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RISK (AND WHEN IS IT NOT)?

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When is Market Incompleteness Irrelevant for the Price of Aggregate Risk (and when is it not)?

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ABSTRACT

In a model with a large number of agents who have constant relative risk aversion (CRRA) preferences, market incompleteness has no effect on the premium for aggregate risk if the distribution of idiosyncratic risk is independent of aggregate shocks and aggregate consumption growth is distributed independently over time. In the equilibrium, which features trade and binding solvency constraints, as opposed to Constantinides and Duffie (1996), households only use the stock market to smooth consumption; there is no trade in bond markets. Furthermore, we show that the cross-sectional wealth and consumption distributions are not affected by aggregate shocks. These results hold regardless of the persistence of idiosyncratic shocks. A weaker irrelevance result survives when we allow for predictability in aggregate consumption growth.

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1 Introduction

This paper provides general conditions under which closing down insurance markets for idiosyncratic risk (our definition of market incompleteness) does not increase the risk premium that stocks command over bonds. We study a standard incomplete markets endowment economy populated by a continuum of agents who have CRRA preferences and who can trade a risk-free bond and a stock, subject to potentially binding solvency constraints. In the benchmark version of our model the growth rate of the aggregate endowment is uncorrelated over time, that is, the logarithm of aggregate income is a random walk. Under these assumptions, the presence of *uninsurable idiosyncratic risk* is shown to lower the equilibrium risk-free rate, but it has no effect on the price of aggregate risk in equilibrium if the distribution of idiosyncratic shocks is statistically independent of aggregate shocks. Consequently, we show analytically that in this class of models, the representative agent Consumption-CAPM (CCAPM) developed by Rubinstein (1974), Breeden (1979) and Lucas (1978) prices the excess returns on the stock correctly. Therefore, as long as idiosyncratic shocks are distributed independently of aggregate shocks, the extent to which household can insure against idiosyncratic income risk as well as the tightness of the borrowing constraints they face are irrelevant for risk premia. These results deepen the equity premium puzzle, because we show that Mehra and Prescott's (1985) statement of the puzzle applies to a much larger class of incomplete market models.¹ We also show that uninsurable income risk does not contribute to any variation in the *conditional* market price of risk.

While earlier work exploring autarchic equilibria by Constantinides and Duffie (1996) is suggestive of the irrelevance of idiosyncratic (consumption) risk, our work derives the exact conditions for the irrelevance of uninsurable idiosyncratic income risk in non-autarchic equilibria for economies with standard labor income processes while allowing for binding borrowing constraints. We show that these conditions depend critically on the properties of aggregate consumption growth once one allows for trade.²

The stark asset pricing implications of our model follow from a crucial result about equilibrium consumption and portfolio allocations. We show that in this class of incomplete market models with idiosyncratic and aggregate risk, the equilibrium allocations and prices can be obtained from the allocations and interest rates of an equilibrium in a stationary model with only idiosyncratic risk (as in Bewley (1986), Huggett (1993) or Aiyagari (1994)). We call this the Bewley model.

¹Weil's (1989) statement of the risk-free rate puzzle, on the contrary, does not.

²There are few general theoretical results in this literature. Levine and Zame (2002) show that in economies populated by agents with infinite horizons, the equilibrium allocations converge in the limit, as their discount factors approach one, to the complete markets allocations. We provide a qualitatively similar equivalence result that applies only to the risk premium. Our result, however, does not depend at all on the time discount factor of households. For examples of numerical solutions of calibrated models, see the important work of Aiyagari and Gertler (1991), Telmer (1993), Lucas (1994), Heaton and Lucas (1996), Krusell and Smith (1998), and Marcet and Singleton (1999). Also see Kocherlakota (1996) for an overview of this literature.

Specifically, we demonstrate that scaling up the consumption allocation of the Bewley equilibrium by the aggregate endowment delivers an equilibrium for the model *with* aggregate risk. In this equilibrium, there is *no* trade in bond markets, only in stock markets. This result is the key to why the history of aggregate shocks has no bearing on equilibrium allocations and prices. Households in equilibrium only trade the stock (i.e. the market portfolio), regardless of their history of idiosyncratic shocks, even in the presence of binding solvency constraints. This invariance of the household's portfolio composition directly implies the irrelevance of the history of aggregate shocks for equilibrium prices and allocations. The wealth and consumption distribution in the model with aggregate risk (normalized by the aggregate endowment) coincides with the stationary wealth and consumption distribution of the Bewley equilibrium. Aggregate shocks therefore have no impact on these equilibrium distributions.

Due to the absence of aggregate risk in the Bewley model, a bond and a stock have exactly the same return characteristics in equilibrium and thus one asset is redundant. Households need not use the bond to smooth consumption. Remarkably, this prediction for portfolio allocations carries over to the model with aggregate risk: at the equilibrium prices households do not trade the uncontingent bond and they only use the stock (i.e. a claim to aggregate consumption), to transfer resources over time. In equilibrium, all households bear the same share of aggregate risk and they all hold the same portfolio, regardless of their history of idiosyncratic shocks and the implied financial wealth holdings. Hence, this class of models does not generate any demand for bonds at the equilibrium interest rate, if bonds are in zero net supply.³ If there was some household that wanted to hold bonds at the equilibrium interest rate, then all households would want to go long in bonds, and the bond market could not be cleared. Hence in equilibrium, no household holds bonds.

Our portfolio result depends on the properties of aggregate consumption growth. In our benchmark model, aggregate consumption growth is not predictable (that is, the growth rate of the aggregate endowment is i.i.d.). This is a standard assumption in the asset pricing literature (e.g. Campbell and Cochrane (1999)), because the empirical evidence for such predictability is rather weak (see e.g. Heaton and Lucas (1996)). However, if there is predictability in aggregate income growth in our model, as in the model with long-run risk developed by Bansal and Yaron (2004), then agents want to hedge their portfolio against interest rate shocks, creating a role for trade in a richer menu of assets. The risk premium irrelevance result goes through only if households can trade a full set of aggregate state-contingent claims and if the solvency constraints do not bind. In this case, there is scope for trade in the contingent bond markets, because the hedging

³In our model, what we refer to as the stock is a claim to aggregate consumption. We show this is the only security households trade. However, if we take equity to be a levered claim to aggregate consumption, then the households effectively hold a fixed portfolio of corporate bonds and stocks. As is easy to verify, the proportion of corporate bonds in the portfolio is 2/3 when corporate leverage is 2. The remaining 1/3 is in equities. A key implication of the equilibrium we derive is that households do not deviate from these fixed proportions.

demands depend on the household's history of idiosyncratic shocks. So, only a weaker form of our result survives predictability in aggregate consumption growth. In contrast, in the autarchic equilibria examined by Constantinides and Duffie (1996), the irrelevance result does not depend on the properties of aggregate consumption growth because households by construction do not use assets to hedge against predictable changes in consumption.

The key ingredients underlying our irrelevance result are (i) a continuum of agents, (ii) CRRA utility, (iii) idiosyncratic labor income risk that is independent of aggregate risk, (iv) a constant capital share of income and (v) solvency constraints or borrowing constraints on total financial wealth that are proportional to aggregate income. We now discuss each of these assumptions in detail to highlight and explain the differences with existing papers in the literature.

First, we need to have a large number of agents in the economy. As pointed out by Denhaan (2001), in an economy with a finite number of agents, each idiosyncratic shock is by construction an aggregate shock because it changes the wealth distribution and, through these wealth effects, asset prices. Second, our results rely on the homotheticity property of the CRRA utility function. Third, in our model labor income grows with the aggregate endowment, as is standard in this literature. In addition, for our results to go through, idiosyncratic income shocks must be distributed independently of aggregate shocks. This explicitly rules out that the variance of idiosyncratic shocks is higher in recessions (henceforth we refer to this type of correlation as countercyclical cross-sectional variance of labor income shocks, or CCV). Fourth, we require that the capital share of income is independent of the aggregate state of the economy. This assumption is standard in macroeconomics, and it is not obviously at odds with the data (see Cooley and Prescott (1995)). Finally, in the main section of the paper, households face either constraints on total net wealth today or state-by-state solvency constraints on the value of their portfolio in each state tomorrow.⁴ These solvency constraints have to be proportional to aggregate income. This assumption is standard in the literature. For example, the liquidity constraints used by Deaton (1991) satisfy this condition, as do the tight borrowing constraints used by Krusell and Smith (1997), and the endogenous solvency constraints derived by Alvarez and Jermann (2000).

The paper is structured as follows. In section 2, we lay out the physical environment of our model. In section 3, we consider the benchmark model without predictability in aggregate consumption growth. We show that the prices and allocations in a model without aggregate risk, called the Bewley model, can be mapped into the equilibrium prices and allocations of an economy with aggregate risk in which households trade only a bond and a stock. This result then immediately implies that market incompleteness is irrelevant for risk premia. In section 4, we allow

⁴However, in the case in which aggregate consumption growth is uncorrelated over time, we can allow for arbitrary separate constraints on the household bond positions, e.g. a short-sale constraint for bonds only. In this case, our result is also robust to the introduction of transaction costs in the bond market (but not to transaction costs in the stock market).

for predictability in aggregate consumption growth. Finally, section 5 concludes; all proofs are contained in the appendix. In a separate appendix, downloadable from the authors' web sites⁵, we show these results survive when idiosyncratic income shocks are permanent, when labor supply is endogenous, and when households have Epstein-Zin preferences.

2 Environment

Our exchange economy is populated by a continuum of individuals of measure 1. There is a single nonstorable consumption good. The aggregate endowment of this good is stochastic. Each individual's endowment depends, in addition to the aggregate shock, also on the realization of an idiosyncratic shock. Thus, the model we study is identical to the one described by Lucas (1994), except that ours is populated by a continuum of agents (as in Bewley (1986), Aiyagari and Gertler (1991), Huggett (1993) and Aiyagari (1994)), instead of just two agents.

2.1 Representation of Uncertainty

We denote the current aggregate shock by $z_t \in Z$ and the current idiosyncratic shock by $y_t \in Y$. For simplicity, both Z and Y are assumed to be finite. Furthermore, let $z^t = (z_0, \dots, z_t)$ and $y^t = (y_0, \dots, y_t)$ denote the history of aggregate and idiosyncratic shocks. As shorthand notation, we use $s_t = (y_t, z_t)$ and $s^t = (y^t, z^t)$. We let the economy start at an initial aggregate node z_0 . Conditional on an idiosyncratic shock y_0 and thus $s_0 = (y_0, z_0)$, the probability of a history s^t is given by $\pi_t(s^t|s_0)$. We assume that shocks follow a first order Markov process with transition probabilities given by $\pi(s'|s)$.

2.2 Preferences and Endowments

Consumers rank stochastic consumption streams $\{c_t(s^t)\}$ according to the following homothetic utility function:

$$U(c)(s_0) = \sum_{t=0}^{\infty} \sum_{s^t \geq s^0} \beta^t \pi(s^t|s_0) \frac{c_t(s^t)^{1-\gamma}}{1-\gamma}, \quad (1)$$

where $\gamma > 0$ is the coefficient of relative risk aversion and $\beta \in (0, 1)$ is the constant time discount factor. We define $U(c)(s^t)$ to be the continuation expected lifetime utility from a consumption allocation $c = \{c_t(s^t)\}$ in node s^t . Lifetime utility can be written recursively as follows:

$$U(c)(s^t) = u(c_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) U(c)(s^t, s_{t+1}),$$

⁵See <http://www.econ.upenn.edu/~dkrueger/research/assetpricingapp.pdf>

where we made use of the Markov property the underlying stochastic processes. The economy's aggregate endowment process $\{e_t\}$ depends only on the aggregate event history; we let $e_t(z^t)$ denote the aggregate endowment at node z^t . Each agent draws a 'labor income' share $\eta(y_t, z_t)$, as a fraction of the aggregate endowment in each period. Her labor income share only depends on the current individual and aggregate event. We denote the resulting individual labor income process by $\{\eta_t\}$, with

$$\eta_t(s^t) = \eta(y_t, z_t)e_t(z^t), \quad (2)$$

where $s^t = (s^{t-1}, y_t, z_t)$. We assume that $\eta(y_t, z_t) > 0$ in all states of the world. The stochastic growth rate of the endowment of the economy is denoted by $\lambda(z^{t+1}) = e_{t+1}(z^{t+1})/e_t(z^t)$. We assume that aggregate endowment growth only depends on the current aggregate state.

