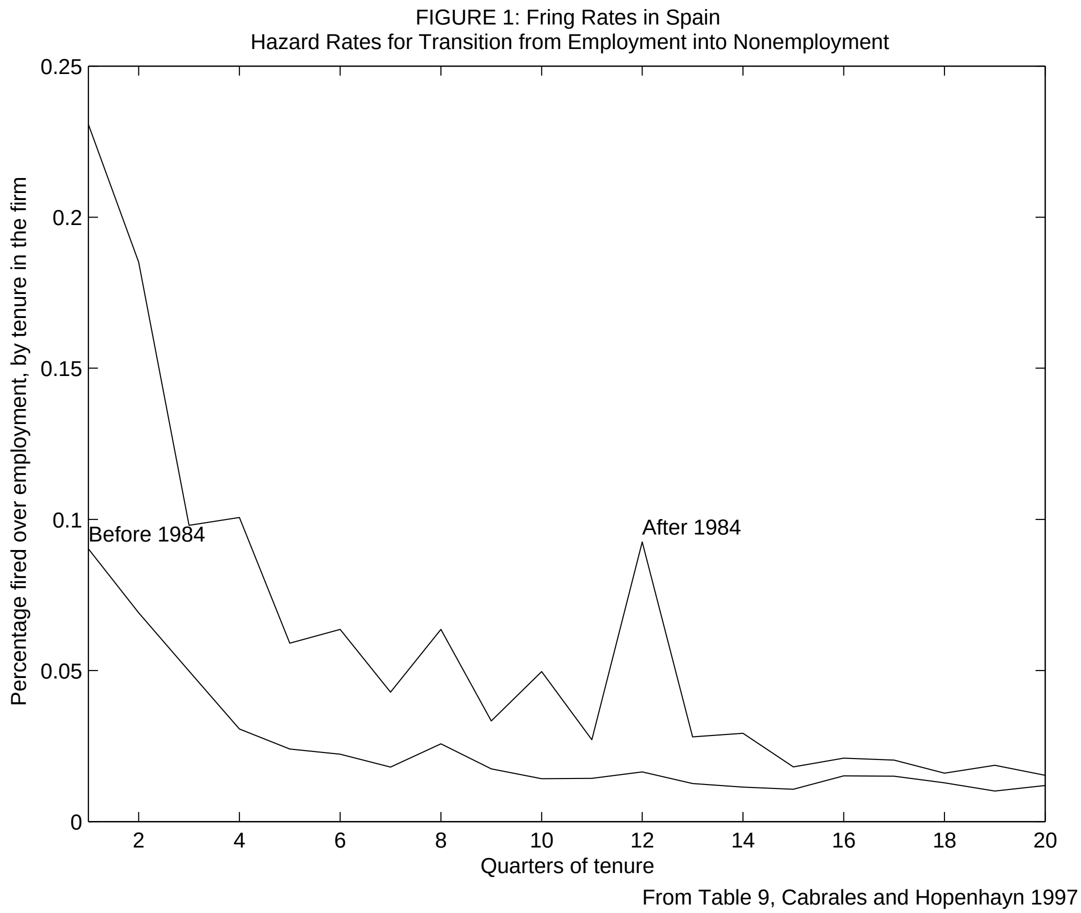


Figure 1:

that the main effects of introducing fixed-term contracts are a substantial increase in the flows from unemployment to employment (i.e., a decrease in the average duration of unemployment), and a significant increase in the flows from employment to unemployment (i.e., an increase in the firing rate) as can be seen, for example, in the literature survey by Dolado et al. (2001). The net effect of these two opposing forces on the unemployment rate is not clear, but the evidence seems to indicate a small increase.

This paper analyzes the effects of fixed-term employment contracts using the equilibrium search model of Alvarez and Veracierto (1999), which is a version of the Lucas and Prescott (1974) island search model with undirected search and variable labor force participation. Production takes place in a large number of locations (or islands). All islands operate the same decreasing returns to scale technology but are subject to island-specific productivity shocks. Changes in the island-specific productivity shock give raise to changes in labor demand across locations. Moving a worker across locations is costly: It requires one period during which the agent neither enjoys leisure nor works. In addition, agents that search arrive randomly to one of the islands in the economy (i.e., search is undirected). Workers that separate from their islands have to choose between two alternatives: to work at home (i.e., to leave the labor force) or to search (i.e., to become unemployed). On each island there are competitive labor markets. In addition, we assume that agents have access to perfect insurance markets, so that firms maximize expected discounted profits and households maximize expected discounted wages.

The employment protection system that we analyze is characterized by two parameters: the firing tax τ and the length of the fixed-term contracts J. In particular, firms must pay a firing tax τ when they reduce their employment of permanent workers (those that have a tenure level equal to or greater than J) but are exempt from paying firing taxes on temporary workers (those with a tenure level less than J). Because firing taxes depend on tenure, the state of an island is described not only by its idiosyncratic productivity level but also by its distribution of workers across tenure levels.

We consider two alternative (yet equivalent) concepts of competitive equilibrium: one with spot labor markets and one with multiperiod employment relations. In the spot labor markets equilibrium, workers and firms solve problems that are natural extensions of McCall's search model and Bentolila and Bertola's firing costs model, respectively. This equilibrium concept has two advantages: It can be easily related to the previous literature and it is simple to analyze. The equilibrium with multiperiod employment relations is more complex but captures the nature of fixed-term employment contracts in a much more realistic way.

In the spot labor market equilibrium, workers are differentiated by their tenure levels, participate in different labor markets, and receive different wages. Given the presence of tenure-dependent firing costs, firms solve a modified (S,s) optimization problem. In turn, workers at each tenure level face a standard search problem: They decide whether to stay on the island where they are currently located or become non-employed. Both firms and workers take as given the island-level law of motion for wages across tenure levels. At equilibrium, this island-level process must be such that the island labor markets clear at each island-wide state. An economy-wide equilibrium is completed by an invariant distribution across island states. This economy-wide distribution is needed to describe the benefit of search and the aggregate demand for labor.

If separation costs are a technological feature of the environment, a version of the first and second welfare theorems holds for our competitive equilibrium. Conversely, if separation costs are taxes rebated lump-sum to households (the most interesting case to consider), the welfare theorems do not apply, but we can still use a modified version of the planning problem to characterize a competitive equilibrium. In particular, we can break the economy-wide planning problem into a series of island-wide planning problems, one for each island. Each of these island-wide social planners solves a similar problem: to maximize the expected discounted value of output by deciding how many workers to keep and take out from the island. In this problem, the island-wide planner takes as given the constant flow U of new arrivals (this flow is independent of the characteristics of the island because of the assumption of undirected search). The island's planner also takes as given the shadow value of returning a worker to non-employment. This shadow value is tenure dependent, taking into account that the separation cost τ applies only to permanent workers. While the state of this problem is the distribution of workers across tenure levels, which is a J dimensional object, we show how to reduce it to a two-dimensional object: the number of temporary workers and the number of permanent workers. We also show that the solution to this control problem is characterized by two-dimensional sets of inaction, one set for each value of the idiosyncratic productivity shock. Given the solution to the island-wide planning problem, the economy-wide equilibrium is obtained by finding two unknowns: the equilibrium shadow value of non-employment and the equilibrium number of agents that searches U.

We take the case with no fixed-term contracts and large firing costs as our benchmark case and calibrate it to reproduce a stylized version of the Spanish economy before the 1984 reform. We use this calibrated version to evaluate to what extent fixed-term contracts of different lengths add flexibility to the labor market. For sufficiently large values of J, introducing fixed-term contracts is equivalent to eliminating all separation costs, since workers never gain permanent status. Thus, we address the question of the added flexibility by computing how much of the gap between the firing-tax case and the laissez-faire case is closed when fixed-term contracts of empirically reasonable length are introduced. We find that, even when the firing tax τ is small, introducing temporary contracts of a short length (i.e., with a small J) sharply increases the average firing rate and sharply decreases the average duration of unemployment. Nevertheless, for firing taxes of about one year of average wages (the value that corresponds to Spain during the 1980s), unemployment, productivity and welfare change smoothly with variations in J. For instance, moving to laissez-faire (i.e., introducing a large J) increases the unemployment rate 2.4 percent relative to the benchmark case. On the contrary, introducing fixed-term contracts of three years duration (the length introduced by the 1984 Spanish reform) increases the unemployment rate by only 1.25 percent relative to the benchmark case. Regarding welfare, we find that moving to laissez-faire increases welfare by 2.5 percent (in perpetual consumption equivalent units) relative to the benchmark case, while introducing fixed-term contracts of three years duration increases welfare by only 1 percent relative to the benchmark case. We thus conclude that fixed-term contracts of three years duration provide significant flexibility: They close about 50 percent of the gap between the benchmark and laissez-faire cases.

Several papers have analyzed the effect of temporary contracts, such as Blanchard and Landier (2001) and Nagypal (2002). The models that are more similar in spirit to our paper, however, are Bentolila and Saint Paul (1992), Hopenhayn and Cabrales (1993), Aguiregabiria and Alonso-Borrego (2004), and Alonso-Borrego et al. (2005), since they all study labor demand models with dynamic adjustment costs. One difference between our model and the models in Bentolila and Saint Paul (1992), Hopenhayn and Cabrales (1993), and Aguiregabiria and Alonso-Borrego (2004) is that these papers consider partial equilibrium models (with exogenous wages) and do not consider unemployment. The paper that is closest to ours is Alonso-Borrego et al. (2005), which performs a general equilibrium analysis in a model with search frictions. However, there are important differences due to the following assumptions. First, agents face incomplete markets. Second, employment contracts are constrained to have a constant wage rate as long as the employment relation lasts. Third, workers under temporary contracts are assumed to be less productive than under ordinary contracts. Fourth, fixed-term contracts last only one model period. Some of these assumptions are meant to provide realism but they substantially complicate the interpretation of the results. For example, it is unclear to what extent the results depend on the rigid wage contracts assumed.[2] We believe that performing the analysis in an economy with efficient contracts, as our paper does, provides easily interpretable results and a useful benchmark for evaluating deviations from the complete markets case.

The paper is organized as follows. Section 2 describes the economy. Section 3 defines efficient

[2] The analysis in Alvarez-Veracierto (2001), on which Alonso and Borrego's paper is based, suggests that rigid wage contracts play a critical role in the results.

allocations. Section 4 characterizes efficient stationary allocations. Section 5 considers the two alternative notions of competitive equilibrium. Finally, Section 6 performs the computational experiments. Seven appendices provide all the proofs and supporting material to the paper.

2 Description of the Economy

Production takes place in a continuum (measure one) of different locations, or "islands." On each island consumption goods are produced according to $F(E, z)$, a neoclassical production function, where E is employment and z is a productivity shock that takes values in the set Z. The process for z is Markov with transition function $Q(z_{t+1}|z_t)$, and realizations are i.i.d. across islands. We let $f(E, z) \equiv \partial F(E, z) / \partial E$ and assume that f is continuous and strictly decreasing in E, strictly increasing in z, and that

$$\lim_{E \to 0} f(E, \underline{z}) = \infty$$

where $\underline{z} \equiv \min \{z : z \in Z\}$.

There is a continuum of agents with mass equal to N. Agents participate in one of the following three activities: work on an island, perform home production (or, equivalently, enjoy leisure), or search. Non-employed agents, whom we sometimes refer to as "agents being at a central location," either work at home (enjoy leisure) or search for a job. If they work at home during the current period, they start the following period as non-employed. If a non-employed agent searches in the current period, she does not produce during the current period but arrives randomly to one of the islands at the beginning of the next period. We assume that search is undirected, so the probability of arriving to an island of any given type is given by the fraction of islands of that type in the economy. An agent located on an island at the beginning of the period can decide whether to stay on the island or become non-employed. If she stays, she works and starts the following period in the same location.

We let L_t be the number of agents engaged in home production at time t, and U_t the fraction engaged in search at time t. The period utility function for the household consuming c_t units of consumption goods and L_t units of leisure is the following:

$$u(c_t, L_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} + \omega L_t.$$

As it is well known, the linearity of leisure in household preferences can represent an economy with indivisible labor and employment lotteries, as in Rogerson (1988). To simplify the description of the planner's problem, we will focus in the case where consumption and leisure are perfect substitutes, which is obtained by setting $\gamma = 0$. In this case we consider home production as an alternative activity that produces ω consumption goods per period, and we let the household's utility function simply be given by

$$E_0 \sum_{t=0}^{\infty} \beta^t c_t.$$

As we explain in Section 4, this assumption is without loss of generality, in the sense that there is a simple mapping between stationary allocations with different values of γ.

Up to this point the environment is a modification of the equilibrium search model of Lucas and Prescott (1974) that introduces household production and undirected search, as in Alvarez and Veracierto (1999). We now introduce a tenure-dependent separation cost. In this section we introduce this separation cost as being a technological feature of the environment. In Section 5 we show how to use the efficient allocation of this economy to construct an equilibrium where the separation cost is a tax levied on firms and rebated to households in a lump-sum way.

The tenure-dependent separation cost works as follows: If an agent has worked for J or more periods in a location, τ consumption goods are lost from the island's production at the time that that worker returns to the central location. If the worker returns to the central location after less than J periods, no separation cost is incurred. In Section 5 we present an equilibrium concept that shows that this tenure-dependent separation cost at the island level captures salient features of the temporary employment contracts used in the real world.

3 Efficient Allocations: A Formal Definition

Since the separation cost depends on tenure levels, a description of an allocation must include the distribution of workers by tenure on each island. We refer to workers with tenure $j = 1, ..., J-1$ in a location as temporary workers and to those with tenure $j \geq J$ as permanent workers. Thus the state of a location is given by its productivity shock z and a J dimensional vector T indicating the number of workers with different tenure levels. In the sequential notation, locations are indexed by their state at time $t = 0$, denoted by $X = T_0$. We use $z^t = (z_0, ..., z_{t-1}, z_t)$ for the history of shocks of length t and index each location at time t by (z^t, X), its history of shocks, and its initial state.

The initial state of the economy is described by a distribution of locations across pairs (z_0, X) and by U_{-1}, the number of agents that searched during $t = -1$. We let $\eta(X|z_0)$ be the fraction of locations with state X conditional on z_0, and $q_0(z)$ the initial distribution of z_0. We assume that q_0 equals the unique invariant distribution associated with the transition function Q. We denote by $q_t(z^t)$ the fraction of islands with history z^t, which by the Law of Large Numbers satisfies

$$q_{t+1}\left(z^t, z_{t+1}\right) = Q\left(z_{t+1}|z_t\right) q_t\left(z^t\right).$$

We indicate the employment of agents with tenure j at a location $(z^t,\ X)$ by $E_{jt}(z^t, X)$, for $j = 0, ..., J,\ z^t \in Z^t$ and $t \geq 0$. Likewise, we denote by $S_{jt}(z^t, X)$ the separations, i.e., the number of agents with tenure j that return to the central location.

Formally, we say that $\{E_{jt}, S_{jt}, U_t, H_t\}$, given η and U_{-1}, is a feasible allocation if the following conditions hold:

i) the island's law of motion

$$E_{j,t}\left(z^t, X\right) = E_{j-1,t-1}\left(z^{t-1}, X\right) - S_{j,t}\left(z^t, X\right),\ j = 1, 2, ..., J-1,$$

$$E_{J,t}\left(z^t, X\right) = E_{J-1,t-1}\left(z^{t-1}, X\right) + E_{J,t-1}\left(z^{t-1}, X\right) - S_{J,t}\left(z^t, X\right),$$

$$E_{0,t}\left(z^t\right) = U_{t-1} - S_{0,t}\left(z^t, X\right),$$

$S_{j,t}(z^t, X) \geq 0$ for $t \geq 0,\ z^t \in Z^t, X \in \text{supp}(\eta)$;

ii) the feasibility constraint for the labor market

$$U_t + \sum_{z^t}\sum_{X}\sum_{j=0}^{J} E_{j,t}\left(z^t, X\right) q_t\left(z^t\right) \eta\left(X|z_0\right) + L_t = N$$

$U_t, H_t \geq 0$ for all $t = 0, 1, ...$;

and iii) the initial conditions

$$E_{j-1,-1} = X_j \text{ for } j = 1, 2, ..., J-1,$$

$$E_{J-1,-1} + E_{J,-1} = X_J,$$

where $E_{0,-1} = U_{-1}$.

The first constraint states that the number of employed workers of tenure $j \leq J-1$ is given by the number of workers of tenure $j-1$ that were employed on the island during the previous period minus the number of these workers taken out of the island during the current period. The second constraint is analogous to the first constraint for workers of tenure J or higher. It differs from the first one because we don't keep track of workers of tenure $j \geq J$ separately (they are all lumped together into tenure J). The third constraint says that the employment of tenure zero workers is given by those just arrived to the island, minus the number of them taken out of the island. The fourth constraint states that the sum of total unemployment, total employment, and agents out of the labor force equals the population N. The fifth and sixth equations define $E_{j,-1}$ in terms of the initial conditions X_j, for $j = 1, ..., J$.

Henceforth we define $T_{j,t}(z^t, X)$ as the number of workers of tenure j available at the beginning of the period t on an island of type (z^t, X), so that

$$T_{j,t}\left(z^t, X\right) = E_{j-1,t-1}\left(z^{t-1}, X\right), \; j = 1, 2, ..., J-1, \tag{1}$$

$$T_{J,t}\left(z^t, X\right) = E_{J-1,t-1}\left(z^{t-1}, X\right) + E_{J,t-1}\left(z^{t-1}, X\right), \tag{2}$$

$$T_{0,t}\left(z^t\right) = U_{t-1} \tag{3}$$

Hence condition i) in the definition of feasibility is equivalent to

$$E_{j,t}\left(z^t, X\right) \leq T_{j,t}\left(z^t, X\right) \text{ for all } j.$$

With these objects at hand we can define a planning problem whose solutions characterize the set of efficient allocations. We say that $\{E_{jt},\ S_{jt},\ T_{j,t},\ U_t,\ L_t\}$ is an efficient allocation if it maximizes

$$\sum_t \beta^t \sum_{z^t} \sum_X F\left(\sum_{j=0}^{J} E_{j,t}\left(z^t, X\right), z_t\right) q_t\left(z^t\right) \eta\left(X|z_0\right)$$
$$+ \sum_t \beta^t \omega L_t - \tau \sum_t \beta^t \sum_{z^t} \sum_X S_{J,t}\left(z^t, X\right) q_t\left(z^t\right) \eta\left(X|z_0\right)$$

over all feasible allocations, given the initial conditions η and U_{-1}.

Given initial conditions η, U_{-1}, a feasible allocation $\{E_{jt},\ S_{jt},\ T_{j,t},\ U_t,\ L_t\}$ is stationary if $U_t,\ L_t$ and the cross sectional distribution η_t are constant over time, where η_t is given by

$$\eta_{t+1}\left(K|z'\right) = \sum_{z^t \in Z^t} \sum_X I_K\left(z^t, X\right) \eta_0\left(X|z_0\right) q_t\left(z^t\right) Q\left(z'|z_t\right),$$

and I_K is an indicator defined as

$$I_K\left(z^t, X\right) = \left\{ \begin{array}{c} 1, \text{ if } \left[T_{1,t}\left(z^t, X\right), ..., T_{J,t}\left(z^t, X\right)\right] \in K \\ 0, \text{ otherwise} \end{array} \right\}$$

for all $z^t \in Z^t$, $X \in \text{supp}(\eta)$, and Borel measurable $K \subset R_+^J$. Finally, we say that $\{L, U, \eta\}$ is a stationary efficient allocation if there is some efficient allocation $\left\{\hat{E}_{jt}, \hat{S}_{jt}, \hat{T}_{j,t}, \hat{U}_t, \hat{L}_t\right\}$ with initial condition $\hat{U}_{-1}$, $\hat{\eta}$ which is stationary and for which

$$\hat{U}_{-1} = \hat{U}_t = U, \quad \hat{L}_t = L, \text{ and } \hat{\eta}_t = \eta$$

for all $t \geq 0$.

4 Characterization of Efficient Stationary Allocations

An efficient allocation is interior if agents are engaged in all three activities: search, home production, and work. Our characterization of interior efficient stationary allocations consists on the solution to two equations in two unknowns: (U, θ), where U is unemployment and θ is the shadow value of being non-employed. One equation states that the shadow value of search equals the expected value of randomly arriving to an island next period, according to the invariant distribution. The second equation ensures that agents are indifferent between doing search and home production. The first equation is quite complex- it involves solving a dynamic programing problem and using the invariant distribution generated by its optimal policies. We refer to this dynamic programing problem as the island planning problem.

The state of this problem is given by (T, z), where T is a vector describing the number of workers across tenure levels $j = 1, 2, ..., J$ at the beginning of the period and z is the current productivity shock. The island planner receives U workers with tenure $j = 0$ every period. The planner decides how many workers to employ at each tenure level and returns workers to the central location at a shadow value given by θ. The planner incurs a cost τ per worker with tenure

J that is returned to the central location. Formally,

$$\begin{aligned} & V(T, z; U, \theta) \\ = & \max_{\{E_j\}} \left\{ F\left(\sum_{j=0}^{J} E_j, z\right) + \theta\left(\left[U - E_0\right] + \sum_{j=1}^{J}\left[T_j - E_j\right]\right) - \tau\left[T_J - E_J\right] \right. \\ & \left. + \beta \sum_{z'} V\left(E_0, E_1, ..., E_{J-2}, E_{J-1} + E_J, z'; U, \theta\right) Q\left(z'|z\right) \right\} \end{aligned}$$

subject to $0 \leq E_j \leq T_j$ for $j = 1, .., J$ and $0 \leq E_0 \leq U$. We let $G(T, z; U, \theta)$ be the optimal employment decision and $T' = A(T, z)$ the implied transition function with $T'_{j+1} = G_j(T, z)$ for $j = 0, ..., J-2$ and $T'_J = G_J(T, z) + G_{J-1}(T, z)$.

It is intuitive to see that if U is the economy-wide efficient unemployment level and θ is the economy-wide shadow value of non-employment, the employment decisions of the island planners' problem recover the economy-wide efficient employment decisions. To see why, notice that each island faces the same value for U, since search is undirected, and the same value of θ, since workers are identical once they leave the island and arrive to the central location.

As stated above, the shadow value of non-employment is equal to the discounted expected value of randomly arriving (with zero tenure) to an island under the invariant distribution. To find the shadow value of workers with tenure zero at each island, we define the problem of an island's planner that faces a flow of unemployed workers equal to $\hat{U}$ for one period but that reverts to the constant flow U thereafter:

$$\begin{aligned} & \hat{V}\left(T, z; \hat{U}, \theta\right) \\ = & \max_{E_j} \left\{ F\left(\sum_{j=0}^{J} E_j, z\right) + \theta\left(\left[\hat{U} - E_0\right] + \sum_{j=1}^{J}\left[T_j - E_j\right]\right) - \tau\left[T_J - E_J\right] \right. \\ & \left. + \beta \sum_{z'} V\left(E_0, E_1, ..., E_{J-2}, E_{J-1} + E_J, z'; U, \theta\right) Q\left(z'|z\right) \right\} \end{aligned} \tag{4}$$

subject to $0 \leq E_j \leq T_j$ for $j = 1, .., J$ and $E_0 \leq \hat{U}$. Using this problem we define the value of an extra zero-tenure worker in a location with state (T, z) as:

$$\lambda(T, z; U, \theta) = \frac{\partial \hat{V}\left(T, z; \hat{U}, \theta\right)}{\partial \hat{U}}|_{\hat{U}=U} \tag{5}$$

where $\partial \hat{V} / \partial \hat{U}$ is a subgradient of $\hat{V}$ (in the case it is not differentiable). The next theorem gives a characterization of the stationary efficient allocations.

Theorem 1 . *Let (U, θ) be an arbitrary pair. Let $V(\cdot; U, \theta)$ be the solution of the island planning problem, and let $G(\cdot; U, \theta)$, $\lambda(\cdot; U, \theta)$ be the the associated optimal policies and shadow value for zero-tenure workers, respectively. Suppose that:*

i) $\mu(\cdot; U, \theta)$ is a stationary distribution for the process (T, z) with transition functions given by $Q(z'|z)$ for z' and $A(T, z)$ for T';

ii) the value of search σ is given by

$$\sigma = \beta \int \lambda(T, z\ ; U, \theta)\, \mu(dT \times dz\ ; U, \theta);$$

iii) the number of agents engaged in home production L satisfies

$$L = N - U - \int \left[\sum_{j=0}^{J} G_j(T, z\ ; U, \theta) \right] \mu(dT \times dz\ ; U, \theta) \geq 0;$$

iv) the labor force participation decisions are optimal, in the sense that

$$\begin{aligned} \theta &= \max\{\sigma,\ \omega + \beta\theta\}, \\ 0 &= L\ [\theta - \omega - \beta\theta]. \end{aligned}$$

Finally, define $\eta(T, z) = \mu(T|z)$ as the distribution of T conditional on z. Then $\{L, U, \eta\}$ is an efficient stationary allocation.

Conditions (i) and (ii) have been explained above. Condition (iii) defines the number of agents doing home production as total population minus the sum of unemployment and employment and states that home production must be nonnegative. The first equation in condition (iv) states that the value of non-employment must be the best of two alternatives: the value of search, which is σ, and the value of doing home production during the current period and being non-employed the following period, which is $\omega + \beta\theta$. The second equation in condition (iv) is a complementary slackness condition for home production.

