OPTIMAL BOND TRADING WITH PERSONAL TAXES:
IMPLICATIONS FOR BOND PRICES AND ESTIMATED
TAX BRACKETS AND YIELD CURVES

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ABSTRACT

The assumption that bondholders follow either a buy-and-hold or a continuous realization trading policy, rather than the optimal trading policy, is at variance with reality and, as we demonstrate, may seriously bias the estimation of the yield curve and the implied tax bracket of the marginal investor. Tax considerations which govern a bondholder's optimal trading policy include the following: realization of capital losses, short term if possible; deferment of the realization of capital gains, especially if they are short term; changing the holding period status from long term to short term by sale of the bond and repurchase, so that future capital losses may be realized short term; and raising the basis through sale of the bond and repurchase in order to deduct from ordinary income the amortized premium. Because of the interaction of these factors, no simple characterization of the optimal trading policy is possible. We can say, however, that it differs substantially from the buy-and-hold policy irrespective of whether the bondholder is a bank, a bond dealer, or an individual. We obtain these strong results even when we allow for transactions costs and explicitly consider numerous IRS regulations designed to curtail tax avoidance.

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1. Introduction

The yield curve implied by the prices of Treasury notes and bonds and corporate bonds is of interest to economists and practitioners alike: it reflects the investors' beliefs about the future course of the short term interest rate. In calculating the yield curve, the tax bracket of the marginal bondholder is either taken to be some given number or is estimated simultaneously with the yield curve. The implied tax bracket of the marginal investor is of independent interest. It provides a direct (but incomplete) test of Miller's (1977) theory on the optimal capital structure of firms. It may also be useful for determining fair prices for other assets.

There are two major problems in estimating pure discount rates (the yield curve of zero coupon bonds) and the implied marginal tax rate. The first problem is that of differing clienteles, studied in detail by Schaefer (1981, 1982a). For a given investor some bonds of particular maturities and coupon rates may be dominated by combinations of other bonds. In this case tax clienteles naturally arise. If there is no one clientele for which every bond remains undominated, then the concept of the "marginal taxable investor" who "sets" all prices may well be meaningless.

The second problem is that of the assumed investment horizon. This is the focus of the present paper. By necessity we ignore the problems of tax clienteles. Extant estimation procedures assume either that the bond is held to maturity, without intermediate realization of capital gains and losses (the buy-and-hold policy), or that capital gains and losses are realized every period as they occur (the continuous realization policy). Both the buy-and-hold policy and the continuous realization policy lead to relatively simple bond pricing formulae. This facilitates the estimation of the yield curve and the implied tax bracket of the marginal investor.

The assumption that bondholders follow either a buy-and-hold or a continuous realization trading policy, rather than the optimal trading policy, is at variance with reality and, as we demonstrate, may seriously bias the estimation of the yield curve and the implied tax bracket of the marginal investor. Perusal of the Wall Street Journal provides convincing evidence that investment advisors--and presumably their clients--are aware of the optimal trading policies which frequently differ sharply from a buy-and-hold or continuous realization policy. By definition, the marginal bondholder is an economic agent (or group of agents) of sufficient stature to set bond prices at the margin. It is illogical then to assert that the marginal investor follows a suboptimal trading policy through ignorance.

The present paper unifies two recent strands of research, the pricing of bonds with stochastically varying interest rates and investment opportunity set and the pricing of stocks in the presence of personal taxes. Cox, Ingersoll and Ross (1981, 1983) presented an equilibrium theory of bond pricing and the term structure of interest rates, in particular explaining the valuation of a deterministic stream of cash flows but with a stochastically varying interest rate and investment opportunity set. Constantinides (1982, 1983) and Constantinides and Scholes (1980) discussed the optimal trading of stocks and options in the presence of personal taxes and presented an equilibrium theory of stock pricing, in particular explaining the effect of optimal realization of capital gains and losses on the pricing of stocks.

Tax considerations which govern a bondholder's optimal trading policy include the following: realization of capital losses, short term if possible; deferment of the realization of capital gains, especially if they are short term; changing the holding period status from long term to short term by sale of the bond and repurchase, so that future capital losses may be realized

short term; and raising the basis through sale of the bond and repurchase in order to deduct from ordinary income the amortized premium. Because of the interaction of these factors, no simple characterization of the optimal trading policy is possible. We can say, however, that it differs substantially from the buy-and-hold policy irrespective of whether the bondholder is a bank, a bond dealer, or an individual. We obtain these strong results even when we allow for transactions costs and explicitly consider numerous IRS regulations designed to curtail tax avoidance.

The paper is organized as follows. In Section 2 we outline the tax provisions in four representative tax scenarios which may apply to the elusive marginal bondholder. The formal model is presented in Section 3. Closed-form solutions for the prices of consol bonds and the value of the timing option are presented in Section 4 for a special case. In Section 5 we derive the optimal trading policies under more general conditions and in Section 6 we illustrate the effect of taxes on the prices of bonds and on the value of the timing option. The estimation of the yield curve and the tax bracket of the marginal investor is grossly biased if the value of the timing option is ignored. This point is illustrated in Section 7. In Section 8 we discuss municipal bonds. Concluding remarks are offered in Section 9.

2. The Tax Environment

To avoid a profusion of details in our discussion we abstract from many of the nuances of the regulations governing the taxation of income, as defined by the tax code and its interpretation by the Internal Revenue Service and the courts. We do emphasize, however, certain important aspects of the code, which, though largely ignored in finance, may materially affect bond prices and the estimation of the yield curve and the marginal tax rate. We also provide some historical perspective to familiarize the reader with major

changes in the tax code which may be reflected in a time series of bond yields.

At least four broad classes of potential marginal investors warrant examination: individuals, banks and bond dealers, corporations, and tax-exempt institutions. Consider first the tax rules applicable to individual investors.

Coupon income (net of interest expense) is taxed at the individual's marginal tax rate on ordinary income, the maximum rate being currently 50%. Between 1970 and 1980, coupon income was classified as "unearned income" and was taxed at a maximum rate of 72%. Prior to the seventies, the top marginal tax rate varied from a low of 7% in 1913 to a high of 95% in 1945. In our calculations we assume that the marginal tax rate on coupon income for an individual is $\tau_c = .5.$[1]

The taxation of capital gains is complex. Unrealized gains and losses remain untaxed. Gains and losses are taxed in the year that they are realized. A realized gain or loss is the difference between the sales price (less cost of sale) and the basis. For most assets the basis is just the purchase price (plus cost of purchase), but for some bonds the purchase price is subject to adjustment.

We consider only original issue par bonds, defined as such by the IRS if the original issue discount does not exceed 1/4 of 1 percent multiplied by the number of full years to maturity. For these bonds, if the purchase price in the secondary market is below par, no adjustment is made and the basis is just the purchase price. If the purchase price is above par, this difference is

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[1] Miller (1977) shows that, under simple tax rules, the marginal bondholder is in the corporate tax bracket, and provides partial justification for our choice of the tax rate. In any case, our qualitative results are insensitive to the assumed rate.

amortized linearly to the maturity date.$\underline{2/}$ The amount amortized in a tax year

is allowed as a deduction against ordinary income and the bond's basis is

correspondingly reduced. There is no specific limitation on this deduction.

In our calculations the amortized amount is (negatively) taxed at the rate

$\tau_c = .5$.

Realized capital gains and losses are either short term or long term.

The required holding period for long term treatment is currently one year.

This has varied many times since capital gains were first differentiated from

ordinary income in 1922. In the years 1942-1977 the holding period was six

months. Prior to that time there were three or more categories of long term

capital gains with required holding periods as long as ten years.

Net short term capital gains are taxed as ordinary income. Net long term

capital gains are currently taxed at 40% of the investor's marginal tax rate

on ordinary income. This treatment has also been changed. The tax rate on

long term gains has varied from 20% to 80% of the tax rate on ordinary

income. In addition there have been periods in which alternate treatment

could be elected (or, was required for large capital gains).

Net short term capital losses and 50% of net long term capital losses are

deducted from ordinary income and may jointly reduce the taxable ordinary

income by a maximum of $3,000 (until 1976, $1,000). Unused losses are carried

forward indefinitely. Short term losses and long term gains, incurred in the

same year, offset each other dollar for dollar, instead of being taxed at

their respective rates.

$\underline{2/}$ Amortization is optional for Treasury and corporate bonds. Since for
practically all individuals the marginal tax rate on ordinary income is no
less than the capital gains tax rate, amortization of the basis dominates
foregoing this option. The amortization method need not be straight line, but
may be that customarily used by the individual, if it is deemed to be
reasonable. If the bond is callable, the basis is amortized to the call price
at the call date or to par at maturity, whichever yields the smaller
amortization. If there are alternate call dates the rule is complex.

We define τ_s to be the <u>marginal</u> tax rate on short term capital gains and losses. This rate is not necessarily equal to the marginal tax rate on ordinary income: if the investor has net short term losses and the deduction limit is binding, $\tau_s = 0$; if the investor has net short term losses but larger long term gains, τ_s is 40% of the marginal tax rate on ordinary income. Likewise, we define τ_L to be the <u>marginal</u> tax rate on long term capital gains and losses.

If an asset is sold at a loss within thirty days before or after the acquisition of "substantially identical" property, the IRS can disallow the loss deduction under the "wash sale" rule. An investor has a high probability of circumventing this rule by purchasing instead another bond with a slightly different coupon or maturity. In any case, this rule is not applicable to dealers or individual taxpayers who are in the business of trading bonds. Consequently we ignore the wash sale rule throughout this paper.

We consider three representative tax scenarios for an individual bondholder and one scenario for banks or bond dealers, as defined by the marginal rates τ_c, τ_s, and τ_L.

I. <u>The marginal investor is an individual. Coupon income is taxed at the rate</u> $\tau_c = .5$. <u>Realized short term and long term gains and losses are taxed at the rate</u> $\tau_s = \tau_L = .25$. <u>The deduction limit is not binding</u>.

This scenario is plausible if the individual is periodically forced to sell some of his portfolio assets by factors beyond his control (or, of more importance than the tax consequences) and, on average, realizes large long term gains. Then the deduction limit is not binding. Since short term losses must be used to offset the long term gains, the marginal tax rate is the long

term gains rate.[3/] We take the long term gains rate to be half of the

investor's marginal tax rate on ordinary income, as it was between 1942 and

1979. We also assume that the investor can always defer the realization of

short term gains until the holding period exceeds one year and then realize

the gains long term.

> II. The marginal investor is an individual. Coupon income is taxed at
> the rate $\tau_c = .5$. Realized short term gains and losses are taxed
> at the rate $\tau_s = .5$. Realized long term gains and losses are taxed
> at the rate $\tau_L = .25$. The deduction limit is not binding.

Scenario II is the least plausible one because it ignores both the

deduction limit and the (unfavorable for the taxpayer) offsetting of long term

gains with short term losses.[4/] Since investors have a tax incentive to

realize losses and defer gains (at least, short term gains), the assumption

that the deduction limit is not binding may be tenuous and is relaxed in the

next scenario.

[3/] Similarly the right to deduct half of long term losses from income, even under the current 40% rule for long term capital gains, could not be used. Losses could only be deducted from other capital gains. Thus, the effective tax rate on both long term gains and losses is the same.

[4/] The individual may mitigate this offset provision of the tax law by realizing long term gains and short and long term losses in alternate tax years; however, we do not explicitly model this. See Constantinides (1982). This procedure may also help to avoid the unfavorable long term gain and loss offset.

III. <u>The marginal investor is an individual. Coupon income is taxed at the rate</u> $\tau_c = .5$. <u>Short and long term gains and losses remain untaxed, i.e.,</u> $\tau_s = \tau_L = 0$.

One justification for this scenario is to assume that the individual realizes losses and defers gains. At the margin losses can only be carried forward as the deduction limit is binding. The only tax "game" permitted under this scenario is to realize a gain on a bond in order to raise its basis above par and start deducting the premium amortization against ordinary income. As we shall see, this policy is profitable.