Condition 2.1. *Aggregate endowment growth is a function of the current aggregate shock only:*

$$\lambda(z^{t+1}) = \lambda(z_{t+1}).$$

Furthermore, we assume that a Law of Large Numbers holds⁶, so that $\pi(s^t|s_0)$ is not only a household's individual probability of drawing a history s^t , but also the deterministic *fraction* of the population drawing that same history.

In addition to labor income, there is a Lucas tree that yields a constant share α of the total aggregate endowment as capital income, so that the total dividends of the tree are given by $\alpha e_t(z^t)$ in each period. The remaining fraction $1 - \alpha$ is labor income. Therefore, by construction, the aggregate labor share of equals the sum over all individual labor income shares:

$$\sum_{y_t \in Y} \Pi_{z_t}(y_t)\eta(y_t, z_t) = (1 - \alpha), \quad (3)$$

for all z_t , where $\Pi_{z_t}(y_t)$ represents the cross-sectional distribution of idiosyncratic shocks y_t , conditional on the aggregate shock z_t . By the law of large numbers, the fraction of agents who draw y in state z only depends on z . An increase in the capital income share α translates into proportionally lower individual labor income shares $\eta(y, z)$ for all (y, z) .⁷

At time 0, the agents are endowed with initial wealth θ_0 . This wealth represents the value of an agent's share of the Lucas tree producing the dividend flow in units of time 0 consumption, as well as the value of her labor endowment at date 0. We use Θ_0 to denote the initial joint distribution of wealth and idiosyncratic shocks (θ_0, y_0) .

⁶See e.g. Hammond and Sun (2003) for conditions under which a LLN holds with a continuum of random variables.

⁷Our setup nests the baseline model of Heaton and Lucas (1996), except for the fact that they allow for the capital share α to depend on z . In their benchmark parameterization, however, fluctuations in $\alpha(z)$ are rather small.

Next, we assume that idiosyncratic shocks are independent of the aggregate shocks. This assumption is crucial for most of the results in this paper.

Condition 2.2. *Individual endowment shares $\eta(y_t, z_t)$ are functions of the current idiosyncratic state y_t only, that is $\eta(y_t, z_t) = \eta(y_t)$. Also, the transition probabilities of the shocks can be decomposed as*

$$\pi(z_{t+1}, y_{t+1} | z_t, y_t) = \varphi(y_{t+1} | y_t) \phi(z_{t+1} | z_t).$$

This means that individual endowment *shares* and the transition probabilities of the idiosyncratic shocks are independent of the aggregate state of the economy z .

In the baseline model, we also assume that the aggregate shocks are independent over time.

Condition 2.3. *The growth rate of the aggregate endowment is i.i.d.:*

$$\phi(z_{t+1} | z_t) = \phi(z_{t+1}).$$

In this case the logarithm of the aggregate endowment follows a random walk with drift.

3 No Predictability in Aggregate Consumption Growth

In this section we assume that condition 2.3 holds so that aggregate endowment and thus consumption growth is i.i.d. aggregate consumption growth shocks. First, analyze the model with Arrow market structure in which households can trade shares in the stock and a complete menu of claims that pay out contingent on the aggregate shock. Idiosyncratic shocks are (as in the entire paper) uninsurable. We first establish an equivalence result: we demonstrate that equilibrium allocations and prices of a stationary version of the model with only idiosyncratic, but no aggregate shocks (which we label the Bewley model) can be transformed, through simple scaling by the aggregate shock, into equilibrium allocations and prices in the Arrow model with aggregate risk. In that equilibrium a zero trade result for all contingent claims is obtained. This zero trade result, in turn, implies that equivalence result for the Arrow model directly extends to the Bond model, in which households can only trade a single uncontingent bond, in addition to the stock. We conclude this section with the risk premium irrelevance result which follows directly from the fact that consumption allocations in the model with aggregate risk are equal to consumption allocations from the model with only idiosyncratic risk, scaled up by the same factor for all households (the stochastic aggregate endowment)..

3.1 The Arrow Model

Let $a_t(s^t, z_{t+1})$ denote the quantity purchased of a security that pays off one unit of the consumption good if aggregate shock in the next period is z_{t+1} , irrespective of the idiosyncratic shock y_{t+1} . Its price today is given by $q_t(z^t, z_{t+1})$. In addition, households trade shares in the Lucas tree. We use $\sigma_t(s^t)$ to denote the number of shares a household with history $s^t = (y^t, z^t)$ purchases today and we let $v_t(z^t)$ denote the price of one share.

An agent starting period t with wealth $\theta_t(s^t)$ buys consumption commodities in the spot market and trades securities subject to the usual budget constraint:

$$c_t(s^t) + \sum_{z_{t+1}} a_t(s^t, z_{t+1})q_t(z^t, z_{t+1}) + \sigma_t(s^t)v_t(z^t) \leq \theta_t(s^t). \quad (4)$$

If next period's state is $s^{t+1} = (s^t, y_{t+1}, z_{t+1})$, her wealth is given by her labor income, the payoff from the contingent claim purchased in the previous period as well as the value of her position in the stock, including dividends:

$$\theta_{t+1}(s^{t+1}) = \eta(y_{t+1}, z_{t+1})e_{t+1}(z_{t+1}) + a_t(s^t, z_{t+1}) + \sigma_t(s^t) [v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1})]. \quad (5)$$

In addition to the budget constraints, the households' trading strategies are subject to solvency constraints of one of two types. The first type of constraint imposes a lower bound on the value of the asset portfolio at the end of the period today,

$$\sum_{z_{t+1}} a_t(s^t, z_{t+1})q_t(z^t, z_{t+1}) + \sigma_t(s^t)v_t(z^t) \geq K_t(s^t), \quad (6)$$

while the second type imposes state-by-state lower bounds on net wealth tomorrow,

$$a_t(s^t, z_{t+1}) + \sigma_t(s^t) [v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1})] \geq M_t(s^t, z_{t+1}) \text{ for all } z_{t+1}. \quad (7)$$

While we can handle potentially very tight solvency constraints we need to assume that they are proportional to the aggregate endowment.

Condition 3.1. *The solvency constraints lie between the natural solvency constraints (the maximum negative value of assets that households can repay by setting consumption to zero in all future periods) and zero*

$$\begin{aligned} K_t^{\text{nat}}(y^t, z^t) &\leq K_t(y^t, z^t) \leq 0, \\ M_t^{\text{nat}}(y^t, z^t, z_{t+1}) &\leq M_t(y^t, z^t, z_{t+1}) \leq 0. \end{aligned}$$

The solvency constraints depend on the aggregate history z^t only through the level of the aggregate endowment:

$$\begin{aligned} K_t(y^t, z^t) &= \hat{K}_t(y^t)e_t(z^t), \\ M_t(y^t, z^t, z_{t+1}) &= \hat{M}_t(y^t)e_{t+1}(z^{t+1}). \end{aligned}$$

The solvency constraints are not so tight as to make the feasible consumption set empty, but they are tight enough to prevent Ponzi schemes. If the constraints did not have the proportionality feature in a stochastically growing economy, they would become more or less binding as the aggregate economy grows, clearly not a desirable feature.⁸ Obviously, the tight standard liquidity constraint $K_t(y^t, z^t) = 0$ satisfies this condition (see Deaton (1991)).

In a competitive equilibrium households optimize, given prices, and for each z^t the goods market, the stock market and all markets for the Arrow securities clear (for a formal definition see section A of the Appendix).

3.2 The De-trended Arrow Model

In order to establish our equivalence result we first show how to transform the Arrow model with idiosyncratic and aggregate risk into a stationary model with only idiosyncratic risk. The mapping from the stochastically growing to a detrended stationary model involves a change of measures that is similar but not identical to the switch to a risk-neutral measure commonly used in asset pricing.

Transformation of the Model with Aggregate Shocks into a Stationary Model We transform the Arrow model with aggregate shocks into a stationary model with a stochastic time discount factor and a growth-adjusted probability matrix, following Alvarez and Jermann (2001). We let $\hat{U}(\hat{c})(s^t)$ denote the expected lifetime continuation utility in node s^t , under the new transition probabilities and discount factor, defined over consumption shares $\{\hat{c}_t(s^t)\} = \left\{ \frac{c_t(s^t)}{e_t(z^t)} \right\}$

$$\hat{U}(\hat{c})(s^t) = u(\hat{c}_t(s^t)) + \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) \hat{U}(\hat{c})(s^t, s_{t+1}), \quad (8)$$

where the *growth-adjusted* probabilities and the growth-adjusted discount factor are defined as:

$$\hat{\pi}(s_{t+1}|s_t) = \frac{\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}} \text{ and } \hat{\beta}(s_t) = \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}.$$

⁸Without this assumption, for example, the risk-free rate would depend on the *level* of aggregate consumption, which in turn follows a random walk. In the incomplete markets asset pricing literature the borrowing constraints usually satisfy this condition (see e.g. Heaton and Lucas (1996)).

$\hat{\pi}$ is a well-defined Markov matrix in that $\sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) = 1$ for all s_t . Given the independence condition (2.2), the growth-adjusted probability matrix $\hat{\pi}$ and the re-scaled discount factor are obtained by adjusting only the transition probabilities for the *aggregate* shock, ϕ , but not the transition probabilities for the idiosyncratic shocks:

$$\hat{\pi}(s_{t+1}|s_t) = \varphi(y_{t+1}|y_t)\hat{\phi}(z_{t+1}|z_t), \text{ and } \hat{\phi}(z_{t+1}|z_t) = \frac{\phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}.$$

Furthermore, the growth-adjusted discount factor only depends on the aggregate state z_t :

$$\hat{\beta}(z_t) = \beta \sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma} \quad (9)$$

We assume that $\max_{z_t} \hat{\beta}(z_t) < 1$ in order to insure that lifetime utility remains finite. Evidently this jointly restricts the time discount factor β , the endowment growth process and the coefficient of relative risk aversion γ . Since aggregate shocks are i.i.d. in the baseline model, in this case the growth-adjusted discount factor is a constant: $\hat{\beta}(z_t) = \hat{\beta}$, since $\hat{\phi}(z_{t+1}|z_t) = \hat{\phi}(z_{t+1})$ for all z_t .

In the appendix we prove that this transformation does not alter the agents' preference ranking over different consumption streams.

Proposition 3.1. *Households rank consumption share allocations $\hat{c} = \frac{c}{e}$ in the de-trended model in exactly the same way as they rank the corresponding consumption allocations c in the original model with growth: for any s^t and any two consumption allocations c, c'*

$$U(c)(s^t) \geq U(c')(s^t) \iff \hat{U}(\hat{c})(s^t) \geq \hat{U}(\hat{c}')(s^t).$$

This result is crucial for showing that we can compute the equilibrium allocations c for the stochastically growing model by solving for equilibrium allocations \hat{c} in the detrended model.