Theorem 1 implies that characterizing efficient stationary allocations is reduced to solving two equations in two unknowns and checking that an inequality is satisfied. Given an arbitrary pair (U, θ), the functions $V(\cdot, U, \theta)$, $G(\cdot, U, \theta)$, $\lambda(\cdot, U, \theta)$, and the distribution $\mu(\cdot, U, \theta)$ can be found using standard recursive techniques. Defining $\sigma(U,\theta)$ and $L(U,\theta)$ as the left-hand sides of conditions (ii) and (iii), respectively, the two equations that U and θ must satisfy are:

$$\begin{aligned} \theta &= \max\left\{\sigma\left(U,\theta\right), \omega+\beta\theta\right\} \\ 0 &= L\left(U,\theta\right)\left[\theta-\omega-\beta\theta\right]. \end{aligned}$$

and the inequality that must be satisfied is that $L(U,\theta) \geq 0$. A consequence of this simple characterization is that Theorem 1 can be used for constructing a computational algorithm and establishing the existence and uniqueness of a stationary efficient allocation.

The households' optimality condition when $\gamma > 0$ and the allocation is interior (i.e., one with a strictly positive amount of time dedicated to leisure and a strictly positive amount of search) is to equate the marginal rate of substitution with the flow value of search:

$$\frac{\omega}{u'(c)} = (1-\beta)\,\sigma, \tag{6}$$

where c is aggregate consumption. In such interior equilibrium the value of search is equal to the value of non-employment (i.e., $\sigma = \theta$).

4.1 Island Planning Problem

We now turn to the analysis of the island planning problem, which is at the center of our characterization. We start by analyzing the derivatives of V, which can be shown to be differentiable. The standard proof by Benveniste and Scheikman does not apply because the optimal choice of E is not interior. In Appendix A we construct an alternative proof and find expressions for the derivatives of V. Intuitively, the marginal value of an additional worker of tenure j is given by the sum of two terms. The first term is the expected discounted sum of the marginal product of labor over periods in which no worker of the same cohort has ever been sent back to the central location. The second term is the expected discounted shadow value (net of any separation costs) the first time that a worker of the same cohort is sent back to the central location. Formally, for $T_j > 0$, $\partial V(T,z)/\partial T_j = V_j^*(T,z)$, where V_j^* is defined as follows. Denote the current date by 0 and define the stopping time n_j as the first date s at which the number of workers with current tenure j is reduced. We let $E_{i,s}^*$ be the optimal employment level s periods from now of workers with tenure level i, and $T_{i,s}$ be the begining-of-period number of workers s periods from now with tenure level i, so that

$$n_j = \text{ first date } s \text{ at which } E^*_{\min\{J,\ j+s\},s} < T_{\min\{J,\ j+s\},s}.$$

Now we are ready to define $V^*_j(T,z)$ as:

$$\begin{aligned} V^*_j(T,z) &= \sum_{s=0}^{\infty} \beta^s E_0 \left[f\left(\sum_{i=0}^{J} E^*_{i,s}, z_s \right) | n_j > s \right] + \\ & \quad E_0 \left[\beta^{n_j}\ \theta \right] - E_0 \left[\beta^{n_j}\ \tau\ | n_j \geq J \right] \end{aligned} \tag{7}$$

This implies that if some workers of tenure j are sent back, i.e., if $E_j = G_j(T,z) < T_j$, then the marginal value of all workers of this tenure level is $V^*_j(T,z) = \theta$ for $j \leq J-1$ and is equal to $\theta - \tau$ for $j = J$.

In Appendix A we show the following three properties of the solution to the island planning problem.

First, we show that if $T_j > 0$, then $\partial V(T,z)/\partial T_j \geq \theta$ for $j \leq J$ and $\geq \theta - \tau$ for $j = J$, since the planner always has the option of sending workers back to the central location (Propositions 10 and 13).

Second, it is easy to see that if some permanent workers are fired, i.e., if $E_J = G_J(T,z) < T_J$, then all the temporary workers must have been fired, i.e., $E_j = G_j(T,z) = 0$ for all $j = 0,...,J-1$. A policy with this property saves on the separation cost τ, which is incurred only by the permanent workers (Proposition 11).

Third, the first workers to be fired are the temporary workers with the longest tenure (Proposition 5). The intuition for this property is that, while all workers are perfect substitutes in production, these workers are the closest to becoming subject to the separation cost τ. Thus this policy saves on potential separation costs. In an economy where all island planners have followed this policy in the past and a constant flow U of tenure $j = 0$ workers has arrived to each island every period, the states T in the ergodic set take a particular form. Formally, the ergodic set is a subset of $\mathcal{E}$, which is given by

$$\mathcal{E} = \left\{ T \in [0,U]^{J-1} \times R_+ : T = (U,...,U,T_j,0,...,0,T_J)\,, \text{ for some } j:\ 1 \leq j \leq J-1 \right\} \tag{8}$$

This property is extremely important: It allows us to reduce the dimensionality of the endogenous state of the island planning problem from J to 2. The next section analyzes the resulting *simplified island planning problem.*

4.2 Simplified Island Planning Problem

States for the island planning problem T that belong to $\mathcal{E}$ can be described by two numbers: t, the total number of temporary workers (workers with tenure less or equal to J), and p, the number of permanent workers (workers with tenure greater than J). We use this feature to consider the island planning problem with a simplified state (t, p, z). In this simplified problem, the choices are employment of temporary workers e_t and employment of permanent workers e_p. The law of motion for the endogenous state is:

$$t' = U + e_t - \max\{e_t - (J-1)U,\ 0\} \text{ and } p' = e_p + \max\{e_t - (J-1)U,\ 0\} \tag{9}$$

The number of temporary workers next period t' is equal to the number of temporary workers employed during the current period e_t plus the arrival of new workers U, minus the temporary workers that will become permanent next period, $\max\{e_t - (J-1)U,\ 0\}$. Likewise, the number of permanent workers next period p' is equal to the number of permanent workers employed during the current period e_p plus the temporary workers that will become permanent next period. The island planner's value function $v : [U, J \cdot U] \times R_+ \times Z \to R$ satisfies the following Bellman equation:

$$\begin{aligned} v(t, p, z) = \max_{e_t, e_p, t', p'} \quad & \{F(e_t + e_p, z) + \theta[t - e_t] + (\theta - \tau)[p - e_p] \\ & + \beta \int v(t',\ p', z') Q(z, dz') \Big\} \end{aligned}$$

subject to

$$0 \leq e_t \leq t, \quad 0 \leq e_p \leq p,$$

and the law of motion (9).

Formally, v is related to V for states $T \in \mathcal{E}$ as follows:

$$v(T_1 + T_2 + \ldots + T_{J-1}, T_J, z) = V(T_1, T_2, \ldots, T_{J-1}, T_J, z).$$

Since v and V are closely related and V is concave, then v is concave in (t, p), even though the graph of the feasible set for this problem is not convex. From the definition of v and the properties of V, we have that v is differentiable with respect to t for all $t > 0$ that are not integer multiples of U, and differentiable with respect to p for all $p > 0$. Thus, for all (t, p, z) with $p > 0$

$$\frac{\partial v(t, p, z)}{\partial p} = \frac{\partial V(T, z)}{\partial T_J}$$

and for all t that can be written as $t = (j-1)\,U + T_j$ with $T_j \in (0, U)$,

$$\frac{\partial v\,(t,p,z)}{\partial t} = \frac{\partial V\,(T,z)}{\partial T_j}.$$

At the points t given by $t = j \times U$ for some $j = 1, ..., J-2$, the right derivative of v with respect to t is $\partial V / \partial T_j$, and its left derivative is $\partial V / \partial T_{j+1}$.

The main result of this section is a characterization of the optimal policy. The optimal policy is defined by a two-dimensional set of inaction $I(z)$. For each z, the optimal policy $(e_t\,(t,p,z)\,, e_p\,(t,p,z))$ is to stay in the set of inaction $I\,(z)$ or otherwise move to its boundary, as explained below. The boundary of the set of inaction is described by two continuous functions, $\hat{p}$ and $\hat{t}$ defined in $\hat{p} : Z \to R_+$ and $\hat{t} : R_+ \times Z \to [0,\ J \cdot U]$. The function $\hat{t}$ is decreasing in p and hits zero at a value of $p \leq \hat{p}\,(z)$. The function $\hat{t}$ is the boundary of the set of inaction for the values t that are strictly positive. Formally, these functions define the set of inaction $I\,(z)$ as follows:

Definition 2 *For each $z \in Z$,*

$$I\,(z) = \left\{ (t,p) \in [0, J \cdot U] \times R_+ : p \leq \hat{p}\,(z)\,,\ \textit{and}\ t \leq \hat{t}\,(p,z)\ \right\} \tag{10}$$

The optimal policy is as follows: if $p \leq \hat{p}\,(z)$ and the state is outside the set of inaction $I\,(z)$, temporary workers are fired until the boundary of $I\,(z)$ is hit, with no change in permanent workers. If $p > \hat{p}\,(z)$, all temporary workers are fired, and permanent workers are fired to hit $\hat{p}\,(z)$. Formally,

$$\begin{aligned} e_t\,(t,p,z) &= \min \left\{ t,\ \hat{t}\,(p,z) \right\}, \\ e_p\,(t,p,z) &= \min \{ p,\ \hat{p}\,(z) \} \end{aligned}$$

Figure 2 illustrates the typical shape of the set of Inaction (for a given value of z).

The threshold $\hat{p}\,(z)$ solves

$$\theta - \tau = f\,(\hat{p}\,(z)\,, z) + \beta \int \frac{\partial v}{\partial p}\,(U, \hat{p}\,(z))\,Q\,(z, dz')$$

That is, $\hat{p}$ is the lowest number of permanent workers for which the marginal value of a permanent worker is equal to $\theta - \tau$ and, thus, any additional permanent worker would be returned to the central location.

Figure 2
Optimal Decision Rule for Employment

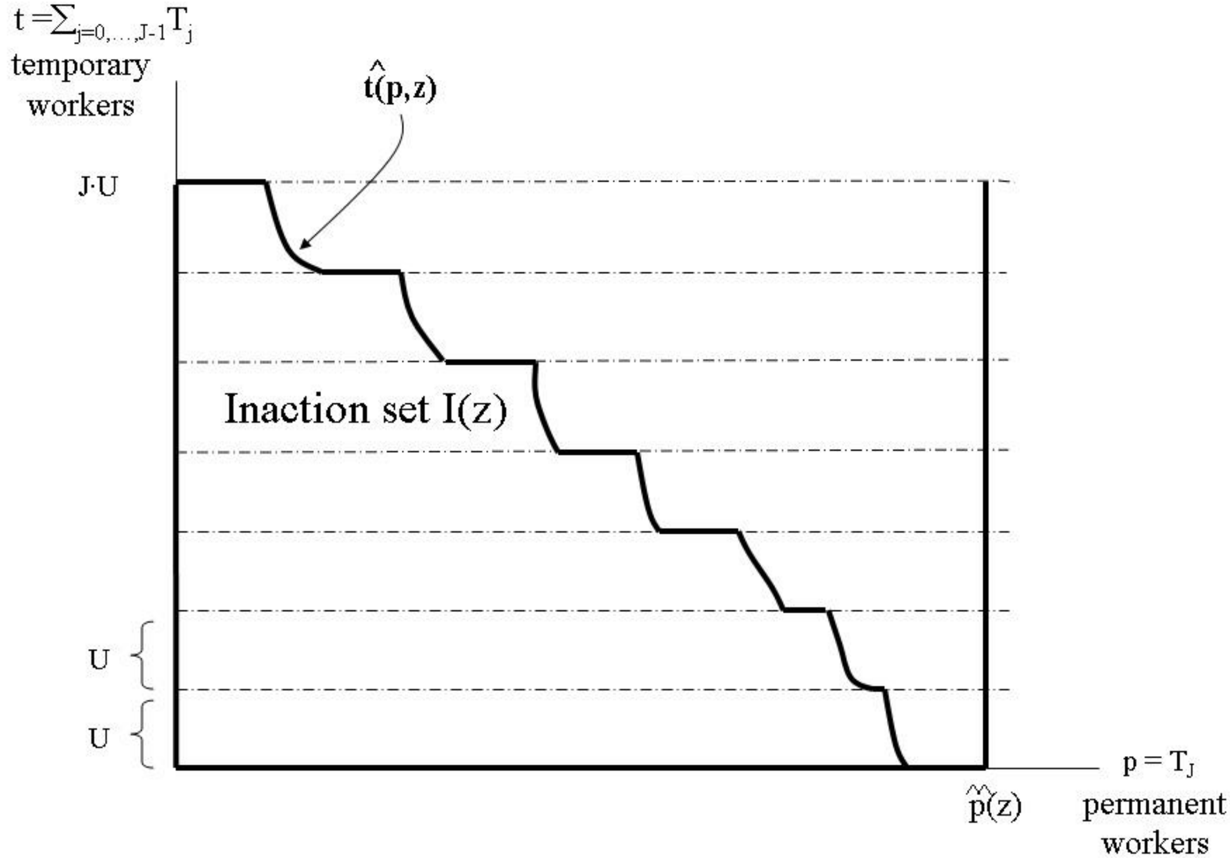


Figure 2:

Given (p, z), the function $\hat{t}(p, z)$ is defined as the lowest number of temporary workers t for which the marginal value of a temporary worker is equal to θ and, thus, any additional temporary worker would be returned to the central location. The function $\hat{t}(p, z)$ solves

$$\theta = f\left(\hat{t}(p, z) + p, z\right) + \beta \int \frac{\partial v}{\partial t}\left(\hat{t}(p, z) + U,\ p\right) Q(z, dz')$$

for $\hat{t}(p, z) \leq (J-1)U$ and

$$\theta = f\left(\hat{t}(p, z) + p, z\right) + \beta \int \frac{\partial v}{\partial p}\left(JU,\ p + \hat{t}(p, z) - (J-1)U\ \right) Q(z, dz')$$

for $\hat{t}(p, z) \in ((J-1)U, JU]$. To simplify the exposition we have written these expressions assuming that v is differentiable. If v is evaluated at integers multiples of U (where v is not differentiable), the expressions would have to be rewritten in terms of the subgradients of v.

The intuition for why the frontier of the set of inaction, given by $\hat{t}$, is decreasing in p is that temporary and permanent workers are perfect substitutes in production. Indeed, it can be shown that $\hat{t}$ is strictly decreasing for values of p such that $\hat{t}(p, z)$ is not an integer multiple of U. At the points on which $\hat{t}$ is an integer multiple of U, this function can be flat: At these points the function v may not be differentiable, as explained above. While all these properties are quite intuitive, the proofs are involved because of the non-differentiability of v. Appendix B provides a formal derivation of these results.

5 Competitive Equilibrium

The representative household has a continuum of members that share their employment risks. Given the perfect consumption pooling at the household level, each household member seeks to maximize her own expected discounted earnings regardless of risk.[3]

In what follows it will be useful to think of each island as a separate economy and define an "island-level equilibrium," taking as given the flow of new workers to the island, U, and the value to a worker of leaving the island, θ. The resulting equilibrium decision rules at the island level define an invariant distribution across island states, which together with U and θ must

[3] An equivalent specification would be to identify each household with one individual and introduce complete markets and employment/search lotteries, as in Prescott and Rios-Rull (1992). However, this alternative would require introducing additional notation.

satisfy certain conditions to constitute an "economy-wide equilibrium." These conditions are that the consumption market clears, the labor market clears, and the marginal rate of substitution between consumption and leisure equals the flow value of search.

In this section we describe two notions of island-level equilibrium: one with spot labor markets (SLM) and one with multiperiod employment relations (MER). The SLM specification has two advantages: 1) It makes a straightforward connection with the island planning problem, and 2) it is closely related to the standard McCall search model (on the workers' side) and to the Bentolila and Bertola firing costs model (on the firms' side). The advantage of the MER specification is that it captures the nature of fixed-term employment contracts in a much more realistic way.

Appendix C gives a formal definition of a SLM equilibrium, relates it to the island planning problem of Section 3 and characterizes the behavior of equilibrium wages. Appendix D provides a similar analysis for a MER equilibrium.

In this section and most of appendices D and E we assume that $\gamma = 0$ (so that consumption and leisure are perfect substitutes) and that the separation cost τ is a feature of the technology (as opposed to a tax rebated lump-sum to households). The first assumption is used to simplify the description of an equilibrium. The second is used to show that the first and second welfare theorems hold. The general case in which $\gamma \geq 0$ and τ is a tax, together with an algorithm for its computation, is described at the end of Appendix C.

5.1 Spot Labor Markets

In an SLM equilibrium, firms and workers participate in competitive labor markets on each island. Wages are indexed by j, the workers' tenure on the island and by (T, z), the island-wide state. As in the previous sections, a permanent worker is defined as having tenure $j \geq J$ on the island. Whenever a firm decreases its employment of permanent workers, it must pay a separation cost τ per unit reduction. Notice that it is the tenure at the island level, as opposed to the tenure at the firm level, that determines if a worker separation is subject to the separation cost τ. This unrealistic assumption affords tractability by allowing a descentralization with spot labor markets. The reason is that, since the separation costs are at the island level, workers are not tied to the firms that hire them.

In an SLM equilibrium, a worker located on an island at the begining of the period solves a very simple problem: whether to stay on the island, receive the wage rate corresponding to her tenure level and start the following period with a higher tenure level, or to leave, in which case she can either search or engage in home production. This problem also applies to workers who have just arrived at an island (i.e., with zero-tenure level). Moreover, in equilibrium searchers contact an island in proportion to its frequency in the invariant distribution. Hence, the problem for searchers is the same as in the classical McCall search model, except that wages depend on tenure levels and the wage distribution is endogenous.

The problem for a firm in an SLM is similar to the one studied by Bertola and Bentolila (1990): Firms take the stochastic process for wages as given, behave as if they could hire and fire any number of workers at these wages, and are subject to a cost τ per reduction of employment. One difference is that in our formulation the separtion cost τ applies only to workers with tenure $j \geq J$ on the island, as opposed to all workers. Another difference is that the process for wages at the island level is endogenously determined: The process for the island-level equilibrium wages must be such that the demand for labor equals its supply at each tenure level and island-wide state.

Since workers and firms take competitive wages as given, the equilibrium pattern of wages across tenure levels must induce firms and workers to follow the employment adjustments described in Figure 2. Proposition 36 in Appendix C shows that there are three equilibrium levels of wages in a given location: one level for temporary workers with tenures $j = 0, ..., J-2$, a second level for workers that are about to become permanent, i.e., those with tenure $J-1$, and a third level for permanent workers, i.e., those with tenure J or higher. Temporary workers with tenures $j = 0$ to $j = J-2$ are paid their marginal productivity. Wages of workers with tenure $J-1$, i.e., those that would become permanent if they were to work during the current period, are (weakly) smaller than their marginal productivity. Wages of permanent workers are (weakly) higher than those with tenure $J-1$. This wage pattern is consistent with the Spanish evidence as described in Aguirregabiria and Alonso-Borrego (2004). Their regressions indicate that the wages of permanent workers are higher than the wages of temporary workers even after controling for firm-specific experience, age, sex, education, and industry effects (Table 5 in Aguirregabiria and Borrego, 2004, reports this difference to be 9.1 percent).

5.2 Multiperiod Employment Relations

In the previous sections the separation cost τ applied to workers with tenure $j \geq J$ at the *island level,* which allowed for a simple competitive structure with spot labor markets. In this section we introduce an alternative and more realistic definition of a competitive equilibrium, where separation costs are determined by the tenure of workers at the *firm level.* This specification ties workers with firms and, hence, requires long-term contracts to achieve efficiency. In fact, we will argue that the competitive equilibrium with long-term contracts and tenure at the firm level supports the same equilibrium allocation as the spot labor market concept of the previous section. This is an important result: There is no loss of realism in specifying that the relevant tenure for temporary contracts is at the island level rather than at the firm level.

To obtain this equivalence result certain restrictions on the type of temporary contracts allowed for are needed. However, this is not a weakness of the model: These restrictions resemble those observed in actual countries. Indeed, since temporary contracts are often introduced with the purpose of increasing flows out of unemployment, their implementation typically includes eligibility clauses. An example is the Spanish reform of 1984, which significantly broadened the scope of fixed-term contracts but specified that workers had to be registered as unemployed to be eligible for temporary employment contracts (see the Appendix in Cabrales and Hopenhayn, 1997). In Portugal temporary contracts could only be used by new firms, or by firms hiring long-term unemployed or first-time job seekers (see Table 1 in Dolado et al., 2001). Another example is the recently approved and later withdrawn CPE ("first employment contract") legislation in France. This type of contract would have allowed an employer to dismiss a worker younger than 25 during the first two years of the contract, provided that he had never been employed at the time of his hiring

To incorporate this type of elegibility restriction we assume that only workers that searched during the previous period (i.e., that were unemployed) can be hired under temporary contracts. If a firm hires a worker that was employed somewhere else on the island during the previous period, the worker immediately becomes subject to regular firing taxes. In this scenario, the market structure would have to be changed to accommodate the fact that workers would try to exploit the bargaining power that they would gain by staying in a same firm. To avoid this, we assume that firms and workers participate in island-wide competitive markets for binding, long-term, state-contingent, wage contracts at the time of the hiring. Below we offer an informal description of the equilibrium using long-term contracts. Appendix D provides a formal treatment.

In this decentralization, firms and workers trade state contingent contracts in competitive labor markets, specifying the periods of time that the worker will supply labor to the firm as a function of the sequence of productivity shocks z^t. Since employment must be continuous over time, each contingent contract is effectively reduced to a stopping time specifying the time of separation. When the realized sequence of productivity shocks triggers a separation, the worker can choose to offer a new stopping time to the market or to leave the island and receive the outside value θ. Each stopping time has its own price, which is taken as given by firms and workers.

There are two types of workers on the island: "incumbent" workers and "newly arrived" workers. An "incumbent" is a worker that has been previously employed by some firm on the island. A "newly arrived" worker is a worker that has just arrived at the island for the first time. The stopping times sold by different types of workers differ in terms of the separation costs involved. In particular, the stopping times sold by "newly arrived" workers are subject to the separation cost τ only if the separation occurs after J periods (the length of the trial periods in the fixed-term contracts). On the contrary, the stopping times sold by "incumbents" are always subject to the separation cost τ. Since the stopping times sold by the different types of workers are different commodities, they have, in general, different prices. Intuitively, a stopping time sold by an "incumbent" worker will have a lower price than the same stopping time sold by a "newly arrived" worker to compensate firms for the potentially higher separation costs.

Taking prices as given, firms decide how many stopping times of each type to purchase from the different types of workers. Their objective is to maximize the expected present value of their profits, net of separation costs.

Despite the unusual commodities traded and the indivisibility in the supply of contracts, the competitive equilibrium considered is standard and, hence, the welfare theorems hold. The equilibrium allocation can then be characterized as the solution to a social planner's problem. In this problem, the planner chooses stopping times for "incumbents" and "newly arrived" workers taking into account that the separation cost τ applies to "incumbent" workers in every separation, but that it applies to "newly arrived" workers only in separations that take place after J periods of employment.

A brief analysis of the planner's problem will help us to understand the equivalence between this type of equilibrium and the SLM equilibrium of the previous section. To start with, note that the social planner will never want to separate a "newly arrived" worker and rehire him as an "incumbent" before the trial period for the fixed-term contracts is over. The reason is that being rehired as an "incumbent" makes the worker liable to separation costs, while maintaining his "newly arrived" status saves on separation costs during the trial period. Also, the social planner will never want to separate a "newly arrived" worker after the trial period is over and rehire him under an "incumbent" contract because this entails incurring the separation cost τ without any benefit. As a consequence, the planner will choose the stopping times for "newly arrived" workers in such a way that they separate only to leave the island (and receive the value θ). This means that the social planner will never use "incumbent" workers. Being left with only "newly arrived" workers, the planner's problem is formally identical to the island planning problem described in Section 4. This has an important implication: The allocation obtained in the MER equilibrium (with tenure at the firm level) is identical to the one obtained in the SLM equilibrium (with tenure at the island level). Moreover, the price of a stopping time sold by a "newly arrived" worker in the MER equilbrium must be equal to the expected discounted value of the spot wages obtained by a "newly arrived" worker in the SLM equilibrium.

6 Computational Experiments

In this section we evaluate to what extent the introduction of temporary contracts adds flexibility to the labor market. To this end we consider as a benchmark the case where $J = 1$ and $\tau > 0$ and calibrate it to an economy with high separation taxes and no temporary contracts similar to the Spanish economy prior to the 1984 reform.[4] Once the benchmark economy (hereafter referred to as the "firing-tax" case) is parameterized , we compute competitive equilibriums under temporary employment contracts of different lengths (i.e., with different values for J) and evaluate their effects.

For comparison purposes we also compute the equilibrium allocation under zero separation taxes, which we refer to as the "laissez-faire" case. This is an interesting case to consider because

[4] When $J = 1$, the dismissal of anyone that has worked, even for one period, triggers the separation tax $\tau > 0$. Thus, there are no temporary workers in this case.

the equilibrium allocation with temporary employment contracts of long duration coincides with the equilibrium allocation under laissez-faire. The reason is quite simple: With a large enough J, firms can perfectly replicate their laissez-faire employment levels by using only temporary workers. Given this property, we address the question of how much flexibility the temporary contracts generate by computing what fraction of the gap between the firing-tax and laissez-faire cases is closed when temporary contracts of different length J are introduced.