Although corporations are taxed differently from individuals, the tax regulations on nonbank corporations that hold bonds for reasons not directly related to their business purposes are sufficiently similar to those applying to individuals that the previous scenarios remain at least qualitatively correct. The primary distinction is that a net capital loss (short and long combined) cannot be deducted in any amount from ordinary income, but may be carried back for three years and forward for five years as a short term loss to offset gains. Banks and (corporate or individual) bond dealers are taxed differently, however.

For banks and dealers, bond coupons and all realized capital gains and losses are treated as ordinary income or loss without explicit limitation. Net operating losses of banks are carried back for ten years and forward for five years. In the following scenario we effectively assume that the bank has positive net earnings in every ten year period so that loss benefits are earned immediately. Corporate earnings and losses are taxed at the corporate rate of 50%. (The current corporate tax rate is 46% on earnings in excess of $100,000. In the past it has been as high as 54%.) The same scenario applies to a bond dealer with marginal personal or corporate tax rate on ordinary income equal to 50%.

IV. <u>The marginal holder is a bank or bond dealer. Coupon income and all</u> <u>capital gains and losses are taxed at the rate</u> $\tau_c = \tau_s = \tau_L = .5.$ <u>There is no deduction limit.</u>

In each of the scenarios, I-IV, the tax rates τ_c, τ_s and τ_L are assumed to remain constant over time because we wish to focus on the long run effect of taxes on bond prices. Certain trading policies not examined here would become optimal at the time that tax provisions were about to change. For example, when the effective maximum rate on long term capital gains was changed from 28% to 20% by the 1978 Tax Revenue Act, individuals paying the 28% rate should have deferred realizing their capital gains, ceteris paribus. Similarly if an investor's income were to change sufficiently to place him in a different tax bracket, the optimal trading policy might be affected.

We also examine bond prices in each of the four tax scenarios under the assumption that the bondholder is (artificially) constrained to follow a buy-and-hold policy and compare the bond prices, tax timing option, and yields to the case when the investor follows optimal policies. The buy-and-hold economy is taken as our primary benchmark in which there are no price effects induced by tax trading.

We do not explicitly examine a scenario in which the marginal bondholder is exempt from all taxes. This might be considered a serious omission because tax-exempt intermediaries currently hold more than one-third of all the outstanding government and corporate bonds and account for an even greater proportion of the trading volume. Furthermore, liberalized tax-deferred retirement plans provide growing opportunities for taxable individuals to defer the tax on coupons, dividends and capital gains until retirement. If the marginal investor is tax-exempt, then there are obviously no tax-induced

"biases" in bond prices.[5] However, since the no-trading policy dominates any other for a tax-exempt investor (in a perfect market), we may assume the buy-and-hold policy. Consequently bond prices should equal the benchmark values, and standard estimation techniques, such as McCulloch's (1975), should verify that the marginal tax bracket is zero.

3. The Model

Our goal is to find the price of a default-free bond with par value one, continuous coupon rate c, and maturity date T. The bond is perfectly divisible and may be bought or sold with zero transactions costs.[6] The bond price is a function of the state vector Y (defined below) and time t, i.e., $P = P(Y, t; c, T)$.

We price the coupon bond relative to short term (instantaneous) lending via a riskless, "single period" bond with maturity dt and before-tax yield $r(Y, t)$. The single period bond is perfectly divisible and may be bought or sold with zero transactions costs. Effectively there is unlimited riskless lending over the time interval dt at the before-tax interest rate r. If an investor's tax rate on ordinary income is τ_c, his after-tax interest rate is $(1 - \tau_c)r$.

We assume that, throughout the term to this coupon bond's maturity, some investor with tax rates τ_c, τ_s, and τ_L (on coupon income, short term

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capital gains and losses, and long term capital gains and losses,

respectively) is indifferent between buying the coupon bond or investing in

the single-period bond. That there exists some tax bracket (τ_c, τ_s, τ_L)

with the property that an investor in this tax bracket is indifferent between

the two investments over a time interval dt, is a weak assumption. The

strong part of our assumption is that <u>investors in the same tax bracket are at</u>

<u>the margin throughout the bond's term to maturity</u>. In a richer model (beyond

our present scope) one might allow for the possibility that the bond is passed

from one tax clientele to another as it approaches maturity or as it becomes a

premium or discount bond due to shifts in interest rates. Since we wish to

focus on the already complex problem of the optimal realization of capital

gains and losses, we abstract from issues related to changing tax

clienteles.[7/] We make the (perhaps unnecessarily strong) assumption that all

marginal investors are in the same tax bracket.

We present first a discussion of the issues relating to the determination

of the equilibrium bond prices and the value to a marginal investor of holding

a particular bond with an established cost basis. In the next section we use

a specific model to discuss the valuation problem from the point of view of

investors not at the margin.

[7/] Tax clienteles for bonds is an important issue which has been
extensively discussed by Schaefer (1981, 1982a, 1982b) under the assumption
that bonds are held to maturity. As we demonstrate below, under tax laws
similar to those in the U.S., a buy-and-hold policy is inferior to trading
schemes which involve (among other things) early realization of capital
losses. Under British regulations, which imposed no long term capital gains
tax on "gilt" securities prior to 1962 or after 1969, such trading schemes
have no direct benefits, so a buy-and-hold policy is not necessarily
inferior. Neither need it be correct, however. Even in Schaefer's world,
future changes in interest rates or the introduction of new bonds may cause an
existing bond to become dominated for its current clientele. The anticipation
of such events should be reflected in the bond's price, and this may mask some
clientele effects. For example, any bond currently selling at a discount that
appears to have a high tax clientele was originally a par bond at which time
it probably appeared to have a lower tax clientele.

The price of a coupon bond, P, is defined to be such that an investor in the given tax bracket (τ_c, τ_s, τ_L) is indifferent between purchasing the coupon bond now or investing P dollars in the single period bond over the time interval dt. This equilibrium condition is formalized below as the after-tax version of the local expectations hypothesis. After purchasing the coupon bond, the investor follows the derived optimal trading policy as opposed to a continuous realization or a buy-and-hold policy.

If an investor's cost basis differs from the prevailing bond price, he may be unwilling (or more than willing) to sell it at that price and thereby realize a capital gain or loss. His reluctance (or eagerness) to sell has nothing to do with the bond price being "unfair." If his cost basis were equal to the price, the same investor would be indifferent to selling at that price because, by definition, the bond price is such that he is indifferent between purchasing the bond (and establishing a basis equal to the price) or investing in the short-term discount bond.

There may be instances when all investors have cost bases such that nobody is willing to sell the bond at the "prevailing price." To avoid this purely technical difficulty, we assume that the government supplies any existing bond at the above-defined price with infinite elasticity. This assumption assures that the market price is well defined and trades may always take place at the reservation price of a marginal purchaser as previously stated.[8]

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[8] Alternatively, we could assume that bonds are fixed in supply and some investors are occasionally forced to trade for reasons unrelated to optimal tax trading. "Liquidity purchasers" will never pay above their reservation price because the discount bond is available. "Liquidity sellers," however, may not be able to hold out for their reservation price.

The value of the bond to an investor, $V(Y, t; c, T; \hat{P}, \hat{t})$, is defined as the present value of the stream of cash flows associated with the bond, assuming that the optimal policy in realizing capital gains and losses is followed. The symbols $\hat{P}$ and $\hat{t}$ denote the current cost basis and the time at which the bond was purchased.

At those "stopping times" at which the investor either by choice or by force sells the bond and realizes a capital gain or loss, the value to the investor is simply the after-tax proceeds from its immediate sale. The bond's maturity date is an obvious stopping time for all investors. At maturity, the capital gain or loss is unavoidably realized, hence

$$V(Y, T; c, T; \hat{P}, \hat{t}) = (1 - \tau_s) + \tau_s \hat{P}, \quad \text{if} \quad T - \hat{t} \leqslant 1$$
$$= (1 - \tau_L) + \tau_L \hat{P}, \quad \text{if} \quad T - \hat{t} > 1 . \qquad (1)$$

A similar result is true at any stopping time prior to maturity when the investor sells his bonds. For the sequence of (possibly random) stopping times, $t = t_1, t_2, \ldots,$ at which the investor realizes a capital gain or loss,

$$V(Y, t; c, T; \hat{P}, \hat{t}) = (1 - \tau_s)P + \tau_s \hat{P}, \quad \text{if} \quad t - \hat{t} \leqslant 1$$
$$= (1 - \tau_L)P + \tau_L \hat{P}, \quad \text{if} \quad t - \hat{t} > 1 \qquad (2)$$
$$\text{at} \quad t = t_1, t_2, \ldots .$$

Stopping times may differ across investors. At the stopping times chosen by the investor the "smooth-pasting" (or "high contact") condition also must hold[9]

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[9] These conditions are formally derived in Grigelionis and Shiryaev
continued

$$\frac{\partial V}{\partial Y_n} = (1 - \tau_s) \frac{\partial P}{\partial Y_n} \; , \quad \text{if} \quad t - \hat{t} \leqslant 1$$

$$= (1 - \tau_L) \frac{\partial P}{\partial Y_n} \; , \quad \text{if} \quad t - \hat{t} > 1 \qquad (3)$$

$$\text{for} \quad n = 1, \; 2, \; \ldots, \; N, \quad \text{at} \quad t_1^o, \; t_2^o, \; \ldots \; .$$

The smooth-pasting condition is not imposed at those stopping times which signify forced realizations. Forced realizations are assumed to be caused by events exogenous to the model, e.g., an unanticipated and unavoidable need for consumption or portfolio revision. Forced realizations are formally modelled as Poisson arrivals with a constant force λ. The Poisson process is independent of the process which generates the movements of the state variables.

For a marginal investor the time of purchase is also an optimal stopping time since, by definition, he is indifferent to the purchase. Thus,

$$V(Y, \; t; \; c, \; T; \; P, \; t) = P \qquad (4a)$$

$$\frac{\partial V}{\partial Y_n} = (1 - \tau_s) \frac{\partial P}{\partial Y_n} \qquad n = 1, \; \ldots, \; N \; . \qquad (4b)$$

This condition of indifference provides the link between the value of the bond and its market price. It may also be interpreted as an alternative description of the marginal investor. As we see below, (4a) need not hold for non-marginal investors who either receive a surplus by purchasing the bond at the prevailing price or find buying the bond to be dominated by lending at the short-term rate.

At all other times, the investor's value of the bond exceeds the after-tax proceeds from immediate sale, and the investor optimally defers the realization of a capital gain or loss. The set of states and times $\{Y, \; t\}$

at which this occurs is referred to as the continuation region, i.e., in the continuation region

$$V(Y, t; c; T; \hat{P}, \hat{t}) > (1 - \tau_s)P + \tau_s\hat{P}, \quad \text{if } t - \hat{t} < 1$$
$$> (1 - \tau_L)P + \tau_L\hat{P}, \quad \text{if } t - \hat{t} > 1 . \tag{5}$$

In the continuation region, the investor's after-tax rate of return on his bond is

$$\frac{dV + (1 - \tau_c)cdt + \max\left[0, (\hat{P} - 1)\tau_c dt/(T - \hat{t})\right]}{V} . \tag{6}$$

The term $(\hat{P} - 1)\tau_c dt/(T - \hat{t})$ is the tax benefit of the linearly amortized premium when the basis is above par.

We assume the after-tax version of the local expectations hypothesis:[10] The after-tax expected rate of return on the coupon bond (measured via the value function) equals the after-tax single period rate of interest over the period $\{t, t + dt\}$, i.e.,

$$E\left[\frac{dV + (1 - \tau_c)cdt + \max[0, (\hat{P} - 1)\tau_c dt/(T - \hat{t})]}{V}\right] = (1 - \tau_c)rdt \tag{7}$$

for all $Y, t, \hat{P}$, and $\hat{t}$.