Trading Using hats $\hat{\cdot}$ to denote the variables in the stationary model we divide the budget constraint (4) by $e_t(z^t)$, and obtain, using equation (5):

$$\begin{aligned} & \hat{c}_t(s^t) + \sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1})\hat{q}_t(z^t, z_{t+1}) + \sigma_t(s^t)\hat{v}_t(z^t) \\ & \leq \eta(y_t) + \hat{a}_{t-1}(s^{t-1}, z_t) + \sigma_{t-1}(s^{t-1}) [\hat{v}_t(z^t) + \alpha], \end{aligned} \quad (10)$$

where we have defined the deflated Arrow positions as $\hat{a}_t(s^t, z_{t+1}) = \frac{a_t(s^t, z_{t+1})}{e_{t+1}(z^{t+1})}$ and prices as $\hat{q}_t(z^t, z_{t+1}) = q_t(z^t, z_{t+1})\lambda(z_{t+1})$. The deflated stock price is given by $\hat{v}_t(z^t) = \frac{v_t(z^t)}{e_t(z^t)}$. Similarly,

deflating the solvency constraints (6) and (7), using condition (3.1), yields:

$$\sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1}) \hat{q}_t(z^t, z_{t+1}) + \sigma_t(s^t) \hat{v}_t(z^t) \geq \hat{K}_t(y^t). \quad (11)$$

$$\hat{a}_t(s^t, z_{t+1}) + \sigma_t(s^t) [\hat{v}_{t+1}(z^{t+1}) + \alpha] \geq \hat{M}_t(y^t) \text{ for all } z_{t+1}. \quad (12)$$

Finally, the goods market clearing condition is given by:

$$\int \sum_{y^t} \varphi(y^t | y_0) \hat{c}_t(\theta_0, s^t) d\Theta_0 = 1. \quad (13)$$

Note that the conditional probabilities simplify due to condition (2.2). In the detrended model, the household maximizes $\hat{U}(\hat{c})(s_0)$ by choosing consumption, Arrow securities and shares of the Lucas tree, subject to the budget constraint (10) and the solvency constraint (11) or (12) in each node s^t . The definition of a competitive equilibrium in the de-trended Arrow model is now straightforward.

Definition 3.1. *For initial aggregate state z_0 and distribution Θ_0 over (θ_0, y_0) , a competitive equilibrium for the de-trended Arrow model consists of trading strategies $\{\hat{a}_t(\theta_0, s^t, z_{t+1})\}$, $\{\hat{\sigma}_t(\theta_0, s^t)\}$, $\{\hat{c}_t(\theta_0, s^t)\}$ and prices $\{\hat{q}_t(z^t, z_{t+1})\}$, $\{\hat{v}_t(z^t)\}$ such that*

1. *Given prices, trading strategies solve the household maximization problem*
2. *The goods market and asset markets clear, that is, equation (13) holds for all z^t and*

$$\int \sum_{y^t} \varphi(y^t | y_0) \hat{\sigma}_t(\theta_0, s^t) d\Theta_0 = 1$$

$$\int \sum_{y^t} \varphi(y^t | y_0) \hat{a}_t(\theta_0, s^t, z_{t+1}) d\Theta_0 = 0 \text{ for all } z_{t+1} \in Z$$

The first order conditions and complementary slackness conditions, together with the appropriate transversality condition, are listed in the appendix in section (A.1). These are necessary and sufficient conditions for optimality on the household side. We will now argue that these conditions are identical to those obtained in a version of the model without any aggregate risk (which we name the Bewley model).

3.3 The Bewley Model

In this model the aggregate endowment is constant and equal to 1. Households face idiosyncratic shocks y that follow a Markov process with transition probabilities $\varphi(y' | y)$. The household's preferences over consumption shares $\{\hat{c}(y^t)\}$ are defined in equation (8), with a constant time discount factor $\hat{\beta}$ as defined in equation (9).

Market Structure In this model without aggregate risk agents trade only a riskless one-period discount bond and shares in a Lucas tree that yields safe dividends of α in every period. The price of the Lucas tree at time t is denoted by \hat{v}_t .⁹ The riskless bond is in zero net supply. Each household is indexed by an initial condition (θ_0, y_0) , where θ_0 denotes its wealth (including period 0 labor income) at time 0.

The household chooses consumption $\{\hat{c}_t(\theta_0, y^t)\}$, bond positions $\{\hat{a}_t(\theta_0, y^t)\}$ and share holdings $\{\hat{\sigma}_t(\theta_0, y^t)\}$ to maximize its normalized expected lifetime utility $\hat{U}(\hat{c})(s^0)$, subject to a standard budget constraint:¹⁰

$$\hat{c}_t(y^t) + \frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t = \eta(y_t) + \hat{a}_{t-1}(y^{t-1}) + \hat{\sigma}_{t-1}(y^{t-1})(\hat{v}_t + \alpha).$$

Finally, each household faces one of two types of borrowing constraints. The first one restricts household wealth at the end of the current period. The second one restricts household wealth at the beginning of the next period:¹¹

$$\frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t \geq \hat{K}_t(y^t) \text{ for all } y^t. \quad (14)$$

$$\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)(\hat{v}_{t+1} + \alpha) \geq \hat{M}_t(y^t) \text{ for all } y^t. \quad (15)$$

The definition of equilibrium in this model is standard:

Definition 3.2. *For an initial distribution Θ_0 over (θ_0, y_0) , a competitive equilibrium for the Bewley model consists of trading strategies $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$, and prices $\{\hat{R}_t, \hat{v}_t\}$ such that*

1. *Given prices, trading strategies solve the household maximization problem*
2. *The goods markets and asset markets clear in all periods t*

$$\int \sum_{y^t} \varphi(y^t|y_0) \hat{c}_t(\theta_0, y^t) d\Theta_0 = 1.$$

$$\int \sum_{y^t} \varphi(y^t|y_0) \hat{a}_t(\theta_0, y^t) d\Theta_0 = 0.$$

$$\int \sum_{y^t} \varphi(y^t|y_0) \hat{\sigma}_t(\theta_0, y^t) d\Theta_0 = 1.$$

⁹The price of the tree is obviously nonstochastic due to the absence of aggregate risk.

¹⁰We suppress dependence on θ_0 for simplicity whenever there is no room for confusion.

¹¹This distinction is redundant in the Bewley model, but it is useful for establishing our equivalence results below.

In the absence of aggregate risk, the bond and the stock are perfect substitutes for households, and no-arbitrage implies that the stock return equals the risk-free rate:

$$\hat{R}_t = \frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t}.$$

In addition, at these equilibrium prices, household portfolios are indeterminate. Without loss of generality one can therefore focus on trading strategies in which households only trade the stock, but not the bond: $\hat{a}_t(\theta_0, y^t) \equiv 0$.

A *stationary equilibrium* in the Bewley model consists of a constant interest rate \hat{R} , share price \hat{v} , optimal household allocations and a time-invariant measure Φ over income shocks and financial wealth.¹² In the stationary equilibrium, households move within the invariant wealth distribution, but the wealth distribution itself is constant over time. Now we are ready to establish the equivalence between equilibria in the Bewley model and in the Arrow model.

3.4 Equivalence I: No Trade in Markets for Arrow Securities

Equilibria in the Bewley model without aggregate risk can be transformed into equilibria of the Arrow model with aggregate risk.

Theorem 3.1. *An equilibrium of the Bewley model $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ and $\{\hat{R}_t, \hat{v}_t\}$ can be implemented as an equilibrium for the Arrow model with growth, $\{a_t(\theta_0, s^t, z_{t+1})\}$, $\{\sigma_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$ and $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$, with*

$$\begin{aligned} c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t)e_t(z^t) \\ \sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t) \\ a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_t(\theta_0, y^t)e_{t+1}(z^{t+1}) \\ v_t(z^t) &= \hat{v}_t e_t(z^t) \\ q_t(z^t, z_{t+1}) &= \frac{1}{\hat{R}_t} * \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})} = \frac{1}{\hat{R}_t} * \frac{\phi(z_{t+1})\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}} \end{aligned} \quad (16)$$

The proof of this result is given in the appendix; here we provide its main intuition. Conjecture that the equilibrium prices of Arrow securities in the de-trended Arrow model are given by:

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t}. \quad (17)$$

An unconstrained household's Euler equation for the Arrow securities in the (detrended) Arrow

¹²See Chapter 17 of Ljungqvist and Sargent (2004) for the standard formal definition and the straightforward algorithm to compute such a stationary equilibrium.

model is given by (see section (A.1) in the appendix)

$$1 = \frac{\hat{\beta}(s_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{s^{t+1}|s^t, z_{t+1}} \hat{\pi}(s_{t+1}|s_t) \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))}.$$

But, in light of conditions (2.2) and (2.3) and given our conjecture that consumption allocations in the de-trended Arrow model only depend on idiosyncratic shock histories y^t and not on $s^t = (y^t, z^t)$, this Euler equation reduces to

$$1 = \hat{\beta} \hat{R}_t \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))},$$

where we used the conjectured form of prices in (17). This is exactly the same Euler equation for bonds in the Bewley model. Since Bewley equilibrium consumption allocations satisfy this condition, they also satisfy the Euler equation in the de-trended Arrow model, if prices are of the form (17). Hence the equivalence.

The proof in the appendix shows that a similar argument applies to the Euler equation for the stock (under the conjectured stock prices). We also show that, in the case of binding solvency constraints, the Lagrange multipliers on the constraints in the Bewley equilibrium are also valid multipliers for the constraints in the de-trended Arrow model. Hence, the equivalence result goes through regardless of how tight the solvency constraints are.

Once one has established that allocations and prices of a Bewley equilibrium are an equilibrium in the de-trended Arrow model, one simply needs to scale up allocation and prices by the appropriate growth factors to obtain the equilibrium prices and allocations in the stochastically growing Arrow model, as stated in the theorem.

It is straightforward to compute the equilibrium risk-free interest rates for the Arrow model. By summing over aggregate states tomorrow on both sides of equation (17), we find that the risk-free rate in the de-trended Arrow model coincides with that of the Bewley model:

$$\hat{R}_t^A = \frac{1}{\sum_{z_{t+1}} \hat{q}_t(z^t, z_{t+1})} = \hat{R}_t. \quad (18)$$

We can then back out the implied interest rate for the original growing Arrow model, using (16) in the previous theorem.

Corollary 3.1. *Equilibrium risk-free interest rates in the Arrow model with aggregate risk are given by*

$$R_t^A = \frac{1}{\sum_{z_{t+1}} q_t(z^t, z_{t+1})} = \hat{R}_t \frac{\sum_{z_{t+1}} \phi(z_{t+1}) \lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}) \lambda(z_{t+1})^{-\gamma}}. \quad (19)$$

The theorem implies that we can solve for equilibria in the Bewley model of section 3.3 (and, in particular, a stationary equilibrium), including risk free interest rates \hat{R}_t , and we can deduce the equilibrium allocations and prices for the Arrow model from those in the Bewley model, using the mapping described in theorem 3.1. The key to this result is that households in the Bewley model face exactly the same Euler equations (at equilibrium prices) as households in the de-trended version of the Arrow model.

In the Arrow equilibrium that we propose, the contingent bond holdings are simply proportional to the aggregate endowment: $a_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)e_{t+1}(z^{t+1})$. Equivalently, in the de-trended Arrow model households choose not to make their Arrow securities purchases contingent on next period's aggregate shock: $\hat{a}_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)$. Furthermore, without loss of generality we can set the bond holdings to zero $\hat{a}_t(\theta_0, y^t) = 0$ in the Bewley model. In that case there is no trade in Arrow securities at all in the equivalent Arrow equilibrium: $a_t(\theta_0, s^t, z_{t+1}) = 0$.