We note that in the laissez-faire case, which is obtained by setting $\tau = 0$, the value of J and the tenure levels of different workers are immaterial because temporary and permanent workers become perfect substitutes. This implies that, while total employment is uniquely determined, the hiring and firing rates across the different tenure levels are undetermined. Despite this, we choose to focus on the employment adjustments obtained as the limit when $\tau \to 0$ (or equivalently, when τ is arbitrarily small). This is useful because it helps emphasize the types of adjustments that temporary contracts lead to even in the case in which they are totally unimportant.

Calibration

We calibrate our model to the Spanish economy prior to the 1984 reform, an economy with high separation costs and no temporary contracts (i.e., with high τ and $J = 1$). The value for τ is selected to reproduce the expected discounted dismissal cost when a worker is hired for the first time, a measure proposed by Heckman and Pages-Serra (2000). Appendix E shows that a value of τ equal to one year of average wages is needed to reproduce this measure under the pre-1984 Spanish regime.

We use $\alpha = 0.64$ for the curvature parameter in the production function $F(E, z) = zE^{\alpha}$, which roughly corresponds to the labor share. This choice implicitely assumes that all other factors, such as capital, are fixed across locations. Since we use a quarterly time period, we choose $\beta = 0.96$ to generate an annual interest rate of 4 percent.

For the idiosyncratic shocks z, we use a discrete Markov chain approximation to the following AR(1) process:

$$\log z' = \rho \log z + \sigma\varepsilon,$$

where ε is a standard normal. We choose the values of ρ and σ so that the unemployment rate is just above 6.75 percent and the duration of unemployment is just above one year. The exact values that we use are $\rho = 0.955$ and $\sigma^2 = 0.075$, which correspond to a discrete approximation that uses six truncated values for z, so that the absolute value of ε never exceeds two standard deviations. The quarterly firing rate (total separations divided by employment) is 1.77 percent (Garcia-Fontes and Hopenhayn, 1996, estimate a firing rate of 1.84 percent per quarter for the years 1978-1984). Our choices are meant to capture the situation in Spain before the 1984 reform. The reason why we chose a lower unemployment rate and a lower duration of unemployment than those observed in Spain is that we are abstracting from the unemployment insurance system.[5]

We consider different values of γ. In each case we pick the value of ω so that labor force participation equals 65 percent for the benchmark case.[6] The rest of the parameters are the same for each pair (γ, ω).

Experiments

We compute equilibriums under different values of J, the length of the temporary contracts, and compare them with the benchmark and laissez-faire cases. Since these two cases correspond to $J = 1$ and $J = \infty$, respectively, these comparisons allow us to determine what fraction of the total potential gains in labor market flexibility is realized by different temporary contracts lengths.

When F is Cobb-Douglass and τ is proportional to average wages, a number of statistics become independent of the intertemporal substitution parameter $1/\gamma$.[7] In particular, the unemployment rate, the average duration of unemployment, and the firing rates are the same in all cases. For this reason, we start by describing the effects of temporary contracts on this set of statistics. Without loss of generality we set $\gamma = 0$. This is the simplest case to interpret because consumption and leisure become perfect substitutes and, as a consequence, the equilibrium value of θ must be equal to $\omega/(1-\beta)$, a parameter independent of policy.

In Figures 3-5, equilibrium values are reported as a function of the length of the temporary

[5] In Alvarez and Veracierto (1999) we analyzed the effects of introducing unemployment insurance benefits into the model with firing taxes. Introducing UI benefits of the magnitude of those in Spain increases the unemployment rate by more than 10% and more than doubles its average duration (see section "UI benefits, firing subsidies, firing taxes and severance payments" and Table 5 of Alvarez and Veracierto, 1999).

[6] The different combinations of (γ, ω) are: $(0,\ 1.3047)$, $(1/2,\ 1.0739)$, $(1,\ 0.883)$ and $(8,\ 0.058)$. With $\gamma = 0$, there are no income effects, since preferences are linear. With $\gamma = 1$, income and substitution effects of a permanent increase in wages cancel. With $\gamma = 8$, the income effect is much higher, so that the uncompensated labor supply elasticity is lower, similar to the values estimated by Nickell (1997).

[7] Henceforth, all experiments will take τ as being proportional to average wages.

contracts J and depicted under the "general equilibrium" label. Observe that the "general equilibrium" values for $J = 1$ correspond to the benchmark case with firing taxes and no temporary contracts. Laissez-faire values are reported under the "laissez-faire" label. In addition, to illustrate the role of general equilibrium effects in generating differences between the benchmark $J = 1$ and the laissez-faire cases, a third set of values is reported under the "partial equilibrium" label. For each $J > 1$, these are the values associated with the solution to the island planner's problem when the U and θ that the planner takes as given are the ones from the benchmark case. Observe that any differences between the "partial" and "general" equilibrium schedules must be due to equilibrium effects on U, since θ is fixed when $\gamma = 0$. Also observe that U will always be higher in the "general equilibrium" case than in the "partial equilibrium" case. The reason for this is that with $J > 1$ there are fewer restrictions to labor mobility. This increases the shadow value of an additional worker at every island and induces a larger fraction of the population to search. For similar reasons, the equilibrium value of U will always be increasing with J. A consequence of this is that for $1 < J < \infty$, the equilibrium value of U will always lie between the benchmark and laissez-faire cases.

FIGURE 3: Unemployment Rate

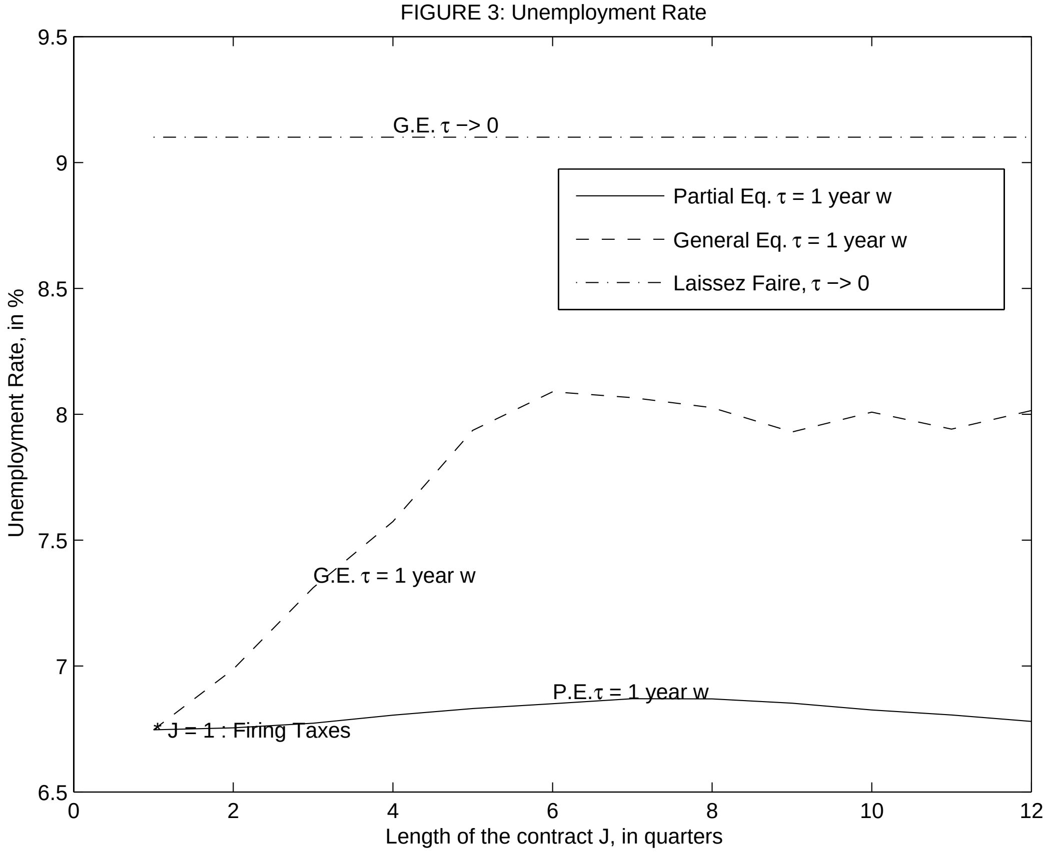


Figure 3 shows the effects on the unemployment rate $ur = U/(U+E)$. We see that the unemployment rate increases with the length of the temporary contracts J and is almost 2.5 percent higher in the laissez-faire case than in the benchmark $J = 1$ case. With temporary contracts of three years duration ($J = 12$), the unemployment rate is 1.3 percent points higher than in the benchmark case. Temporary contracts of this length, which are similar to those introduced by the 1984 Spanish reform, are thus able to close about half of the gap with the laissez-faire case.[8] Figure 3 also shows that the equilibrium effects on U are crucial for generating the higher unemployment rates: The effects on the unemployment rate are non-monotonic and small in the partial equilibrium case.[9]

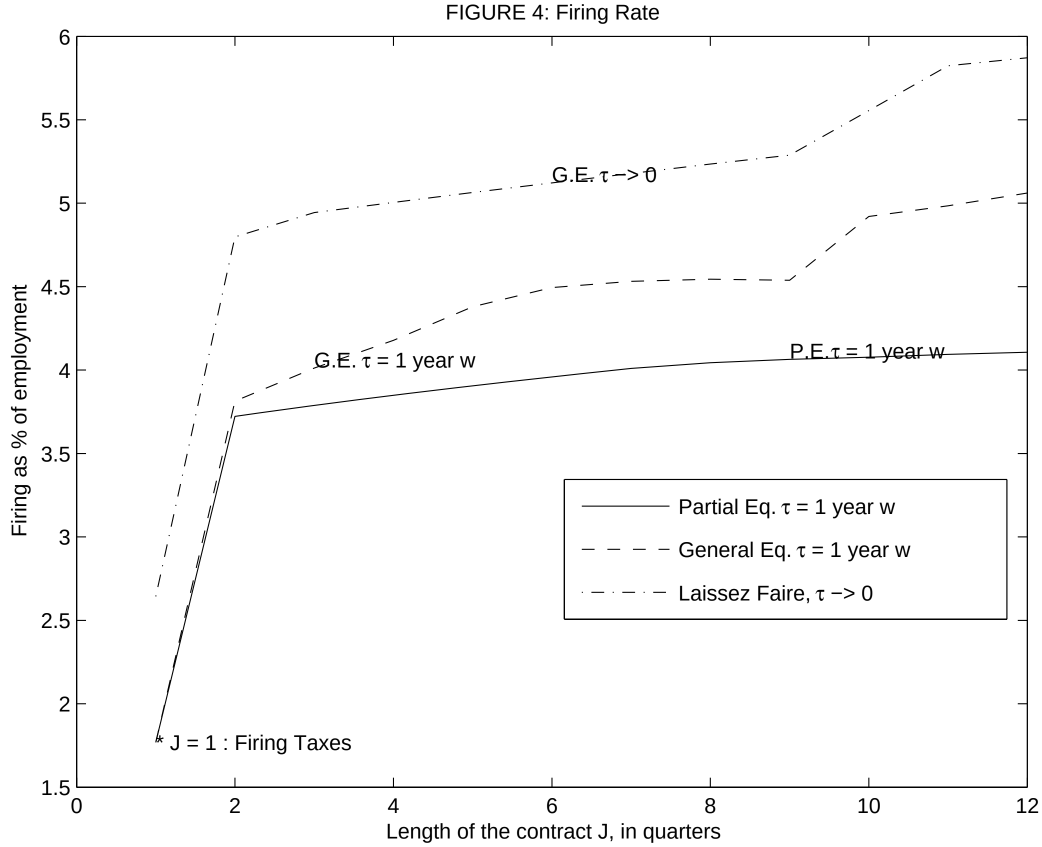
To better understand the effects on the unemployment rate (Figure 3) it is helpful to decompose them into firing rate effects (Figure 4) and average duration of unemployment effects (Figure 5). Figure 4 shows the effects on the firing rate fr, defined as total firing over total employment. Recall that for the laissez-faire and the partial equilibrium cases, the values of U and θ are the same across all J. As it should be expected, the firing rates for laissez-faire are higher than the ones for the partial equilibrium case for all values of J. Notice that the firing rates in these two cases are increasing in J, with a large jump at $J = 2$. To understand this pattern we concentrate on the laissez-faire case where the employment on each island stays constant. Recall that we compute employment by tenure in the laissez-faire case as the limit for an equilibrium with $\tau \rightarrow 0$. The increase in the firing rate helps to avoid the (arbitrarily small) separation tax. The firing rate jumps between $J = 1$ and $J = 2$ because, with $J = 2$, the temporary workers with longest tenure are fired and replaced by newly arrived workers. This reshuffling cannot be done with $J = 1$. The smooth increase in the firing rate with J is due to the fact that with higher J, firms can accumulate a larger proportion of their work force as temporary workers. With this larger proportion, if they need to decrease total employment they can do so at the same time that they hire newly arrived

[8] In the data the relationship between unemployment and temporary contracts is not as clear. However, Dolado et al. (2001) survey the literature and conclude that the introduction of temporary contracts in Spain had a "neutral or slightly positive effect on unemployment."

[9] These partial equilibrium effects are consistent with previous findings in the literature. The island planner problem is similar to the standard problem of a firm facing firing costs (except that its hiring is bounded above by the arrival of new workers U) and we know at least since Bentolila and Bertola (1990) that the effects of firing costs on average employment are ambiguous in that setting.

workers. Notice that the pattern of firing rates as a function of J for the partial equilibrium case where the separation costs are substantial (one year of average wages) is the same as in the laissez-faire case, with essentially zero firing taxes.

FIGURE 4: Firing Rate



The value for the firing rate in the general equilibrium case lies in between the value for the partial equilibrium case and the one for the laissez-faire case, and it gets closer to the one for the laissez-faire case as J increases. Since in general equilibrium firms receive a higher flow of newly arrived workers (i.e., a higher U), they can engage more in the replacement of temporary workers with high tenure for newly arrived workers to save on separation costs.

The quarterly firing rate for the general equilibrium case goes from 1.77 percent for $J = 1$ to 5.1 percent for $J = 12$, roughly similar to the values for Spain before and after 1984: Garcia-Fontes and Hopenhayn (1996) estimate quarterly firing rates of 1.84 percent during the six years prior to the extension of temporary contracts and 4.8 percent for the six years after. The model slightly overestimate these effects, since comparing the effect in the model for $J = 1$ with $J = 12$ does not correspond exactly to Spain before and after 1984 — before 1984 some temporary contracts were allowed, as we explain below.

Figure 5 shows the average duration of unemployment d, defined as $(1/fr)\ ur\ /\ (1-ur)$. The three cases display similar values. There is a large drop in the average duration between the benchmark case and $J=2$. This is the result of the increase in hiring of newly arrived workers, as explained in the case of Figure 4. Since d is similar for the three cases, the effects on unemployment are accounted by the behavior of firing rates discussed above. Notice that, as opposed to the jumps at $J=2$ for the firing rate and average duration of unemployment, the increase in the unemployment rate for the general equilibrium is smooth (compare Figure 3 with Figures 4 and 5). This is because for $J=2$, the sharp decrease in the average duration of unemployment coincides with a sharp increase in the firing rate.

FIGURE 5: Average Duration of Unemployment

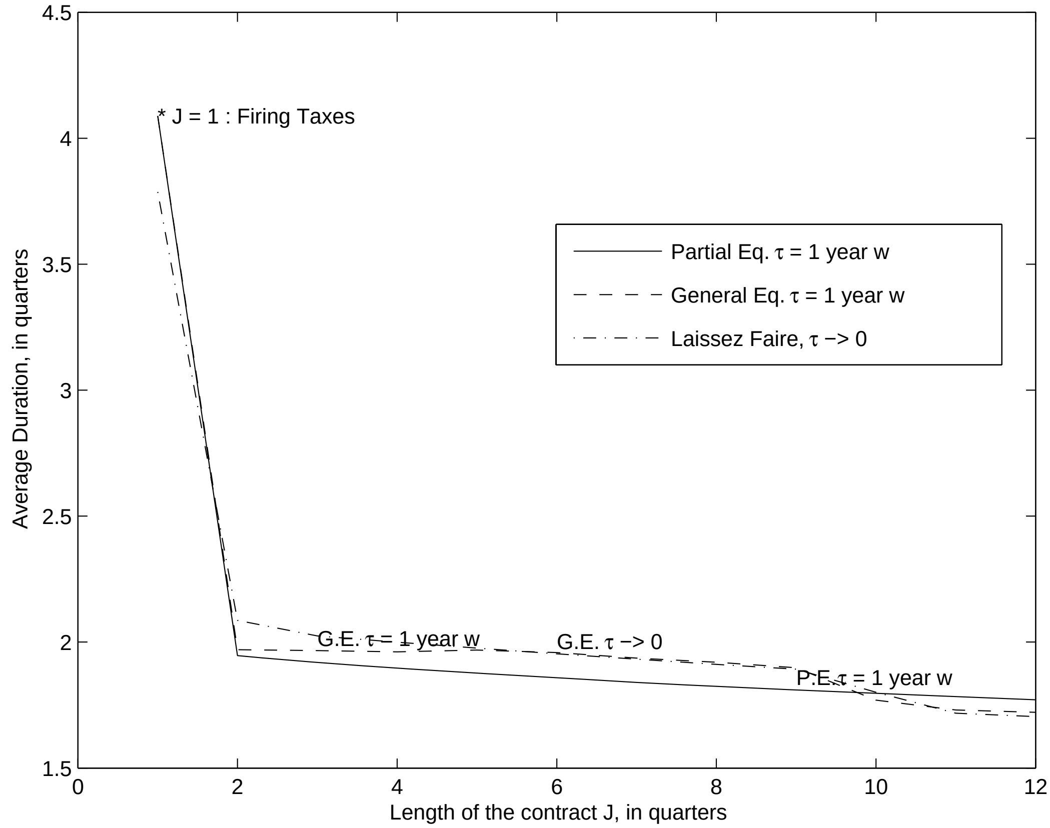
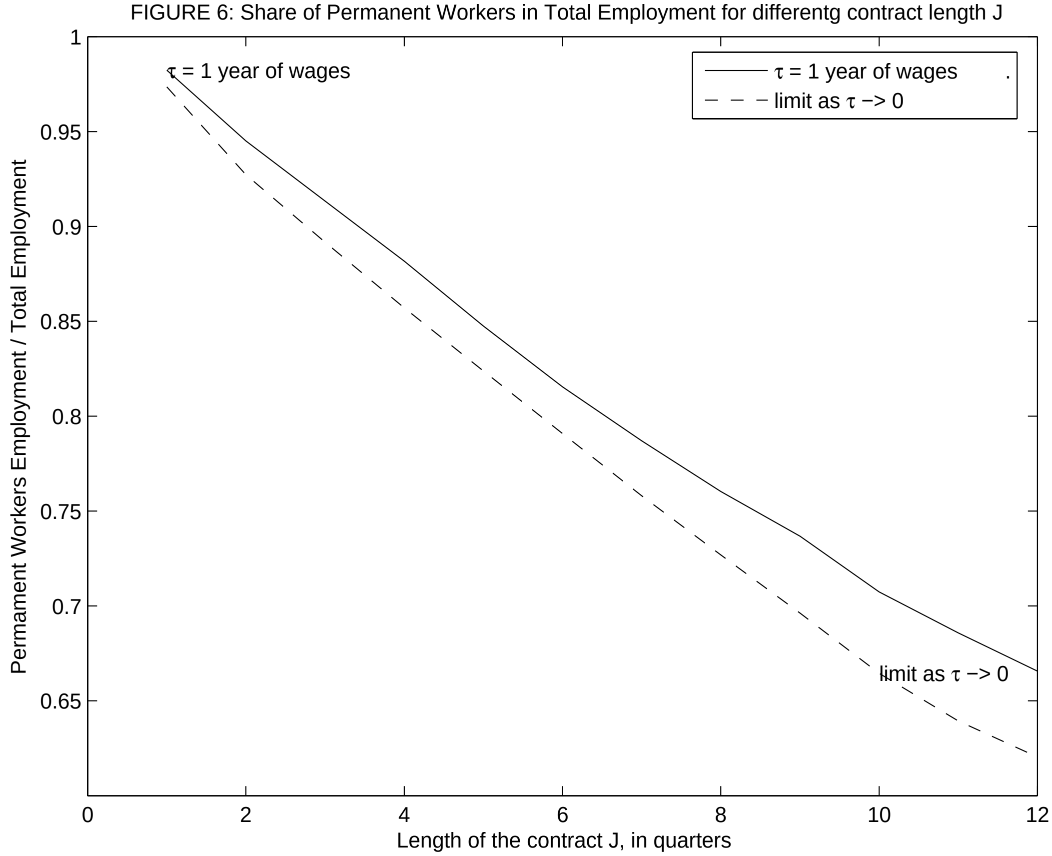


Figure 6 displays the fraction of permanent workers in total employment for the general equilibrium and laissez-faire cases. The fraction of permanent workers is higher for the general equilibrium case than for the laissez-faire case, since in the general equilibrium case firms retain more permanent workers to avoid the high separation cost. Nevertheless, the fraction of permanent workers is very similar in the two cases. Notice also that as J increases, the fraction of permanent workers decreases steadily. For $J = 12$, which corresponds to temporary contracts of three years, 33 percent of workers are in temporary contracts. In Europe in the 1990s, the fraction of workers with temporary contracts increased steadily over time to about 12 percent, reaching its highest value in Spain; there this fraction went from 11 percent before 1984 to an average of 33 percent during the 1990s.

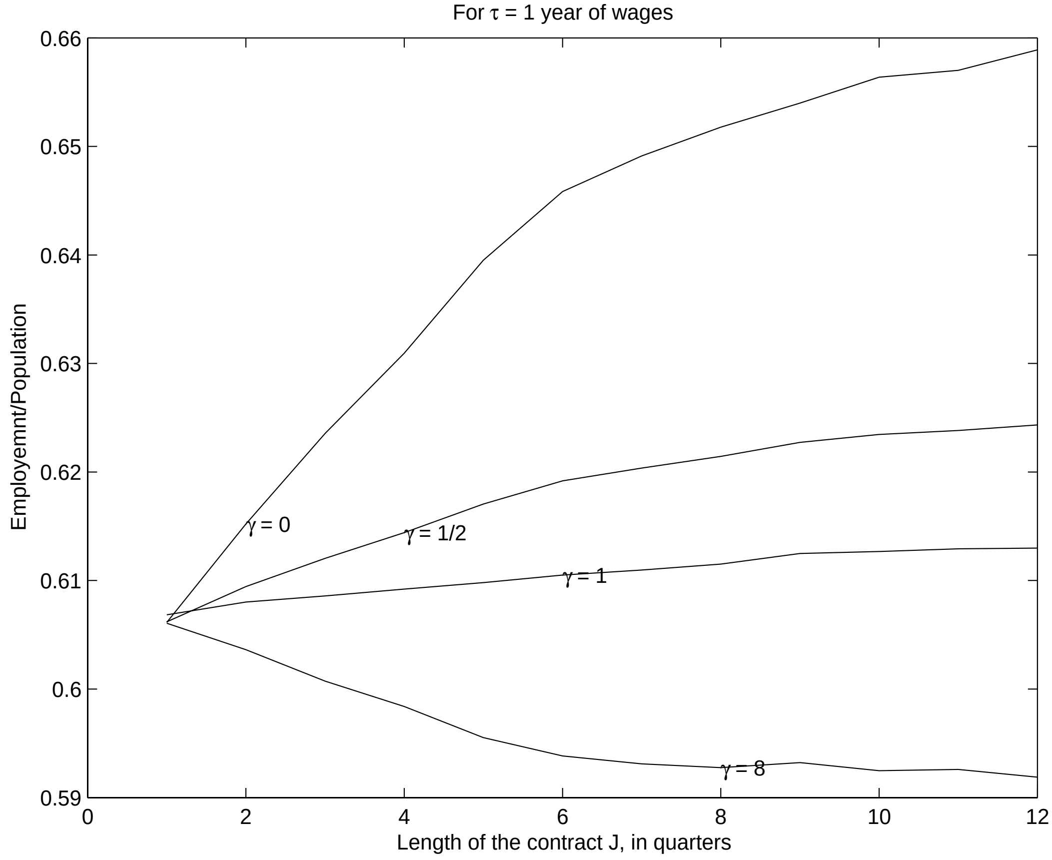
FIGURE 6: Share of Permanent Workers in Total Employment for differentg contract length J



Notice that the patterns displayed in Figures 5 and 6 for the average duration of unemployment and the share of permanent workers in total employment are similar to the ones found in Spain after the mid-1980s and have typically being interpreted as evidence that temporary contracts play an important role. However, in our model similar patterns are obtained for τ equal to one year of average wages as well as for τ arbitrarily small, which shows that by itself, large changes in turnover do not necesarilly entail large changes in welfare relevant variables, such as employment, unemployment, aggregate consumption and productivity. We obtain this result under the extreme assumption that workers with different tenure are perfect substitutes. Under a different specification, such as on-the-job learning, this result will not be obtained. In particular, if the effect of on-the-job learning is large enough, a small separation cost may have a very small effect on turnover rates. Nevertheless, we interpret the spike in Figure 1 for tenure of about three years, as evidence that the effects of separation taxes are not completely outweighed by learning.[10] We leave the examination of a model that incorporates both features for future work.

Figure 7 shows the behavior of employment for the general equilibrium case for different values of γ. As J increases, there are both income and substitution effects. The substitution effect is due to the fact that as J increases, firms have more flexibility and thus working in the market is more attractive, i.e., the equilibrium value of θ increases. The income effect is due to the fact that the economy is more productive. For low values of γ, the substitution effect dominates and thus aggregate employment increases with J. For high values of γ, the income effect dominates and thus aggregate employment decreases with J.