We assume that the state of the economy at time t is summarized by a vector $\{Y_n(t)\}_{N \times 1}$. This vector also summarizes the history of the economy, $Y(\tau)$, $\tau < t$, to the extent that the history is of current economic relevance. The state variables are jointly Markov with movements determined by the system of stochastic differential equations

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[10] See Cox, Ingersoll, and Ross (1981) for a discussion of the different forms of the expectations hypothesis. In another paper (1983) they show how this assumption may be weakened by incorporating a risk premium into the drift terms for the state variables.

$$dY_n(t) = \mu_n(Y, t)dt + \sigma'_n(Y, t)dw(t) \qquad (n = 1, 2, \ldots, N) . \qquad (8)$$

μ_n is a scalar, σ_n is a K-dimensional vector, $K \leqslant N$, and $dw(t)$ is the increment of the Wiener process $w(t)$ in R^K. The variance-covariance matrix $\{\sigma'_n \sigma_m\}$ is positive semidefinite and of rank K (positive definite, if $K = N$).

If $\{Y, t\}$ lies in the continuation region, the expected value of dV due to the movement of the state variables Y, t, is given by Ito's Lemma, as the first three terms of equation (9) below. The expected value of dV due to a stochastic forced realization is $[P - \tau(t, \hat{t})(P - \hat{P}) - V]\lambda dt$ where $\tau(t, \hat{t})$ is the short term or long term rate, depending on the bond's status. The term in the brackets is the loss incurred when the investor is forced to deviate from his optimal policy. The term λdt is the probability of a forced realization over $[t, t + dt]$. Also, the expected value of dV due to the amortization of the premium is $-\dfrac{\partial V}{\partial \hat{P}} \max[0, (\hat{P} - 1)dt/(T - \hat{t})]$. Then equation (7) becomes

$$\frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{\partial^2 V}{\partial Y_n \partial Y_m} \sigma'_n \sigma_m + \sum_{n=1}^{N} \frac{\partial V}{\partial Y_n} \mu_n + \frac{\partial V}{\partial t} + [P - \tau(t, \hat{t})(P - \hat{P}) - V]\lambda$$

$$(9)$$

$$+ (1 - \tau_c)c + \left(\tau_c - \frac{\partial V}{\partial \hat{P}}\right) \max[0, (\hat{P} - 1)/(T - \hat{t})] - rV = 0 .$$

The solution to this differential equation, subject to the boundary conditions (1) through (5), provides the bond price, P, the value of a bond to the

investor, V, and the optimal policy for the realization of capital gains and losses.[11/]

For general functions $\sigma_n(Y, t)$ and $\mu_n(Y, t)$, a closed-form solution does not typically exist. In Section 4 we illustrate the solution procedure in a simplified version of this problem and discuss the economic implications. In Section 5 we provide numerical solutions to the general problem.

4. An Example

In this section we begin to examine the value of the timing option regarding the realization of capital gains and losses on bonds and to analyze the effect of the capital gains tax on their pricing. To discuss these issues in the simplest possible setting and through closed-form solutions, we make a number of simplifying assumptions.

We assume that there is only one state variable, the short term rate of interest, r, with movements determined by the stochastic differential equation

$$dr = \alpha r^2 dt + s r^{3/2} dw(t) \tag{10}$$

where $dw(t)$ is the increment of the scalar Wiener process $w(t)$.[12/] The price, $P(r; c)$, of an infinite maturity coupon bond is then a function of the

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[11/] The now-familiar American put pricing problem provides a useful analogy. Let $G(S, K, T)$ denote the value of a put with striking price K and time to maturity T on a stock with price S. Equation (2) is analogous to the condition at exercise, $G(S^*, K, T) = K - S^*$. The "smooth pasting" condition analogous to (3) is $G_S(S^*, K, T) = -1$. Together these relations are sufficient to derive both the pricing function G and the optimal exercise policy $S^*(T)$. Similarly here we derive both the value function and the optimal realization policy Y^* conditional on the bond price function. Equation (4) then provides the closure finally giving all three.

[12/] We may alternatively consider (10) as the risk-adjusted interest rate dynamics with $\alpha = \mu + \pi$ where μ measures the expected change in the short rate and π captures the risk premium due on interest-rate-sensitive securities. Cox, Ingersoll, and Ross (1980) discuss this interpretation for the stochastic process in (10).

short term interest rate, r, but is independent of the current time, t, because the process generating interest rate movements is stationary.

We also assume that the tax rates on all capital gains and losses are equal, i.e., $\tau_s = \tau_L \equiv \tau$. Thus the length of time over which the bond has been held is irrelevant, and the consol's value to an investor, $V(r; c; \hat{P})$, is also independent of the current time, t. Finally, we assume away forced realizations, i.e., $\lambda = 0$.

It is easy to prove that any investor's optimal policy is to realize capital losses immediately and defer capital gains indefinitely.[13] Given the basis, $\hat{P}$, the continuation region is defined by the range of interest rates such that $P(r; c) > \hat{P}$. In the continuation region the differential equation (9) becomes

$$1/2 \ s^2 r^3 \frac{\partial^2 V}{\partial r^2} + \alpha r^2 \frac{\partial V}{\partial r} - (1 - \tau_c)rV + (1 - \tau_c)c = 0 \ , \quad P(r; c) > \hat{P} \ . \quad (11)$$

The boundary condition (4a) becomes

$$V(r; c; \hat{P}) = \hat{P}, \quad \text{at} \quad P(r; c) = \hat{P} \ , \quad (12)$$

and the "smooth-pasting" conditions (3) and (4a) become

$$\frac{\partial V(r; c; \hat{P})}{\partial r} = (1 - \tau) \frac{\partial P(r, c)}{\partial r} \ , \quad \text{at} \quad P(r; c) = \hat{P} \ . \quad (13)$$

The bond price $P(r; c)$ is a function of the interest rate, r, and of the parameters c, α, s, τ, and τ_c. Inspection of equation (10) indicates that the parameters α and s are dimensionless as are the parameters τ and τ_c. The units of the coupon yield c are dollars per unit of time and the unit of

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[13] This statement is formally proved in Constantinides (1983). If the tax rates τ_s and τ_L are unequal the optimal policy is a great deal more complex. Under these circumstances, the optimal policy for trading stocks is discussed in Constantinides (1982) and the optimal policy for trading bonds is discussed in Section 5 of this paper.

the interest rate is the inverse of the time unit. The bond price must be proportional to the coupon rate, and since it is invariant to changes in the unit of time, it must also be inversely proportional to r. Hence

$$P(r; c) = Hc/r \qquad (14)$$

where H is a function of only the parameters α, s, τ and τ_c.

Since we have determined the functional form of P, we can simplify (11) with the aid of equation (14) to eliminate r and its derivatives, obtaining

$$\frac{s^2}{2} P^2 V_{PP} + (s^2 - \alpha)P V_P - (1 - \tau_c)V + \frac{(1 - \tau_c)P}{H} = 0, \qquad P > \hat{P} . \qquad (15)$$

The general solution to equation (15) is given below:[14]

$$V = \frac{(1 - \tau_c)P}{(1 - \tau_c + \alpha - s^2)H} + A \hat{P}^{1-\eta} P^\eta + A'\hat{P}^{1-\eta'}P^{\eta'}, \qquad P > \hat{P} \qquad (16)$$

where A, A' are arbitrary constants to be determined, and η, η', ($\eta < 0 < \eta'$) are the roots of the quadratic equation

$$\frac{s^2}{2} \eta(\eta - 1) + (s^2 - \alpha)\eta - (1 - \tau_c) = 0 . \qquad (17)$$

By homogeneity, the coefficients of P^η and $P^{\eta'}$ must be proportional to the parameters $\hat{P}^{1-\eta}$ and $\hat{P}^{1-\eta'}$, respectively. Thus, A and A' depend only on the parameters α, s, τ, and τ_c.

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[14] For a meaningful solution the parameters of the interest rate process must satisfy $s^2 - \alpha < 1 - \tau_c$. From (22) and (23) the expected rate of price appreciation and the limit (as r goes to zero) of the expected rate of appreciation of the value function are both $(s^2 - \alpha)r$. Thus, if the stated condition is violated the expected rates of return including coupons must exceed the after-tax rate of interest $(1 - \tau_c)r$ and the expectations hypothesis cannot obtain as was assumed. Furthermore, given that the dynamics may be interpreted in a risk-adjusted sense, as discussed in footnote 12, no other equilibrium is possible either.

The following argument determines A'. Since the optimal trading policy involves no sales at any price above the basis, $\hat{P}$ must have a negligible effect on the value function whenever $P \gg \hat{P}$. Formally

$$\lim_{\hat{P}/P \to \infty} \frac{\partial V}{\partial \hat{P}} = 0 \ .$$

(18)

This condition is satisfied only if A' = 0. The remaining two constants can be determined using (12) and (13). Substituting (16) into (12) and setting A' = 0, we obtain

$$\frac{(1 - \tau_c)P}{(1 - \tau_c + \alpha - s^2)H} + A\ P = P\ .$$

(19)

Similarly, substituting (16) into (13) and setting A' = 0, we obtain

$$\frac{1 - \tau_c}{(1 - \tau_c + \alpha - s^2)H} + A\ \eta = 1 - \tau\ .$$

(20)

We solve for H and A and obtain

$$P(r;\ c) = \frac{(1 - \tau_c)c}{(1 - \tau_c + \alpha - s^2)\{1 - \tau/(1 - \eta)\}r}$$

(21)

and

$$V(r;\ c;\ \hat{P}) = (1 - \tau)P + \tau\ \hat{P}\ , \qquad P \leqslant \hat{P}$$

(22)

$$= \left(1 - \frac{\tau}{1 - \eta}\right)P + \left(\frac{\tau}{1 - \eta}\right)P^\eta\ \hat{P}^{1-\eta}, \qquad P > \hat{P}$$

where

$$\eta \equiv -\ [s^2/2 - \alpha + \{(s^2/2 - \alpha)^2 + 2\ s^2(1 - \tau_c)\}^{1/2}]/s^2$$

is the negative root of (17).

Equation (21) shows that the price of a consol bond is increasing in the capital gains tax rate of the marginal investor. A high capital gains rate does not hurt the investor because he is never forced to realize gains and his

optimal policy is to defer indefinitely the realization of capital gains. In fact a high capital gains rate is a benefit because it enables him to obtain larger tax rebates by realizing a capital loss whenever such a loss occurs. Provided that forced realizations are not too frequent, the same conclusion also applies to a finite maturity par bond, as indeed is demonstrated in the numerical solutions of Section 6. If the bond currently sells at par, the investor can be neutral to the capital gains tax by following the naive policy of deferring both gains and losses. The intelligent policy of deferring gains and realizing losses can only turn the taxation to his advantage, and he therefore benefits by a high capital gains tax rate.

We find the expected rate of price appreciation to be

$$E\left[\frac{dP}{P}\,\Big|\,P\right] = (s^2 - \alpha)rdt \quad (\equiv \gamma\, rdt) \tag{23}$$

by using Ito's Lemma and equations (10) and (21). Since $s^2 - \alpha \gtrless 0$, the expected capital gains rate can be either positive or negative. We write the bond price in terms of γ and obtain

$$P(r;\ c) = \frac{c/r}{\{1 - \gamma/(1 - \tau_c)\}\{1 - \tau/(1 - n)\}} . \tag{21$'$}$$

We observe that the bond price is increasing (decreasing) in the ordinary income tax rate, if capital gains (losses) are expected. This indeterminancy is due to the light taxation of capital gains relative to interest and coupon income in this model. If capital gains are expected, the consol's current coupon yield is less than the interest rate, and an increase in τ_c represents a greater loss for potential holdings in the instantaneous bond than in the consol.