We will use this equivalence result to show below that asset prices in the Arrow model are identical to those in the representative agent model, except for a lower risk-free interest rate (and a higher price/dividend ratio for stocks).¹³

3.5 Equivalence II: No Trade in Uncontingent Bond Markets

The equivalence result stated in Theorem 3.1 carries over to a model with aggregate risk in which only a discount bond and a stock are traded. We call this the Bond model. The no-trade result for contingent claims directly implies that agents in the Bond model optimally choose bond positions in equilibrium that are given by:

$$b_t(\theta_0, y^t, z^t) = \hat{a}_t(\theta_0, y^t)e_t(z^t) = 0.$$

Equilibria in the Bewley model can then be transformed into equilibria of the stochastically growing Bond model, just like for the Arrow model.¹⁴

Corollary 3.2. *An equilibrium of the Bewley model $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ and $\{\hat{R}_t, \hat{v}_t\}$ can be implemented as an equilibrium for the Arrow model with growth, $\{b_t(\theta_0, s^t)\}$, $\{\sigma_t(\theta_0, s^t)\}$,*

¹³Our equivalence theorem has two other implications. First, the existence proofs in the literature for stationary equilibria in the Bewley model directly carry over to the stochastically growing model. Second, the moments of the wealth distribution vary over time but proportionally to the aggregate endowment. If the initial wealth distribution in the de-trended model corresponds to an invariant distribution in the Bewley model, then -for example- the ratio of the mean to the standard deviation of the wealth distribution is constant over time and across states of the world in the Arrow model with aggregate risk.

¹⁴See the separate appendix available at <http://www.econ.upenn.edu/~dkrueger/research/assetpricingapp.pdf> for a complete description of the model and formal proofs of the results..

$\{c_t(\theta_0, s^t)\}$ and $\{R_t(z^t)\}$, $\{v_t(z^t)\}$, with

$$\begin{aligned}
c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t)e_t(z^t) \\
\sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t) \\
b_t(\theta_0, s^t) &= \hat{a}_t(\theta_0, y^t)e_{t+1}(z^{t+1}) = 0 \\
v_t(z^t) &= \hat{v}_t e_t(z^t) \\
\frac{1}{R_t} &= \sum_{z_{t+1}} q_t(z^t, z_{t+1}) = \frac{1}{\hat{R}_t} \sum_{z_{t+1}} * \frac{\phi(z_{t+1})\lambda(z_{t+1})^{-\gamma}}{\sum_{z_s} \phi(z_s)\lambda(z_s)^{1-\gamma}}.
\end{aligned} \tag{20}$$

There is one difference between this result and the corresponding result for the Arrow model. In the Arrow model we demonstrated that contingent claims positions were in fact uncorrelated: $\hat{a}_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)$ and equal to the Bond position in the Bewley equilibrium. It was not necessary, however, to set them to zero. Since bonds in the Bewley equilibrium are a redundant asset, one can without loss of generality restrict attention to the situation where $\hat{a}_t(\theta_0, y^t) = 0$, but this is not necessary for our results in the Arrow model. In the Bond model bond positions *have to be* zero. This becomes apparent from the deflated household budget constraint in the Bond model:

$$\hat{c}_t(y^t) + \frac{\hat{b}_t(y^t)}{R_t} + \sigma_t(y^t)\hat{v}_t \leq \eta(y_t) + \frac{\hat{b}_{t-1}(y^{t-1})}{\lambda(z_t)} + \sigma_{t-1}(y^{t-1})[\hat{v}_t + \alpha],$$

where the deflated bond position is $\hat{b}(y^t) = b(y^t)/e_t(z^t)$. Bond holdings $\hat{b}_t(y^{t-1}) = 0$ need to be zero for all y^{t-1} , even if all the consumption, portfolio choices and prices, are independent of the aggregate history z^t , simply because the bond return in the deflated economy depends on the aggregate shock z_t through $\lambda(z_t)$. This demonstrates the irrelevance of the history of idiosyncratic shocks y^t for portfolio choices in this model. There is no link between financial wealth and the share of this wealth being held in equity.

3.6 Asset Pricing Implications

By making use of our equivalence results it is now straightforward to derive our main asset pricing result: the multiplicative risk premium on a claim to aggregate consumption in both the Bond model and the Arrow model equals the risk premium in the representative agent model. Uninsurable idiosyncratic income risk only lowers the risk-free rate, but does not affect the risk premium at all.

Consumption-CAPM The benchmark model of consumption-based asset pricing is the representative agent Breeden-Lucas (*BL*) model. The representative agent owns a claim to the aggregate ‘labor’ income stream $\{(1 - \alpha)e_t(z^t)\}$ and she can trade a stock (a claim to the dividends $\alpha e_t(z^t)$) of

the Lucas tree), a bond and a complete set of Arrow securities. First, we show that the *Breeden-Lucas Consumption-CAPM* also prices excess returns on the stock in the Bond model and the Arrow model. Let R^s denote the return on a claim to aggregate consumption. We have

Lemma 3.1. *The BL Consumption-CAPM prices excess returns in the Arrow model and the Bond model. In equilibrium in both models*

$$E_t [(R_{t+1}^s - R_t) \beta (\lambda_{t+1})^{-\gamma}] = 0$$

where $\beta \lambda_{t+1} (z_{t+1})^{-\gamma} = m_{t+1}^{RE} (z^{t+1})$ is the stochastic discount factor in the BL representative agent (RE) model.

This result follows directly from subtracting the Euler equations for the bond from that for the stock in both models. This result has important implications for empirical work in asset pricing. First, despite the existence of market incompleteness and binding solvency constraints, an econometrician can estimate the coefficient of risk aversion (or the intertemporal elasticity of substitution) directly from aggregate consumption data and the excess return on stocks, as in Hansen and Singleton (1982), *if* the Bond model or the Arrow model is the true model of the real world. Second, the result provides a strong justification for work trying to explain the cross-section of excess returns using the CCAPM, without necessarily trying to match the risk-free rate. As we show here, the implications of the *BL*, the Arrow and the Bond model are identical with respect to excess returns, but not with respect to the risk-free rate.

Risk Premia We now show that the equilibrium risk premium in the Arrow and the Bond model is identical to the one in the *BL* representative agent model. While the risk-free rate is higher in the *BL* representative agent model than in the Arrow and Bond model, and consequently the price of the stock is correspondingly lower, the multiplicative risk premium is the same in all three models and it is nonstochastic.

In order to demonstrate our main asset pricing result we first show that the stochastic discount factors that price stochastic payoffs in the *BL* representative agent model and the Arrow (and thus the Bond) model only differ by a non-random multiplicative term, equal to the ratio of (growth-deflated) risk-free interest rates in the two models.

Proposition 3.2. *The equilibrium stochastic discount factor in the Arrow and the Bond model given by*

$$m_{t+1}^A = m_{t+1}^{RE} \kappa_t$$

where the non-random multiplicative term is given

$$\kappa_t = \frac{\hat{R}_t^{RE}}{\hat{R}_t} \geq 1.$$

The proof that risk premia are identical in the representative agent model and the Arrow as well as the Bond model follows directly from the previous decomposition of the stochastic discount factor. Let $R_{t,j}[\{d_{t+k}\}]$ denote the j -period holding return on a claim to the endowment stream $\{d_{t+k}\}_{k=0}^{\infty}$ at time t . Consequently $R_{t,1}[1]$ is the gross risk-free rate and $R_{t,1}[\alpha e_{t+k}]$ is the one-period holding return on a k -period strip of the aggregate endowment (a claim to α times the aggregate endowment k periods from now). Thus $R_{t,1}[\{\alpha e_{t+k}\}]$ is the one period holding return on an asset (such as a stock) that pays α times the aggregate endowment in all future periods. Finally, we define the multiplicative risk premium as the ratio of the expected return on stocks and the risk-free rate:

$$1 + \nu_t = \frac{E_t R_{t,1}[\{\alpha e_{t+k}\}]}{R_{t,1}[1]}.$$

With this notation in place, we can state now our main asset pricing result.

Theorem 3.2. *The multiplicative risk premium in the Arrow model and Bond model equals that in the representative agent model*

$$1 + \nu_t^A = 1 + \nu_t^B = 1 + \nu_t^{RE},$$

and is constant across states of the world.

Thus, the extent to which households smooth idiosyncratic income shocks in the Arrow model or in the Bond model has absolutely no effect on the size of risk premia; it merely lowers the risk-free rate. Market incompleteness does not generate any dynamics in the conditional risk premia either: the conditional risk premium is constant.

4 Predictability in Aggregate Consumption Growth

In this section, we investigate how robust our results are to the assumption that endowment growth is *i.i.d* over time. We show that our results remain valid even when aggregate endowment growth is serially correlated, but only for the Arrow model, and only if the solvency constraints (i) do not bind or (ii) are judiciously chosen (reverse-engineered).

Assume that the aggregate shock z follows a first order Markov chain characterized by the transition matrix $\phi(z'|z) > 0$. So far we studied the special case in which $\phi(z'|z) = \phi(z')$. Recall

that the growth-adjusted Markov transition matrix and time discount factor are given by

$$\hat{\phi}(z'|z) = \frac{\phi(z'|z)\lambda(z')^{1-\gamma}}{\sum_{z'} \phi(z'|z)\lambda(z')^{1-\gamma}} \text{ and } \hat{\beta}(z) = \beta \sum_{z'} \phi(z'|z)\lambda(z')^{1-\gamma}.$$

Thus if ϕ is serially correlated, so is $\hat{\phi}$, and the discount factor $\hat{\beta}$ now does depend on the current aggregate state of the world. This indicates (and we will show this below) that the aggregate endowment shock acts as an aggregate taste shock in the de-trended model which renders all households simultaneously more or less impatient. Since this shock affects all households in the same way, they will not be able to insure against it. As a result, this shock affects the price/dividend ratio and the interest rate, but it leaves the risk premium unaltered. In contrast to our previous results, however, now there is trade in contingent bonds in equilibrium, so the equivalence between equilibria in the Arrow and the Bond model breaks down.

4.1 The Bewley Model

We apply the same strategy to show our results as before. The equilibrium allocations and prices in the Bewley model can be implemented, after appropriate scaling by the aggregate endowment, as equilibria in the stochastically growing model. However, since the time discount factors are subject to an aggregate shock, we first have to choose an appropriate nonstochastic time discount factor for this Bewley model. We choose a sequence of non-random time discount factors such that the Bewley equilibrium allocations satisfy the time zero budget constraint in the model with aggregate shocks when the initial wealth distribution Θ_0 in the two models coincide.

Let

$$\hat{\beta}_{0,\tau}(z^\tau|z_0) = \hat{\beta}(z_0)\hat{\beta}(z_1) \dots \hat{\beta}(z_\tau)$$

denote the time discount factor between period 0 and period $\tau + 1$, given by the product of one-period stochastic time discount factors. We define the average (across aggregate shocks) time discount factor between period 0 and t as:

$$\tilde{\beta}_t = \sum_{z^{t-1}|z_0} \hat{\phi}(z^{t-1}|z_0)\hat{\beta}_{0,t-1}(z^{t-1}|z_0), t \geq 1, \quad (21)$$

where $\hat{\phi}(z^{t-1}|z_0)$ is the (conditional on z_0) probability distribution over z^{t-1} induced by $\phi(z'|z)$. If aggregate shocks are *i.i.d.*, then we have that $\tilde{\beta}_t = \hat{\beta}^t$, as before. Since z_0 is a fixed initial condition, we chose not to index $\tilde{\beta}_t$ by z_0 to make sure it is understood that $\tilde{\beta}_t$ is nonstochastic.

In order to construct equilibrium allocations in the model with aggregate shocks, we will show that equilibrium allocations and interest rates in the Bewley model with a sequence of non-random time discount factors $\left\{ \tilde{\beta}_t \right\}_{t=1}^{\infty}$ can be implemented as equilibrium allocations and interest rates for

the *actual* Arrow model with stochastic discount factors. The crucial adjustment in this mapping is to rescale the risk-free interest rate in proportion to the taste shock $\hat{\beta}(z)$.

To understand the effect of these aggregate taste shocks on the time discount factor to be used in the Bewley model, consider the following simple example.