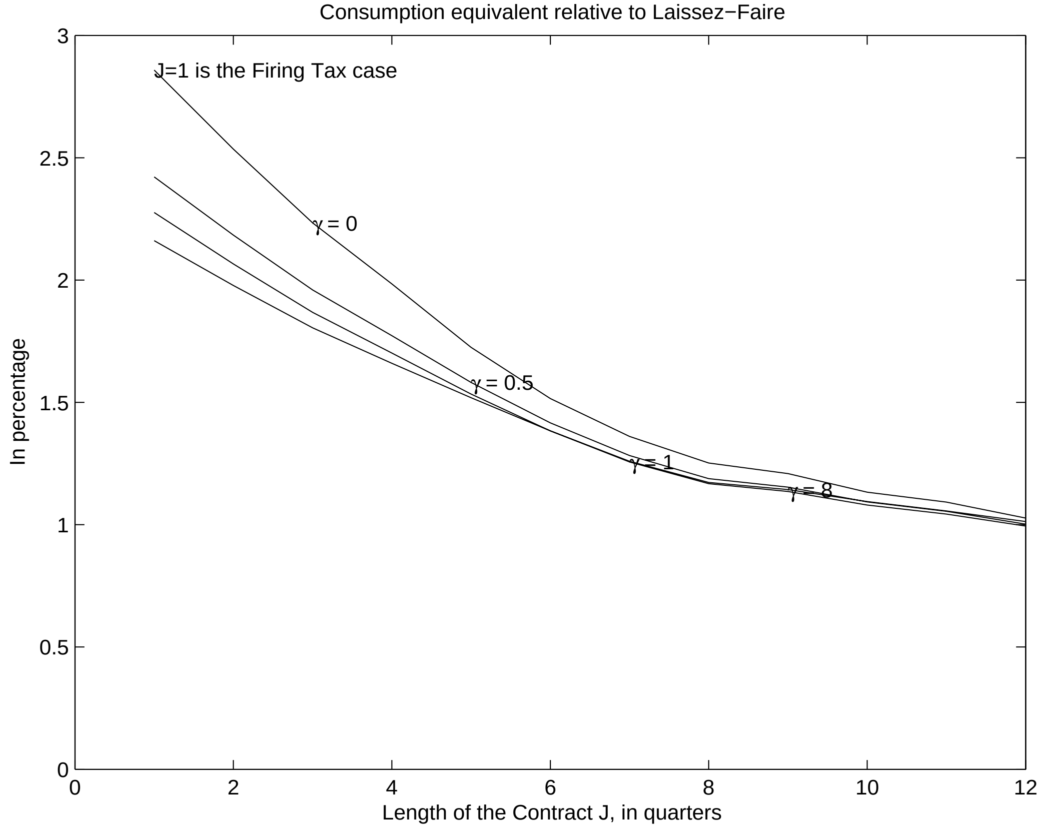
FIGURE 7: Equilibrium Employment for different values of γ
For τ = 1 year of wages



[10] With on-the-job learning and firing costs, if the effect of learning is strong enough, it will not be optimal for the firm to fire first the temporary workers with higher tenure. In this case, the spike at the end of the fixed-term contracts shown for Spain in Figure 1 will not obtain.

Figure 8 displays the welfare cost of temporary contracts of different lengths. This figure plots the extra perpetual consumption flow needed to make the representative household indifferent between living in the economy with temporary contracts of length J and living in the laissez-faire economy. This calculation compares the stationary equilibrium of the two economies and hence does not take into account the transition after a change in policy. For the same J, the welfare cost is higher for smaller γ, since in this case there is more substitution between consumption and leisure. For $J = 1$, Figure 8 shows the welfare cost of firing taxes, which are about 2.5 percent. This number is similar to the one found by Hopenhayn and Rogerson (1993) and Veracierto (2001). As J increases the welfare cost decreases: It goes from about 2.5 percent for a contract length of one quarter and decreases smoothly with J until a value of 1 percent for a contract length of three years, or $J = 12$. Thus, even if some characteristics of the allocation (such as employment in Figure 7) do not converge monotonically to their laissez-faire values as J increases, the welfare cost, which in a sense takes all the relevant features into consideration, does converges monotonically.

Figure 8: Welfare Cost of Contracts
Consumption equivalent relative to Laissez−Faire



An important quantitative finding of this paper is that introducing temporary employment contracts of three-year length provides 50 percent as much flexibility as moving to laissez-faire. We focus on two statistics: the unemployment rate, which summarizes labor reallocation, and welfare, which summarizes overall effects. The differences in these statistics between the laissez-faire case ($\tau = 0$) and the benchmark case ($J = 1, \quad \tau > 0$) measure the flexibility that can be gained by a labor market reform that achieves the first-best allocation. We found that a labor market reform that replaces the benchmark firing taxes with temporary contracts of three-year length ($J = 12, \ \tau > 0$) closes about 50 percent of those differences.

This result holds for two different parameterizations of the economy, in addition to the baseline calibration. The first one has low separation taxes, with τ of half the value used so far ($\tau = 1/2$ year of average wages). This case is interesting because, as explained in Appendix E, we set the model firing taxes to the value of separation taxes that include transfers from firms to workers (severance payments), but a fraction of these transfers could potentially be undone by private contracting between firms and workers. The second parameterization has a linear production function ($\alpha = 1$, so $F(E, z) = z \ E$). This case is interesting because we use a production function where labor is the only mobile factor, and hence the labor share and its returns to scale cannot be separately identified. If there were other factors besides labor that were mobile across locations (such as capital), the degree of decreasing returns would be much lower. For each of the two additional cases, we recalibrate the economy with firing taxes ($J = 1$) to the same observations as before, introduce temporary contracts of three-year length ($J = 12$), and evaluate the effects on the unemployment rate and welfare. Rows 1–3 and 5-6 of Table 1 report the values for the unemployment rate and welfare for each of the cases. Rows 4 and 7 report the fraction of the gap (i.e., the difference in the statistics between the benchmark and laissez-faire cases) that the introduction of the temporary contracts closes. Rows 4 and 7 show that, in all cases, a labor market reform that replaces firing taxes with temporary contracts of three-year length achieves more than 40 percent of the potential welfare gains of labor market reform and more than 50 percent of the increase in the unemployment rate.

Table 1

	Baseline $\tau = 1,\ \alpha = .64$	Low firing tax $\tau = .5\ ,\ \alpha = .64$	Linear F $\tau = 1,\ \alpha = 1$
(1) Unempl. rate F.T.	6.75%	6.76 %	6.72%
(2) Unempl. rate T.C.	8.02%	7.30%	9.54%
(3) Unempl. rate L.F.	9.10%	7.79%	12.16%
(4) % Gap Unempl. rate	54 %	52 %	52 %
(5) Welfare cost F.T.	2.28%	0.70%	8.4%
(6) Welfare cost T.C.	1.0%	0.28%	3.7%
(7) % Gap Welfare	44 %	40 %	44 %

F.T. stands for firing taxes ($\tau > 0, J = 1$); T.C. stands for temporary contracts ($\tau > 0,\ J = 12$); and L.F. stands for "laissez-faire" ($\tau = 0$). Welfare cost is the consumption equivalent relative to laissez-faire for $\gamma = 1$. Row (4) = [(2)-(1)] /[(3)-(1)] and (7) = (6)/(5). τ is measured in years of economy-wide, average wages.

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Technical appendix to:

Fixed Term Employment Contracts

in an Equilibrium Search Model

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This document contains 7 appendices:
Appendix A: Analysis of the Island Planning Problem.
Appendix B: Analysis of the Simplified Island Planning Problem.
Appendix C: Analysis of Spot Labor Market ("SLM").
Appendix D: Analysis of Multiperiod Employment Relations ("MER"): Binding contracts and tenure at the firm level (a formal description).
Appendix E: Calibration of τ.
Appendix F: Proofs.
Appendix G: Definition of Auxiliary Competitive Equilibrium ("ACE").

Appendix A: Analysis of the Island Planning Problem

Consider the problem of the planner of an island that receives U workers per period and that starts with workers $(T_1, T_2, ..., T_{J-1}, T_J)$ where T_i is the number of workers with tenure $i = 1, 2, ..., J$. Define E as the set of possible workers tenure profiles, $E = [0, U]^{J-1} \times R_+$. The planners value function $V : E \times Z$ solves

$$\begin{aligned} & H[V](T_1, T_2, ..., T_{J-1}, T_J, z) \qquad (11) \\ = & \max_{\{E_i\}_{i=0}^{J}} \left\{ F\left(\sum_{i=0}^{J} E_i, z\right) + \sum_{i=0}^{J-1} \theta [T_i - E_i] + (\theta - \tau)[T_J - E_J] \right. \\ & \left. +\beta \int V(E_0, E_1, ..., E_{J-2}, E_{J-1} + E_J, z') Q(z, dz') \right\} \end{aligned}$$

subject to

$$\begin{aligned} 0 &\leq E_0 \leq U, \\ 0 &\leq E_i \leq T_i \text{ for } i = 1, 2, ..., J. \end{aligned}$$

The fixed point of H gives the stationary version of the island planning problem stated in Definition 38.

Proposition 3 *H maps concave functions into concave ones.*

We use the following notation for subgradients. Let $G : X \to R$ be a concave function. We use $\partial G(x)$ to denote its subgradient at x (if it is clear the value of x from the context we simply use ∂G). In our case $X \subset R^n$, we use $\partial G_{x_i}(x)$ for $i = 1, 2, ..., n$ (and ∂G_{x_i} when it is clear) to denote the projection of $\partial G(x)$ into the subspace of the $x_i' s$. Abusing notation, we use $G_{x_i}(x)$ (and G_i when it is clear) to denote a generic element of $\partial G_{x_i}(x)$, so that $G_{x_i}(x) \in \partial G_{x_i}(x)$.

The next proposition gives a useful result, ordering the subgradients of V

Proposition 4 *Consider a function V satisfying*

$$\begin{aligned} V_{T_1} &\geq V_{T_2} \geq \cdots \geq V_{T_{J-1}} \geq V_{T_J}, \qquad (12) \\ V_{T_1} &\leq V_{T_J} + \tau \qquad (13) \end{aligned}$$

for all z and $T > 0$, where

$$\left(V_{T_1}, V_{T_2}, \ldots, V_{T_{J-1}}, V_{T_J}\right) \in \partial V(T, z).$$

Then,

$$\begin{aligned} H[V]_{T_1} &\geq H[V]_{T_2} \geq \cdots \geq H[V]_{T_{J-1}} \geq H[V]_{T_J}, \qquad (14) \\ H[V]_{T_1} &\leq H[V]_{T_J} + \tau \qquad (15) \end{aligned}$$

for all z and $T > 0$, where

$$\left(H[V]_{T_1}, H[V]_{T_2}, \ldots, H[V]_{T_{J-1}}, H[V]_{T_J}\right) \in \partial H[V](T, z).$$

Intuitively it follows from the assumption that workers are perfect substitutes and from the fact that $\tau > 0$.

The following proposition and corollaries are important to characterize the solution of the problem and reduce its dimensionality.

Proposition 5 *Let V satisfy (12). Then the policies for $H[V]$ satisfy the following. Let $E = (E_0, E_1, ..., E_{J-1}, E_J) \in [0, U]^J \times R_+$ be feasible given T. Consider an alternative $\tilde{E} = \left(\tilde{E}_0, \tilde{E}_1, ..., \tilde{E}_{J-1}, \tilde{E}_J\right)$ such that: i) it is feasible for T, ii)*

$$\sum_{j=0}^{J-1} E_j = \sum_{j=0}^{J-1} \tilde{E}_j \text{ and } E_J = \tilde{E}_J,$$

and iii) there is a j' such that $\tilde{E}_j \geq E_j$ for all $j \leq j' \leq J-1$ and that $\tilde{E}_j = 0$ for all j, $j' < j \leq J-1$. Then $\tilde{E}$ is weakly preferred to E.

Corollary 6 *The optimal policy can be chosen with the following property:*
(*) If $E_j < T_j$ for some j, $1 \le j \le J-1$, then $E_{j'} = 0$ for all $j' : j < j' \le J-1$.

The following corollary states that the ergodic set is a subset of $\mathcal{E}$, which is given by equation (8).

Corollary 7 *If $T \in \mathcal{E}$ and T' is given by the optimal policy*

$$T' = (T_1', T_2', ..., T_J') = (E_0, E_1, ..., E_{J-2}, E_{J-1} + E_J)$$

then $T' \in \mathcal{E}$.

The next set of results establish that the fixed point $V = H[V]$ is differentiable and its derivatives are indeed given by V_j^* in equation (7). The results in the next three lemmas and two propositions are analogous to standard manipulations of first order conditions, except for the fact that V may not be differentiable.

Let define the function $\hat{R}(E,z)$, as follows: $\hat{R} : R_+^{J+1} \times Z \to R$

$$\begin{aligned} \hat{R}(E,z) &= F\left(\sum_{i=0}^{J} E_i, z\right) - \theta \sum_{i=0}^{J-1} E_i - (\theta - \tau) E_J \\ &\quad + \beta \int V(E_0, E_1, \ldots, E_{J-2}, E_{J-1} + E_J, z') Q(z, dz'). \end{aligned}$$

The first lemma shows a standard saddle-type result for the problem defining $H[V]$.

Lemma 8 *Let V be concave. Fix T, z and let*

$$\begin{aligned} H[V](T,z) &= \max_E \left\{ \hat{R}(E,z) + \hat{\theta} T : 0 \le E \le T \right\}, \\ E(T,z) &= \arg\max_E \left\{ \hat{R}(E,z) : 0 \le E \le T \right\}. \end{aligned} \tag{16}$$

Then

$$\hat{\theta} + \lambda^* = (H[V]_0, H[V]_1, ..., H[V]_J) \in \partial H[V](T,z)$$

if and only if λ^* is a Lagrange multiplier, i.e.

$$\begin{aligned} \hat{R}(E^*,z) + \lambda (T - E^*) &\ge \hat{R}(E^*,z) + \lambda^* (T - E^*) \\ &\ge \hat{R}(E,z) + \lambda^* (T - E) \end{aligned} \tag{17}$$

for all non-negative E, λ, where $\hat{\theta} = (\theta, ..., \theta, \theta - \tau)$, $E^* = E(T,z)$ and $U = T_0$.

Notice that since $\hat{R}$ is concave and the restrictions are linear, $E(T,z)$ solves problem (16) if and only if there is a saddle (E^*, λ^*) as in equation (17) -see, for example, "Analytical Method in Economics", Takayama, Theorem 2.9-.

The next lemma shows the Kuhn-Tucker conditions for this problem.

Lemma 9 *Let V be concave. A necessary and sufficient condition for $E^* = \{E_i^*\}_{i=0}^J$ to solve*

$$E^* \in \arg\max_E \hat{R}(E,z) \;\; s.t. \;\; 0 \le E \le T$$

given T, z, and there exists a $\left\{\hat{R}_i\right\}_{i=0}^J \in \partial \hat{R}(E^*, z)$ such that (E^*, λ^*) is a saddle where

$$\lambda_i^* = \hat{R}_i^*. \tag{18}$$

Given our previous results we can now write the analogous to the Euler equations.

Proposition 10 *Let V be concave. Fix T, z. Then, $0 \leq E^* \leq T$ is an optimal choice given T, z if and only if for all $\{H[V]_i(T,z)\}_{i=0}^J \in \partial H[V](T,z)$, there is a $\left\{\hat{R}_i\right\}_{i=0}^J \in \partial \hat{R}(E^*, z)$ such that*

$$\begin{aligned}
H[V]_i(T,z) &= \hat{R}_i(E^*,z) + \theta \text{ for } i = 0,\ldots,J-1 \\
\hat{R}_i(E^*,z) &\geq f\left(\sum_{i=0}^J E_i^*, z\right) - \theta \\
&\quad + \beta \int V_{i+1}\left(E_0^*,\ldots,E_{J-2}^*, E_{J-1}^* + E_J^*, z'\right) Q(z,dz') \\
\text{with} &= \text{ if } E_i^* > 0 \\
\hat{R}_i(E^*,z) &\geq 0, \\
0 &= \left(H[V]_i(T,z) - \theta\right)(T_i - E_i^*), \text{ and} \\
H[V]_J(T,z) &= \hat{R}_J(E^*,z) + \theta - \tau\ , \\
0 &= \left(H[V]_J(T,z) - (\theta - \tau)\right)(T_J - E_J^*), \\
\hat{R}_J(E,z) &\geq f\left(\sum_{i=0}^J E_i^*, z\right) - (\theta - \tau) \\
&\quad + \beta \int V_J\left(E_0^*,\ldots,E_{J-2}^*, E_{J-1}^* + E_J^*, z'\right) Q(z,dz') \\
\text{with} &= \text{ if } E_J^* > 0 \\
\hat{R}_J(E^*,z) &\geq 0
\end{aligned}$$

where we let $U = T_0$.

The following proposition shows that if some permanent workers are fired then all temporary workers must have been fired.

Proposition 11 *$E_J^* < T_J \Rightarrow E_i^* = 0$ for every $i \neq J$.*

The next lemma shows that employment is bounded below, and hence marginal productivity is bounded above.

Lemma 12 *There is an $e > 0$ such that for all T, z*

$$\sum_{i=0}^J E_i(T,z) \geq e > 0.$$

By this lemma, the solution for V_j^* in equation (7) is well defined because $f\left(\sum_{i=0}^J E_{i,s}^*, z_s\right)$ is uniformly bounded.

Proposition 13 *Let V be the fixed point of H. Assume that $U > 0$. Then V is differentiable with respect to T_i when $T_i > 0$.*

Appendix B: Analysis of the Simplified Island Planning Problem

The planner's value function $v : [0, J \cdot U] \times R_+ \times Z$ has to satisfy the functional equation h:

$$\begin{aligned} & h[v](t,p,z) \\ = \ & \max_{e_t,e_p} \{F(e_t+e_p,z)+\theta[t-e_t]+(\theta-\tau)[p-e_p] \\ & +\beta \int v(t',p',z') Q(z,dz')\Big\} \end{aligned} \tag{19}$$

subject to

$$\begin{aligned} 0 &\leq e_t \leq t, \\ 0 &\leq e_p \leq p, \end{aligned}$$

and where the law of motion is given by

$$\begin{aligned} t' &= \min\{U+e_t, JU\} \\ p' &= e_p + \max\{U+e_t-JU, 0\} \end{aligned}$$

Proposition 14 *Consider V and v such that*

$$v(T_1+T_2+...+T_{J-1}, T_J, z) = V(T_1, T_2, ..., T_{J-1}, T_J, z) \tag{20}$$

for all $(T_1, T_2, ..., T_{J-1}, T_J) \in \mathcal{E}$. Then

$$h[v](T_1+T_2+...+T_{J-1}, T_J, z) = H[V](T_1, T_2, ..., T_{J-1}, T_J, z) \tag{21}$$

for all $(T_1, T_2, ..., T_{J-1}, T_J) \in \mathcal{E}$.

Lemma 15 *Assume that V satisfies (12). Consider T and $\hat{T}$ and V such that*

$$T_1+T_2+...+T_{J-1} = \hat{T}_1+\hat{T}_2+...+\hat{T}_{J-1} \text{ and } T_J = \hat{T}_J. \tag{22}$$

for any $\hat{T} \in \mathcal{E}$ and $T \in E$ then

$$H[V](T,z) \leq H[V]\left(\hat{T}, z\right).$$

Proposition 16 *Let v be the function corresponding to V as in (20) defined for $T \in \mathcal{E}$. Assume that $V(\cdot, z)$ is concave, and V satisfies (12). Then $h[v](\cdot, z)$ is concave in t, p.*

Remark 17 *The previous proposition is not obvious since the feasible set of the problem defined by the right hand side of $h[v]$ in (19) is not convex.*

We now introduce the R, which is the objective function being maximized in $h[v]$. The "derivatives" of R are used to define the functions $\hat{t}$ and $\hat{p}$.

Definition 18 *Given v, define $R(e_t, e_p, z)$ as*

$$R(e_t, e_p, z) = F(e_t+e_p, z) - \theta e_t - (\theta-\tau) e_p$$
$$+\beta \int v(U+\min\{e_t,(J-1)U\}, e_t+e_p-\min\{e_t,(J-1)U\}, z') Q(z, dz')$$

Consider an island planner with no temporary workers ($t = 0$) and a given z. The quantity $\hat{p}(z)$ is the number of permanent workers that leaves the island's planner indifferent between firing "one" permanent worker and keeping all $\hat{p}(z)$ of them.

Definition 19 *Let R be defined as in definition (18). For each z define $\hat{p}(z)$, such that*

$$0 \in \partial R_{e_p}(0, \hat{p}(z), z).$$

Consider an island planner with $0 < p < \hat{p}(z)$, so it does it not want to fire any permanent worker for that z. The quantity $\hat{t}(p,z)$ is the number of temporary workers that leaves the island's planner indifferent between firing "one" transitory worker and keeping all $\hat{t}(p,z)$ of them. Formally:

Definition 20 *Let R be defined as in definition (18). For each p, z define $\hat{t}(p,z)$ as follows:*
(i) if $R_{et} > 0$ for all $R_{et} \in \partial R_{et}(U \cdot J, p, z)$, then $\hat{t}(p,z) = J \cdot U$,
(ii) if $R_{et} < 0$ for all $R_{et} \in \partial R_{et}(0, p, z)$, then $\hat{t}(p,z) = 0$,
(iii) otherwise $\hat{t}(p,z)$ solves $0 \in \partial R_{e_t}\left(\hat{t}(p,z), p, z\right)$.

The remaining of this appendix shows that $\hat{p}, \hat{t}$ exist, they are unique, and $\hat{t}$ is decreasing in p. The proofs are complicated by the fact that R is not differentiable.

Proposition 21 *Let v be given by V as in (20). Assume that V is concave and satisfies (12). The function $R(\cdot, z)$ is strictly concave.*

Define $M : [0, U \cdot J] \to R_+$ as

$$M(e_t) \equiv \min\{e_t, (J-1)U\}$$

notice that

$$\begin{aligned} & e_p + \max\{e_t - (J-1)U, 0\} \\ = \; & e_p + e_t - \min\{e_t, (J-1)U\} \\ = \; & e_p + e_t - M(e_t). \end{aligned}$$

Remark 22 *It is standard to show that $h[v]$ is increasing in t, p and z if v has these properties.*

Remark 23 *Assume that V satisfies (12) and (13). Let v be defined as in (20). Denote by $\partial h[v]$ the subgradient of $h[v](t,p,z)$ when v is considered as a function of t and p. A corollary of Proposition (14) and Proposition (4) is that*

$$h[v]_p \le h[v]_t \le h[v]_p + \tau,$$

for all $\left(h[v]_t, h[v]_p\right) \in \partial h[v](t,p,z)$.

Proposition 24 *Fix t, p, z. Assume that V satisfies (12), (13), and is concave. Define v as in (20). Let $\left(h[v]_t, h[v]_p\right) \in \partial h[v](t,p,z)$. Then $h[v]_p \ge \theta - \tau$. Moreover, there exists a $\bar{p}(z)$ such that for all $p \ge \bar{p}(z)$ and t, $h[v]_p = \theta - \tau$ for any $h[v]_p \in \partial h[v]_p(t,p,z)$.*

Given v define

$$b(e_t, e_p, z) \equiv \int v\left(U + M(e_t), e_t + e_p - M(e_t), z'\right) Q(z, dz')$$

as a function of e_t and e_p and z. Let ∂B be its subgradient with respect to (e_t, e_p).

Lemma 25 *Define v as in (20). Assume that v is concave and satisfies*

$$v_p \le v_t \le v_p + \tau,$$

for all t, p, z. Fix any z, e_t, e_p. Let $\left(b_{e_t}, b_{e_p}\right) \in \partial b(e_t, e_p, z)$. Then

$$b_{e_p} \le b_{e_t} \le b_{e_p} + \tau.$$

Let $\partial R(e_t, e_p, z)$ be the subgradient of R when considered as a function of (e_t, e_p).

Lemma 26 *Assume that v is concave and satisfies*

$$v_p \le v_t \le v_p + \tau,$$

for all t, p, z. Fix any z, e_t, e_t. For all $\left(R_{e_p}, R_{e_t}\right) \in \partial R(e_t, e_p, z)$

$$R_{e_p} \ge R_{e_t} + \tau(1-\beta).$$

Corollary 27 *Let e_p, e_t be the optimal choice of employment for Problem (19). If $e_p < p$ and $t > 0$, then $e_t = 0$. If this were not true, i.e. if $e_p < p$ and $e_t > 0$, then $R_{e_p} = R_{e_t} = 0$, which contradicts Lemma 26.*

Lemma 28 *Let v be given by V as in (20), assume that V is concave and satisfies (12). Let R be defined as in definition (18).*
For each z there is a unique $\hat{p}$ satisfying (19). Moreover, $0 < \hat{p}(z) < \bar{p}(z) < +\infty$.

Using the concavity of R and strict concavity of F we define $\hat{t}$ as follows.

Lemma 29 *Let v be given by V as in (20), assume that V is concave and satisfies (12). Let R be defined as in definition (18).*
Then for each (p, z), $0 < p < \hat{p}(z)$, there exists a unique $\hat{t}$ that satisfies (20).