We use two benchmarks to measure the value of the tax timing option. The first is the price of the consol, P_H, in an economy where the marginal

investor follows a buy-and-hold policy. This benchmark is also the consol's

price in an economy with zero capital gains tax.[15/]   Hence, we write

$$P_H = \frac{(1 - \tau_c)c}{(1 - \tau_c - \gamma)r} \; .$$ (24)

The second benchmark is the consol price, P_C, in an economy where the

marginal bondholder realizes all gains and losses continuously. Proceeding as

before, we find that this consol price satisfies the equation

$$(1 - \tau)\left\{\frac{s^2}{2} r^3 P_C'' + \alpha r^2 P_C'\right\} - (1 - \tau_c)rP_C + (1 - \tau_c)c = 0$$ (25)

with solution

$$P_C = \frac{(1 - \tau_c)c}{\{1 - \tau_c - \gamma(1 - \tau)\}r} \; .$$ (26)

Note that $P_C \gtrless P_H$, as $\gamma \lessgtr 0$: A continuous realization policy dominates the

buy-and-hold policy whenever capital losses are expected.

The tax effect on the consol's price is expressed relative to the two

benchmarks as follows:

$$\frac{P - P_H}{P} = \frac{\tau}{1 - \eta}$$ (27a)

$$\frac{P - P_C}{P} = \frac{\tau}{1 - \eta}\left[1 + \frac{\gamma(1 - \eta - \tau)}{1 - \tau_c - \gamma(1 - \tau)}\right] \; .$$ (27b)

When the buy-and-hold benchmark is used, the timing option's value derives

from the right to realize capital losses early. When the continuous-

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realization benchmark is used, the timing option measures the value of deferring capital gains.

A similar analysis can be performed for a non-marginal investor in the tax bracket (τ_{ci}, τ_i). The valuation equation (15) and general solution (16) retain the same form with the same constant H but with different constants A, A' and with different tax rates substituted. The optimal policy again consists of realizing all losses immediately and holding all gains; therefore, (13) is still a valid boundary condition. Equation (12), however, no longer obtains.[16/] Non-marginal investors need not value a potential purchase at the price they pay for it. The valuation can be either more or less. Of course, if they do buy, they must value the bond in excess of its price.

Since the bond's price is set by the marginal investor, only two coefficients need to be determined for the non-marginal investor's problem. These are determined by (13) and (18) giving

$$V_i(r; c; \hat{P}) = \theta_i(1 - \frac{\tau}{1 - \eta})P + \frac{1}{\eta}(1 - \tau_i - \theta_i(1 - \frac{\tau}{1 - \eta}))P^{\eta} \hat{P}^{1-\eta}$$

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where (28)

$$\theta_i = \frac{1 - \gamma/(1 - \tau_c)}{1 - \gamma/(1 - \tau_{ci})} \cdot \underline{17/}$$

Any investor who finds $V_i(r; c; P) > P$ enjoys a tax-induced surplus in buying the bond. This is true for those tax brackets characterized by

$$1 - \eta - \tau_i < \theta_i(1 - \eta - \tau) . \qquad (29)$$

There are two distinct tax effects. First, the larger the investor's capital gains tax rate the greater is the surplus since losses provide larger benefits. Second, when capital gains (losses) are expected, $\gamma > 0$ $(\gamma < 0)$, the larger (smaller) his ordinary tax rate, the greater the surplus. This effect is the standard one of high tax investors' preference for capital gains, observed even in buy-and-hold economies. Since the two tax rates are positively related in a cross section of investors, the effects are enhancing when capital gains are expected and opposing when capital losses are expected.

If the marginal holder of bonds is tax exempt, then there are no tax effects on the consol's price. Nevertheless, even in this case equations (28) and (29) reveal that many taxed investors may have a valuable timing option.

To measure the magnitude of the timing option we require estimates of the parameters α and s and the marginal tax bracket. The current annual riskless rate is approximately 11%. In using the Ibbotson and Sinquefield (1982) study the reported annualized standard deviation of changes in the short rate over the period 1926-1981 is 2.2%. Using equation (10) we set $s = (0.022)(0.11)^{-3/2} = 0.604$. In the same study the reported standard deviation of annualized returns on long term U.S. Treasury bonds is 5.7%. If we take

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<u>17/</u> If there are instantaneous tax exempt instruments with an interest rate of $(1 - \tau_c)r$, then this is the appropriate after-tax opportunity cost to investors in higher tax brackets $\tau_{ci} > \tau_c$. In this case $\theta_i \equiv (1 - \tau_{ci})/(1 - \tau_c)$, and only investors in higher than marginal capital gains tax brackets can enjoy a surplus. Those with a lower capital gains tax rate buy short term taxable (tax exempt) bonds if τ_{ci} is less (greater) than τ_c.

this number as an estimate of the standard deviation of returns on a consol,
then using (23) we can approximate $s = (0.057)/\sqrt{0.11} = .172.$ [18/]

Ibbotson and Sinquefield do not report the average change in the interest
rate so we somewhat arbitrarily examine the two cases $\alpha = 0$ and $\alpha = s^2$
which correspond to no expected change in the interest rate and in the consol
price, respectively. If we choose to interpret α as a risk premium measure,
then under the assumption of no drift in the interest rate, the expected rate
of return on a consol is $r(1 + \alpha)$. Ibbotson and Sinquefield estimate that
investors expected on average a premium of 131 basis points on twenty-year
bonds. This gives an estimate for α of 0.44 based on the average interest
rate.

Table 1 displays the value of the timing option as a percentage of price
for these parameter values (equations (29a, b)). It also presents the surplus
to non-marginal investors as a proportion of price. The values presented are
for the marginal tax rates $\tau_c = .5$ and $\tau = .25$.

For the higher variance process the timing option contributes a signi-
ficant portion of the bond's value as measured against either benchmark. For
the lower variance process the timing option remains important except in the
case when large capital losses are expected and the continuous realization

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[18/] The Ibbotson and Sinquefield estimate based on a portfolio of long-
term bonds may be a downwardly biased estimate of the standard deviation of a
consol's rate of return for two reasons:
    (a) They considered a portfolio of bonds with an average maturity of 20
years (not infinite).
    (b) The variability of a portfolio of bonds generally underestimates the
return variability of each bond. For example, a shock in the economy which
raises the price of ten-year bonds and lowers the price of thirty-year bonds
may leave the portfolio's price essentially unchanged and contribute little to
the variability of the portfolio's return. The same shock, however, may have
a significant impact on the consol's return.
    Both of our estimates of s, particularly the first, may also be
negatively biased because the interest rate was substantially less than 11%
for most of this period.

benchmark is employed. We conclude that the potential effect of tax trading on bond prices cannot safely be ignored in practice.

Examination of the levels of investor surplus indicates that clienteles can be either sharp or weak, depending on the parameter values of the stochastic interest rate process. The first column illustrates a very precise marginal clientele when large capital gains are expected and the two tax effects are strongly enhancing. The final column shows a somewhat weaker clientele structure with a reversed ordering when the two tax effects are opposing. The impact of the ordinary tax is greater in this case because the required coupon yield is over forty percent in excess of the interest rate. The remaining columns are examples of broad marginal clienteles. Columns two and five show the effect of differences in the capital gains tax rate alone.

5. Optimal Bond Trading—the General Case

We examine a discrete-time version of the model outlined in Section 3, focusing on the distinction between short and long term gains and losses, the effect of the amortization deduction and transactions costs. Since our primary concern is on how optimal trading affects the bond prices, we confine our attention to the marginal bondholder.

We assume that the trading interval is one year.[19] If an asset is sold one year after purchase, we assume that the holding period is short term or long term at the investor's discretion. Since the cutoff point is one year after purchase, the investor can make the holding period long or short term,

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[19] The choice of one year is primarily a matter of convenience, coinciding with both the maximum holding period for short term losses and the tax year. If the holding period were shorter, as it was until recently, and the offset provision is assumed to be binding, then an additional complication results. The value of short-term losses in the first part of the year cannot be determined until it is known if there are later offsetting long term capital gains. This problem affects only tax scenario II, below.

by delaying or advancing the bond sale by only one day. By a simple dominance argument, all capital gains are realized long term. Similarly whenever the investor realizes a capital loss one year after purchasing the bond, he does so short term.

We maintain our assumption that there are no forced realizations. On each trading date the investor either holds his bond, deferring the realization of a capital gain or loss, or sells his bond and immediately repurchases it, thereby realizing a capital gain or loss and re-establishing a short term status. The following set of factors determines whether the investor holds his bond or executes a wash sale:

(a) If the bond price is below the basis, the investor would like to sell the bond and receive the tax deduction immediately. The reason becomes more compelling if the bond was purchased one year earlier, so that this is the only chance to realize the capital loss short term.

(b) If the bond price is above the basis, the investor would like to defer the realization of the capital gain and thereby defer the tax liability. As stated previously, he never realizes a short term gain because he can wait one more day. Nevertheless, he may wish to realize a long term gain as explained in (c) and (d).

(c) A short term holding status is beneficial to the investor. This status helps when he realizes a capital loss, because he realizes it short term, and it never hurts, even when the investor realizes a capital gain, because he can always wait one more day and convert to the long term status. The short term status turns out to be a very important factor governing the optimal liquidation policy. Under certain realistic conditions, an investor may realize a capital gain solely to convert to the beneficial short term status.

(d) The peculiar amortization rules on bonds introduce another twist to this already complex problem. If the bond's basis is above par, this difference is linearly amortized over its remaining term to maturity with the "loss" applied against the investor's ordinary income. The present value of this tax deduction is high for short maturity bonds, but decreases with longer maturity, because of the linearity of the amortization rule. For short maturity bonds the benefit in establishing a basis above par may be sufficiently large to make it optimal to realize a capital gain.

We assume that the short term rate of interest, r, is the only state variable and that it follows a driftless binomial random walk with two reflecting barriers. We consider two specifications for the interest rate process. In the low-variance process, the interest rate takes one of the twenty-one values, .04, .05, .06, ..., .24. If the interest rate equals any of the interior values then at the next date it has the value $r \pm .01$, each with probability one half. If the interest rate is equal to one of the end-point values .04 or .24 then at the next date it remains unchanged or takes on the value .05 or .23, respectively with probability one half. The reader may verify that the unconditional distribution of r is uniform over the twenty-one points. The standard deviation of changes in the interest rate is

$\sigma_r \equiv std(r(t + 1) | r(t)) = .01$ per year, independent of the state (except in the end-point states). In the high-variance process, the interest rate takes one of the eleven values, .04, .06, ..., .24. The probabilities are as in the low-variance process and the reflecting barriers are at the levels .04 and .24. The standard deviation of the changes in the interest rate in the high-variance process is $\sigma_r = .02$ per year. From the Ibbotson and Sinquefield (1982) study, the annualized standard deviation of the short term rate is .022. The low-variance process then underestimates the interest rate

variability, while the high-variance process reflects the average variability in the period 1926-1981.

As we shall see, the low-variance process implies, on average, that the standard deviation of the annual rate of return of 20-year Treasury Bonds is 5.66%, if priced under the buy-and-hold policy, and 5.82% if priced under the optimal policy with τ_s = .25 (scenario I). For the high-variance process, the corresponding numbers are 9.47% and 8.73%. From the Ibbotson and Sinquefield (1982) study this standard deviation is 5.7% for long term Treasury bonds over the 1926-1981 period. Therefore the low-variance process reflects the actual initial variability over that period.[20] The high-variance process may be more representative of recent history.

In discrete time, the differential equation (9) becomes a difference equation which we may solve numerically subject to the boundary conditions. Equivalently and more directly we obtain the bond price and the value of a bond to the marginal investor by dynamic programming, at dates T, T - 1, T - 2, ..., etc.

Equations (30) and (31) establish the bond price and value of a bond to the investor at maturity, i.e., at t = T. At maturity the ex-coupon bond is priced at par which we take to be unity.

$$P(r, T; c, T) = 1 . \tag{30}$$

The value of the bond to an investor is the after-tax sale proceeds. By the maturity date, the bond basis cannot exceed one, because the excess will have

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[20] See, however, the caveat in footnote (18). In addition, when the low-variance process is used, the simulated volatility of a 20-year bond over its life will be lower than the historic average because in the last part of its life its price is more strongly affected by the short term rate so the volatility is underestimated.

been completely amortized by then. Thus only a gain can be realized at maturity, and the appropriate capital gains tax rate is the long term rate. Thus,

$$V(r, T; c, T; \hat{P}, \hat{t}) = 1 - \tau_L + \tau_L \hat{P} . \qquad (31)$$

With the terminal values established, the bond's price and its value to a given investor at points in time prior to maturity can now be obtained through dynamic programming. We distinguish between the cases in which amortization is and is not utilized.