Example 4.1. *Suppose that $\hat{\beta}(z) = e^{-\hat{\rho}(z)}$ is lognormal and i.i.d, where $\hat{\rho}(z)$ has mean $\hat{\rho}$ and variance σ^2 . Define the average t -period time discount rate $\tilde{\rho}_t$ by the equation $\tilde{\beta}_t = e^{-\tilde{\rho}_t}$. Then the average one-period discount rate used in the Bewley model is given by:*

$$\frac{\tilde{\rho}_t}{t} = \hat{\rho} - \frac{1}{2}\sigma^2 \text{ for any } t \geq 1$$

The presence of taste shocks ($\sigma^2 > 0$) in the de-trended Arrow model induces a discount rate $\tilde{\rho}$ in the Bewley model that is lower than the mean discount rate $\hat{\rho}$ because of the risk associated with taste shocks.

This example shows that the taste shocks (originating from serially correlated endowment growth rates) tend to lower the risk-free interest rates compared to a model without taste shocks.

As before, we denote the Bewley equilibrium allocations by $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ and prices by $\{\hat{R}_t, \hat{v}_t\}$, for a given sequence of time discount factors $\{\tilde{\beta}_t\}$. Only the total wealth positions in the Bewley model are uniquely pinned down. Without loss of generality, we focus on the case where $\hat{a}_t(\theta_0, y^t) = 0$ for all y^t . We now argue that the allocation $\{\hat{c}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ can be made into an Arrow equilibrium, and in the process we show why we need to choose the specific discount factor sequence in (21) for the Bewley model. We introduce some additional notation for state prices. Let

$$\tilde{Q}_{t,\tau} = \prod_{j=0}^{\tau-t-1} \hat{R}_{t+j}^{-1} = \frac{1}{\hat{R}_{t,\tau}} \quad (22)$$

denote the Bewley equilibrium price of one unit of consumption to be delivered at time τ , in terms of consumption at time t . By convention $\tilde{Q}_\tau = \tilde{Q}_{0,\tau}$ and $\tilde{Q}_{\tau,\tau} = 1$ for all τ . In other words, $\hat{R}_{t,\tau}$ is the gross risk-free interest rate between period t and τ in the Bewley equilibrium.

4.2 The Detrended Arrow Model

In contrast to the Bewley model, the de-trended Arrow model features aggregate shocks to the time discount factor $\hat{\beta}$. These need to be reflected in prices. We therefore first conjecture state-dependent equilibrium prices for the de-trended Arrow model. Then we show that the Bewley equilibrium allocations satisfy the Euler equations of the detrended Arrow model when evaluated at these prices, and that they also satisfy the intertemporal budget constraint in the de-trended Arrow model. This implies that, absent binding solvency constraints, the Bewley equilibrium can

be implemented as an equilibrium of the de-trended Arrow model, and thus, after the appropriate scaling, as an equilibrium of the original Arrow model. Finally, we discuss potentially binding solvency constraints and the Bond model.

We conjecture that the Arrow-Debreu prices in the deflated Arrow model are given by

$$\hat{Q}_t(z^t|z_0) = \hat{\phi}(z^t|z_0)\tilde{Q}_t \frac{\hat{\beta}_{0,t-1}(z^{t-1}|z_0)}{\tilde{\beta}_t} = \hat{\phi}(z^t|z_0) \frac{\hat{\beta}_{0,t-1}(z^{t-1}|z_0)}{\tilde{\beta}_t \hat{R}_{0,t}}, \quad (23)$$

where \tilde{Q}_t was defined above as the time 0 price of consumption in period t in the Bewley model. The prices of the (one-period ahead) Arrow securities are then given by:

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{Q}_{t+1}(z^{t+1}|z_0)}{\hat{Q}_t(z^t|z_0)} = \hat{\beta}(z_t)\hat{\phi}(z_{t+1}|z_t) \frac{1}{\hat{R}_t} \frac{\tilde{\beta}_t}{\tilde{\beta}_{t+1}}, \quad (24)$$

where we used the fact that $\frac{1}{\hat{R}_t} = \frac{\hat{R}_{0,t}}{\hat{R}_{0,t+1}}$. Arrow prices are Markovian in z_t , since \hat{R}_t and $(\tilde{\beta}_t, \tilde{\beta}_{t+1})$ are all deterministic. Equation (24) implies that interest rates in the de-trended Arrow model are given by

$$\hat{R}_t^A(z_t) \equiv \sum_{z_{t+1}} \hat{q}_t(z_{t+1}|z_t) = \hat{R}_t \frac{\tilde{\beta}_{t+1}}{\hat{\beta}(z_t)\tilde{\beta}_t}. \quad (25)$$

Interest rates in the detrended Arrow model now depend on the current aggregate state of the world z_t . Finally, we conjecture that the stock price in the de-trended Arrow model satisfies:

$$\hat{v}_t(z^t) = \hat{v}_t(z_t) = \sum_{z_{t+1}} \hat{\phi}(z_{t+1}|z_t) \left(\frac{\hat{v}_{t+1}(z_{t+1}) + \alpha}{\hat{R}_t^A(z_t)} \right) \quad (26)$$

Armed with these conjectured prices we can now prove the following result.

Lemma 4.1. *Absent solvency constraints, the household Euler equations are satisfied in the Arrow model at the Bewley allocations $\{\hat{c}_{t+1}(y^t, y_{t+1})\}$ and prices for Arrow securities $\{\hat{q}_t(z_{t+1}|z_t)\}$ given by (24).*

4.3 Trade in Arrow Securities Markets

Next, we spell out which asset trades support the Bewley equilibrium consumption allocations in the de-trended Arrow model. The Arrow security positions implied by the Bewley allocations are not zero, but they are zero on average across households and hence clear the asset markets.

At any point in time and any node of the event tree, the Arrow securities position, plus the value of the stock position cum dividends, has to finance the value of excess demand from today into the infinite future. Thus, the Arrow securities position implied by the Bewley equilibrium

allocation $\{\hat{c}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ is given by:

$$\begin{aligned} \hat{a}_{t-1}(\theta_0, y^{t-1}, z^t) &= \hat{c}_t(\theta_0, y^t) - \eta(y_t) + \sum_{\tau=t+1}^{\infty} \sum_{z^\tau, y^\tau} \hat{Q}_\tau(z^\tau | z_t) \varphi(y^\tau | y^t) (\hat{c}_\tau(\theta_0, y^\tau) - \eta(y_\tau)) \\ &\quad - \hat{\sigma}_{t-1}(\theta_0, y^{t-1}) [\hat{v}_t(z_t) + \alpha] \end{aligned} \quad (27)$$

Proposition 4.1. *The Arrow securities positions implied by the Bewley allocations in (27) clear the Arrow securities markets, that is*

$$\int \sum_{y^{t-1}} \varphi(y^{t-1} | y_0) \hat{a}_{t-1}(\theta_0, y^{t-1}, z^t) d\Theta_0 = 0 \text{ for all } z^t.$$

Since the wealth from stock holdings at the beginning of the period

$$\hat{\sigma}_{t-1}(\theta_0, y^{t-1}) [\hat{v}_t + \alpha] = \alpha \hat{\sigma}_{t-1}(\theta_0, y^{t-1}) \sum_{\tau=t}^{\infty} \tilde{Q}_{t,\tau}$$

has to finance future excess consumption demand in the Bewley equilibrium, we can state the Arrow securities positions as:

$$\begin{aligned} \hat{a}_{t-1}(\theta_0, y^{t-1}, z_t) &= \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau | z_t) - \tilde{Q}_{t,\tau} \right) \sum_{y^\tau} \varphi(y^\tau | y^t) (\hat{c}_\tau(\theta_0, y^\tau) - \eta(y_\tau)) \\ &\quad - \hat{\sigma}_{t-1}(\theta_0, y^{t-1}) \alpha \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau | z_t) - \tilde{Q}_{t,\tau} \right) \end{aligned}$$

The Arrow security portfolios held by households are used to hedge against the interest rate shocks that govern the difference between the stochastic state prices $\hat{Q}_\tau(z^\tau | z_t)$ and the deterministic prices $\tilde{Q}_{t,\tau}$. If aggregate endowment growth is i.i.d, there are no taste shocks in the detrended Arrow model, and from (25) we see that then interest rates are deterministic. The gap between $\hat{Q}_\tau(z^\tau | z_t)$ and $\tilde{Q}_{t,\tau}$ is zero and no Arrow securities are traded in equilibrium, confirming the results in section 3.1.

In order to close our argument, we need to show that no initial wealth transfers between individuals are required for this implementation. In other words, the initial Arrow securities position $\hat{a}_{-1}(\theta_0, y^{-1}, z_0)$ implied by (27) at time 0, is zero for all households.¹⁵

To do so we proceed in two steps. First, we show that average time zero state prices in the Arrow model coincide with the state prices in the Bewley model. For this result to hold our

¹⁵Without this argument we merely would have shown that a Bewley equilibrium for initial condition Θ_0 can be implemented as equilibrium of the de-trended Arrow model with initial conditions z_0 and some initial distribution Ψ_0 of wealth, but not necessarily Θ_0 .

particular choice of time discount factors $\{\tilde{\beta}_t\}$ for the Bewley model is crucial.

Lemma 4.2. *The conjectured prices for the Arrow model in (23) and the prices in the Bewley model defined in (22) satisfy*

$$\sum_{z^\tau} \hat{Q}_\tau(z^\tau|z_0) = \tilde{Q}_\tau$$

Finally, using this result we can establish that no wealth transfers are necessary to implement the Bewley equilibrium as equilibrium in the de-trended Arrow model.

Lemma 4.3. *The Arrow securities position at time 0 given in (27) is zero:*

$$\hat{a}_{-1}(\theta_0, y^{-1}, z_0) = 0.$$

The Bewley equilibrium therefore is an equilibrium for the de-trended Arrow model with the same initial wealth distribution Θ_0 . This result then immediately implies the following theorem.

Theorem 4.2. *An equilibrium of the Bewley model $\{\hat{c}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ and $\{\hat{R}_t, \hat{v}_t\}$ where households have a sequence of time discount factors $\{\tilde{\beta}_t\}$ can be implemented as an equilibrium for the Arrow model with growth, $\{a_t(\theta_0, s^t, z_{t+1})\}$, $\{\sigma_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$ and $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$, with*

$$\begin{aligned} c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t) e_t(z^t) \\ \sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t) \\ a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_t(\theta_0, y^t, z_{t+1}) e_{t+1}(z^{t+1}) \text{ with } \hat{a}_t \text{ defined in (27)} \\ v_t(z_t) &= \sum_{z_{t+1}} \frac{\hat{\phi}(z_{t+1}|z_t)}{\lambda(z_{t+1})} \left[\frac{v_{t+1}(z_{t+1}) + \alpha e_{t+1}(z_{t+1})}{\hat{R}_t^A(z_t)} \right] \\ \hat{R}_t^A(z_t) &= \frac{\hat{R}_t \tilde{\beta}_{t+1}}{\hat{\beta}(z_t) \tilde{\beta}_t} \\ q_t(z^t, z_{t+1}) &= \frac{\hat{q}_t(z^t, z_{t+1})}{\lambda(z_{t+1})} = \frac{1}{\hat{R}_t^A(z_t)} * \frac{\phi(z_{t+1}|z_t) \lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t) \lambda(z_{t+1})^{1-\gamma}} \end{aligned}$$

Risk Premia Of course, this result implies that our baseline irrelevance result for risk premia survives the introduction of non-i.i.d. aggregate shocks, provided that a complete menu of aggregate-state-contingent securities is traded. Note that

$$\kappa_t(z_t) = \frac{\hat{R}_t^{RE}(z_t)}{\hat{R}_t^A(z_t)} = \frac{\tilde{\beta}_t}{\hat{R}_t \tilde{\beta}_{t+1}} = \kappa_t$$

is still deterministic, and thus the proofs of section 3.6 go through unchanged. These aggregate taste shocks only affect interest rates and price/dividend ratios, not risk premia. When agents in

the transformed model become more impatient, the interest rate rises and the price/dividend ratio falls, but the conditional expected excess return is unchanged.