Proposition 30 *Assume that V is concave and satisfies (12) and (13). Let v be given by V as in (20). Assume, without loss of generality that v is concave in (t, p). Then,*
i) The optimal decision rules of $h[v]$ are described by the set of Inaction for R as

$$\begin{aligned} e_t(t,p,z) &= \min\left\{t, \hat{t}(p,z)\right\}, \\ e_p(t,p,z) &= \min\left\{p, \hat{p}(z)\right\} \end{aligned}$$

for all t, p, z.
ii) $H[V]$ is concave, and satisfies (12) and (13).
iii) $h[v]$ and $H[V]$ satisfy (20), and $h[v]$ is concave.

Lemma 31 *Let V be concave, and satisfy (12) and (13). Let v be defined as in (20). Let $\hat{p}$, $\hat{t}$ and I be defined as in (18), (19), and equation (10), respectively. Then, the subgradients of $h[v]$ are as follows:*
If $t \neq iU$ for $i = 1, 2, .., J-1$, then $h[v](t,p,z)$ is differentiable with respect to t.
If $(t,p) \in Int(I(z))$:

$$h[v]_t(t,p,z) = f(t+p,z) + \beta \int b_{e_t}(t,p,z') Q(z,dz') > \theta,$$

If $(t,p) \in Int\left(I(z)^C\right)$:

$$h[v]_t(t,p,z) = \theta > f(t+p,z) + \beta \int b_{e_t}(t,p,z') Q(z,dz'),$$

If (t,p): $t = \hat{t}(p,z) < JU$:

$$\left[\underline{h}[v]_t(t,p,z), \bar{h}[v]_t(t,p,z)\right] = \left[\theta,\ f(t+p,z) + \beta \bar{b}_{et}(t,p,z)\right]$$

Definition 32 *We say that $\partial v_t\ (t,p,z)$ is decreasing in p if it satisfies the following property. If $p < p'$, define $\underline{v}'_t, \bar{v}'_t, \underline{v}_t$ and $\bar{v}_t$ satisfying*

$$\left[\underline{v}'_t, \bar{v}'_t\right] = \partial v_t(t,p',z),$$

and

$$\left[\underline{v}_t, \bar{v}_t\right] = \partial v_t(t,p,z).$$

Then

$$\underline{v}'_t \leq \underline{v}_t \text{ and } \bar{v}'_t \leq \bar{v}_t$$

Notice that if v is differentiable at (t,p,z), this property simply says that $\partial v(t,p,z)/\partial t$ is decreasing in p.

Lemma 33 . *Let V be concave, and satisfy (12) and (13). Let v be defined as in (20). Assume that the subgradient of v_t is decreasing in p, i.e. it satisfies definition (32). Let $\hat{t}(p,z)$ be defined as in (19) for the optimal rule that attains the right hand side of $h[v]$. Then, the subgradient of $h[v]_t$ is decreasing in p too, i.e. it satisfies definition (32) and $\hat{t}(p,z)$ is weakly decreasing in p.*

Finally

Proposition 34 *Let v be the fixed point of h. Let $\hat{t}$ be defined as in definition (18). Then $\hat{t}(p,z)$ is decreasing in p. Moreover, if $\hat{t}$ is not a multiple of U, then $\hat{t}$ is strictly decreasing in p.*

Appendix C: Analysis of Spot Labor Markets ("SLM")

Current wages across tenure levels are given by

$$w(T,z) = (w_0(T,z), w_1(T,z), ..., w_{J-1}(T,z), w_J(T,z)),$$

a function of the island-wide state (T,z). The law of motion for wages can then be obtained from the island-wide equilibrium employment rule and the associated law of motion for the island-wide state. The equilibrium employment rule is denoted by

$$G(T,z) \equiv (G_0(T,z), G_1(T,z), ..., G_{J-1}(T,z), G_J(T,z)).$$

The law of motion for the endogenous state $T' = A(T,z)$ is then given by

$$A(T,z) = (G_0(T,z), G_1(T,z), ..., G_{J-2}(T,z), G_{J-1}(T,z) + G_J(T,z)).$$

The problem for a worker with tenure j on an island of state (T,z) is to decide whether to become non-employed or stay and work. Becoming non-employed entails a value given by θ. By staying, the worker receives a wage rate w_j during the current period and gains tenure $\min\{j+1, J\}$ for the following period. We denote the value function for a j-tenure worker in a (T,z)-island as $W_j(T,z)$. This value function must solve

$$W_j(T,z) = \max\left\{\theta,\ w_j(T,z) + \beta \int W_{\min\{j+1,J\}}(A(T,z), z') Q(z, dz')\right\}$$

for all (T,z) and $j = 0, ..., J$.

The value function $B(p;T,z)$ of a firm that employed p permanent workers during the previous period on an island with state (T,z) solves:

$$\begin{aligned} & B(p;T,z) \\ = & \max_{\{g_j \geq 0\}_{j=0}^{J}} \left\{ F\left(\sum_{j=0}^{J} g_j,\ z\right) - \sum_{i=0}^{J} w_j(T,z) g_j - \tau \max\{p - g_J, 0\} \right. \\ & \left. + \beta \sum_{z'} B(g_J + g_{J-1};\ A(T,z), z') Q(z|z') \right\} \end{aligned}$$

The optimal decision rule is denoted by

$$g_j = m_j(p;T,z),$$

for $0 \leq j \leq J$, describing the optimal employment level at each tenure j. For future reference, notice that $B(p;T,z)$ is decreasing in p, since having employed more permanent workers in the previous period makes the firm subject to higher potential separation costs. Thus, provided that B is differentiable, $-\tau \leq \partial B/\partial p \leq 0$, and $\partial B/\partial p = -\tau$ if some permanent workers are fired, i.e. if $g_J = m_J(p;T,z) < p$.

A spot labor market equilibrium (SLM) is given by numbers $\{\theta, U, \sigma\}$ and functions $\{w, G, B, m, W\}$ that satisfy the following conditions:

i) Given wages $w(\cdot)$, employment $G(\cdot)$, and the law of motion $A(\cdot)$, the representative firm is representative

$$m_j(T_J;T,z) = G_j(T,z),$$

for all (T,z) and all $0 \leq j \leq J$;

ii) Given wages $w(\cdot)$, employment $G(\cdot)$ and law of motion $A(\cdot)$, the decision of the representative worker is representative

$$\begin{aligned} W_j(T,z) &> \theta \Rightarrow G_j(T,z) = T_j, \quad \text{for } j > 0 \text{ and} \\ W_0(T,z) &> \theta \Rightarrow G_0(T,z) = U, \end{aligned}$$

and if $G_j(T,z) > 0$, then

$$W_j(T,z) = w_j(T,z) + \beta \int W_{\min\{J,j+1\}}(A(T,z), z') Q(z, dz');$$

iii) The law of motion A defines an invariant distribution μ across states (T, z) as follows

$$\mu(D, z') = \sum_{z \in Z} \left[\int_{\{T,z:\ A(T,z) \in D\}} \mu(dT \times z) \right] Q(z'|z);$$

iv) Feasibility in the labor market is satisfied

$$N - U - \int G(T, z)\, \mu(dT \times dz) \geq 0, \ \ U \geq 0,$$

v) The value of search σ and the value of becoming non-employed θ satisfy

$$\sigma = \beta \int W_0(T, z)\, \mu(dT \times dz), \quad \theta = \max\{\omega + \beta\theta,\ \sigma\}; \text{ and}$$

vi) The labor force participation decision is optimal

$$\begin{aligned} 0 &= \left[N - U - \int G(T, z)\, \mu(dT \times dz) \right] [\theta - \omega - \beta\theta] \\ 0 &= U\ [\theta - \sigma]. \end{aligned}$$

The next theorem establishes the first and second welfare theorems for this economy and provides a partial characterization of the SLM equilibrium.

Theorem 35 *Welfare Theorems and equilibrium characterization:*
i) Let $\{U, \theta, w, G, B, m, W, \mu\}$ be a spot labor market equilibrium (SLM). Then, there is an island planner value function V, for which $\{V, G, U, \theta, \mu\}$ is an stationary efficient allocation.
ii) Conversely, let $\{V, G, U, \theta, \mu\}$ be a stationary efficient allocation. Then, there are wages and value functions $\{w$, B, m, $W\}$ for which $\{U, \theta, w, E, B, m, W, \mu\}$ is a spot labor market equilibrium (SLM).
iii) The functions B, W and V related as in i) and ii) satisfy

$$\begin{aligned} W_j(T, z) &= \partial V(T, z) / \partial T_j \text{ for } j = 0, ..., J-1 \\ \partial B(T_J, T, z) / \partial p + W_J(T, z) &= \partial V(T, z) / \partial T_J \end{aligned} \tag{23}$$

The reasons for the equivalence in i) and ii) are the same as in the Prescott and Mehra (1980) result about equivalence between SLM equilibrium and efficient allocations. Our set up does not directly maps into theirs, so in Appendix F we offer a constructive proof of i) and ii).

Condition iii) is obtained by comparing the first order conditions for the planning problem with the optimality conditions for the workers and firms in the spot labor market equilibrium (SLM). These conditions give some intuition on how prices decentralize the efficient allocation. Recall that $\partial V/\partial T_j$ is the shadow value of a tenure j worker in the island planning problem. Condition iii) says that the shadow value of an extra temporary worker for the planner is the same as the equilibrium value function W_j. Instead the shadow value of a permanent worker for the planer, $\partial V/\partial T_j$, is lower than the equilibrium value function for a worker W_J. This difference is exactly the shadow value of an extra permanent worker for the firm, $\partial B/\partial p$, which, due to the separation cost, is a number between $-\tau$ and 0.

The next proposition gives a partial characterization of equilibrium wages.

Proposition 36 *Let $\{U, \theta, w, G, B, m, W, \mu\}$ be a spot labor market equilibrium (SLM). Without loss of generality, the equilibrium wage w can be chosen to satisfy*
a) for all $j = 0, 1, ..., J-2$

$$w_j(T, z) = f\left(\sum_{i=0}^{J} G_i(T, z), z \right),$$

b) for all $j = 0, 1, ..., J-2$

$$\begin{aligned} w_j(T, z) - \beta\tau &\leq w_{J-1}(T, z) \leq w_j(T, z) \\ w_j(T, z) - \beta\tau &\leq w_J(T, z) \leq w_j(T, z) + \tau \\ w_{J-1}(T, z) &\leq w_J(T, z) \leq w_{J-1}(T, z) + \tau \end{aligned}$$

and if $E_J(T,z) < T_J$:

$$w_{J-1}(T,z) \leq w_j(T,z) < w_J(T,z),$$

c) and the equilibrium value function W *for workers can be chosen so that they satisfy:*

$$\begin{aligned} W_0(T,z) &\geq W_1(T,z) \geq \cdots \geq W_{J-1}(T,z) \\ W_J(T,z) &\geq W_{J-1}(T,z) \ . \end{aligned}$$

The proof of Proposition 36 follows, essentially, from the analysis of the first order conditions of the firm problem. Appendix F contains a joint proof of Theorem 35 and Proposition 36.

Stationary Equilibrium with separation taxes and $\boldsymbol{\gamma > 0}$, and computational algorithm

In an equilibrium when the separation cost τ is a tax rebated lump-sum to households and $\gamma > 0$, aggregate consumption is given by

$$c = \int F\left(\sum_{j=0}^{J} G_j(T,p),\ z\right)\mu(dT \times dz) \tag{24}$$

Given this change, the equilibrium allocation can be described by $\{V, G, U, \theta, \mu\}$, where V is the value function and G the optimal policy for the island planning problem for (U,θ), and where μ is the invariant distribution for $\{(T,z)\}$ generated by (G,Q) such that:

a) The value of search is generated by $\hat{V}, \mu$

$$\sigma = \beta \int \left[\frac{\partial \hat{V}\left(T, z; \hat{U}, \theta\right)}{\partial \hat{U}}\Big|_{\hat{U}=U}\right] \mu(dT \times dz\ ; U, \theta)$$

where $\hat{V}$ is defined in terms of V as in (4); and

b) The marginal condition (6) holds with aggregate consumption given by (24).

We define the equilibrium in terms of the objects of an stationary efficien allocations because it provides an algorithm for its computation: a) and b) can be regarded as two equations in two unknowns, (U,θ). Alternatively we could have define the equilibrium using the objects in the firms and workers problems, so that $W_0(T,z)$ would have taken the place of $\partial \hat{V}\left(T, z; \hat{U}, \theta\right)/\partial \hat{U}$ in a). Using the arguments in Theorem 35, it is easy to show that the two definitions would have been equivalent.

Appendix D: Analysis of Multiperiod Employment Relations ("MER"): Binding contracts and tenure at the firm level (a formal description)

There are competitive markets on the island. At each date t, history z^t, the set of commodities traded is $S(z^t)$. A commodity $s \in S(z^t)$ is a stopping time indicating the time at which a worker will be dismissed under each possible continuation sequence $z_{t+1}^{\infty} = \{z_{t+1}, z_{t+2}, ...\}$ following the history z^t. Formally, $S(z^t)$ is the set of all functions

$$s\left(z^t; z_{t+1}^{\infty}\right) : Z^{\infty} \to \{t+1, t+2, ..., \infty\}$$

satisfying

$$\begin{aligned} s\left(z^t; z_{t+1}^{\infty}\right) &= k \Rightarrow s\left(z^t; \hat{z}_{t+1}^{\infty}\right) = k, \\ \text{for all } \hat{z}_{t+1}^{\infty} \text{ such that:} & \quad \{z_{t+1}, z_{t+2}, ..., z_k\} = \{\hat{z}_{t+1}, \hat{z}_{t+2}, ..., \hat{z}_k\}. \end{aligned}$$

When a worker arrives for the first time to the island at date t, history z^t, he is a "newly arrived" worker and can supply only one stopping time in the set $S(z^t)$. The worker cannot supply a new stopping time before the previous stopping time is actually executed, i.e. before the worker is separated from his previous employer. The first time that the worker separates he becomes an "incumbent" worker for the rest of his stay on the island. An "incumbent" worker at date t, history z^t, can also supply one stopping time in the set $S(z^t)$ as long as he has no outstanding stopping time from a previous sale. "Newly arrived" workers and "incumbent" workers sell different commodities, though. The stopping time sold by an "incumbent" worker at date t, history z^t, entails a cost τ at date $s\left(z^t; z_{t+1}^{\infty}\right)$, for every possible realization z_{t+1}^{∞}. On the contrary, the stopping time sold by a "newly arrived" worker at date t, history z^t, entails a cost τ at date $s\left(z^t; z_{t+1}^{\infty}\right)$, only if the realization z_{t+1}^{∞} is such that $s\left(z^t; z_{t+1}^{\infty}\right) \geq t + J$.

Each stopping time, being a different commodity, has a different price associated with it. We express the price of the stopping times traded at date t, history z^t, in terms of the final consumption good at that time and event, and denote them for each $s \in S(z^t)$ by $P^A(z^t, s)$ and $P^I(z^t, s)$ for the "newly arrived" and "incumbent" stopping times, respectively. Workers and firms take the prices $P^A(z^t, s)$ and $P^I(z^t, s)$ for all $t \geq 0,\ z^t \in Z^t$, and $s \in S(z^t)$ as given.

The problem of an "incumbent" worker at date t, history z^t, if she has no outstanding stopping times, is the following:

$$I\left(z^t\right) = \max\left\{\theta,\ \max_{s \in S(z^t)} \left\{P^I\left(z^t, s\right) + E\left[\beta^{s-t}\ I\left(z^s\right)\right]\right\}\right\} \tag{25}$$

where the expectation is taken with respect to all possible realizations $z_{t+1}^{\infty} = \{z_{t+1}, z_{t+2}, ...\}$, conditional on z^t. This equation states that an incumbent worker can choose to leave the island, obtaining θ, or sell the stopping time $s \in S(z^t)$ that provides the highest value. A stopping time $s \in S(z^t)$ provides $P^I(z^t, s)$ units of the consumption good during the current period and the value $I(z^s)$ of being an incumbent worker at the (random) stopping time s. Observe that, since the worker maximizes the present expected value of his earnings, equation (25) implicitly assumes linear preferences.[11]

The problem of a "newly arrived" worker at time t state z^t is given by

$$A\left(z^t\right) = \max\left\{\theta,\ \max_{s \in S(z^t)} \left\{P^A\left(z^t, s\right) + E\left[\beta^{s-t}\ I\left(z^s\right)\right]\right\}\right\}.$$

This problem is analogous to the "incumbent" worker problem, except that the "newly arrived" worker faces a different price for the stopping time that she sells and becomes an "incumbent" worker at the end of the stopping time (i.e. she changes its type).

We let $N^A(z^t, s)$ be the quantity of newly arrived workers hired with contract $s \in S(z^t)$ at date t, history z^t. Likewise, we let $N^I(z^t, s)$ be the quantities of incumbent workers hired with contract $s \in S(z^t)$ at date t, history z^t. The firm chooses $N^A(z^t, s)$ and $N^I(z^t, s)$ for every z^t and $s \in S(z^t)$ to maximize expected discounted profits, taking as given the prices $P^A(z^t, s)$ and $P^I(z^t, s)$, and the fact that the stopping times of the different types of workers entail potentially different separation costs at termination. Without loss of generality, we assume that the firm never employed any worker previous to $t = 0$. This will have no consequence in the analysis given our focus on steady state equilibria.

The problem of the representative firm is the following:

$$\max_{N^A, N^I} \sum_{t=0} \sum_{z^t \in Z^t} \beta^t \left[F(n_t\left(z^t\right), z_t) - \sum_{s \in S(z^t)} \left(P^A\left(z^t, s\right) N^A\left(z^t, s\right) + P^I\left(z^t, s\right) N^I\left(z^t, s\right)\right) - T_t\left(z^t\right)\right] \mu_t\left(z^t\right)$$

[11] The linear preferences assumption in this "island-economy" is justified by the existence of perfect insurance markets in the original economy.

subject to:

$$n_t\left(z^t\right)=\sum_{i=0}^{t}\left\{\sum_{s\in S\left(z_0^i\right):\ s\left[z_0^i;\left(z_{i+1}^t,z_{t+1}^\infty\right)\right]\ >\ t,\ \text{for every}\ z_{t+1}^\infty}\left[N^A\left(z^i,s\right)+N^I\left(z^i,s\right)\right]\right\} \tag{26}$$

$$\begin{aligned} T_t\left(z^t\right) &= \tau\sum_{i=0}^{t-1}\left\{\sum_{s\in S\left(z_0^i\right):\ s\left[z_0^i;\left(z_{i+1}^t,z_{t+1}^\infty\right)\right]\ =\ t,\ \text{for every}\ z_{t+1}^\infty} N^I\left(z^i,s\right)\right\} \\ &+\tau\sum_{i=0}^{t-J}\left\{\sum_{s\in S\left(z_0^i\right):\ s\left[z_0^i;\left(z_{i+1}^t,z_{t+1}^\infty\right)\right]\ =\ t,\ \text{for every}\ z_{t+1}^\infty} N^A\left(z^i,s\right)\right\} \end{aligned} \tag{27}$$

where z_j^i in equations (26) and (27) denotes the partial history $\{z_j, z_{j+1}, ..., z_{i-1}, z_i\}$ embodied in z^t. The firm maximizes the expected discounted value of profits, which are given by output minus the purchase of the stopping times supplied both by "newly arrived" and "incumbent" workers, minus separation costs. The employment of the firm at date t, history z^t, is given by equation (26). This equation says that total employment is the sum of all the workers, both "newly arrived" and "incumbents", that were hired between periods zero and t and have been never fired along the history z^t. Equation (27) describes the separation costs at time and event z^t as the sum of two terms. The first term is the sum of all "incumbent" workers that have been hired between periods 0 and $t-1$, which have been contracted to separate at date t if event z^t took place. The second term is the sum of all "newly arrived" workers that have been hired between periods 0 and $t-J$, which have been contracted to separate at date t if event z^t took place. Observe that those "newly arrived" workers that have been hired between periods $t-J+1$ and $t-1$ and separate at date t and event z^t are not included in equation (27) because they separate during the trial period stipulated by the fixed term contracts and, thus, are not subject to separation costs.

The market clearing conditions are as follows. If $N^A\left(z^t,s\right)>0$ at some time t and event z^t and some $s\in S(z^t)$, then

$$A\left(z^t\right)=P^A\left(z^t,s\right)+E\left[\beta^{s-t}\ I\left(z^s\right)\right]$$

Also,

$$\sum_{s\in S(z^t)} N^A\left(z^t,s\right)<U\ \Rightarrow A\left(z^t\right)=\theta.$$

The conditions for "incumbent" workers are similar. If $N^I\left(z^t,s\right)>0$ at some time t and event z^t and some $s\in S(z^t)$, then

$$I\left(z^t\right)=P^I\left(z^t,s\right)+E\left[\beta^{s-t}\ I\left(z^s\right)\right].$$

Also,

$$\sum_{s\in S(z^t)} N^I\left(z^t,s\right)<X^I\left(z^t\right)\Rightarrow I\left(z^t\right)=\theta,$$

where $X^I\left(z^t\right)$ is the number of "incumbent" workers available for hiring at the beginning of time t and event z^t, which is given by:

$$X^I\left(z^t\right)=\sum_{i=0}^{t-1}\left\{\sum_{s\in S\left(z_0^i\right):\ s\left[z_0^i;\left(z_{i+1}^t,z_{t+1}^\infty\right)\right]\ =\ t,\ \text{for every}\ z_{t+1}^\infty}\left[N^I\left(z^i,s\right)+N^A\left(z^i,s\right)\right]\right\} \tag{28}$$

Finally, the hiring of each type of workers cannot exceed the amount initially available:

$$\sum_{s\in S(z^t)} N^A\left(z^t,s\right)\le U \tag{29}$$

$$\sum_{s\in S(z^t)} N^I\left(z^t,s\right)\le X^I\left(z^t\right) \tag{30}$$

Observe that the supply of stopping time is indivisible: Workers can only supply one stopping time $s\in S(z^t)$, only in the case that the worker has no previous outstanding stopping time. However, the linear preferences assumed, together with the convex production possibility set of the firm, guarantee that the welfare theorems hold. The competitive allocation is then obtained as the solution to the social planner's problem, which is to maximize

$$\sum_{t=0} \sum_{z^t \in Z^t} \beta^t \left[F(n_t\left(z^t\right), z_t) + \theta \left(U - \sum_{s \in S(z^t)} N^A\left(z^t, s\right) \right) \right.$$
$$\left. + \theta \left(X^I\left(z^t\right) - \sum_{s \in S(z^t)} N^I\left(z^t, s\right) \right) - T_t\left(z^t\right) \right] \mu_t\left(z^t\right)$$

subject to equations (26), (27), (28), (29) and (30).

A few remarks are in order. Clearly, the social planner will never want to separate a "newly arrived" worker and rehire him as a an "incumbent" before the trial period for the fixed-term contracts is over. The reason is that being rehired as "incumbent" makes the worker liable to separation costs, while maintaining his "newly arrived" status saves on separation costs during the trial period. Also, the social planner will never want to separate a "newly arrived" worker after the trial period is over and rehire him under an "incumbent" contract because this entails incurring the separation cost τ without any benefit. As a consequence, the planner will choose the stopping times for "newly arrived" workers in such a way that they separate only when they are to leave the island (and receive the value θ). This means that $N^I\left(z^t, s\right) = 0$ for every z^t and every $s \in S\left(z^t\right)$.

Being left with only "newly arrived" workers, the planner's problem is formally identical to the island planning problem described in Section 4.1.[12] This has an important implication: The competitive equilibrium with long-term contracts and tenure at the firm level described in this appendix is equivalent to the competitive equilibrium with spot labor contracts and tenure at the island level described in Appendix C. Moreover, for every z^t and $s \in S(z^t)$ such that $N^A\left(z^t, s\right) > 0$, the price $P^A\left(z^t, s\right)$ must be equal to the expected discounted value of the spot wages obtained (in the equilibrium with spot labor contracts and tenure at the island level) by a worker that arrives to the island at time t and event z^t, and follows the employment plan described by the stopping time s. Hence, $A\left(z^t\right) = W_0\left(T\left(z^t, X\right), z_t\right)$, where T is defined by equations (1)-(3) and the Pareto optimal employment process $\{E_t\}$. Given this property, an economy-wide equilibrium can be constructed from the island-level competitive equilibrium described thus far in the same way as it is done in the "spot labor market" competitive equilibrium described in Appendix C.

[12] In particular, it is identical to the problem of an island's planner endowed with no worker of positive tenure at $t = 0$.

Appendix E: Calibration of τ.

Heckman and Pages-Serra (2000) propose to summarize employment protection policies into a single statistic. The measure they use is the expected discounted cost at the time that a worker is hired of dismissing that worker in the future. Their index I is given by

$$I = \sum_{t=1}^{T} \beta^t \delta^{t-1} (1-\delta) \left\{ b_t + a S_t^j + (1-a) S_t^u \right\}$$

where T is the maximum tenure consider in the index, β a time discount factor, δ is the survival rate (probability of remaining employed next period if employed during the current period), b_t is wage earned during the advance notice period for a worker of tenure t, S_t^j is the severance payment to a worker of tenure t if the dismissal is classified as "justified" (i.e. "fair" or "objective"), and S_t^u is the severance payment to a worker of tenure t if the dismissal is "not justified".