The bond price is what a marginal investor is willing to pay for it. His alternative is investing in the short term asset over the next year in which case his investment increases at the prevailing after-tax interest rate. He is indifferent to buying the bond, therefore, only if the after-tax coupon and amortization benefit plus the expectation of the value function next period is greater than the current bond price by exactly the after tax foregone interest. If the bond is selling today for less than par, then its price is the appropriate basis in the value function. If the bond is priced above par, then $(P - 1)/(T - t)$ will be amortized in the next year, and the basis in the value function next year is less than the prevailing price by this amount. Thus at time t

$$P = [1 + (1 - \tau_c)r]^{-1}\{(1 - \tau_c)c +$$

$$E_t[V(\hat{r(t} + 1), t + 1; c, T; P, t]\}, \quad \text{if} \quad P < 1 + \qquad (32a)$$

$$P = [1 + (1 - \tau_c)r]^{-1}\{(1 - \tau_c)c + (P - 1)\tau_c/(T - t)$$

$$E_t[V(\hat{r(t} + 1), t + 1; c, T; P - (P - 1)/(T - t), t]\}, \quad \text{if} \quad P > 1 . \qquad (32b)$$

The bond price, P, is the solution to (32a, b).[21]

The value of a long position in the bond is the greater of the after-tax proceeds from immediate sale and the discounted value of the benefits if the bond is retained. The after-tax proceeds from immediate sale are

$$(1 - \tau_s)P + \tau_s\hat{P}, \quad \text{if} \quad P \leqslant \hat{P} \quad \text{and} \quad t = \hat{t} = 1 \tag{33}$$

$$(1 - \tau_L)P + \tau_L\hat{P}, \quad \text{otherwise.}$$

As in (32) the discounted benefits, if the bond is retained, are

$$[1 + (1 - \tau_c)r]^{-1}\{(1 - \tau_c)c + \tag{34a}$$

$$E_t[V(\tilde{r}(t + 1), t + 1; c, T; \hat{P}, \hat{t})]\}, \quad \text{if} \quad \hat{P} \leqslant 1$$

and

$$[1 + (1 - \tau_c)r]^{-1}\{(1 - \tau_c)c + (\hat{P} - 1)\tau_c/(T - \hat{t}) + \tag{34b}$$

$$E_t[V(\tilde{r}(t + 1), t + 1; c, T; \hat{P} - (\hat{P} - 1)/(T - \hat{t}), \hat{t})]\}, \quad \text{if} \quad \hat{P} > 1 .$$

In comparing equations (32) and (34) we note that P = V(r, t; c, T; P, t) so the relation in (4a) is satisfied.

We illustrate the optimal trading policies for a bond with a 14 percent stated coupon payable annually, in the four tax scenarios.

I. <u>Treasury bond held by a high-tax-bracket individual, with</u> τ_c = .5, $\tau_s = \tau_L$ = .25.

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[21] The right-hand side of (32a) is positive at P = 0 since the first term is, and the right-hand side of (32b) is less than P for large values (since the maximum benefit of future tax losses is $\tau_c(P - 1)$). These expressions are continuous at P = 1. Therefore a solution to (32a, b) exists. For the dynamics assumed, the solution is also unique.

Table 2 reports the bond prices and values, V, for the high variance interest rate process, for a range of interest rates and bases, and for maturities 1, 5 and 20 years.[22/] Asterisks and daggers mark the states in which a wash sale is optimal. Asterisks indicate the realization of a capital gain establishing a new or higher amortizable basis. Daggers denote the realization of a capital loss. If both the basis and the bond price are less than one, the amortization feature is not in effect and the simple trading rule is to realize a loss and defer a gain as indicated by daggers. If the basis or the bond price exceeds one, the amortization feature becomes relevant and complicates the rule.

For one-year bonds, if the bond sells at a premium, $P > 1$, and the basis is below the bond price, $\hat{P} < P$, the investor realizes a capital gain in order to establish a higher basis and benefit from the amortization of the basis. For five-year bonds, the amortization benefit is reduced and large capital gains may be deferred. For example, if $P = 1.10$ and $\hat{P} = .8$ or .9, the investor realizes a capital gain; but if $\hat{P} = .7$ the investor defers the capital gain. The amortization benefit becomes negligible for twenty-year bonds. For example, if $P = 1.25$ and $\hat{P} < 1.2$, the investor optimally defers the realization of a capital gain and thereby foregoes the amortization benefit of increasing the basis to 1.25. In fact, if $P = .88$ and $\hat{P} = 1.1$, 1.2, or 1.3 the investor foregoes the amortization benefit and realizes the capital loss.

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[22/] Some of the entries in this table as well as those in Table 3 give the value function for states which could never arise along the optimal path. For example, since losses are always realized when the basis is below par, the basis can never be substantially in excess of the current price in this situation. These entries, therefore, give the value of changing to the optimal policy from a suboptimal position.

II. <u>Treasury bond held by a high-tax-bracket individual with</u> $\tau_C =$
.5, τ_s = .5, τ_L = .25.

Table 3 reports the bond price and values, V, for the high variance

interest rate process, for a range of interest rates and bases, and for

maturities 1, 5, and 20 years. Asterisks, daggers and double daggers mark the

states where the optimal policy is to perform a wash sale. Panel A reports

results when the bond has been held for longer than one year, $t - \hat{t} > 1$,

while panel B reports results when the bond has been held for just one year,

$t - \hat{t} = 1$. Note that the value function in the two panels can differ only

when a wash sale is executed and a capital loss is realized. When a gain is

realized, it is presumed to be long term so the taxes paid are the same. When

no wash sale occurs, the ensuing status must be long term regardless of the

current status.

These tables indicate that the investor performs a wash sale of long term

bonds practically every year in order to revert to the short term status.

This is emphasized by the double daggers which mark states in which a wash

sale is executed to this end alone. The desirability of the short term status

seems to dominate all other considerations.

III. <u>Treasury bond held by a high-tax-bracket individual with</u> τ_c = .5,
$\tau_s = \tau_L = 0$.

The optimal trading policy is quite simple and need not be illustrated in

a table. Whenever the bond price is above par and the basis, the investor

makes a wash sale to establish a higher basis and deduct from future ordinary

income the premium amortization. The investor has no tax incentives to

perform any other trades.

IV. <u>Treasury bond held by a bank or bond dealer with</u> $\tau_c = \tau_s = \tau_L = .5$.

Again the optimal policy can be described without a table. The investor optimally realizes all capital losses and defers the realization of capital gains. He never realizes a capital gain in order to establish a higher basis with the benefit of the amortization deduction. The tax rate on ordinary income is the same as that on capital gains so amortization "losses" at best exactly offset the capital gain and occur later. Neither does he defer the realization of a capital loss in order to maintain the benefit of the amortization deduction.

6. Bond Prices and the Tax Timing Option

Table 4 displays simulated Treasury bond prices that would be established by the marginal investor, following the optimal trading policy under each of the four tax scenarios. We assume that the current value of the short term interest rate is 14%. For comparison, the 14% coupon bond would be priced just above par if the marginal investor followed a buy-and-hold policy. The exact buy-and-hold prices range from 1.002 to 1.071 for the high-variance process and from 1.001 to 1.030 for the low-variance process.[23/]

The prices which prevail under tax scenario II are uniformly higher than those under scenario I since investors are not subject to the restrictive offset provision of the tax code but can exploit in full their short term losses. Furthermore, except for the bonds of five-year maturity, tax scenario II

[23/] Buy-and-hold prices are computed with the formula (35) below. Even though the interest rate is not expected to increase or decrease from 14%, the yield curve is slightly downward sloping due to Jensen's inequality, and prices are above par. For these and other premium bonds the buy-and-hold policy assumes that the excess above par is amortized and deducted year by year. Thus no capital losses (or gains) are earned on premium bonds under the buy-and-hold policy. Therefore this benchmark price is the same for all scenarios.

typically results in the highest price. We would expect the second scenario to

yield high prices because short term losses provide valuable rebates and a short

term holding period is relatively cheap to establish. This advantage is least

valuable for short maturity bonds because they are the least volatile.

Consequently all of the five-year bonds and a few of the other short maturity

bonds are priced highest under tax scenario III. There are two distinct

reasons.

First, for discount bonds the buy-and-hold price is highest under tax

scenario III since the guaranteed capital gain escapes all taxation. Second,

with zero capital gains tax rate, it is costless to establish above-par

amortizable basis. For sufficiently short maturities these two effects

dominate.

A comparison of the pricing under scenarios I, III, and IV is also of

interest. While their interpretation is radically different, they actually

differ in only one respect. The capital gain tax rates, both long and short

term, are .25, 0, and .5 respectively. All other taxes are the same. Scenario

III with the lowest tax rate has prices which are uniformly highest;

nevertheless, the high tax rate in scenario IV does not always induce the lowest

price. Again there is a tradeoff between the value of capital losses and the

cost of capital gains. The former is more important for the volatile longer

maturity bonds. The latter is more important for the shorter maturity bonds,

particularly those selling below par.

It is frequently asserted that discount bond prices are higher than what is

justified by the term structure of interest rates, reflecting the fact that a

portion of the return is realized as a lightly taxed capital gain. Our

discussion of Table 4 demonstrates that the above is just one of several tax

effects on bond prices. The direction and magnitude of the tax effect depends

critically on the tax scenario applicable to the marginal investor and on whether the marginal investor follows a passive or optimal trading policy.

We now turn our attention to the tax timing option, defined as the difference between the bond prices under the optimal and buy-and-hold policies.[24/] Table 5 reports the value of the timing option as a percentage of the bond price under the optimal policy. In each case the buy-and-hold price is calculated using the corresponding long term capital gains tax rate (.25, .25, 0, .5). If this price is above par, the amortization is deducted every year. Thus no capital losses (or gains) are earned under the buy-and-hold policy for premium bonds, and the benchmark price is the same for all scenarios. For discount bonds the buy-and-hold prices vary across the scenarios and are inversely related to the long term capital gains tax rate. The timing option varies widely for different coupon rates, maturities, and tax scenarios, but in most cases it represents a substantial fraction of the bond price just as the example in section 4 illustrates.

The one exception is deep discount bonds under tax scenario III. Here the timing option is worth little since there is only a small probability of ever amortizing a premium and no other tax trading benefit is possible. For bonds selling near or above par, however, the timing option is more important under tax scenario III than under scenarios I or IV. The binding deduction limit under scenario III is a mixed blessing. On the one hand the individual may not obtain tax rebates from the government by realizing capital losses. On the

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[24/] An alternative definition of the timing option is the difference between the bond prices under the optimal and continuous realization policies. The two definitions are compared in the example of Section 4. Since the interest rate dynamics employed here are without drift, the results are most similar to the case $\alpha = 0$ in the continuous time model. The buy-and-hold benchmark resulted in smaller timing options in that case so our choice is conservative.

other hand he can costlessly realize capital gains in order to raise the basis and take advantage of the amortization deduction.

The reported values show that the tax timing option is typically increasing in maturity. This is due to both the increased value of standard options when their maturities are lengthened and the greater volatility of the longer maturity bonds underlying these options. This feature explains why the 25- and 30-year 10% coupon bonds are more expensive than those with 10- to 20-year maturities even though the interest rate is above the coupon rate at 14% and the yield curve is essentially flat.

Although longer maturity bonds generally have more valuable timing options, it does not follow that a larger tax subsidy flow is available on long bonds. For example, holding two 15-year bonds in succession may provide greater total tax benefit than a single 30-year bond provides. One way to compare the benefits of different maturity bonds is to express the timing option on an annualized basis. The maturity of bonds with the largest annualized benefits would then represent the natural "habitat" of investors particularly concerned with tax benefits. The annualized tax subsidy on a T-year bond is approximately $r(1 - \tau_c)/[1 - \exp(-r(1 - \tau_c)T)]$ per dollar value of the timing option. Using this approximation we establish that the lowest annual subsidy is on short maturity bonds. On bonds with ten or more years to maturity the benefits are fairly constant, regardless of the tax scenario.