Solvency Constraints So far, we have abstracted from binding solvency constraints. Previously, we assumed that the solvency constraints satisfy $K_t(s^t) = \hat{K}_t(y^t)e_t(z^t)$ and $M_{t+1}(s^{t+1}) = \hat{M}_t(y^t)e_t(z^{t+1})$. The allocations in theorem 4.2 derived from the stationary model using $\hat{K}_t(y^t)$ and $\hat{M}_t(y^t)$ as solvency constraints do not necessarily satisfy the solvency constraints given by $K_t(s^t) = \hat{K}_t(y^t)e_t(z^t)$ and $M_{t+1}(s^{t+1}) = \hat{M}_t(y^t)e_t(z^{t+1})$, but they satisfy modified solvency constraints.

Proposition 4.2. *The allocations from theorem 4.2 satisfy the modified solvency constraints:*

$$K_t^*(s^t) = K_t(s^t) + \sum_{z_{t+1}} q_t(z^t, z_{t+1}) a_t(\theta_0, s^t, z_{t+1}) + \sigma_t(\theta_0, s^t) [v_t(z_t) - \hat{v}_t e_t(z^t)] \quad (28)$$

$$M_{t+1}^*(s^{t+1}) = M_{t+1}(s^{t+1}) + a_t(\theta_0, s^t, z_{t+1}) + \sigma_t(\theta_0, s^t) [v_{t+1}(z_{t+1}) - \hat{v}_{t+1} e_{t+1}(z^{t+1})] \quad (29)$$

where \hat{v}_t is the (deterministic) Bewley equilibrium stock price.

If the allocations satisfy the constraints in the stationary Bewley model, they satisfy the modified solvency constraints (28) or (29) in the actual Arrow model, but not the ones we originally specified, because of the nonzero state-contingent claims positions. These violations of the original constraints are completely due to the impact of interest rate shocks on the value of the asset portfolio.

Finally, in the Bond model, our previous equivalence result no longer holds, since with predictability in aggregate consumption growth households trade state-contingent claims in the equilibrium of the Arrow model. Unless the aggregate shock can only take two values, the market structure of the Bond model prevents them from doing so, and thus our implementation and irrelevance results in this model are not robust to the introduction of non-i.i.d aggregate endowment growth.

5 Conclusion

We have derived conditions under which the history of a household's idiosyncratic shocks has no effect on his portfolio choice, even in the presence of binding solvency constraints. This portfolio irrelevance result directly implies our risk premium irrelevance result. Since all households bear the same amount of aggregate risk in equilibrium, the history of aggregate shocks does not affect equilibrium prices and allocations. The equilibrium risk-free rate, the risk premium and the price/dividend ratio are all deterministic, at least in the benchmark model without predictability in aggregate consumption growth. However, only a weaker version of this irrelevance result

survives in economies with a predictable component in aggregate consumption growth (see e.g. Bansal and Yaron (2004)), because the associated interest rate risk gives rise to heterogeneous hedging demands that do depend on the household’s history of idiosyncratic shocks. Surprisingly, the impact of idiosyncratic risk on risk premia depends critically on the predictability of aggregate consumption growth, but it does not depend at all on the persistence or variance of household labor income.

Other than predictability in aggregate consumption growth shocks, there are two main avenues around our results. One approach is to concentrate aggregate risk by forcing some households out of the stock market altogether. This is the approach adopted in the literature on limited participation (see e.g. Guvenen (2003), Vissing-Jorgensen (2002), Gomes and Michaelides (2007) and Attanasio, Banks, and Tanner (2002)). The second approach consists of concentrating labor income risk in recessions. Recently Krusell and Smith (1997) and Storesletten, Telmer, and Yaron (2007) have argued that models with idiosyncratic income shocks and incomplete markets can generate an equity premium that is substantially larger than the CCAPM if there is counter-cyclical cross-sectional variance (CCV) in labor income shocks. Our paper demonstrates analytically that CCV in labor income, limited stock market participation or some form of predictability in aggregate consumption growth is *necessary* to make uninsurable idiosyncratic income shocks potentially useful for explaining the equity premium.

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A Equilibrium in Arrow Model

The definition of an equilibrium in the Arrow model is standard. Each household is assigned a label that consists of its initial financial wealth θ_0 and its initial state $s_0 = (y_0, z_0)$. A household of type (θ_0, s_0) then chooses consumption allocations $\{c_t(\theta_0, s^t)\}$, trading strategies for Arrow securities $\{a_t(\theta_0, s^t, z_{t+1})\}$ and shares $\{\sigma_t(\theta_0, s^t)\}$ to maximize her expected utility (1), subject to the budget constraints (4) and subject to solvency constraints (6) or (7).

Definition A.1. *For initial aggregate state z_0 and distribution Θ_0 over (θ_0, y_0) , a competitive equilibrium for the Arrow model consists of household allocations $\{a_t(\theta_0, s^t, z_{t+1})\}$, $\{\sigma_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$ and prices $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$ such that*

1. *Given prices, household allocations solve the household maximization problem*

2. The goods market clears for all z^t ,

$$\int \sum_{y^t} \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t | z_0)} c_t(\theta_0, s^t) d\Theta_0 = e_t(z^t)$$

3. The asset markets clear for all z^t

$$\begin{aligned} \int \sum_{y^t} \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t | z_0)} \sigma_t(\theta_0, s^t) d\Theta_0 &= 1 \\ \int \sum_{y^t} \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t | z_0)} a_t(\theta_0, s^t, z_{t+1}) d\Theta_0 &= 0 \text{ for all } z_{t+1} \in Z \end{aligned}$$

A.1 Optimality Conditions for De-trended Arrow Model

Define the Lagrange multiplier

$$\hat{\beta}(s^t) \hat{\pi}(s^t | s_0) u'(\hat{c}_t(s^t)) \hat{\mu}(s^t) \geq 0$$

for the constraint in (11) and

$$\hat{\beta}(s^t) \hat{\pi}(s^t | s_0) u'(\hat{c}_t(s^t)) \hat{\kappa}_t(s^t, z_{t+1}) \geq 0$$

for the constraint in (12). The Euler equations of the de-trended Arrow model are given by:

$$\begin{aligned} 1 &= \frac{\hat{\beta}(s_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{s^{t+1} | s^t, z_{t+1}} \hat{\pi}(s_{t+1} | s_t) \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\ &+ \hat{\mu}_t(s^t) + \frac{\hat{\kappa}_t(s^t, z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \quad \forall z_{t+1}. \end{aligned} \quad (30)$$

$$\begin{aligned} 1 &= \hat{\beta}(s_t) \sum_{s^{t+1} | s^t} \hat{\pi}(s_{t+1} | s_t) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right] \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\ &+ \hat{\mu}_t(s^t) + \sum_{z_{t+1}} \hat{\kappa}_t(s^t, z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right]. \end{aligned} \quad (31)$$

Only one of the two Lagrange multipliers enters the equations, depending on which version of the solvency constraint we consider. The complementary slackness conditions for the Lagrange multipliers are given by

$$\begin{aligned} \hat{\mu}_t(s^t) \left[\sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1}) \hat{q}_t(z^t, z_{t+1}) + \hat{\sigma}_t(s^t) \hat{v}_t(z^t) - \hat{K}_t(y^t) \right] &= 0 \\ \hat{\kappa}_t(s^t, z_{t+1}) \left[\hat{a}_t(s^t, z_{t+1}) + \hat{\sigma}_t(s^t) [\hat{v}_{t+1}(z^{t+1}) + \alpha] - \hat{M}_t(y^t) \right] &= 0. \end{aligned}$$

The appropriate transversality conditions read as

$$\begin{aligned} \lim_{t \rightarrow \infty} \sum_{s^t} \hat{\beta}(s^{t-1}) \hat{\pi}(s^t | s_0) u'(\hat{c}_t(s^t)) [\hat{a}_{t-1}(s^{t-1}, z_t) - \hat{M}_{t-1}(y^{t-1})] &= 0. \\ \lim_{t \rightarrow \infty} \sum_{s^t} \hat{\beta}(s^{t-1}) \hat{\pi}(s^t | s_0) u'(\hat{c}_t(s^t)) [\hat{\sigma}_{t-1}(s^{t-1}) (\hat{v}_t(z^t) + \alpha) - \hat{M}_{t-1}(y^{t-1})] &= 0, \end{aligned}$$

and

$$\begin{aligned} \lim_{t \rightarrow \infty} \sum_{s^t} \hat{\beta}(s^{t-1}) \hat{\pi}(s^t | s_0) u'(\hat{c}_t(s^t)) \left[\sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1}) \hat{q}_t(z^t, z_{t+1}) - \hat{K}_t(y^t) \right] &= 0. \\ \lim_{t \rightarrow \infty} \sum_{s^t} \hat{\beta}(s^{t-1}) \hat{\pi}(s^t | s_0) u'(\hat{c}_t(s^t)) \left[\hat{\sigma}_t(s^t) \hat{v}_t(z^t) - \hat{K}_t(y^t) \right] &= 0. \end{aligned}$$

Since the household optimization has a concave objective function and a convex constraint set the first order conditions and complementary slackness conditions, together with the transversality condition, are necessary and sufficient conditions for optimality of household allocation choices.

B Proofs

- Proof of Proposition 3.1:

Proof. We use $U(c)(s^t)$ to denote the continuation utility of an agent from consumption stream c , starting at history s^t . This continuation utility follows the simple recursion

$$U(c)(s^t) = u(c_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1} | s_t) U(c)(s^t, s_{t+1}),$$

where it is understood that $(s^t, s_{t+1}) = (z^t, z_{t+1}, y^t, y_{t+1})$. Divide both sides by $e_t(s^t)^{1-\gamma}$ to obtain

$$\frac{U(c)(s^t)}{e_t(z^t)^{1-\gamma}} = u(\hat{c}_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1} | s_t) \frac{e_{t+1}(z^{t+1})^{1-\gamma}}{e_t(z^t)^{1-\gamma}} \frac{U(c)(s^t, s_{t+1})}{e_{t+1}(z^{t+1})^{1-\gamma}}.$$

Define a new continuation utility index $\hat{U}(\cdot)$ as follows:

$$\hat{U}(\hat{c})(s^t) = \frac{U(c)(s^t)}{e_t(z^t)^{1-\gamma}}.$$

It follows that

$$\begin{aligned} \hat{U}(\hat{c})(s^t) &= u(\hat{c}_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1} | s_t) \lambda(z_{t+1})^{1-\gamma} \hat{U}(\hat{c})(s^t, s_{t+1}). \\ &= u(\hat{c}_t(s^t)) + \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1} | s_t) \hat{U}(\hat{c})(s^t, s_{t+1}). \end{aligned}$$

Thus it follows, for two consumption streams c and c' , that

$$U(c)(s^t) \geq U(c')(s^t) \text{ if and only if } \hat{U}(\hat{c})(s^t) \geq \hat{U}(\hat{c}')(s^t),$$

i.e., the household orders original and growth-deflated consumption streams in exactly the same way. \square

- Proof of Theorem 3.1:

Proof. The proof consists of two parts. In a first step, we argue that the Bewley equilibrium allocations and prices can be transformed into an equilibrium for the de-trended Arrow model, and in a second step, we

argue that by scaling the allocations and prices by the appropriate endowment (growth) factors, we obtain an equilibrium of the stochastically growing Arrow model.