Heckman and Pages-Serra (2000) use a year as the time period, and the following values: $\beta = 0.92$ (an 8 percent interest rate), $\delta = 0.88$ (a turnover rate of 12 percent, based on data for the US), a value of T of 20 years, and for Spain they advocate to use $a = 0.2$ for the period before 1997, based instead on the information on Bertola, Boeri and Cazes (2000), "Employment protection, the case of Industrialized countries: the case for new indicators", International Labor Review, 139(1):2000. Heckman and Pages-Serra (2000) compute their job security index for Spain in the late nineties. Since we calibrate our model to the period before the broadening in the applicability of temporary contracts, we recompute their index for the policies in place before the 1984 reform. We use the following values:

- b_t : one month of wages for tenure 1 and 2 and 3 months for higher tenure (from Chapter 2 of OECD Labor Outlook, 1999, Table 2.2)
- a : 0.2 (since their argument applies prior to 1984)
- S_t^j : 2/3 months per year up to a maximum of 12 months (from Chapter 2 of OECD Labor Outlook, 1999, page 96)
- S_t^u : 1 1/2 months per year up to a maximum of 42 months (from Chapter 2 of OECD Labor Outlook, 1999, page 101).

We consider two cases. Case 1: With these choices for b_t, a, S_t^u and S_t^j, and using the values for β and δ used by Heckman and Pages-Serra (2000), we obtain that prior to 1984 I equals to 0.42 as a fraction of annual average wages. Case 2: if instead we use $\beta = 0.96$, which is the value we use in our paper, and $\delta = 0.93$, which is closer to the one for Spain prior to 1984 according to Hopenhayn and Cabrales (1997), we obtain a value of I prior to 1984 of 0.56 as a fraction of annual wages.

Finally, since in our benchmark case the firing taxes do not depend on the tenure of the workers, we select the value of τ so that the value of the index above will give the value we calibrate for Spain prior to the reform. This value solves the equation:

$$I = \sum_{t=1}^{T} \beta^t \delta^{t-1} (1-\delta) \ \tau = \tau \ (1-\delta) \beta \frac{1-(\beta\delta)^T}{1-\beta\delta}$$

or

$$\tau = I \frac{1-\beta\delta}{(1-\delta) \ \beta \ \left(1-(\beta\delta)^T \right)}$$

The value of τ that corresponds to the first case is 0.74, and to the second case is 0.98 of annual wages. We think that for our purposes the choices of the second case better reflect the situation prior to 1984 in Spain and hence we calibrate the model to τ equivalent to one year of average wages.

Appendix F: Proofs

Proof of Theorem 1. To show this theorem we characterize the competitive equilibrium of a particular decentralization of the economy. Since the first welfare theorem holds, characterizing this equilibrium gives us a characterization of the efficient allocations. We call this equilibrium "auxiliary competitive equilibrium" or "ACE" for short. See Appendix G below for a definition of the ACE. The characterization of a stationary ACE coincides with conditions i) to iv) of Theorem 1.

We start by providing some of the necessary conditions that an ACE must satisfy.

Lemma 37 *Let $\{\theta_t,\ \lambda_t(z^t,X),\ E_{j,t}(z^t,X),\ T_{j,t}(z^t,X),\ S_{j,t}(z^t,X),\ U_t,\ L_t;\ \text{all } t, z^t, j, X\}$ be an ACE. Then, there is sequence $\{\sigma_t\}$ where σ_t is the value of search at t, for which:*
i) without loss of generality, $\theta_t(z^t,X)=\theta_t$,
ii)

$$\begin{aligned}\sigma_t &= \beta\sum_X\sum_{z^{t+1}}\lambda_{t+1}\left(z^{t+1},X\right)\eta\left(X|z_0\right)q_t\left(z^t\right)\\ \theta_t &= \max\left\{\omega+\beta\theta_{t+1},\sigma_t\right\}\\ 0 &= L_t\left[\theta_t-\omega-\beta\theta_{t+1}\right]\end{aligned}$$

and
iii) for all z^t, X,

$$\begin{aligned}\theta_t &\leq \lambda_t\left(z^t,X\right),\\ 0 &= \left[\lambda_t\left(z^t,X\right)-\theta_t\right]\left[T_{0,t}\left(z^t,X\right)-E_{0,t}\left(z^t,X\right)\right].\end{aligned}$$

The proof of this Lemma follows directly from the linearity of the problem of firms type II.

This Lemma shows, among other things, that the value to a firm type I of reallocating (firing) a worker does not depend on the characteristic of the island, so that θ_t does not depend on (z^t,X) and the value of search σ_t is related to the value of "selling" (assigning) a worker to a different island randomly, i.e. in proportion to the number of islands of each type.

We will show that the ACE allocation can be obtained by solving a particular dynamic programing problem given two numbers (θ,U), and by checking two appropriate equilibrium conditions. We develop this characterization in a sequence of results.

The solution of the dynamic programing problem will give the equilibrium quantities chosen by firms type I and the equilibrium prices $\lambda_t(z^t,X)$. This problem has the interpretation of the maximization problem solved for a coalition of firms type I that are endowed with a flow $\mathbf{U}=\{U_t\}_{t=0}^{\infty}$ of newly arrived workers. We refer to this problem as the "island planning problem", i.e. the problem of a planner in charge of the island employment decision by tenure. The planner chooses how many workers to employ at each tenure level and how many to send back, obtaining θ_t for each of them, net of the cost τ.

Definition 38 *Let $V_t: R_+^J\times Z\times R_+^\infty\to R$*

$$\begin{aligned}& V_t\left(T_1,...T_J;z_t,\mathbf{U}\right)\\ =\ & \max_{E_j,j=0,...J}\Big\{\ F\left(\sum_{j=0}^J E_j,z_t\right)\\ & +\sum_{j=0}^J\left[T_j-E_j\right]\theta_t-\tau\left[T_J-E_J\right]\\ & \beta\sum_{z_{t+1}\in Z}V_{t+1}\left(E_0,...,E_{J-1}+E_J;z_{t+1},\mathbf{U}\right)\ \Big\}Q\left(z_{t+1}|z_t\right)\ \Big\}\end{aligned}$$

subject to

$$\begin{aligned}T_0 &= U_t\\ E_j &\leq T_j\ j=0,1,...,J.\end{aligned}$$

where $\mathbf{U}=\{U_t;\ \text{all } t\geq 0\}\in R_+^\infty$.

The next Lemma links the island planning problem with the equilibrium quantities chosen by firms type I and the prices $\{\lambda_t\}$.

Lemma 39 *Let* $\left\{\theta_t^*,\ \lambda_t^*(z^t,X),\ E_{j,t}^*(z^t,X),\ T_{j,t}^*(z^t,X),\ S_{j,t}^*(z^t,X),\ U_t^*,\ L_t^*;\ all\ t,z^t,j,X\right\}$ *be an auxiliary competitive equilibrium given initial conditions* $U_{-1}^*, \eta^*(X|z_0)$*. Define* $\hat{V}_t$ *for* $\{U_t^*, \theta_t^*\}$*, and let* $\hat{E}_{j,t}(T,z)$ *be its optimal policy. Then,* $\left\{E_{j,t}^*(z^t,X)\right\}$ *solves* V_t *for all the initial conditions* X*, i.e.*

$$\hat{E}_{j,t}\left(T_t^*\left(z^t,X\right),z_t\right)=E_{j,t}^*\left(z^t,X\right)\ \textit{for all}\ t,z^t,X$$

and

$$\lambda_t^*\left(z^t,X\right)=\partial V_t\left(T_t^*\left(z^t,X\right),z_t;\mathbf{U}^*\right)\ \textit{for all}\ t,z^t,X$$

where $\partial V_t(T,z_t;\mathbf{U}^*)$ *is and element of the subgradient of* $V_t\left(T,z_t;\left\{U_0^*,...,U_{t-1}^*,\cdot,U_{t+1}^*,...\right\}\right)$ *with respect to* U_t^**.*

The proof of this Lemma follows from comparing the island planning problem with the problem of firms type I in a competitive equilibrium, and from the definition of a subgradient.

The next Lemma gives the characterization of an ACE.

Lemma 40 *. Let some arbitrary initial distribution* $\eta^*(X|z_0)$ *be given. Let also some arbitrary sequence* $\{U_t^*,\theta_t^*:$*all* $t\}$ *be given. Let* $\hat{E}_{j,t}(T,z)$ *be the optimal policy of the island planning problem in condition (38) defined for* $\{U_t^*,\theta_t^*\}$*. Define* $\left\{E_{j,t}^*\right\}$ *as*

$$E_{j,t}^*\left(z^t,X\right)=\hat{E}_j\left(T_t^*\left(z^t,X\right),z_t\right)$$

where $T_{t,j}^*$ *has been generated by* $\left\{\hat{E}\right\}$ *and the initial condition* X*, i.e.*

$$\begin{aligned}T_{0,j}^* &= X_j\\ T_{t,j}^*\left(z^t,X\right) &= \hat{E}_{j-1}\left(T_{t-1}^*\left(z^{t-1},X\right),z_{t-1}\right)\ \textit{for}\ j=1,...,J-1\\ T_{t,J}^*\left(z^t,X\right) &= \hat{E}_{J-1}\left(T_{t-1}^*\left(z^{t-1},X\right),z_{t-1}\right)+\hat{E}_J\left(T_{t-1}^*\left(z^{t-1},X\right),z_{t-1}\right)\end{aligned}$$

and let $\partial V_t(T,z_t;U^*)$ *be an element of the subgradient of* $V_t\left(T,z_t;\left\{U_0^*,...,U_{t-1}^*,\cdot,U_{t+1}^*,...\right\}\right)$ *with respect to* U_t^**.*

i) Define $\left\{\lambda_{t+1}^*\right\}$ *as*

$$\lambda_t^*\left(z^t,X\right)=\partial V_t\left(T_t^*\left(z^t,X\right),z_t;U^*\right).$$

ii) Define the value of search $\{\sigma_t\}$ *as*

$$\sigma_t=\beta\sum_{z^{t+1}}\sum_X\lambda_{t+1}^*\left(z^{t+1},X\right)q_{t+1}\left(z^{t+1}\right)\eta^*\left(X|z_0\right)\ \textit{for all}\ t.$$

iii) Define $\{L_t^*\}$ *as*

$$L_t^*=N-U_t^*-\sum_{z^t}\sum_X E_{j,t}^*\left(z^t,X\right)q_t\left(z^t\right)\eta^*\left(X|z_0\right)\geq 0\ \textit{for all}\ t.$$

iv) Suppose that the following optimal labor force participation conditions are satisfied

$$\theta_t^*=\max\left\{\sigma_t,\omega+\beta\theta_{t+1}^*\right\},$$

$$L_t^*\ \left[\theta_t-\omega+\beta\theta_{t+1}\right]=0$$

for all t*. Then* $\left\{\theta_t^*,\ \lambda_t^*(z^t,X),\ E_{j,t}^*(z^t,X),\ T_{j,t}^*(z^t,X),\ S_{j,t}^*(z^t,X),\ U_t^*,\ L_t^*;\ all\ t,z^t,j,X\right\}$ *is an auxiliary competitive equilibrium (ACE) given the initial conditions* U_{-1}^* *and* η^**.*

The proof of this Lemma follows by construction, the definition of competitive equilibrium, and the properties of the island planning problem defined in (38).

Since the first welfare theorem holds for this economy, the characterization of the allocation for an ACE in the previous Lemma applies to the efficient allocations.

Now we define a stationary ACE in terms of the objects used our previous characterization of the ACE.

Definition 41 *We say that the ACE* $\{\theta_t, \lambda_t, L_t, U_t, E_{jt}, S_{jt}\}$*, for an initial measure* η*, is a stationary equilibrium if there are constants,* θ, U, L*, and functions,* $E_j^* : R_+^{J+1} \times Z \to R,\ j = 0, ..., J$*,* $\lambda^* : R_+^{J+1} \times Z \to R$*, for which*

$$\begin{aligned}
\theta_t &= \theta,\ \textit{all } t \\
U_t &= U,\ \textit{all } t \\
L_t &= N,\ \textit{all } t \\
E_{i,t}\left(z^t, X\right) &= E_j^*\left(T_t\left(z^t, X\right), z_t\right),\ \textit{all } t, z^t \\
\lambda_t\left(z^t, X\right) &= \lambda^*\left(T_t\left(z^t, X\right), z_t\right),\ \textit{all } t, z^t
\end{aligned}$$

and where defining $T_j' : R_+^{J+1} \times Z \to R$ *as*

$$\begin{aligned}
T_0'(T, z) &= U, \\
T_j'(T, z) &= E_{j-1}^*(T, z) \ \textit{for } j = 1, ..., J-1, \\
T_J'(T, z) &= E_J^*(T, z) + E_{J-1}^*(T, z)
\end{aligned}$$

and letting μ *be an invariant distribution of the joint process* (T, z)*, with transition given by* (T', Q)*, we have*

$$\eta(T|z)\,\zeta(z) = \mu(T, z)$$

where $\zeta(z)$ *is the invariant distribution of* z*.*

Finally, since a stationary ACE is a particular type of ACE, then by the previous application of the first welfare theorem, the stationary version of conditions i) to iv) in Lemma 40 characterizes a stationary efficient allocation. Since the stationary version of conditions i) to iv) in Lemma 40 coincide with conditions i) to iv) of Theorem 1, we have finished its proof. ■

Proof of Proposition 3 The proof is standard, so we omit it. ■

Proof of Proposition 4 We first show that (15) holds. Consider two states $T > 0$ and $T' > 0$, where T' is obtained from T by increasing the number of workers with tenure J by δ and by decreasing the number of workers with tenure 1 by δ:

$$\begin{aligned}
T_j' &= T_j \text{ for } j = 2, ..., J-1 \\
T_1' &= T_1 - \delta \text{ and } T_J' = T_J + \delta
\end{aligned}$$

It suffices to show that there is a feasible policy for T' that produces a reduction in total payoff at most by τ and thus

$$H[V(T', z)] - H[V(T, z)] \geq -\tau.$$

To establish this, consider two cases, depending on whether in the original plan more than δ workers with tenure 1 were fired or not. Let δ be a positive number smaller than $T_1/2$. In the case where more than δ workers with tenure 1 were fired, reduce the firing of workers with tenure 1 by δ and increase the firing of workers with tenure J by δ. Then there is a reduction in current payoff of τ, and no change in the future state. In the second case, let

$$\begin{aligned}
& \frac{1}{\delta}\left(H[V(T', z)] - H[V(T, z)]\right) \\
\geq\ & \frac{1}{\delta}\beta E\left[V\left(\tilde{E}_0, ..., \tilde{E}_{J-2}, \tilde{E}_{J-1} + \tilde{E}_J, z'\right)|z\right] \\
& -\frac{1}{\delta}\beta E\left[V\left(E_0, ..., E_{J-2}, E_{J-1} + E_J, z'\right)|z\right]
\end{aligned}$$

where

$$\begin{aligned}
\tilde{E}_j &= E_j \ \text{ for } j = 2, ..., J-1 \\
\tilde{E}_1 &= E_1 - \delta \\
\tilde{E}_J &= E_J + \delta
\end{aligned}$$

which is feasible given the stated assumptions. Thus using the properties of directional derivatives and subgradients of concave functions

$$\begin{aligned}
& \lim_{\delta \to 0} \frac{1}{\delta} V\left(\tilde{E}_0, ..., \tilde{E}_{J-2}, \tilde{E}_{J-1} + \tilde{E}_J, z'\right) - V\left(E_0, ..., E_{J-2}, E_{J-1} + E_J, z'\right) \\
= & \min_{(V_1,...,V_J) \in \partial V} \left\{ (V_J - V_1)\left(E_0, ..., E_{J-2}, E_{J-1} + E_J, z'\right) \right\} \\
= & -\tau + \min_{(V_1,...,V_J) \in \partial V} \left\{ (V_J - V_1)\left(E_0, ..., E_{J-2}, E_{J-1} + E_J, z'\right) + \tau \right\} \\
= & -\tau + \min_{(V_1,...,V_J) \in \partial V} \left\{ \lim_{\varepsilon \downarrow 0} (V_J - V_1)\left(E_0 + \varepsilon, ..., E_{J-2} + \varepsilon, E_{J-1} + E_J + \varepsilon, z'\right) + \tau \right\} \\
= & -\tau + \lim_{\varepsilon \downarrow 0} \min_{(V_1,...,V_J) \in \partial V} \left\{ (V_J - V_1)\left(E_0 + \varepsilon, ..., E_{J-2} + \varepsilon, E_{J-1} + E_J + \varepsilon, z'\right) + \tau \right\} \\
\geq & -\tau
\end{aligned}$$

where we use theorem 24.4, page 233, of Rockafellar (1997) which shows that the graph of ∂f is closed for a concave function on R^n, the hypothesis that (13) holds for all subgradients with $T > 0$, and where we denote

$$\begin{aligned}
& (V_J - V_1)\left(E_0, ..., E_{J-2}, E_{J-1} + E_J, z'\right) \\
\equiv & V_J\left(E_0, ..., E_{J-2}, E_{J-1} + E_J, z'\right) + \tau - V_1\left(E_0, ..., E_{J-2}, E_{J-1} + E_J, z'\right).
\end{aligned}$$

Finally since for all subgradients

$$H[V]_J(T, z) - H[V]_1(T, z) \geq \lim_{\delta \to 0} \frac{1}{\delta}\left(H[V(T', z)] - H[V(T, z)]\right),$$

then

$$H[V]_J(T, z) - H[V]_1(T, z) \geq -\tau.$$

The argument to show that (14) holds follows a similar argument, where we let

$$T_j' = T_j + \delta \text{ and } T_{j+1}' = T_{j+1} - \delta$$

for $j = 1, ..., J-1$. ■

Proof of Proposition 5 Replacing any policy by one with these properties can not decrease output but can decrease the separation cost τ. ■

Proof of Lemma 8. Let λ^* be a Lagrange multiplier, then $\lambda^*(T - E^*) = 0$. Consider T', and $E' = E(T', z)$, then

$$\begin{aligned}
& H[V](T, z) - \hat{\theta} T \\
= & \hat{R}(E(T), z) \\
\geq & \hat{R}(E(T'), z) + \lambda^*(E(T) - E(T')) \\
\geq & \hat{R}(E(T'), z) + \lambda^*(T - T') \\
= & H[V](T', z) - \hat{\theta} T' + \lambda^*(T - T')
\end{aligned}$$

thus $\hat{\theta} + \lambda^*$ is a subgradient of $H[V]$. Let $\hat{\theta} + \lambda^*$ be a subgradient of $H[V](T, z)$. Since workers can always be sent back and get $\hat{\theta}$, then $\lambda^* \geq 0$. Also,

$$H[V](T, z) = H[V](E^*, z) + \hat{\theta}[T - E^*]$$

for $E^* = E(T, z)$. Then, by definition of subgradient

$$\hat{\theta}[T - E^*] = H[V](T, z) - H[V](E^*, z) \geq \left(\hat{\theta} + \lambda^*\right)(T - E^*)$$

or

$$0 = \hat{R}(T, z) - \hat{R}(E^*, z) \geq \lambda^*(T - E^*)$$

but $E^* \leq T$ so $\lambda^*(T - E^*) = 0$. This equality, together with the definition of a subgradient, implies that λ^* is a Lagrange multiplier. ■

Proof of Lemma 9. Let (E^*, λ^*) be a saddle satisfying (18). Then, by Theorem 2.9 in Takayama, E^* is optimal. Let E^* be optimal. Then, by Theorem 2.9 in Takayama there are $\lambda^* \geq 0$ such that (E^*, λ^*) is a saddle. It remains to show that $\lambda_i^* = \hat{R}_i^*$ for some subgradient. From the definition of a saddle,

$$\hat{R}(E^*, z) + \lambda^* (T - E^*) \geq \hat{R}(E, z) + \lambda^* (T - E)$$

or

$$\hat{R}(E^*, z) \geq \hat{R}(E, z) + \lambda^* (E^* - E).$$

which is the definition of a subgradient. ■

Proof of Proposition 10. Let E^* be optimal. Take any $\{H[V]_i (T, z)\}_{i=0}^J \in \partial H[V](T, z)$. By Lemma 8 λ^* is a Lagrange multiplier, where

$$\{H[V]_i (T, z)\}_{i=0}^J = \lambda^* + \hat{\theta}.$$

By Lemma (9), $\lambda_i^* = \hat{R}_i^*$ for some subgradient. Then $\hat{R}_i \geq 0$, and

$$0 = \hat{R}_i^* (T_i - E_i^*) = (H[V]_i (T, z) - \theta)(T_i - E_i^*).$$

Let $\{H[V]_i (T, z)\}_{i=0}^J \in \partial H[V](T, z)$, and let R_i^* be a subgradient of $\hat{R}^*$ evaluated at some $0 \leq E^* \leq T$ such that the above conditions are satisfied. Define λ^* as

$$\lambda^* = \{H[V]_i (T, z)\}_{i=0}^J - \hat{\theta} = \{R_i^*\}_{i=0}^J$$

where the last equality follows from the assumed properties. We will show that (E^*, λ^*) is a saddle. From the above conditions,

$$0 = \lambda_i^* (T_i - E_i^*)$$

Hence,

$$\hat{R}(E^*, z) + \lambda (T - E^*) \geq \hat{R}(E^*, z) + \lambda^* (T - E^*), \text{ for every } \lambda \geq 0$$

Since, by the above conditions, λ^* is a subgradient of $\hat{R}^*$ evaluated at $0 \leq E^* \leq T$, it follows that

$$\hat{R}(E, z) \leq \hat{R}(E^*, z) + \lambda^* (E - E^*), \text{ for every } E$$

Hence,

$$\hat{R}(E^*, z) + \lambda^* (T - E^*) \geq \hat{R}(E, z) + \lambda^* (T - E), \text{ for every } E$$

so that E^* is optimal.

If $E_i^* > 0$,

$$\begin{aligned} \hat{R}_i (E^*, z) &= f\left(\sum_{i=0}^J E_i^*, z\right) - \theta \\ &\quad + \beta \int V_{i+1} \left(E_0^*, \ldots, E_{J-2}^*, E_{J-1}^* + E_J^*, z'\right) Q(z, dz') \end{aligned}$$

follows since $\partial (g + h)(x) = \partial g(x) + \partial h(x)$ (see Rockafeller, Theorem 23.8), and F is differentiable with derivative f. When $E_i^* = 0$, the subgradient of F is any number greater than f, and hence the previous expression holds with inequality. ■

Proof of Proposition 11. From Proposition **10** and $E_J^* < T_J$ we have that

$$H[V]_J (T, z) = f\left(\sum_{i=0}^J E_i^*, z\right) + \beta \int V_J \left(E_0^*, \ldots, E_{J-2}^*, E_{J-1}^* + E_J^*, z'\right) Q(z, dz') = \theta - \tau.$$

Suppose that $E_i^* > 0$. Then,

$$H[V]_i (T, z) = f\left(\sum_{i=0}^J E_i^*, z\right) + \beta \int V_{i+1} \left(E_0^*, \ldots, E_{J-2}^*, E_{J-1}^* + E_J^*, z'\right) Q(z, dz')$$

Therefore,

$$\begin{aligned} H[V]_i(T,z) - H[V]_J(T,z) &= \beta \int \left[V_{i+1}\left(E_0^*,\ldots,E_{J-2}^*,E_{J-1}^*+E_J^*,z'\right)\right. \\ &\quad \left. - V_J\left(E_0^*,\ldots,E_{J-2}^*,E_{J-1}^*+E_J^*,z'\right)\right] Q(z,dz') \end{aligned}$$

If $i = J-1$ it follows that $H[V]_i(T,z) = H[V]_J(T,z) = \theta - \tau$. But from Lemma 8, $H[V]_i(T,z) \geq \theta$.
If $i < J-1$ it follows that

$$\begin{aligned} H[V]_i(T,z) - \theta + \tau &= \beta \int \left[V_{i+1}\left(E_0^*,\ldots,E_{J-2}^*,E_{J-1}^*+E_J^*,z'\right)\right. \\ &\quad \left. - V_J\left(E_0^*,\ldots,E_{J-2}^*,E_{J-1}^*+E_J^*,z'\right)\right] Q(z,dz') \\ &\leq \beta\tau \end{aligned}$$

where the inequality follows from equations (14) and (15).
Therefore,

$$H[V]_i(T,z) \leq \theta - (1-\beta)\tau$$

But from Lemma 8, $H[V]_i(T,z) \geq \theta$.. ∎

Proof of Lemma 12. By contradiction , for all $e > 0$, there is a T, z such that

$$\sum_{i=0}^{J} E_i(T,z) \leq e,$$

Take $e < U$ such that

$$f(e,\underline{z}) > \theta.$$

where $\underline{z} = min\{z : z \in Z\}$. Since $T_0 = U > 0,\ E_0(T,z) < T_0$. From 10

$$0 = [H[V]_0(T,z) - \theta][T_0 - E_0]$$

thus

$$H[V]_0(T,z) = \theta$$

but

$$H[V]_0(T,z) = \hat{R}_0(E^*,z) + \theta$$

so $\hat{R}_0(E^*,z) = 0$. Since

$$\begin{aligned} 0 &= \hat{R}_0(E^*,z) \geq f\left(\sum_{i=0}^{J} E_i^*, z\right) - \theta \\ &\quad + \beta \int V_1\left(E_0^*,\ldots,E_{J-2}^*,E_{J-1}^*+E_J^*,z'\right) Q(z,dz') \\ &> \beta \int V_1\left(E_0^*,\ldots,E_{J-2}^*,E_{J-1}^*+E_J^*,z'\right) Q(z,dz') \\ &\geq 0. \end{aligned}$$

∎

Proof of Proposition 13. Let T, z be such that $T_i > 0$. Assume that $\left\{E_{j,s}^*\right\}$ is optimal. Take a subgradient $V_i(T,z) = H[V]_i(T,z)$.
First consider the case where $E_{i,s}^* = 0$. Then

$$[H[V]_i(T,z) - \theta_i(s)]\left[T_{i,s} - E_{i,s}^*\right] = 0$$

and its unique solution is $H[V]_i(T,z) = \theta_i(s)$, provided that $T_{i,s} > 0$. Thus, as an special case, if $E_{i,0}^* = 0,\ \theta_i(0)$ is the derivative of V.