Annualizing the timing option also permits us to normalize the tax benefits relative to the rate of return earned on the bond. For example, under the four scenarios tax benefits provide on average 7, 32, 18 and 10 percent, respectively, of the total return expected on the 25-year, 14% coupon bond.

We have so far ignored transactions costs. A bid-ask spread or other costs of trading will reduce the value of the timing option since the optimal policies

involve substantially more trading than the buy-and-hold policy. Constantinides has examined the optimal tax trading policy on stocks in the presence of proportional transactions costs.[25/] In a simple continuous-time lognormal model he found that investors should not realize losses immediately, but should wait until the price falls to a specific fraction of the basis. A similar rule applies to our model in section 4. The modifications to the optimal trading policies of the models here are more complicated, but the basic idea remains the same: Trades are deferred until capital gains and losses are larger than in the absence of transactions costs.

Table 6 displays the value of the timing option when trading is costly. The round-trip transaction cost is represented by a bid-ask spread of 0.2, 0.5 or 1.0 percent of par.[26/] The timing option retains a large fraction of its value even with sizeable transactions costs. Bonds of ten or more years to maturity retain at least half of the original timing option even with one percent transactions costs. The reduction may not be as large as we might have expected because transactions costs are not entirely a dead weight loss. The cost of purchase is added to the basis while the cost of sale is deducted from the sales proceeds. Effectively, the taxing authority subsidizes the costs of trading.

Transactions costs decrease the value of the timing option on short maturity bonds more than they do on long maturity bonds. With one point bid-ask spread, the five-year bond loses 71, 64, 46 or 75 percent of its timing option under the four tax scenarios while bonds of at least fifteen years to maturity

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25/ In an earlier version of Constantinides (1983).

26/ U.S. Treasury bonds are typically quoted with spreads of one-quarter to one-half of a point in the Wall Street Journal. A few have spreads of one-eighth of one point. Treasury note spreads are usually one-eighth to one-quarter of a point.

never give up more than 40 percent. At thirty-year maturities the examined

bonds always retain at least two-thirds of the value of their timing options.

7. The Tax-Adjusted Yield Curve and Implied Tax Rates

We demonstrated that bond prices set by the marginal investors following

the optimal trading policy are markedly different from those set under a buy-

and-hold or continuous-realization policy. In this section we explore the

implications of these differences when interest rate and tax bracket estimates

are inferred from market prices.

Previous authors typically have assumed that a particular marginal investor

holds the bond to maturity.[27/] Under this assumption the price of a bond with

maturity T, coupon rate c, and par value one is the solution to

$$P = (1 - \tau_c)c \sum_{t=1}^{T} \pi_t + (1 - \tau_L + \tau_L P(\pi_T , \qquad P < 1 \qquad (35a)$$

$$P = [(1 - \tau_c)c + \tau_c(P - 1)/T] \sum_{t=1}^{T} \pi_t + \pi_T , \qquad P > 1 \qquad (35b)$$

where π_t is the price at time zero of one dollar after tax at time t.

Given a set of bond prices, the resulting set of equations (35) can be

inverted to solve for the discount factors and the tax rates.

27/
See, for example, Caks (1977), McCulloch (1975), Robichek and Niebuhr
(1970), and Schaefer (1981). McCulloch explicitly and Caks implicitly use
equations identical to (35a). Robichek and Niebuhr impose the additional
assumption that all of the π_t are related through a single interest rate.
Only McCulloch recognizes the premium amortization feature embodied in (35b).

Robichek and Niebuhr (1970) do this by imposing the additional assumptions $\tau_L = .5 \tau_c$ and a flat term structure, $\pi_t = (1 + y)^{-t}$. They then solve for the remaining unknowns, τ_c and y, using just two bonds. Their estimates of the marginal tax bracket for the year 1968 range from 37.5% to 50%, depending on the pair of bonds used (and disregarding the cheapest flower bond).

McCulloch (1975) also assumes $\tau_L = .5 \tau_c$; however, he does not require a flat term structure. Instead, he estimates the tax bracket and a cubic spline for the discount function to minimize a weighted sum of the squared deviations between actual and modeled prices. Using data from 1963-1966 he concludes that "the effective tax rate that best explains the prices of U.S. Treasury securities lies somewhere in the range .22 to .33." For new data from 1973 the best estimate of the tax rate is only .19.

Pye (1969) estimates the tax bracket of the marginal bondholder using various combinations of discount and par, taxable and exempt bonds. The analysis closest to ours compares par and moderately discounted taxable bonds. Pye concludes that the effective tax rate at the margin varies between 10% and 36% over the period 1967-1968.

Our analysis provides a possible explanation of this finding. If bond prices are set by investors who follow an optimal trading policy, estimates of the yield curve and the marginal tax bracket obtained under the assumption of a naive buy-and-hold policy may be biased. To test for bias, we generate a sample of simulated bond prices under the assumption of optimal trading policies with known tax rates and yield curves. We then estimate the yield curve and tax rate from this sample by a procedure which is in the spirit of the methods discussed.

Since our "data" is simulated and, therefore, not subject to measurement error, there is no statistical advantage in using many prices. Thus, like Robichek and Niebuhr we use an exact "estimation" requiring only a few bonds. We eliminate the need of assuming a flat term structure, however, by using four rather than two bonds. In fact with four bonds no smoothness requirement for the yield curve even of the weak type assumed by McCulloch is required.

For each estimation we use two different coupon bonds from each of two adjacent maturities. Under an assumed buy-and-hold policy, the two longer bonds with maturity $T + 1$ are priced according to

$$P' = (1 - \tau_c)c \sum_{t=1}^{T} \pi_t + \{1 - \tau_L + \tau_L P' + (1 - \tau_c)c\}\pi_{T+1} , \qquad P < 1 \qquad (36a)$$

$$P' = [(1 - \tau_c)c + \tau_c(P' - 1)/T] \sum_{t=1}^{T} \pi_t + [(1 - \tau_c)c + \tau_c(P' - 1)/T + 1]\pi_{T+1} ,$$

$$P' > 1 \qquad (36b)$$

while the shorter maturity bonds are priced by (35).

Substituting the four bond prices into (35) and (36) gives four equations in the five unknowns, $\sum \pi_t$, π_T, π_{T+1}, τ_c, and τ_L. If we assume $\tau_L = \tau_c/2$, the system of equations is now fully specified. We eliminate π_T, π_{T+1}, τ_c and τ_L to obtain a quadratic equation in the variable $\sum \pi_t$. Solving for this unknown and then the others yields two solution sets. Only one of these satisfies the constraints $0 \leqslant \pi_{T+1} \leqslant \pi_T \leqslant 1$ and $\tau_c \leqslant 100\%$, and this is the one chosen.[28]

Tables 7 and 8 report the errors in the estimated forward rates and the estimated tax brackets on coupon income (correct tax bracket $\tau_c = 50\%$ in

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[28] In some cases this forces negative estimates for the tax rates.

each case) for different maturities, tax scenarios, coupon rates and interest rate variances.[29/] The errors are usually opposite in sign since an increase in the tax rate decreases the effective discount rate and errors of opposite signs have partially offsetting effects. In most cases the interest rate is overestimated while the tax bracket is underestimated. In the extreme, the tax rate is estimated to be negative.

The errors are usually smaller for the low-variance process, as we would expect, since the timing option then has less value and buy-and-hold prices are more accurate. For the same reason, errors are smaller when the deep discount bonds are used in the estimation.

The estimates are generally most accurate under tax scenario I. Again this corresponds to the case when the timing option has the least value. Tax scenario II yields very poor results as does scenario III when near par bonds are used. Tax scenario IV is interesting because the implied tax bracket is about the same for all maturities. It ranges between 20% to 30%, disturbingly reminiscent of the tax rate estimated by McCulloch. (By construction, the actual tax rates in this case are all 50%.)

While the errors in the forward rates are often large, the computed numbers are almost invariably within one standard deviation of both the true forward rate and the single-period rate expected to prevail at the forecast time. Consequently, verifying the induced tax trading bias in the forward rates would require a large sample of data. Furthermore, even with large amounts of data available, the errors probably could not be distinguished from liquidity or other term premia. It is interesting to note

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[29/] The error in the estimated forward rate is the deviation between the estimate and the true forward rate calculated from the binomial model. The true forward rate is not equal to the future expected spot rate, 14% in this case, due to Jensen's inequality.

that the positive errors are at least qualitatively consistent with the usually claimed upward bias in the yield curve.

We also tried estimation under the buy-and-hold assumption with $\hat{\tau}_L = 0$ and $\hat{\tau}_L = \hat{\tau}_c$. These rates are correct for tax scenarios III and IV respectively, but the estimates are not noticeably improved, probably because the buy-and-hold policy is "too far" from optimal.

8. Municipal Bonds

The tax treatment of municipal bonds differs from the tax treatment of Treasury and corporate bonds in two important respects. First, coupon income on municipal bonds is exempt from Federal tax. Second, if the purchase price in the secondary market is above par this excess must be amortized but the amortized amount is not allowed as a deduction, even though the bond's tax basis is correspondingly reduced. In effect the taxation of bond coupons and of premium amortization are symmetric: for Treasury bonds, the coupons and premium amortization are taxed at the individual's marginal tax rate on ordinary income; for municipal bonds the coupons and premium amortization remain untaxed.

Coupon income on municipal bonds may be subject to state tax, but in our calculations we ignore state taxes. We consider this a good first approximation for two reasons. Many states exempt from state tax the coupons on bonds issued by municipalities within the state so the marginal holders of such bonds may well be exempt from taxes. Also, while state tax rates vary widely across states, they are generally very low relative to the Federal tax rates of investors who would consider holding municipal bonds.[30/]

30/ As of 1980 seven states had no individual income tax on interest. More than half the states had maximum marginal tax rates at or below 7%. In only three states was the maximum tax rate above 11%. The highest rate was Minnesota's 16%.

The main difference between the optimal trading policies for municipal and taxable bonds is that no trades are ever made at a price above par since there is no advantage in establishing an amortizeable basis. Since this is the only trading advantage of taxable bonds under tax scenario III, the value of the timing option on municipal bonds is zero in this scenario. At the opposite extreme is tax scenario IV. In this case it is never optimal to establish an above par basis on a taxable bond, so the right to amortize such a basis contributes nothing to the value of the timing option. Thus under scenario IV, the value of the timing option on a municipal bond is equal to that on a taxable bond with the same after-tax coupons (i.e., before-tax coupons $1/(1 - \tau_c)$ times as large). Under tax scenarios I and II the timing option on municipal bonds is less valuable than the option on coupon-equivalent taxable bonds.

Table 9 presents the value of the timing option on municipal bonds. When municipals are deep discount, the timing option under scenarios I and II is nearly as valuable as on coupon-equivalent taxable bonds. The same is true on short maturity municipals even if the discount is small. These, of course, are the cases when the right to amortize the basis in the future has negligible value. On premium municipal bonds the timing option is substantially smaller than on coupon equivalent taxables, especially if the comparison is made between short maturity bonds. For example, under tax scenario II the timing option on short-maturity municipals is one-third as large as the timing option on short-term taxables; the timing option on long-term municipals is one-half as large as the timing option on long-term taxables.

9. Concluding Remarks

This paper extended the work of Cox, Ingersoll, and Ross (1981, 1983) on valuing bonds and combined it with the work of Constantinides (1982, 1983) and Constantinides and Scholes (1980) on optimal trading policies. We determined that of the tax timing option is an important fraction of the bond price.

We also discussed how the price distortion affects standard estimation techniques for extracting interest rates and marginal tax brackets from observed bond prices. We found the implied errors to be substantial.