Step 1: Take allocations and prices from a Bewley equilibrium, $\{\hat{c}_t(y^t), \hat{a}_t(y^t), \hat{\sigma}_t(y^t)\}, \{\hat{R}_t, \hat{v}_t\}$ and let the associated Lagrange multipliers on the solvency constraints be given by

$$\hat{\beta}^t \varphi(y^t | y_0) u'(\hat{c}_t(y^t)) \hat{\mu}(y^t) \geq 0,$$

for the constraint in (14) and

$$\hat{\beta}^t \varphi(y^t | y_0) u'(\hat{c}_t(y^t)) \hat{\kappa}_t(y^t) \geq 0,$$

for the constraint in (15). The first order conditions (which are necessary and sufficient for household optimal choices together with the complementary slackness and transversality conditions) in the Bewley model, once combined to the Euler equations, are given by:

$$1 = \hat{R}_t \hat{\beta} \sum_{y^{t+1} | y^t} \varphi(y_{t+1} | y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}_t(y^t) + \hat{R}_t \hat{\kappa}_t(y^t). \quad (32)$$

$$= \hat{\beta} \left[\frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t} \right] \sum_{y^{t+1} | y^t} \varphi(y_{t+1} | y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}_t(y^t) + \left[\frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t} \right] \hat{\kappa}_t(y^t). \quad (33)$$

The corresponding Euler equations for the de-trended Arrow model, evaluated at the Bewley equilibrium allocations and Lagrange multipliers $\hat{\mu}(y^t)$ and $\hat{\kappa}_t(y^t) \hat{\phi}(z_{t+1})$, read as (see (30) and (31)):

$$\begin{aligned} 1 &= \frac{\hat{\beta}(s_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{s^{t+1} | s^t, z_{t+1}} \hat{\pi}(s_{t+1} | s_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\ &\quad + \hat{\mu}_t(y^t) + \frac{\hat{\kappa}_t(y^t) \hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \quad \forall z_{t+1}. \\ 1 &= \hat{\beta}(s_t) \sum_{s^{t+1} | s^t} \hat{\pi}(s_{t+1} | s_t) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right] \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\ &\quad + \hat{\mu}_t(y^t) + \hat{\kappa}_t(y^t) \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right]. \end{aligned}$$

Evaluated at the conjectured prices,

$$\hat{v}_t(z^t) = \hat{v}_t \quad (34)$$

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t}, \quad (35)$$

and using the independence and i.i.d. assumptions, which imply

$$\begin{aligned} \hat{\pi}(s_{t+1} | s_t) &= \varphi(y_{t+1} | y_t) \hat{\phi}(z_{t+1}) \\ \hat{\beta}(s_t) &= \hat{\beta} \end{aligned}$$

these Euler equations can be restated as follows:

$$1 = \frac{\hat{\beta}\hat{R}_t}{\hat{\phi}(z_{t+1})} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \hat{\phi}(z_{t+1}) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}_t(y^t) + \hat{R}_t \hat{\kappa}_t(y^t). \quad (36)$$

$$1 = \hat{\beta} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t} \right] \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}_t(y^t) + \hat{\kappa}_t(y^t) \left[\frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t} \right] \sum_{z_{t+1}} \hat{\phi}(z_{t+1}), \quad (37)$$

which are, given that $\sum_{z_{t+1}} \hat{\phi}(z_{t+1}) = 1$, exactly the Euler conditions (32) and (33) of the Bewley model and hence satisfied by the Bewley equilibrium allocations. A similar argument applies to the complementary slackness conditions, which for the Bewley model read as

$$\hat{\mu}_t(y^t) \left[\frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t) \hat{v}_t - \hat{K}_t(y^t) \right] = 0, \quad (38)$$

$$\hat{\kappa}_t(y^t) \left[\hat{a}_t(y^t) + \hat{\sigma}_t(y^t) (\hat{v}_{t+1} + \alpha) - \hat{M}_t(y^t) \right] = 0, \quad (39)$$

and for the de-trended Arrow model, evaluated at Bewley equilibrium allocations and conjectured prices, read as

$$\hat{\mu}_t(y^t) \left[\frac{\hat{a}_t(y^t)}{\hat{R}_t} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) + \hat{\sigma}_t(y^t) - \hat{K}_t(y^t) \right] = 0,$$

$$\hat{\kappa}_t(y^t) \left[\hat{a}_t(y^t) + \hat{\sigma}_t(y^t) [\hat{v}_{t+1} + \alpha] - \hat{M}_t(y^t) \right] = 0/\hat{\phi}(z_{t+1}).$$

Again, the Bewley equilibrium allocations satisfy the complementary slackness conditions in the de-trended Arrow model. The argument is exactly identical for the transversality conditions. Finally, we have to check whether the Bewley equilibrium allocation satisfies the de-trended Arrow budget constraints. Plugging in the allocations yields:

$$\hat{c}_t(s^t) + \frac{\hat{a}_t(y^t)}{\hat{R}_t} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) + \hat{\sigma}_t(y^t) \hat{v}_t \leq \eta(y_t) + \hat{a}_{t-1}(y^{t-1}) + \sigma_{t-1}(y^{t-1}) [\hat{v}_t + \alpha], \quad (40)$$

which is exactly the budget constraint in the Bewley model. Thus, given the conjectured prices, the Bewley equilibrium allocations are optimal in the de-trended Arrow model.

Since the market clearing conditions for assets and consumption goods coincide in the two models, Bewley allocations satisfy the market clearing conditions in the de-trended Arrow model. Thus we conclude that the Bewley equilibrium allocations, together with prices (34) and (35) are an equilibrium in the de-trended Arrow model.

Step 2: Now, we need to show that an equilibrium of the de-trended Arrow model is, after appropriate scaling, an equilibrium in the stochastically growing model, but this was established in section 3.2, in which we showed that by with the transformations $\hat{c}_t(s^t) = \frac{c_t(s^t)}{e_t(z^t)}$, $\hat{a}_t(s^t, z_{t+1}) = \frac{a_t(s^t, z_{t+1})}{e_{t+1}(z_{t+1})}$, $\hat{\sigma}_t(s^t) = \sigma_t(s^t)$, $\hat{q}_t(z^t, z_{t+1}) = q_t(z^t, z_{t+1})\lambda(z_{t+1})$, $\hat{v}_t(z^t) = \frac{v_t(z^t)}{e_t(z^t)}$ household problems and market clearing conditions in the de-trended and the stochastically growing Arrow model coincide. \square

- Proof of Lemma 3.1:

Proof. The stock return is defined as:

$$R_{t+1}^s(z^{t+1}) = \frac{v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1})}{v_t(z^t)}.$$

Subtracting the two Euler equations (36)-(37) in the Arrow model and (65)-(66) in the Bond model yields, in both cases

$$\hat{\beta} \sum_{z^{t+1}|z^t} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} - \hat{R}_t(z^t) \right] = 0.$$

Using the fact that $\hat{v}_{t+1}(z^{t+1}) = v_{t+1}(z^{t+1})/e_{t+1}(z_{t+1})$ and the definition of $\hat{\phi}(z_{t+1})$ and $\hat{\beta}$, as well as (18) yields

$$\beta \sum_{z^{t+1}|z^t} \phi(z_{t+1}) \lambda(z_{t+1})^{-\gamma} [R_{t+1}^s(z^{t+1}) - R_t(z^t)] = 0$$

or in short

$$E_t \{ \beta \lambda(z_{t+1})^{-\gamma} [R_{t+1}^s - R_t] \} = 0$$

Thus the representative agent stochastic discount factor $\beta \lambda(z_{t+1})^{-\gamma}$ prices the excess return of stocks over bonds in both the Arrow and the Bond model. Note that in the Arrow model (but not in the Bond model) this stochastic discount factor any excess return $R_{t+1}^i - R_t$ as long as the returns only depend on the aggregate state z_{t+1} . \square

- Proof of Proposition 3.2:

Proof. From theorem 3.1 we know that in the Arrow model equilibrium prices for Arrow securities are given by:

$$q_t^A(z^t, z_{t+1}) = \frac{\hat{q}_t^A(z^t, z_{t+1})}{\lambda(z_{t+1})} = \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1}) \hat{R}_t^A}$$

whereas in the representative agent model equilibrium prices for Arrow securities are given by:

$$q_t(z^t, z_{t+1}) = \frac{\hat{q}_t(z^t, z_{t+1})}{\lambda(z_{t+1})} = \hat{\beta} \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})}$$

so that

$$\frac{q_t^A(z^t, z_{t+1})}{q_t(z^t, z_{t+1})} = \frac{1}{\hat{\beta} \hat{R}_t^A} = \frac{\hat{R}^{RE}}{\hat{R}_t^A} = \kappa_t \geq 1 \quad (41)$$

where

$$\hat{R}_t^{RE} = \hat{R}^{RE} = \frac{1}{\sum_{z_{t+1}} \hat{q}_t(z^t, z_{t+1})} = \frac{1}{\sum_{z_{t+1}} \hat{\beta} \hat{\phi}(z_{t+1})} = \frac{1}{\hat{\beta}}$$

is the risk-free interest rate in the de-trended representative agent model. Note that the multiplicative factor κ_t may depend on time since \hat{R}_t^A may, but is nonstochastic, since $\hat{R}_t^A = \hat{R}_t$ (the risk-free interest rate in the de-trended Arrow model equals that in the Bewley model, which is evidently nonstochastic). Since interest rates in the Bewley model are (weakly) smaller than in the representative agent model, $\kappa_t \geq 1$. Equation (41) implies that the stochastic discount factor in the Arrow model equals the SDF in the representative agent model, multiplied by κ_t :

$$m_{t+1}^A(z^{t+1}) = m_{t+1}^{RE}(z^{t+1}) \kappa_t$$

Finally, since the stochastic discount factor for the Arrow model is also a valid stochastic discount factor in the Bond model (although not necessarily the unique valid stochastic discount factor), the previous result also applies to the Bond model. \square

- Proof of Theorem 3.2:

Proof. Remember that we defined the multiplicative risk premium in the main text as

$$1 + \nu_t = \frac{E_t R_{t,1} [\{e_{t+k}\}]}{R_{t,1} [1]}$$

We use $m_{t,t+k} = m_{t+1} \cdot m_{t+2} \dots \cdot m_{t+k}$ to denote the k -period ahead pricing kernel (with convention that $m_{t,t} = 1$), such that $E_t(d_{t+k} m_{t,t+k})$ denotes the price at time t of a random payoff d_{t+k} . Note that whenever there is no room for confusion we suppress the dependence of variables on z^t .

First, note that the multiplicative risk premium on a claim to aggregate consumption can be stated as a weighted sum of risk premia on strips (as shown by Alvarez and Jermann (2001)). By definition of $R_{t,1} [\{e_{t+k}\}]$ we have

$$R_{t,1} [\{e_{t+k}\}] = \frac{\sum_{k=1}^{\infty} E_{t+1} m_{t+1,t+k} \alpha e_{t+k}}{\sum_{k=1}^{\infty} E_t m_{t,t+k} \alpha e_{t+k}} \quad (42)$$

$$= \frac{1}{\sum_{j=1}^{\infty} E_t m_{t,t+j} e_{t+j}} * \sum_{k=1}^{\infty} \frac{E_{t+1} m_{t+1,t+k} e_{t+k}}{E_t m_{t,t+k} e_{t+k}} * E_t m_{t,t+k} e_{t+k} \quad (43)$$

$$= \sum_{k=1}^{\infty} \frac{E_{t+1} m_{t+1,t+k} e_{t+k}}{E_t m_{t,t+k} e_{t+k}} \frac{E_t m_{t,t+k} e_{t+k}}{\sum_{j=1}^{\infty} E_t m_{t,t+j} e_{t+j}} \\ = \sum_{k=1}^{\infty} \omega_k R_{t,1} [e_{t+k}] \quad (44)$$

where the nonrandom weights ω_k are given by

$$\omega_k = \frac{E_t m_{t,t+k} e_{t+k}}{\sum_{j=1}^{\infty} E_t m_{t,t+j} e_{t+j}}$$

Thus

$$1 + \nu_t = \frac{E_t R_{t,1} [\{e_{t+k}\}]}{R_{t,1} [1]} = \sum_{k=1}^{\infty} \omega_k \frac{E_t R_{t,1} [e_{t+k}]}{R_{t,1} [1]} \quad (45)$$

and it is sufficient to show that the multiplicative risk premium $E_t R_{t,1} [e_{t+k}] / R_{t,1} [1]$ on all k -period strips of aggregate consumption (a claim to the Lucas tree's dividend in period k only, not the entire stream) is the same in the Arrow model as in the representative agent model. First, we show that the one-period ahead conditional strip risk premia are identical:

$$\frac{E_t \frac{E_{t+1} e_{t+1}}{E_t [m_{t+1}^A e_{t+1}]}}{\frac{1}{E_t [m_{t+1}^A]}} = \frac{E_t \frac{E_{t+1} \lambda_{t+1}}{E_t [m_{t+1}^A \lambda_{t+1}]}}{\frac{1}{E_t [m_{t+1}^A]}} = \frac{E_t \frac{E_{t+1} \lambda_{t+1}}{E_t [m_{t+1}^{RE} \lambda_{t+1}]}}{\frac{1}{E_t [m_{t+1}^{RE}]}}$$

The first equality follows from dividing through by e_t . The second equality follows from the expression for m^A in Proposition 3.2: $m_{t+1}^A = m_{t+1}^{RE} \kappa_t$.