Now consider the case where $E^*_{i,s} > 0$. We use the formula in Proposition 10 and replace its value repeatedly, solving it forward until $\hat{\tau}_j = s$, the first time that for this cohort, employment is smaller than the number of workers present at the location. Since $E^*_{i,s} > 0$ at each iteration

$$\begin{aligned} H[V]_i(T,z) &= f\left(\sum_{i=0}^{J} E^*_i, z\right) \\ &+\beta \int V_{i+1}\left(E^*_0, \ldots, E^*_{J-2}, E^*_{J-1} + E^*_J, z'\right) Q(z, dz'). \end{aligned}$$

Notice that in this case, we argue above that $V_i = \theta_i(s)$. Thus, we find that the unique solution for $H[V]_i(T,z)$ is $V^*_i(T,z)$. Hence the subgradient is unique, and thus $V(T,z)$ is differentiable. ■

Proof of Proposition 14 By Proposition 5 and its corollaries, $h[v] = H[V]$ in $\mathcal{E}$. ■

Proof of Lemma 15 If follows directly from the definition of $\mathcal{E}$ and the assumed property (12). ■

Proof of Proposition 16. Take (t_1, p_1) and (t_2, p_2) and consider their convex combination $(t_\lambda, p_\lambda) = (\lambda t_1 + (1-\lambda) t_2,\ \lambda p_1 + (1-\lambda) p_2)$. Let the unique corresponding elements in $\mathcal{E}$ for (t_1, p_1) and (t_2, p_2) be T_1 and T_2. Consider $T_\lambda = \lambda T_1 + (1-\lambda) T_2$, which is not necessarily on $\mathcal{E}$. Let $\hat{T}_\lambda$ be the unique element in $\mathcal{E}$ that corresponds to T_λ. Note that (t_λ, p_λ) satisfies

$$t_\lambda = \sum_{j=1}^{J-1} \hat{T}_\lambda \text{ and } p_\lambda = \hat{T}_J.$$

Then,

$$\begin{aligned} & \lambda h[v](t_1, p_1, z) + (1-\lambda) h[v](t_2, p_2, z) \\ = & \lambda H[V](T_1, z) + (1-\lambda) H[V](T_2, z) \\ \leq & H[V](T_\lambda, z) \\ \leq & H[V]\left(\hat{T}_\lambda, z\right) \\ = & h[v](t_\lambda, p_\lambda, z), \end{aligned}$$

where the first equality follows from Proposition 14, the first inequality follows from concavity of V and Proposition 3 , the second inequality follows from Lemma 15, and the last equality follows from Proposition 14. ■

Proof of Proposition 21. First define

$$\begin{aligned} \hat{R}(E,z) &= F\left(\sum_{j=0}^{J} E_j, z\right) - \theta \sum_{j=0}^{J-1} E_j - (\theta - \tau) E_J \\ &+\beta \int V(U, E_0, \ldots, E_{J-2}, E_{J-1} + E_J, z') Q(z, dz'). \end{aligned}$$

Since V and F are concave, then $\hat{R}$ is concave. Now take $\left(e^i_t, e^i_p\right)$ for $i = 1, 2$ and consider $\left(e^\lambda_t, e^\lambda_p\right) = (\lambda e^1_t + (1-\lambda) e^2_t, \lambda e^1_p + (1-\lambda) e^2_p)$. Let the unique corresponding elements to $\left(e^i_t, e^i_p\right)$ in $[0,U]^J \times R_+$ that satisfies property (*) in Corollary 6 be denoted by $\tilde{E}^i$ for $i = 1, 2$. Define $E^\lambda = \lambda \tilde{E}^1 + (1-\lambda) \tilde{E}^2$. Note that

$$\sum_{j=0}^{J-1} E^\lambda_j = e^\lambda_t \text{ and } E_J = e^\lambda_p.$$

Define $\tilde{E}^\lambda$ as the unique element in $[0,U]^J \times R_+$ that satisfies property (*) and such that

$$\sum_{j=0}^{J-1} E^\lambda_j = \sum_{j=0}^{J-1} \tilde{E}^\lambda_j \text{ and } E_J = \tilde{E}_J.$$

Then

$$\begin{aligned}
& \lambda R\left(e_p^1, e_t^1, z\right)+(1-\lambda) R\left(e_p^2, e_t^2, z\right) \\
= & \lambda \hat{R}\left(\tilde{E}^1, z\right)+(1-\lambda) \hat{R}\left(\tilde{E}^2, z\right) \\
\leq & \hat{R}\left(E^\lambda, z\right) \\
\leq & \hat{R}\left(\tilde{E}^\lambda, z\right) \\
= & R\left(e_p^\lambda, e_t^\lambda, z\right),
\end{aligned}$$

where the first equality follows from construction of $\tilde{E}^i$ and since by assumption v and V satisfy (20), the first inequality follows from the concavity of $\hat{R}$, the second inequality follows by assumption (12) and Proposition 5 and its corollaries, and the last equality follows from the same argument as the one in Proposition 14. ■

Proof of Proposition 24 Define the operator $\bar{h}$ as

$$\begin{aligned}
\bar{h}[v](t, p, z) \quad = & \max_{0 \leq e_t, 0 \leq e_p}\left\{F\left(e_t+e_p, z\right)+\theta\left[t-e_t\right]+(\theta-\tau)\left[p-e_p\right]+\right. \\
& \left.+\beta \int v\left(U+M\left(e_t\right), e_p+e_t-M\left(e_t\right), z^{\prime}\right) Q\left(z, d z^{\prime}\right)\right\}
\end{aligned}$$

Comparing this problem with (19) the constraints $e_t \leq t$ and $e_p \leq p$ were removed, hence

$$h[v](t, p, z) \leq \bar{h}[v](t, p, z) .$$

The optimal policies e_t, e_p do not depend on t and p, thus the function $\bar{h}[v]$ is linear with derivatives

$$\begin{aligned}
\bar{h}[v]_p(t, p, z) & =\theta-\tau \\
\bar{h}[v]_t(t, p, z) & =\theta
\end{aligned}$$

for all t, p, z. By concavity of $h[v]$,

$$\begin{aligned}
h[v](t, 0, z) & \leq h[v](t, p, z)+h[v]_p(0-p) \text { or } \\
h[v](t, p, z) & \geq h[v]_p p+h[v](t, 0, z)
\end{aligned}$$

where $\left(h[v]_t, h[v]_p\right) \in \partial h[v](t, p, z)$. Rearranging and using the linearity of $\bar{h}[v]$:

$$h[v]_p p+h[v](t, 0, z) \leq h[v](t, p, z) \leq \bar{h}[v](t, p, z)=\bar{h}[v](t, 0, z)+[\theta-\tau] p$$

for all p. Thus by monotonicity of $h[v]$ and $\bar{h}[v]$ on t :

$$h[v]_p(t, p, z) p+h[v](0,0, z) \leq \bar{h}[v]((J-1) U, 0, z)+[\theta-\tau] p$$

$$\sup_{t \in[0, U(J-1)]} h[v]_p(t, p, z) p+h[v](0,0, z) \leq \bar{h}[v]((J-1) U, 0, z)+[\theta-\tau] p$$

Hence

$$\begin{aligned}
& \liminf_{p \rightarrow \infty} \frac{h[v](0,0, z)-\bar{h}[v](U(J-1), 0, z)}{p} \\
= & 0 \leq \liminf_{p \rightarrow \infty}\left([\theta-\tau]-\sup_{t \in[0, U(J-1)]} h[v]_p(p, t, z)\right)
\end{aligned}$$

or

$$\limsup_{p \rightarrow \infty}\left[\sup_{t \in[0, U(J-1)]} h[v]_p(p, t, z)\right] \leq \theta-\tau .$$

On the other hand, for the original problem (19), for $\left(p_0, t, z\right)$, a feasible policy for $p \geq p_0$ is to set $e_p^0=e_p\left(p_0, t, z\right)$, in which case each additional unit of p yields $\theta-\tau$. Hence the right derivative of $h[v]\left(t, p_0, z\right)$ is greater or equal than $\theta-\tau$. Since $h[v]$ is concave, then $h[v]_p\left(t, p_0, z\right) \geq \theta-\tau$ for all $\left(t, p_0, z\right)$.

Combining the two inequalities, for large enough p, $h[v]_p(p,t,z) = \theta - \tau$ for all t. ■

Proof of Lemma 25. Consider two cases. First $e_t < (J-1)U$. In this case $M(e_t) = e_t$, which implies that

$$b(e_t, e_p, z) = \int v(U + e_t, e_p, z')\, Q(z, dz'),$$

thus

$$\begin{aligned} b_{e_t} &= \int v_t dQ, \\ b_{e_p} &= \int v_p dQ \end{aligned}$$

where

$$(v_t, v_p) \in \partial v(U + e_t, e_p, z')$$

for the corresponding elements. Second, if $e_t > (J-1)U$,

$$b(e_t, e_p, z) = \int v(JU, e_p + e_t - (J-1)U, z')\, Q(z, dz'),$$

thus

$$\begin{aligned} b_{e_t} &= \int v_p dQ, \\ b_{e_p} &= \int v_p dQ. \end{aligned}$$

Since, by assumption,

$$v_p \le v_t \le v_p + \tau,$$

we have shown the required result, except for the case where $e_t = (J-1)U$. This case follows by continuity, since the graph of the subgradient of a concave function is closed (Rockafellar, 1997, Theorem 24.4, page 233). ■

Proof of Lemma 26. By definition of R:

$$\begin{aligned} R_{e_p} &= f(e_t, e_p, z) - (\theta - \tau) + \beta b_{e_p}, \\ R_{e_t} &= f(e_t, e_p, z) - \theta + \beta b_{e_t} \end{aligned}$$

where

$$\left(b_{e_t}, b_{e_p}\right) \in \partial b(e_t, e_p, z).$$

Then

$$R_{e_p} - R_{e_t} = \tau + \beta\left[b_{e_p} - b_{e_t}\right] \ge \tau(1-\beta)$$

where the inequality follows from the previous lemma. ■

Proof of Lemma 28. The existence of $\hat{p}$ follows by the concavity of R with respect to p, the Inada conditions on F, and from Proposition 24, which shows that $v_p = \theta - \tau$ for large p. The uniqueness of p follows by the strict concavity of F. That $\hat{p} < \bar{p}$ follows from concavity of R with respect of e_p and Lemma 26. ■

Proof of Lemma 29 The existence and uniqueness of $\hat{t}$ follows from the strict concavity of R. ■

Proof of Proposition 30 It follows from the definition of $\hat{t}$, $\hat{p}$ and various of the previous results. ■

Proof of Lemma 31. The first statement follows by considering the case where $T \in \mathcal{E}$ so that there is a $i \in \{1, 2, ..., J-1\}$ and T_i such that

$$T_1, ..., T_J = (U, .., U, T_i, 0, ..., 0, T_J)$$

for $T_i \in (0, U)$

$$V(U, ..., U, T_i, ..., 0, T_J) = v((i-1)U + T_i, T_J) \text{ for all } T_i \in (0, U)$$

Thus

$$V_i(U,...,U,T_i,...,0,T_J) = v_t((i-1)U + T_i, T_J) \text{ for } i = 1,2,...,J-1.$$

The second and third claim follows from the form of the optimal decision rules, i.e. the definition of the range of inaction and the strict concavity of R. The third follows since, for $t \geq \hat{t}(p,z)$ it is feasible to fire any extra temporary workers, so that we know the right derivative of $h[v]$ with respect to t. ■

Proof of Lemma 33. First we establish that $\hat{t}(p,z)$ is decreasing in p. Then we use this result, to show that $h[v]_t$ is decreasing in p.

By definition of $\hat{t}$,

$$0 \in R_{et}\left(\hat{t}(p,z), p, z\right)$$

for the case where $0 < \hat{t} < JU$. The main idea is to show that $R_{et}(t,p,z)$ is decreasing in p, and then use that, by concavity, $R_{et}(t,p,z)$ is decreasing in t.

The subgradient R_{et} is given by

$$R_{e_t}(t,p,z) = f(t+p,z) - \theta + \beta b_{et}(t,p,z)$$

where $b(t,p,z)$ is given by

$$b(t,p,z) = \int v(U + \min\{t, U(J-1)\}, t+p-\min\{t,U(J-1)\}, z') Q(z,dz')$$

We can then write b by cases as

$$\begin{aligned} b(t,p,z) &= \int v(U+t,p,z') Q(z,dz') \text{ if } t \leq U(J-1) \\ b(t,p,z) &= \int v(UJ, t+p-U(J-1), z') Q(z,dz') \text{ for } t \geq U(J-1) \end{aligned}$$

and hence its subgradients are

$$\begin{aligned} b_{et}(t,p,z) &= \int v_t(U+t,p,z') Q(z,dz') \text{ if } t < U(J-1) \\ b_{et}(t,p,z) &= \left[\int \underline{v}_p(UJ,p,z') Q(z,dz'), \int \bar{v}_t(UJ,p,z') Q(z,dz')\right] \text{ if } t = U(J-1) \\ b_{et}(t,p,z) &= \int v_p(UJ, t+p-U(J-1), z') Q(z,dz') \text{ for } t > U(J-1) \end{aligned}$$

Now we are ready to show that $R_{et}(t,p,z)$ is strictly decreasing in p. Consider first the case where $t < U(J-1)$. In this case the result follows from the hypothesis that v_t is decreasing in p and f strictly concave. Consider the case where $t > (J-1)U$. In this case the result follows from the concavity of v, so that v_p is decreasing, and the strict concavity of f. Finally, for the case where $t = (J-1)U$, we combine the previous two arguments for the right and left derivatives.

Having established that $R_{et}(t,p,z)$ is strictly decreasing in p, then it follows that $\hat{t}$ is decreasing in p since R_{et} is decreasing in t by concavity of R.

The cases where $\hat{t} = UJ$ or $\hat{t} = 0$ are similar.

Now we turn to show that $h[v]_t$ is decreasing in p. We consider three cases. First, let $(t,p) \in Int(I(z))$. In this case,

$$\begin{aligned} h[v]_t(t,p,z) &= f(t+p,z) + \beta b_{et}(t,p,z) \\ &= R_{et}(t,p,z) + \theta \end{aligned}$$

and thus $h[v]_t$ is decreasing in p since, as shown above, $R_{et}(t,p,z)$ is decreasing in p. In the case where $(t,p) \in Int\left(I(z)^C\right)$, then $h[v]_t = \theta$, and hence $h[v]$ is differentiable, and its derivative constant, so that it is weakly decreasing in p. Finally, consider the case where (t,p,z) is such that $t = \hat{t}(p,z)$. As shown above, $\hat{t}$ is weakly decreasing in p, thus for $p' > p$, $t \geq \hat{t}(p',z)$. Also, the subgradient of $h[v]_t$ is

$$\left[\underline{h}[v]_t(t,p,z), \bar{h}[v]_t(t,p,z)\right] = \left[\theta,\ f(t+p,z) + \beta\bar{b}_{et}(t,p,z)\right]$$

If $\hat{t}(p', z) = \hat{t}(p, z)$, then, using the expression for the left derivative of $h[v]_t$, it follows since f is concave and, as shown above, b_{et} is decreasing in p. If $\hat{t}(p', z) < \hat{t}(p, z)$, then, it must be that the point $\left(\hat{t}(p, z), p', z\right)$ is in the interior of the complement of the range of inaction, and thus $h[v]_t\left(\hat{t}(p, z), p', z\right) = \theta$. Thus $h[v]_t$ has decreased in this case too, since the subgradient has collapsed to its right derivative. ■

Proof of Proposition 34. That $\hat{t}$ is decreasing in p follows from using Lemma 33. Notice that starting with $V^0 = 0$ and $v^0 = 0$ satisfies all the hypothesis of this lemma. Since all these properties are preserved in the limit, they hold for the fixed point. To see that $\hat{t}$ is strictly decreasing, consider first the case where $t < U(J-1)$. In this case it follows from the fact that in a neighborhood of those points $v(t + U, p, z')$ is differentiable with respect to t, satisfies

$$\theta = f\left(\hat{t}(p, z) + p, z\right) + \beta \int v_t\left(\hat{t}(p, z) + U, p, z'\right) Q(z, dz'),$$

is decreasing in p, and f is strictly decreasing. A similar argument holds when $t > U(J-1)$, where

$$\theta = f\left(\hat{t}(p, z) + p, z\right) + \beta \int v_p\left(JU, p + \hat{t}(p, z) - JU, z'\right) Q(z, dz').$$

■

Proof of Theorem 35 and Proposition 36 (Appendix C).

To prove the theorem it is convenient to define a sequential economy that corresponds to the island planning problem taking as given U, θ. This economy has a firm whose problem corresponds to the one of the firm with value function B in the recursive competitive equilibrium (RCE) and a family whose problem has solution that gives the workers value function W in the RCE.

I) We define this economy in a standard Arrow-Debreu sequential way. This definition allows to use the first and second welfare theorems to link the allocation that solves the sequential island planning problem with an allocation that solves the firms and workers problem in the island economy as well as with the equilibrium wages w.

The commodities for the sequential island economy with initial state (X, z_0) is given by processes for employment by tenure E and consumption C

$$(E, C) = \left\{C_t\left(z^t\right), E_{jt}\left(z^t\right) : j = 0, ..., J,\ z^t \in Z\right\}.$$

We use g_{jt} to denote the labor choice of the firms in a sequential island problem. We use the h_t and s_t for hiring and firing of permanent workers. The net output of firms is to produce the following date t history z^t amount of consumption good

$$F\left(\sum_{j=0}^{J} g_{jt}\left(z^t\right), z_t\right) - \tau\ s_t\left(z^t\right). \tag{31}$$

The choices of g for the firms are subject to the restrictions that $g_{j,-1}(z_{-1}) = X_j$ for $j = 0, ..., J$, the law of motion for the permanent workers

$$g_{Jt}\left(z^t\right) = g_{Jt-1}\left(z^{t-1}\right) + g_{J-1t-1}\left(z^{t-1}\right) - s_t\left(z^t\right) + h_t\left(z^t\right), \tag{32}$$

and the non-negativity constraints for hiring and firing

$$s_t\left(z^t\right) \geq 0\ ,\ h_t\left(z^t\right) \geq 0, g_{jt}\left(z^t\right) \geq 0$$

for all $j = 0, ..., J,\ z^t, t \geq 0$.

We use e to denote the labor choice and c the consumption choice of the household in the sequential island problem. This household "owns" as an endowment a stream of U unemployed workers per period, that arrives to the island every period. The household is risk neutral in terms of consumption $c_t\left(z^t\right)$. Its decision is to assign a worker to work on the island or permanently work outside the island, which gives value θ per worker, in units of the final good. The utility function of the household is

$$\sum_{t=0} \sum_{z^t} \beta^t Q\left(z^t | z_0\right) \times$$

$$\times \left[c_t\left(z^t\right) + \theta\left(\sum_{j=0}^{J-1}\left[e_{j-1t-1}\left(z^{t-1}\right) - e_{jt}\left(z^t\right)\right] + \left[e_{Jt-1}\left(z^{t-1}\right) + e_{J-1t-1}\left(z^{t-1}\right) - e_{Jt}\left(z^t\right)\right]\right)\right].$$

The household is subject to the following restrictions:

$$e_{j,-1}(z_{-1}) = X_j \text{ for } j = 0, ..., J$$

and for all t, z^t, non-negative e_{jt} is subject to:

$$\begin{aligned} e_{0,t}(z^t) &\leq U, \\ e_{jt}(z^t) &\leq e_{j-1t-1}(z^{t-1}) \text{ for } j = 1, 2, ..., J-1 \\ e_{Jt}(z^t) &\leq e_{Jt-1}(z^{t-1}) + e_{J-1t-1}(z^{t-1}). \end{aligned} \tag{33}$$

Market clearing for the sequential economy is given by

$$\begin{aligned} e_{jt}(z^t) &= g_{jt}(z^t) \\ c_t(z^t) &= F\left(\sum_{i=0}^{J} g_{it}(z^t),\ z_t\right) - \tau s_t(z^t) \end{aligned}$$

for all $j = 0, ..., J$, and all $t,\ z^t$. Prices in this sequential island economy are given by intertemporal consumption prices, $P_t(X, z^t)$ and wages by tenure $w_{jt}(X, z^t)$ in terms of date t, history z^t consumption goods. Given the household preferences for consumption we impose

$$P_t(X, z^t) = \beta^t Q(z^t|z_0).$$

With these prices the problem of the firm is to maximize profits, i.e.

$$\begin{aligned} & B_0(x_J, X, z_0) \\ = \; & \max_{\{g\}} \sum_{t=0} \beta^t \sum_{z^t} \left[F\left(\sum g_{jt}(z^t), z_t\right) - \sum_{j=0}^{J} g_{jt}(z^t) w_{jt}(X, z^t) - \tau s_t(z^t) \right] Q(z^t|z_0) \end{aligned}$$

subject to

$$g_{J-1} = x_J,$$

and the laws of motion for s, h and g. Let $\beta^t \xi_t(z^t) Q(z^t|z_0)$ be the multiplier of the restriction (32). The first order conditions for the firm's problem are:

$$f\left(\sum g_{jt}(z^t), z_t\right) - w_{jt}(X, z^t) \leq 0$$

for $j = 0, ..., J-2$ with equality if $g_{jt}(z^t) > 0$. For $j = J-1$

$$f\left(\sum g_{jt}(z^t), z_t\right) - w_{J-1t}(X, z^t) + \beta \sum_{z_{t+1}} \hat{\xi}_{t+1}(z^t, z_{t+1}) Q(z_{t+1}|z_t) \leq 0$$

with equality if $g_{J-1t}(z^t) > 0$. For $j = J$

$$\hat{\xi}_t(z^t) = f\left(\sum_{j=0}^{J} g_{jt}(z^t), z_t\right) - w_{Jt}(X, z^t) + \beta \sum_{z_{t+1}} \hat{\xi}_{t+1}(z^t, z_{t+1}) Q(z_{t+1}|z_t) \tag{34}$$

if $g_{Jt}(z^t) > 0$. The first order conditions for $h_t(z^t)$ is

$$\hat{\xi}_t(z^t) \leq 0$$

with equality if $h_t(z^t) > 0$. The first order conditions for $s_t(z^t)$ is

$$-\tau - \hat{\xi}_t(z^t) \leq 0$$

with equality if $s_t(z^t) > 0$. Equation (32) thus gives:

$$g_{J,t-1}(z^{t-1}) + g_{J-1,t-t}(z^{t-1}) > g_{J,t}(z^t) \text{ then } \hat{\xi}_t(z^t) = -\tau$$

$$g_{J,t-1}\left(z^{t-1}\right)+g_{J-1,t-t}\left(z^{t-1}\right)<g_{J,t}\left(z^{t}\right) \text{ then } \hat{\xi}_t\left(z^t\right)=0.$$

Now we turn to the household problem in a sequential island economy. Letting $\beta^t Q\left(z^t|z_0\right)\hat{\nu}_{j,t}\left(z^t\right)$ be the Lagrange multiplier for (33), the first order conditions for the household problem are equivalent to

$$W_{jt}\left(X,z^t\right)=\max\left\{\theta,\ w_{jt}\left(X,z^t\right)+\beta\sum_{z_{t+1}}W_{j+1t+1}\left(X,z^t,z_{t+1}\right)Q\left(z_{t+1}|z_t\right)\right\},$$

for $j=0,...,J-1$ and

$$W_{Jt}\left(X,z^t\right)=\max\left\{\theta,\ w_{Jt}\left(X,z^t\right)+\beta\sum_{z_{t+1}}W_{Jt+1}\left(X,z^t,z_{t+1}\right)Q\left(z_{t+1}|z_t\right)\right\}$$

where

$$W_{jt}\left(X,z^t\right)=\hat{v}_{jt}\left(X,z^t\right)+\theta$$

with slackness

$$e_{jt}\left(z^t\right)<e_{j-1t-1}\left(z^{t-1}\right) \text{ then } W_{jt}\left(X,z^t\right)=\theta,$$

and $e_{jt}\left(z^t\right)>0$,

$$W_{jt}\left(X,z^t\right)=w_{jt}\left(X,z^t\right)+\beta\sum_{z_{t+1}}W_{j+1t+1}\left(X,z^t,z_{t+1}\right)Q\left(z_{t+1}|z_t\right)$$

for $j=0,...,J-1$ and analogously for $j=J$.