Although we focused on the value of the timing option to the marginal bondholder, we did illustrate in one case that the optimal trading policy differs from the buy-and-hold or continuous realization policy for all taxed investors. Thus, even if the marginal investor is tax exempt, the timing option has positive value to other bondholders.

Our paper only examined the case when the tax bracket of the marginal bondholder remains unchanged. That is, investors may buy and sell the bond in the course of the optimal trading policy, but the bond remains in the hands of investors in the same tax bracket throughout its term to maturity. The next step should be to recognize the existence of tax clienteles as in Schaefer (1981); but unlike Schaefer, to explore the implications of the bondholders' following optimal trading policies and of the bond being passed from one tax bracket investor to another as its maturity shortens or as it changes from a discount to a premium bond.

TABLE 1

The Value of the Timing Option and the Non-Marginal
Investor's Surplus as a Percentage
of the Consol Price[a]

		s = 0.604			s = 0.172		
		$\alpha = 0$	$\alpha = s^2$	$\alpha = .44$	$\alpha = 0$	$\alpha = s^2$	$\alpha = .44$
Timing option:							
buy and hold benchmark		7.7%	11.2	11.9	3.4	3.9	11.7
continuous realization benchmark		44.9	11.2	9.0	4.9	3.9	0.5
Investor Surplus:							
τ_{ci}	τ_i						
0.5	0.5	11.2	20.3	22.7	3.9	4.7	22.1
0.6	0.25	277.1	0.0	−5.3	1.8	0.0	−16.8
0.4	0.2	−43.7	−4.1	−0.8	−1.9	−0.9	9.1
0.3	0.15	−62.7	−8.1	−2.6	−3.6	−1.9	15.7
0.0	0.0	−88.0	−20.3	−11.1	−7.4	−4.7	26.3

[a]Computed for infinitely lived investors. Interest rate follows the risk-adjusted stochastic process $dr = \alpha r^2 \, dt + sr^{3/2} \, dw$. Marginal bond-holder's tax rates are $\tau_c = .5$ on coupon income and $\tau = .25$ on all capital gains. Non-marginal holders' tax rates are as given.

TABLE 2

Treasury Bond Prices and Values of a Long Position Under Tax
Scenario I, $\tau_c = .5$, $\tau_s = \tau_L = .25$ [a]

Interest Rate	Bond Price	Basis						
		.7	.8	.9	1.0	1.1	1.2	1.3

		Maturity = 1 year						
0.06	1.0755	*0.98	*1.01	*1.03	*1.06	1.09	1.14	1.18
0.10	1.0364	*0.95	*0.98	*1.00	*1.03	1.07	1.11	1.16
0.14	1.0000	0.93	0.95	0.98	*1.00	1.05	1.09	1.14
0.18	0.9762	0.91	0.94	0.96	†0.98	1.03	1.07	1.12
0.22	0.9535	0.90	0.92	0.94	†0.97	1.01	1.05	1.10
		Maturity = 5 years						
0.06	1.3363	*1.18	*1.20	*1.23	*1.25	*1.28	*1.30	*1.33
0.10	1.771	*1.06	*1.08	*1.11	*1.13	*1.16	1.18	1.22
0.14	1.0197	0.95	0.97	0.99	*1.01	1.05	1.08	1.12
0.18	0.9110	0.88	0.89	0.91	†0.93	0.96	1.00	1.04
0.22	0.8354	0.81	0.83	†0.85	†0.88	†0.90	0.93	0.97
		Maturity = 20 years						
0.06	1.7358	1.48	*1.50	*1.53	*1.55	*1.58	*1.60	*1.63
0.10	1.4187	1.26	1.27	1.29	1.31	1.34	1.36	*1.39
0.14	1.1044	1.03	1.04	1.06	1.08	1.10	†1.13	†1.15
0.18	0.8793	0.85	0.86	†0.88	†0.91	†0.93	†0.96	†0.98
0.22	0.7429	0.74	†0.76	†0.78	†0.81	†0.83	†0.86	†0.88

[a]Coupon rate on bond is 0.14 paid annually. Interest rate follows high-variance process with standard deviation of .02 per year. Asterisks and daggers mark the states in which the optimal policy is to perform a wash sale. Asterisks indicate the realization of a long term capital gain establishing a new or higher amortizable basis. Daggers denote the realization of a long term capital loss.

TABLE 3

Treasury Bond Prices and Values of a Long Position Under Tax
Scenario II $\tau_c = .5$, $\tau_s = .5$, $\tau_L = .25$ [a]

Interest Rate	Bond Price	Basis						
		.7	.8	.9	1.0	1.1	1.2	1.3

Panel A: Long Term Status
Maturity = 1 year

Interest Rate	Bond Price	.7	.8	.9	1.0	1.1	1.2	1.3
0.06	1.0755	*0.98	*1.01	*1.03	*1.06	1.09	1.14	1.18
0.10	1.0364	*0.95	*0.98	*1.00	*1.03	1.07	1.11	1.16
0.14	1.0000	0.93	0.95	0.98	*1.00	1.05	1.09	1.14
0.18	0.9762	0.91	0.94	0.96	†0.98	1.03	1.07	1.12
0.22	0.9535	0.90	0.92	0.94	†0.97	1.01	1.05	1.10

Maturity = 5 years

Interest Rate	Bond Price	.7	.8	.9	1.0	1.1	1.2	1.3
0.06	1.3476	*1.19	*1.21	*1.24	*1.26	*1.29	*1.31	*1.34
0.10	1.1919	*1.07	*1.09	*1.12	*1.14	*1.17	†1.19	1.22
0.14	1.0368	0.95	*0.98	*1.00	*1.03	†1.05	1.08	1.12
0.18	0.9177	0.88	0.89	‡0.91	†0.98	†1.00	1.04	1.04
0.22	0.8357	0.81	0.83	†0.85	†0.88	†0.90	0.93	0.97

Maturity = 20 years

Interest Rate	Bond Price	.7	.8	.9	1.0	1.1	1.2	1.3
0.06	1.9200	*1.62	*1.64	*1.67	*1.69	*1.72	*1.74	*1.77
0.10	1.6138	*1.39	*1.41	*1.44	*1.46	*1.49	*1.51	*1.54
0.14	1.2672	*1.13	*1.15	*1.18	*1.20	*1.23	*1.25	†1.28
0.18	0.9791	‡0.91	‡0.93	†0.96	†0.98	†1.01	†1.03	†1.06
0.22	0.7856	‡0.76	†0.79	†0.81	†0.84	†0.86	†0.89	†0.91

Panel B: Short Term Status
Maturity = 1 year

Interest Rate	Bond Price	.7	.8	.9	1.0	1.1	1.2	1.3
0.06	1.0755	*0.98	*1.01	*1.03	*1.06	†1.09	†1.14	†1.19
0.10	1.0364	*0.95	*0.98	*1.00	*1.03	†1.07	†1.12	†1.17
0.14	1.0000	0.93	0.95	0.98	*1.00	†1.05	†1.10	†1.15
0.18	0.9762	0.91	0.94	0.96	†0.99	†1.04	†1.09	†1.14
0.22	0.9535	0.90	0.92	0.94	†0.98	†1.03	†1.08	†1.13

Maturity = 5 years

Interest Rate	Bond Price	.7	.8	.9	1.0	1.1	1.2	1.3
0.06	1.3476	*1.19	*1.21	*1.24	*1.26	*1.29	*1.31	*1.34
0.10	1.1919	*1.07	*1.09	*1.12	*1.14	*1.17	†1.20	†1.25
0.14	1.0368	0.95	*0.98	*1.00	*1.03	†1.07	†1.12	†1.17
0.18	0.9177	0.88	0.89	‡0.91	†0.96	†1.01	†1.06	†1.11
0.22	0.8715	0.84	0.86	†0.87	†0.92	†0.97	†1.02	†1.07

Maturity = 20 years

Interest Rate	Bond Price	.7	.8	.9	1.0	1.1	1.2	1.3
0.06	1.9200	†1.62	†1.64	†1.67	*1.69	*1.72	*1.74	*1.77
0.10	1.6138	*1.39	*1.41	*1.44	*1.46	*1.49	*1.51	*1.54
0.14	1.2672	*1.13	*1.15	*1.18	*1.20	*1.23	*1.23	†1.28
0.18	0.9791	‡0.91	‡0.93	‡0.96	†0.99	†1.04	†1.09	†1.14
0.22	0.7856	‡0.76	†0.79	†0.84	†0.89	†0.94	†0.99	†1.04

[a]Coupon rate on bond is c = 0.14, paid annually. Interest rate follows the high-variance process with standard deviation of 0.02 per year. Asterisks, daggers, and double daggers mark the states in which the optimal policy is to perform a wash sale. In each case, one of the benefits is reestablishing a short term holding status. Asterisks and double daggers indicate the realization of a long term capital gain. The former also denote the establishing of a new or higher amortizable basis. Daggers indicate the realization of a long or short term capital loss in panels A and B respectively.

TABLE 4

Treasury Bond Prices Established by Optimal Trading
Policies Under Each Tax Scenario[a]

Maturity	High-variance Process $\sigma_r = .02$ per year				Low-variance Process $\sigma_r = .01$ per year			
Scenario:[b]	I	II	III	IV	I	II	III	IV
Coupon rate c = .06								
5	0.802	0.803	0.837	0.748	0.801	0.801	0.836	0.746
10	0.690	0.708	0.728	0.642	0.681	0.683	0.721	0.628
15	0.624	0.664	0.655	0.592	0.607	0.613	0.641	0.568
20	0.584	0.644	0.605	0.566	0.561	0.578	0.586	0.535
25	0.558	0.633	0.570	0.551	0.531	0.560	0.548	0.516
30	0.540	0.627	0.545	0.542	0.512	0.553	0.523	0.505
Coupon rate c = .10								
5	0.904	0.912	0.923	0.878	0.901	0.901	0.918	0.874
10	0.861	0.903	0.889	0.840	0.844	0.855	0.864	0.821
15	0.841	0.923	0.874	0.833	0.812	0.842	0.831	0.800
20	0.832	0.947	0.865	0.836	0.796	0.846	0.812	0.792
25	0.828	0.969	0.859	0.841	0.787	0.857	0.801	0.791
30	0.825	0.986	0.853	0.847	0.783	0.869	0.796	0.792
Coupon rate c = .14								
5	1.020	1.037	1.039	1.010	1.009	1.017	1.018	1.004
10	1.054	1.118	1.103	1.043	1.023	1.054	1.048	1.019
15	1.082	1.199	1.153	1.080	1.035	1.096	1.075	1.037
20	1.104	1.267	1.188	1.113	1.047	1.137	1.097	1.055
25	1.120	1.320	1.209	1.139	1.057	1.172	1.114	1.071
30	1.132	1.359	1.221	1.159	1.067	1.201	1.127	1.085
Coupon rate c = .18								
5	1.161	1.176	1.184	1.147	1.150	1.157	1.163	1.142
10	1.276	1.344	1.342	1.255	1.244	1.275	1.282	1.231
15	1.350	1.484	1.453	1.339	1.301	1.367	1.365	1.292
20	1.401	1.593	1.527	1.402	1.337	1.440	1.422	1.337
25	1.436	1.667	1.572	1.450	1.363	1.496	1.460	1.370
30	1.460	1.707	1.595	1.485	1.382	1.536	1.484	1.396

[a]Computed at the midpoint of the interest rate range, $r = .14$.

[b]Tax scenarios are described by their capital gains tax rates. If the
investor is an individual, these depend on whether the short-term loss/long-term
gain offset rule and the $3,000 deduction limit are binding. For banks and
dealers these rules are not applicable. In each case the ordinary tax rate is τ_c
= .5.