Next we repeat the argument for the risk premium of a k -period strip:

$$\frac{E_t R_{t,1}^A [e_{t+k}]}{E_t R_{t,1}^A [1]} = \frac{E_t \frac{E_{t+1}[m_{t+1,t+k}^A e_{t+k}]}{E_t[m_{t,t+k}^A e_{t+k}]}}{\frac{1}{E_t[m_{t+1}^A]}} = \frac{E_t \frac{\kappa_{t+1} \dots \kappa_{t+k-1} E_{t+1}[m_{t+1,t+k}^{RE} e_{t+k}]}{\kappa_t \kappa_{t+1} \dots \kappa_{t+k-1} E_t[m_{t,t+k}^{RE} e_{t+k}]}}{\frac{1}{\kappa_t E_t[m_{t+1}^{RE}]}} = \frac{E_t R_{t,1}^{RE} [e_{t+k}]}{R_{t,1}^{RE} [1]}$$

and thus risk premia on all k -period consumption strips in the Arrow model coincide with those in the representative agent model. But then (45) implies that the multiplicative risk premium in the two models coincide as well. \square

- Proof of Lemma 4.1:

Proof. Absent binding solvency constraints the Euler equation in the Bewley model read as

$$\begin{aligned} 1 &= \frac{\tilde{\beta}_{t+1}}{\tilde{\beta}_t} \hat{R}_t \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\ &= \frac{\tilde{\beta}_{t+1}}{\tilde{\beta}_t} \frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \end{aligned}$$

while in the Arrow model the Euler equations for Arrow securities are given by

$$1 = \frac{\hat{\beta}(z_t) \hat{\phi}(z_{t+1}|z_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}.$$

With conjectured Arrow securities prices $\hat{q}_t(z^t, z_{t+1}) = \hat{\beta}(z_t) \hat{\phi}(z_{t+1}|z_t) \frac{1}{\hat{R}_t} \frac{\tilde{\beta}_t}{\tilde{\beta}_{t+1}}$ these equations obviously coincide with the bond Euler equation in the Bewley model, and thus the Bewley equilibrium allocation satisfies the Euler equations for Arrow securities. A similar argument applies to the Euler equation for stocks:

$$\begin{aligned} 1 &= \hat{\beta}(z_t) \sum_{z^{t+1}|z^t} \hat{\phi}(z_{t+1}|z_t) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right] \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\ &= \frac{\tilde{\beta}_{t+1}}{\tilde{\beta}_t} \hat{R}_t \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \end{aligned}$$

The state-contingent interest rate in this model is given by:

$$\frac{1}{\hat{R}_t^A(z_t)} = \hat{\beta}(z_t) \frac{\tilde{\beta}_t}{\hat{R}_t \tilde{\beta}_{t+1}}$$

which can easily be verified from equation (24). \square

- Proof of Proposition 4.1:

Proof. We need to check that Arrow securities positions defined in (27) satisfy the market clearing condition

$$\int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0) \hat{a}_{t-1}(\theta_0, y^{t-1}, z^t) d\Theta_0 = 0 \text{ for all } z^t.$$

for each z_t . By the goods market clearing condition in the Bewley model we have, since total labor income makes up a fraction $1 - \alpha$ of total income

$$\begin{aligned} & \int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0) (\hat{c}_t(y^t, \theta_0) - \eta(y_t)) d\Theta_0 \\ &= \int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0) \sum_{y^t|y^{t-1}} \varphi(y^t|y^{t-1}) (\hat{c}_t(y^t, \theta_0) - \eta(y_t)) d\Theta_0 \\ &= \sum_{y^t} \varphi(y^t|y_0) (\hat{c}_t(y^t, \theta_0) - \eta(y_t)) d\Theta_0 = \alpha \end{aligned}$$

Similarly

$$\int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0) \sum_{y^\tau} \varphi(y^\tau|y^t) [\hat{c}_\tau(\theta_0, y^\tau) - \eta(y_\tau)] d\Theta_0 = \alpha \text{ for all } \tau > t$$

Since the stock market clears in the Bewley model we have

$$\int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0) \hat{\sigma}_{t-1}(\theta_0, y^{t-1}) d\Theta_0 = 1.$$

Since the stock is a claim to α times the aggregate endowment in all future periods its (ex-dividend) price has to satisfy

$$\hat{v}_t(z_t) = \alpha \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_\tau(z^\tau|z_t)$$

Combining these results implies that

$$\begin{aligned} & \int \sum_{y^t} \varphi(y^t|y_0) \hat{a}_{t-1}(\theta_0, y^{t-1}, z^t) d\Theta_0 \\ &= \alpha + \alpha \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_\tau(z^\tau|z_t) - 1 \left(\alpha + \alpha \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_\tau(z^\tau|z_t) \right) = 0 \end{aligned}$$

for each z_t . Thus, each of the Arrow securities markets clears if households hold portfolios given by (27). \square

- Proof of Lemma 4.2:

Proof. By (23)

$$\hat{Q}_t(z^t|z_0) = \hat{\phi}(z^t|z_0) \tilde{Q}_t \frac{\hat{\beta}_{0,t-1}(z^{t-1}|z_0)}{\tilde{\beta}_t}$$

and thus

$$\begin{aligned} \sum_{z^t} \hat{Q}_t(z^t|z_0) &= \frac{\tilde{Q}_t}{\tilde{\beta}_t} \sum_{z^t|z_0} \hat{\phi}(z^t|z_0) \hat{\beta}_{0,t-1}(z^{t-1}|z_0) \\ &= \frac{\tilde{Q}_t}{\tilde{\beta}_t} \sum_{z^t|z^{t-1}} \hat{\phi}(z^t|z^{t-1}) \sum_{z^{t-1}|z_0} \hat{\phi}(z^{t-1}|z_0) \hat{\beta}_{0,t-1}(z^{t-1}|z_0) = \tilde{Q}_t. \end{aligned}$$

by definition of $\tilde{\beta}_t$ in equation (21) and the fact that $\sum_{z^t|z^{t-1}} \hat{\phi}(z^t|z^{t-1}) = 1$. \square

- Proof of Lemma 4.3:

Proof. The Arrow securities position at time zero needed to finance all future excess consumption mandated by the Bewley equilibrium is given by

$$\begin{aligned}\hat{a}_{-1}(\theta_0, y_0, z_0) &= \hat{c}_0(\theta_0, y_0) - \eta(y_0) + \sum_{\tau=1}^{\infty} \sum_{z^\tau, y^\tau} \hat{Q}_\tau(z^\tau | z_0) (\hat{c}_\tau(\theta_0, y^\tau) - \eta(y_\tau)) \\ &\quad - \hat{\sigma}_0(\theta_0, y_0) [\hat{v}_0(z_0) + \alpha],\end{aligned}$$

where we substituted indexes $_{-1}$ by 0 to denote initial conditions. In particular, $\hat{\sigma}_0(\theta_0, y_0)$ is the initial share position of an individual with wealth θ_0 . But

$$\begin{aligned}&\hat{a}_{-1}(\theta_0, y_0, z_0). \\ &= \hat{c}_0(\theta_0, y_0) - \eta(y_0) + \sum_{\tau=1}^{\infty} \sum_{z^\tau, y^\tau | z_0, y_0} \hat{Q}_\tau(z^\tau | z_0) \varphi(y^\tau | y_0) (\hat{c}_\tau(\theta_0, y^\tau) - \eta(y_\tau)) - \hat{\sigma}_0(\theta_0, y_0) [\hat{v}_0(z_0) + \alpha]. \\ &= \hat{c}_0(\theta_0, y_0) - \eta(y_0) + \sum_{\tau=1}^{\infty} \sum_{y^\tau | y_0} \varphi(y^\tau | y_0) (\hat{c}_\tau(\theta_0, y^\tau) - \eta(y_\tau)) \sum_{z^\tau | z_0} \hat{Q}_\tau(z^\tau | z_0) \\ &\quad - \hat{\sigma}_0(\theta_0, y_0) \alpha \left[1 + \sum_{\tau=1}^{\infty} \sum_{z^\tau | z_0} \hat{Q}_\tau(z^\tau | z_0) \right]. \\ &= \hat{c}_0(\theta_0, y_0) - \eta(y_0) + \sum_{\tau=1}^{\infty} \tilde{Q}_\tau \sum_{y^\tau | y_0} \varphi(y^\tau | y_0) (\hat{c}_\tau(\theta_0, y^\tau) - \eta(y_\tau)) - \hat{\sigma}_0(\theta_0, y_0) \alpha \left[\sum_{\tau=0}^{\infty} \tilde{Q}_\tau \right] \\ &= 0,\end{aligned}$$

where the last equality comes from the intertemporal budget constraint in the standard incomplete markets Bewley model and the fact that the initial share position in that model is given by $\hat{\sigma}_0(\theta_0, y_0)$. \square

- Proof of Proposition 4.2:

Proof. The stationary Bewley allocation $\{\hat{a}_t^B(y^t) = 0, \hat{\sigma}_t(y^t)\}$ satisfies the constraint

$$\frac{\hat{a}_t^B(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t) \hat{v}_t \geq \hat{K}_t(y^t). \quad (46)$$

Using the fact that $\hat{a}_t^B(y^t) = 0$ and adding

$$\sum_{z_{t+1}} \hat{q}_t(z_{t+1} | z_t) \hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t) \hat{v}_t(z_t)$$

to both sides of (46) yields

$$\begin{aligned}&\sum_{z_{t+1}} \hat{q}_t(z_{t+1} | z_t) \hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t) \hat{v}_t(z_t) \\ &\geq \hat{K}_t(y^t) + \sum_{z_{t+1}} \hat{q}_t(z_{t+1} | z_t) \hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t) [\hat{v}_t(z_t) - \hat{v}_t] \\ &\equiv \hat{K}_t^*(y^t, z_t)\end{aligned}$$

where $\hat{K}_t^*(y^t)$ is the modified constraint for the de-trended Arrow model. Multiplying both sides by $e_t(z^t)$ gives the modified constraint for the Arrow model with growth stated in the main text. For the alternative

constraint, we know that the Bewley equilibrium allocation satisfies

$$\hat{a}_t^B(y^t) + \hat{\sigma}_t(y^t)[\hat{v}_{t+1} + \alpha] \geq \hat{M}_t(y^t). \quad (47)$$

Again using $\hat{a}_t^B(y^t) = 0$ and adding

$$\hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t)[\hat{v}_{t+1}(z_{t+1}) + \alpha]$$

to both sides of (46) yields

$$\begin{aligned} & \hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t)[\hat{v}_{t+1}(z_{t+1}) + \alpha] \\ \geq & \hat{M}_t(y^t) + \hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t)[\hat{v}_{t+1}(z_{t+1}) - \hat{v}_{t+1}] \\ \equiv & \hat{M}_t^*(y^t, z_{t+1}) \end{aligned}$$

Multiplying both sides by $e_{t+1}(z^{t+1})$ again gives rise to the constraint stated in the main text. \square