To see why this is the case, write the Lagrangian of the household problem as

$$\sum_{t=0}\beta^t\sum_{z^t}Q\left(z^t|z_0\right)\times$$

$$\Bigg\{\sum_{j=0}^{J}e_{jt}\left(z^t\right)w_{jt}\left(X,z^t\right)+$$
$$\theta\left(\sum_{j=0}^{J-1}\left[e_{j-1t-1}\left(z^{t-1}\right)-e_{jt}\left(z^t\right)\right]+\left[e_{J\ t-1}\left(z^{t-1}\right)+e_{J-1t-1}\left(z^{t-1}\right)-e_{Jt}\left(z^t\right)\right]\right)$$
$$+\hat{\nu}_{0t}\left(z^t\right)\left[U-e_{0t}\left(z^t\right)\right]+$$
$$+\sum_{j=1}^{J-1}\hat{\nu}_{jt}\left(z^t\right)\left[e_{j-1t-1}\left(z^{t-1}\right)-e_{jt}\left(z^t\right)\right]+\hat{\nu}_{jt}\left(z^t\right)\left[e_{J-1t-1}\left(z^{t-1}\right)+e_{Jt-1}\left(z^{t-1}\right)-e_{Jt}\left(z^t\right)\right]\Bigg\}$$

The first order conditions for this problem are as follows. For $e_{jt}\left(z^t\right)$:

$$w_{jt}\left(X,z^t\right)-\left(\theta+\hat{\nu}_{jt}\left(z^t\right)\right)+\beta\sum_{z_{t+1}}\left(\theta+\hat{v}_{j+1\ t+1}\left(z^t,z_{t+1}\right)\right)Q\left(z_{t+1}|z_t\right)\leq 0$$

with = if $e_{jt}\left(z^t\right)>0$ for $j=0,1,...,J-1$ and

$$w_{Jt}\left(X,z^t\right)-\left(\theta+\hat{\nu}_{Jt}\left(z^t\right)\right)+\beta\sum_{z_{t+1}}\left(\theta+\hat{v}_{J\ t+1}\left(z^t,z_{t+1}\right)\right)Q\left(z_{t+1}|z_t\right)\leq 0$$

with = if $e_{Jt}\left(z^t\right)>0$.

The slackness conditions are: if $e_{jt}\left(z^t\right)<e_{j-1t-1}\left(z^{t-1}\right)$ then $\hat{\nu}_{jt}\left(z^t\right)=0$ for $j=0,...,J-1$, and if $e_{Jt}\left(z^t\right)<e_{J-1t-1}\left(z^{t-1}\right)+e_{J-1t-1}\left(z^{t-1}\right)$ then $\hat{\nu}_{Jt}\left(z^t\right)=0$.

To compare a competitive equilibrium with the planning problems it helps defining a sequential island planning problem. In this problem the planner maximizes the expected discounted value of net output (31) subject to the feasibility constraints (32) and (33). This is the sequential version of the recursive island planning problem. Let $V_0\left(X,z_0\right)$ be the value attained by this planning problem.

Let $\beta^t \xi_t(z^t) Q(z^t|z_0)$ be the multiplier of the constraint (32) and $\beta^t \nu_{jt}(z^t) Q(z^t|z_0)$ the multiplier of the constraints (33). The first order conditions for the sequential island planning problem are equivalent to:

$$\theta + \nu_{jt}(z^t) = \max\left\{\theta,\ f\left(\sum E_{jt}(z^t), z_t\right) + \beta \sum_{z_{t+1}} (\theta + v_{j+1\ t+1}(z^t, z_{t+1})) Q(z_{t+1}|z_t)\right\}$$

with $\nu_{jt}(z^t) = 0$ if $E_{jt}(z^t) < E_{j-1t-1}(z^{t-1})$ and

$$\theta + \nu_{jt}(z^t) = f\left(\sum E_{jt}(z^t), z_t\right) + \beta \sum_{z_{t+1}} (\theta + v_{j+1\ t+1}(z^t, z_{t+1})) Q(z_{t+1}|z_t)$$

if $E_{jt}(z^t) > 0$ for $j = 0, .., J-2$. For $j = J-1$ we have

$$\begin{aligned} & \theta + \nu_{J-1t}(z^t) \\ = \ & \max\left\{\theta,\ f\left(\sum E_{jt}(z^t), z_t\right) + \beta \sum_{z_{t+1}} (\theta + v_{J\ t+1}(z^t, z_{t+1}) + \xi_{t+1}(z^t, z_{t+1})) Q(z_{t+1}|z_t)\right\} \end{aligned}$$

with $v_{J-1t}(z^t) = 0$ if $E_{J-1t}(z^t) < E_{J-1t-1}(z^{t-1})$, and

$$\begin{aligned} & \theta + \nu_{J-1t}(z^t) \\ = \ & f\left(\sum E_{jt}(z^t), z_t\right) + \beta \sum_{z_{t+1}} (\theta + v_{J\ t+1}(z^t, z_{t+1}) + \xi_{t+1}(z^t, z_{t+1})) Q(z_{t+1}|z_t) \end{aligned}$$

if $E_{J-1t}(z^t) > 0$. For $j = J$ we have

$$\begin{aligned} & \theta + \nu_{Jt}(z^t) + \xi_t(z^t) \\ = \ & \max\left\{\theta,\ f\left(\sum E_{jt}(z^t), z_t\right) + \beta \sum_{z_{t+1}} (\theta + v_{J\ t+1}(z^t, z_{t+1}) + \xi_{t+1}(z^t, z_{t+1})) Q(z_{t+1}|z_t)\right\} \end{aligned}$$

with $v_{Jt}(z^t) = 0$ if $E_{Jt}(z^t) < E_{Jt-1}(z^{t-1}) + E_{J-1t-1}(z^{t-1})$, and

$$\begin{aligned} & \theta + \nu_{Jt}(z^t) + \xi_t(z^t) \\ = \ & f\left(\sum E_{jt}(z^t), z_t\right) + \beta \sum_{z_{t+1}} (\theta + v_{J\ t+1}(z^t, z_{t+1}) + \xi_{t+1}(z^t, z_{t+1})) Q(z_{t+1}|z_t) \end{aligned}$$

if $E_{Jt}(z^t) > 0$.

To see why this is the case, write the Lagrangian for the planning problem as:

$$V_0(X, z_0) = \max_{\{E\}} \sum_{t=0} \beta^t \sum_{z^t} Q(z^t|z_0) \times$$

$$\begin{aligned} & \left\{F\left(\sum E_{jt}(z^t), z_t\right) - \tau S_t(z^t)\right. \\ & +\theta\left(\sum_{j=0}^{J-1}\left[E_{j-1t-1}(z^{t-1}) - E_{jt}(z^t)\right] + \left[E_{J\ t-1}(z^{t-1}) + E_{J-1t-1}(z^{t-1}) - E_{Jt}(z^t)\right]\right) \\ & +\nu_{0t}(z^t)\left[U - E_{0t}(z^t)\right] + \sum_{j=1}^{J-1} \nu_{jt}(z^t)\left[E_{j-1}(z^{t-1}) - E_{jt}(z^t)\right] \\ & +\nu_{Jt}(z^t)\left[E_{J-1t-1}(z^{t-1}) + E_{Jt-1}(z^{t-1}) - E_{Jt}(z^t)\right] \\ & \left.\xi_t(z^t)\left[-E_{Jt}(z^t) + E_{Jt-1}(z^{t-1}) + E_{J-1t-1}(z^{t-1}) - S_t(z^t) + H_t(z^t)\right]\right\} \end{aligned}$$

The first order conditions are:

$$f\left(\sum E_{jt}(z^t), z_t\right) - \theta - \nu_{jt}(z^t) + \beta \sum_{z_{t+1}} (\theta + v_{j+1\ t+1}(z^t, z_{t+1})) Q(z_{t+1}|z_t) \leq 0$$

with equality if $E_{jt}\left(z^{t}\right)>0$,

$$f\left(\sum E_{jt}\left(z^{t}\right),z_{t}\right)-\theta-\nu_{J-1t}\left(z^{t}\right)+$$

$$\beta\sum_{z_{t+1}}\left(\theta+v_{J\ t+1}\left(z^{t},z_{t+1}\right)\right)Q\left(z_{t+1}|z_{t}\right)+\beta\sum_{z_{t+1}}\xi_{t+1}\left(z^{t},z_{t+1}\right)Q\left(z_{t+1}|z_{t}\right)\leq 0$$

with equality if $E_{J-1t}\left(z^{t}\right)>0$,

$$f\left(\sum E_{jt}\left(z^{t}\right),z_{t}\right)-\theta-\xi_{t}\left(z^{t}\right)-\nu_{Jt}\left(z^{t}\right)+$$

$$\beta\sum_{z_{t+1}}\left(\theta+v_{J\ t+1}\left(z^{t},z_{t+1}\right)\right)Q\left(z_{t+1}|z_{t}\right)+\beta\sum_{z_{t+1}}\xi_{t+1}\left(z^{t},z_{t+1}\right)Q\left(z_{t+1}|z_{t}\right)\leq 0$$

with equality if $E_{Jt}\left(z^{t}\right)>0$.

The first order condition for $H_{t}\left(z^{t}\right)$ is

$$\xi_{t}\left(z^{t}\right)\leq 0$$

with equality if $H_{t}\left(z^{t}\right)=0$. The first order condition for $S_{t}\left(z^{t}\right)$ is

$$-\tau-\xi_{t}\left(z^{t}\right)\leq 0$$

with equality if $S_{t}\left(z^{t}\right)>0$.

(II) We now show i), the first welfare theorem, and iii). We start with an island RCE $\{w,W,B,G\}$. Pick an arbitrary state $(T,z)=(X,z_{0})$ in the support of μ. We construct the sequential competitive equilibrium with (X,z_{0}) as initial condition as follows. Let wages be:

$$w_{jt}\left(X,z^{t}\right)=w_{j}\left(D^{t-1}\left(X,z^{t-1}\right),z_{t}\right).$$

and let multipliers and employment be

$$\begin{aligned}\theta+\hat{v}_{jt}\left(X,z^{t}\right)&=W_{j}\left(D^{t-1}\left(X,z^{t-1}\right),z_{t}\right)\\ e_{jt}\left(X,z^{t}\right)&=G_{j}\left(D^{t-1}\left(X,z^{t-1}\right),z_{t}\right)\end{aligned}$$

where

$$\begin{aligned}D^{t}\left(X,z^{t}\right)&=G\left(T,z_{t}\right)\quad\text{for}\\ T&=\left(U,D_{0}^{t-1}\left(X,z^{t-1}\right),...,D_{J-2}^{t-1}\left(X,z^{t-1}\right),D_{J-1}^{t-1}\left(X,z^{t-1}\right)+D_{J}^{t-1}\left(X,z^{t-1}\right)\right)\ \text{and}\\ D^{-1}\left(X,z_{0}\right)&=X\end{aligned}\tag{35}$$

It is immediate to verify that $\{e,\hat{\nu}\}$ solves the first order conditions for the household problem in a sequential island equilibrium, and hence it solves the household problem. For future reference we define

$$W_{j0}\left(X,z_{0}\right)=\hat{\nu}_{j0}\left(z_{0},X\right)+\theta.$$

Define the Lagrange multiplier and employment for the firms problem as:

$$\begin{aligned}\hat{\xi}_{t}\left(X,z^{t}\right)&=B_{p}\left(D_{J}^{t-1}\left(X,z^{t-1}\right),D^{t-1}\left(X,z^{t-1}\right),z_{t}\right)\\ g_{jt}\left(X,z^{t}\right)&=G_{j}\left(D^{t-1}\left(X,z^{t-1}\right),z_{t}\right)\end{aligned}$$

It is immediate to verify that $\left\{g,\hat{\xi}\right\}$ solves the first order conditions for the firm's problem in an sequential island competitive equilibrium, and hence it solves the firm's problem. Let B_{0} be the value of the firm in the sequential island CE. For future reference, from the envelope theorem, we have

$$\partial B_{0}\left(x_{J},X,z_{0}\right)/\partial x_{J}=\hat{\xi}_{0}\left(X,z_{0}\right)$$

evaluated at $x_{J}=X_{J}$.

By the first welfare theorem applied to the sequential island economy, $\{e\}=\{g\}$ is a Pareto optimal allocation, and hence solves the sequential island planning problem. By inspection, the Lagrange multipliers $\{\xi_{t},\nu_{jt}\}$ that satisfy the first order conditions for the sequential planning problem are identical to the Lagrange multipliers for the firm's problem $\left\{\hat{\xi}_t\right\}$ and the Lagrange multipliers $\{\hat{\nu}_{jt}\}$ for the households problem in the sequential competitive equilibrium. >From these first order conditions:

$$W_{j0}^{*}(X, z_0) = \hat{\nu}_{j0}(z_0, X) + \theta = \nu_{j0}(z_0, X) + \theta = \partial V_0(X, z_0) / \partial X_j$$

for $j = 0, ..., J-1$ and

$$\begin{aligned} W_{J0}(X, z_0) + \partial B_0(x_J, X, z_0) / \partial x_J &= \hat{\nu}_{J0}(X, z_0) + \theta + \hat{\xi}_0(X, z_0) \\ &= \nu_{J0}(X, z_0) + \theta + \xi_0(X, z_0) = \partial V_0(X, z_0) / \partial X_J \end{aligned}$$

evaluated at $x_J = X_J$. The allocation described by G is, by hypothesis, recursive, so it solves the recursive island planning problem with initial condition (X, z_0). Repeating this argument for each initial condition (X, z_0) we show that

$$\begin{aligned} V_0(T, z) &= V(T, z), \\ B_0(T_J, T, z) &= B(T_J, T, z), \\ W_{j0}(T, z) &= W_j(T, z) \end{aligned}$$

for all $(T, z.)$ Hence we have shown the first welfare theorem for the recursive representation of the island problem, and that (23), condition iii) of the theorem, holds.

(III). We now show ii), the second welfare theorem, and condition iii) of Theorem 35. We start with a solution of the recursive planning problem, $\nu(T, z)$ and $\xi(T, z)$ which, by the envelope theorem, satisfy

$$\frac{\partial V(T, z)}{\partial T_j} = \theta + \nu_j(T, z).$$

for $j = 0, ..., J-1$, and

$$\frac{\partial V(T, z)}{\partial T_J} = \theta + \nu_J(T, z) + \xi(T, z)$$

If there were more than one pair ν_J, ξ for a given T, z, select the one that only depends on (T, z). From the principle of optimality, the solution to the recursive island problem V is the same as the value function for the sequential island problem V_0, so that $V(T, z) = V_0(T, z)$.

Choose any initial state (X, z_0) to be used as initial condition to the sequential island problem. Define

$$\begin{aligned} \nu_{jt}(X, z^t) &= \nu_j\left(D^{t-1}(X, z^{t-1}), z_t\right) \\ \xi_t(X, z^t) &= \xi\left(D^{t-1}(X, z^{t-1}), z_t\right) \\ E_{jt}(X, z^t) &= G\left(D^{t-1}(X, z^{t-1}), z_t\right) \end{aligned}$$

where D^{t-1} is defined as in (35) using the optimal decision rule from the recursive planning problem. By comparing the first order conditions for the recursive island planing problem with the first order conditions for the sequential island planning problem, it can be seen that $\{E_t, \nu_{jt}, \xi_t\}$ so defined solve the first order conditions for the sequenctial island planning problem. Next define wages as follows:

$$w_{jt}(X, z^t) = f\left(\sum_{j=0}^{J} E_{jt}(z^t, X), z_t\right) \tag{36}$$

for $j = 0, 1, 2, ..., J-2$; for $j = J-1$ let

$$w_{J-1t}(X, z^t) = f\left(\sum_{j=0}^{J} E_{jt}(X, z^t), z_t\right) + \beta \sum_{z^{t+1}} \xi_{t+1}(X, z^t, z_{t+1}) Q(z_{t+1}|z_t); \tag{37}$$

finally, for $j = J$

$$w_{Jt}(X, z^t) = f\left(\sum_{j=0}^{J} E_{jt}(X, z^t), z_t\right) - \xi_t(X, z^t) + \beta \sum_{z^{t+1}} \xi_{t+1}(X, z^t, z_{t+1}) Q(z_{t+1}|z_t). \tag{38}$$

The function $B_0(x_J, X, z_0)$ is defined as the solution to the firm's problem for wages w_{jt} in the sequential island equilibrium. Given wages w_{jt}, the functions W_{jt} are defined as:

$$W_{jt}(X, z^t) = \nu_{jt}(X, z^t) + \theta \tag{39}$$

for $j = 0, ..., J$.

Define the candidate multipliers for the sequential firm problem as $\hat{\xi}_t = \xi_t$. Given wages w_{jt}, and multipliers $\hat{\xi}_t$, it is easy to verify that the allocation $g_{jt} = E_{jt}$, and its implied $\{s_t, h_t\}$ solve the first order conditions for the firms in the sequential island economy. To verify this, one uses the first order conditions for the island planning problem in the island sequential economy. From the envelope condition it is immediate that

$$\partial B_0(x_J, X, z_0) / \partial x_J = \hat{\xi}_0(X, z_0)$$

where $x_J = X_J$.

Define the candidate multipliers for the sequential household problem $\hat{v}_{jt} = v_{jt}$. Given wages w_{jt} and multipliers $\hat{v}_{jt}$, it is easy to verify that the allocation $e_{jt} = E_{jt}$ solves the first order conditions for the household's problem in the sequential island economy. To verify this, use the first order conditions for the island planner problem in the sequential island economy.

Thus we have established that the sequential allocation constructed out of the solution of the recursive island planning problem from an initial state (X, z_0) can be decentralized as a sequential island competitive equilibrium. Finally, we define the elements of the recursive island competitive equilibrium as follows:

$$\begin{aligned} w_j(X, z_0) &= w_{j0}(X, z_0), \\ W_j(X, z_0) &= W_{j0}(X, z_0), \\ B(X_J, X, z_0) &= B_0(X_J, X, z_0) \end{aligned}$$

By repeating this construction for all (X, z_0) in the support of μ, we construct the functions w, W and B. These functions constitute a RCE since they are constructed from the sequential island competitive equilibrium.

>From the previous arguments we have:

$$\begin{aligned} \frac{\partial V(T, z)}{\partial T_j} &= \theta + \nu_j(T, z) = \theta + v_{j0}(z, T) \\ &= \theta + \hat{\nu}_{j0}(z, T) = W_{j0}(T, z). \end{aligned} \tag{40}$$

for $j = 0, ..., J-1$ and

$$\begin{aligned} \frac{\partial V(T, z)}{\partial T_J} &= \theta + \nu_J(T, z) + \xi(T, z) = \theta + \nu_{J0}(, z) + \xi_0(z, T) \\ &= \theta + \hat{\nu}_{J0}(z, T) + \hat{\xi}_0(z, T) = W_{J0}(T, z) + \frac{\partial}{\partial x_J} B_0(T_J, T, z), \end{aligned} \tag{41}$$

and thus condition iii) is satisfied.

(IV) We establish condition (b) of Proposition 36. Since in (II) and (III) we have shown the first and second welfare theorems, we can, without loss of generality, start with an efficient allocation and examine the equilibrium wages w that we constructed in (III), equations (36), (37) and (38). The multiplier $\xi \in [-\tau, 0]$ and $\xi_t(X, z^t) = -\tau$ if $S_t(z^t) > 0$, i.e. if permanent workers are being fired. Thus, the inequalities in (b) follow from these definitions and the properties of ξ.

(V). We establish condition c) of Proposition 36. Since in (II) and (III) we have shown the first and second welfare theorems, we can, without loss of generality, start with an efficient allocation and examine the equilibrium value function for workers W that we constructed in (III), equation (39). Using equations (40), (41) we have that

$$\begin{aligned} \frac{\partial V(T, z)}{\partial T_j} &= W_j(T, z), \text{ for } j = 0, ..., J-1 \\ \frac{\partial V(T, z)}{\partial T_J} &= W_J(T, z) + \xi(T, z) \ . \end{aligned}$$

In Proposition (4) we have shown that

$$V_{T_1} \geq V_{T_2} \geq \cdots \geq V_{T_{J-1}},$$

Thus

$$W_1 \geq W_2 \geq \cdots \geq W_{J-1}.$$

Finally since W is part of an equilibrium, it satisfies

$$\begin{aligned} W_{J-1}(T,z) &= w_{J-1}(T,z) + \beta E\left[W_J\left(A(T,z), z'\right)|z\right] \\ W_J(T,z) &= w_J(T,z) + \beta E\left[W_J\left(A(T,z), z'\right)|z\right] \end{aligned}$$

and since we have already established (c), $w_J \geq w_{J-1}$, and thus we have $W_{J-1} \leq W_J$. This finishes the proof of V.
■

Appendix G: Definition of Auxiliary Competitive Equilibrium ("ACE")

This appendix defines the auxiliary competitive equilibrium (ACE) used in the proof of Theorem 1. There are two types of firms, type I and II, and families. There are as many markets to "buy" and "sell" workers as islands of type (X, z^t).

Preferences of the family.

The families own all firms of both type and consume final consumption goods. They are risk neutral, and discount future at rate β.

$$\sum_t \beta^t \sum_{z^t} C_t\left(z^t\right) q_t\left(z^t\right)$$

"Labor" is allocated initially to the two types of firms.

To simplify the notation we anticipate that, given the risk neutrality of households, the price for final goods sold at date t, state z^t, is $\beta^t q_t\left(z^t\right)$.

Problem of firms type I.

There is a continuum of firms type I on each island of type X that "buys" workers from a central location at price $\lambda_t\left(z^t, Z\right)$. They start at period $t = 0$ with a profile of workers given by their type X. Workers that are "bought" in this period are given tenure $j = 0$. They operate the technology F. If they sale workers to the central location, they obtain a price $\theta_t\left(z^t, X\right)$. If they "sale" workers with tenure J or higher at the island level, they loose τ per worker.

The sequence problem for firms type I who "buy" workers at price λ_t, sell them at price θ_t, and also pay the separation cost τ, is the following:

For each (X, z_0), they maximize:

$$\sum_{t=0} \beta^t \sum_{z^t} \left\{ F\left(\sum_{j=1}^{J} E_{j,t}\left(z^t, X\right), z_t \right) - T_{0,t}\left(z^t, X\right) \lambda_t\left(z^t, X\right) \right\} q_t\left(z^t\right) \tag{42}$$
$$+ \sum_{t=0} \beta^t \sum_{z^t} \left\{ \sum_{j=0}^{J} \left[T_{j,t}\left(z^t, X\right) - E_{j,t}\left(z^t, X\right)\right] \theta_t\left(z^t, X\right) - \left[T_{J,t}\left(z^t, X\right) - E_{J,t}\left(z^t, X\right)\right] \tau \right\} q_t\left(z^t\right)$$

by choice of $\{E_{jt}, T_{jt}\}_{t\geq 0}$ subject to to the technological constraints in hiring and firing:

$$E_{j,t}\left(z^t, X\right) \leq T_{j,t}\left(z^t, X\right) \text{ for } j = 0, 1, .., J$$

$$T_{j,t}\left(z^t, X\right) = E_{j-1,t-1}\left(z^{t-1}, X\right) \text{ for } j = 1, 2, ..., J-1$$
$$T_{J,t}\left(z^t, X\right) = E_{J-1,t-1}\left(z^{t-1}, X\right) + E_{J,t-1}\left(z^{t-1}, X\right),$$

given initial conditions

$$T_{j,0}\left(z^0, X\right) = X_j \text{ for } j = 1, 2, ..., J.$$

Problem of firms type II.

There is a continuum of firms type II that sells workers to each island, subject to the undirected search technology, and buys them back from islands. Firms also operate the home production technology. "Purchases" are denoted by $S_{jt}\left(z^t, X\right)$ with price $\theta_t\left(z^t, X\right)$ and "sales" are denoted by $Y_t\left(z^t, X\right)$ with price $\lambda_t\left(z^t, X\right)$.

The sequence problem for firms type II that produce home goods and reallocate workers is the following:

For each (X, z_0), they maximize:

$$\sum_t \beta^t \omega L_t + \sum_t \beta^t \sum_{z^t} \sum_X Y_t\left(z^t, X\right) \lambda_t\left(z^t, X\right) \eta\left(X|z_0\right) q_t\left(z^t\right) \tag{43}$$
$$- \sum_t \beta^t \sum_{z^t} \sum_X \left[\sum_{j=0}^{J} S_{j,t}\left(z^t, X\right) \right] \theta_t\left(z^t, X\right) \eta\left(X|z_0\right) q_t\left(z^t\right)$$

by choice of $\{Y_t, S_{jt}, U_t\}_{t\geq 0}$ subject to the undirected search technology, so that they cannot sell different quantities to different islands, which is written as

$$U_{t-1} = Y_t\left(z^t, X\right) \text{ for all } t, z^t, X,$$

and the flow constraint stating that workers "bought" can be allocated to either increase the stock producing at home or to search:

$$U_t + L_t - L_{t-1} \leq \sum_x \sum_{z^t} \sum_{j=0}^{J} S_{jt}\left(z^t, X\right) \eta\left(X|z_0\right) q_t\left(z^t\right) \text{ for all } t,$$

where U_{-1} and L_{-1} are given.

Market clearing.

For final goods:

$$\begin{aligned} & \tau \sum_x \sum_{z^t} \left[T_{J,t}\left(z^t, X\right) - E_{J,t}\left(z^t, X\right)\right] q_t\left(z^t\right) \eta\left(X|z_0\right) + C_t \\ = \quad & L_t \omega + \sum_x \sum_{z^t} F\left(\sum_{j=1}^{J} E_{j,t}\left(z^t, X\right), z_t\right) q_t\left(z^t, X\right) \eta\left(X|z_0\right); \end{aligned}$$

for the market of new (tenure $j = 0$) workers:

$$U_{t-1} = T_{0,t}\left(z^t, X\right) \text{ for all } t, z^t, X;$$

for the market of incumbent (tenure $j > 0$) workers:

$$S_{j,t}\left(z^t, X\right) = E_{j,t}\left(z^t, X\right) - T_{j,t}\left(z^t, X\right) \text{ for all } j, t, z^t, X.$$