Scenario I: Offset rule binding, deduction limit not binding, $\tau_s = \tau_L$
= .25

II: Neither rule binding, $\tau_s = .5$, $\tau_L = .25$

III: Deduction limit binding, offset rule irrelevant $\tau_s = \tau_L = 0$

IV: Bank or dealer at margin, $\tau_s = \tau_L = .5$

TABLE 5

Value of the Timing Option on Treasury Bonds
as Reflected by the Percentage Difference
Between the Prices under the Optimal
and Buy-and-Hold Policies[a]

Maturity	High-variance Process $\sigma_r = .02$ per year				Low-variance Process $\sigma_r = .01$ per year			
Tax scenario:[b]	I	II	III	IV	I	II	III	IV
				Coupon rate c = .06				
5	0.0%	0.1	0.0	0.1	0.0	0.0	0.0	0.0
10	0.5	2.9	0.2	1.6	0.1	0.3	0.0	0.4
15	1.3	7.2	0.4	3.4	0.4	1.4	0.0	1.3
20	2.0	11.1	0.5	5.1	0.9	3.7	0.1	2.4
25	2.6	14.2	0.4	6.3	1.3	6.5	0.2	3.4
30	3.0	16.4	0.4	7.1	1.8	8.9	0.3	4.2
				Coupon rate c = .10				
5	0.2	1.1	0.4	0.4	0.0	0.1	0.0	0.1
10	1.5	6.1	2.5	2.2	0.3	1.6	0.2	0.9
15	2.6	11.2	4.3	4.2	0.8	4.4	0.9	2.0
20	3.4	15.1	5.5	5.8	1.4	7.4	1.6	3.1
25	3.9	17.9	6.9	6.1	2.0	10.0	2.4	4.0
30	4.1	19.7	6.2	7.7	2.4	12.1	3.0	4.7
				Coupon rate c = .14				
5	1.7	3.3	3.5	0.8	0.8	1.6	1.7	0.4
10	3.9	9.4	8.1	2.8	1.9	4.8	4.2	1.5
15	5.0	14.2	10.8	4.7	2.6	8.0	6.1	2.7
20	5.4	17.6	12.0	6.1	3.0	10.7	7.4	3.7
25	5.5	19.8	12.4	7.0	3.3	12.7	8.2	4.5
30	5.4	21.2	12.2	7.6	3.4	14.3	8.6	5.0
				Coupon rate c = .18				
5	1.7	2.9	3.6	0.5	0.9	1.5	2.0	0.2
10	3.5	8.4	8.3	2.0	1.9	4.3	4.8	0.8
15	4.2	12.8	11.0	3.4	2.3	7.0	6.9	1.6
20	4.4	15.9	12.3	4.5	2.3	9.3	8.2	2.3
25	4.4	17.6	12.7	5.3	2.4	11.1	8.9	2.9
30	4.3	18.2	12.4	5.9	2.5	12.2	9.2	3.4

[a]Computed at the midpoint of the interest rate range, r = .14.

[b]Tax scenarios are described by their capital gains tax rates. If the investor is an individual, these depend on whether the short-term loss/long-term gain offset rule and the $3000 deduction limit are binding. For banks and dealers these rules are not applicable. In each case the ordinary tax rate is τ_c = .5.

Scenario I: Offset rule binding, deduction limit not binding, $\tau_s = \tau_L$ = .25

II: Neither rule binding, τ_s = .5, τ_L = .25

III: Deduction limit binding, offset rule irrelevant, $\tau_s = \tau_L$ = 0

IV: Bank or dealer at margin, $\tau_s = \tau_L$ = .5

TABLE 6

Effects of Transactions Costs on the Timing Option[a]

	Tax Scenario I: $\tau_s = \tau_L = .25$[b]				**Tax Scenario II:** $\tau_s = .5, \quad \tau_L = .25$[b]			
Maturity	Value of Timing Option				Value of Timing Option			
	k = 0.0	0.2	0.5	1.0	k = 0.0	0.2	0.5	1.0
5	1.7%	1.4	1.0	0.5	3.3%	2.8	2.0	1.2
10	3.9	3.3	2.7	2.0	9.4	8.4	7.0	5.0
15	5.0	4.4	3.8	3.0	14.2	13.1	11.4	8.7
20	5.4	4.8	4.2	3.5	17.6	16.3	14.4	11.4
25	5.5	4.9	4.3	3.7	19.8	18.4	16.5	13.2
30	5.4	4.8	4.2	3.6	21.1	19.9	17.8	14.4

	Tax Scenario III: $\tau_s = \tau_L = 0$[b]				**Tax Scenario IV:** $\tau_s = \tau_L = .5$[b]			
Maturity	Value of Timing Option				Value of Timing Option			
	k = 0.0	0.2	0.5	1.0	k = 0.0	0.2	0.5	1.0
5	3.5%	3.2	2.7	1.9	0.8%	0.6	0.4	0.2
10	8.1	7.6	6.9	5.9	2.8	2.6	2.3	1.9
15	10.8	10.1	9.3	8.2	4.7	4.5	4.2	3.7
20	12.0	11.3	10.4	9.2	6.1	5.8	5.5	5.0
25	12.4	11.6	10.7	9.4	7.0	6.7	6.4	5.9
30	12.2	11.4	10.5	9.2	7.6	7.4	7.0	6.6

[a]Computed at midpoint of interest rate range, r = .14. Interest rate follows high-variance process with standard deviation of .02 per year. Coupon rate on bond c = .14. k measures the transactions costs (bid-ask spread) as a percent of par. Tax scenarios are as outlined in Table 4.

[b]Tax scenarios are described by their capital gains tax rates. If the investor is an individual, these depend on whether the short-term loss/long term gain offset rule and the $3,000 deduction limit are binding. For banks and dealers these rules are not applicable. In each case the ordinary tax rate is $\tau_c = .5$.

Scenario I: Offset rule binding, $\tau_s = \tau_L = .25$
II: Neither rule binding, $\tau_s = .5, \tau_L = .25$
III: Deduction limit binding, offset rule irrelevant $\tau_s = \tau_L = 0$
IV: Bank or dealer at margin, $\tau_s = \tau_L = .5$

TABLE 7

Errors (Basis Points) in Estimated Forward Rates under the Buy-and-Hold Assumption with $\tau_L = .5\ \tau_c$ [a]

Forecast Period	High-variance Process $\sigma_r = .02$ per year					Low-variance Process $\sigma_r = .01$ per year				
	Std. Dev.[b]	Tax Scenario[c]				Std. Dev.[b]	Tax Scenario[c]			
		I	II	III	IV		I	II	III	IV
(Coupon rates on bonds used for estimation .08, .10)										
5	400	39	203	-136	466	200	5	27	-201	470
10	549	85	266	22	357	300	17	201	-134	350
15	595	98	255	126	287	403	33	243	-64	284
20	620	105	283	243	238	426	42	271	10	246
25	625	98	296	391	198	468	50	292	87	212
30	631	82	338	657	179	497	57	318	175	192
(Coupon rates on bonds used for estimation .04, .06)										
5	400	5	30	-200	467	200	0	0	-201	420
10	549	7	129	-146	328	300	13	45	-167	338
15	595	6	112	-125	260	403	6	65	-138	262
20	620	4	97	-105	211	426	7	141	-114	222
25	625	0	67	-97	151	468	7	149	-94	188
30	631	-23	47	-83	94	497	0	127	-79	138

[a]Computed at midpoint of interest rate range $r = .14$. Errors reported in basis points.

[b]Standard deviation of single period interest rate being forecasted.

[c]Tax scenarios are described by their capital gains tax rates. If the investor is an individual, these depend on whether the short-term loss/long-term gain offset rule and the $3,000 deduction limit are binding. For banks and dealers these rules are not applicable. In each case the ordinary tax rate is $\tau_c = .5$.

Scenario I: Offset rule binding, deduction limit not binding, $\tau_s = \tau_L = .25$

II: Neither rule binding, $\tau_s = .5$, $\tau_L = .25$

III: Deduction limit binding, offset rule irrelevant, $\tau_s = \tau_L = 0$

IV: Bank or dealer at margin, $\tau_s = \tau_L = .5$

TABLE 8

Estimated Tax Brackets under the Buy-and-Hold Assumption with $\tau_L = .5 \tau_c$ [a]

Maturity	High-variance Process $\sigma_r = .02$ per year				Low-variance Process $\sigma_r = .01$ per year			
	Tax Scenario[b]							
	I	II	III	IV	I	II	III	IV
(Coupon rates on bonds used in estimation .08, .10)								
5	44%	17	52	20	49	45	63	18
10	36	17	28	23	47	11	53	27
15	35	0	14	25	44	7	42	31
20	35	-17	-6	26	42	-2	30	31
25	38	-26	-36	29	40	-15	16	31
30	44	-48	-108	27	38	-33	-4	30
(Coupon rates on bonds used in estimation .04, .06)								
5	49	44	63	18	50	50	63	26
10	48	20	55	27	47	39	59	28
15	47	20	56	29	48	30	56	32
20	47	18	54	30	47	8	54	32
25	48	24	56	39	47	5	52	33
30	55	27	55	50	49	11	51	43

[a]Computed at the midpoint of interest rate range, $r = .14$. True tax bracket in each case is $\tau_c = 50\%$.

[b]Tax scenarios are described by their capital gains tax rates. If the investor is an individual, these depend on whether the short-term loss/long-term gain offset rule and the $3,000 deduction limit are binding. For banks and dealers these rules are not applicable. In each case the ordinary tax rate is $\tau_c = .5$.

Scenario I: Offset rule binding, deduction limit not binding, $\tau_s = \tau_L = .25$

II: Neither rule binding, $\tau_s = .5$, $\tau_L = .25$

III: Deduction limit binding, offset rule irrelevant, $\tau_s = \tau_L = 0$

IV: Bank or dealer at margin, $\tau_s = \tau_L = .5$

TABLE 9

Value of the Timing Option on Municipal Bonds as Reflected
by the Percentage Difference between the Prices
under the Optimal and Buy-and-Hold Policies[a]

Maturity	High-variance Process $\sigma_r = .02$ per year			Low-variance Process $\sigma_r = .01$ per year		
	Tax Scenario[b]					
	I	II	IV	I	II	IV
	Coupon rate $c = .03$					
5	0.0%	0.1	0.1	0.0	0.0	0.0
10	0.5	2.6	1.6	0.1	0.3	0.4
15	1.2	6.6	3.4	0.4	1.0	1.3
20	1.9	10.5	5.1	0.7	1.9	2.4
25	2.5	13.6	6.3	1.0	3.0	3.4
30	3.0	15.9	7.1	1.2	4.1	4.2
	Coupon rate $c = .05$					
5	0.1	0.8	0.4	0.0	0.1	0.1
10	0.7	3.8	2.2	0.2	1.2	0.9
15	1.5	7.2	4.2	0.6	2.9	2.0
20	2.2	10.0	5.8	1.0	4.5	3.1
25	2.8	12.2	6.1	1.3	5.8	4.0
30	3.2	13.8	7.7	1.5	6.8	4.7
	Coupon rate $c = .07$					
5	0.2	1.0	0.8	0.1	0.5	0.4
10	0.9	3.3	2.8	0.5	1.7	1.5
15	1.8	5.7	4.7	0.9	2.9	2.7
20	2.4	7.9	6.1	1.2	3.9	3.7
25	3.0	9.6	7.0	1.5	4.8	4.5
30	3.4	11.1	7.6	1.7	5.4	5.0
	Coupon Rate $c = .09$					
5	0.6	1.2	0.5	0.3	0.6	0.2
10	1.4	3.1	2.0	0.7	1.5	0.8
15	2.1	4.9	3.4	1.0	2.2	1.6
20	2.6	6.6	4.5	1.3	2.7	2.3
25	2.9	8.1	5.3	1.4	3.2	2.9
30	3.2	9.4	5.9	1.5	3.7	3.4

[a]Computed at the midpoint of the interest rate range $r = .14.$ [b]Tax
scenarios are described by their capital gains tax rates. If the investor is
an individual, these depend on whether the short-term loss/long-term gain
offset rule and the $3,000 deduction limit are binding. For banks and dealers
these rules are not applicable. In each case the ordinary tax rate is
$\tau_c = .05.$

Scenario I: Offset rule binding, deduction limit not binding, $\tau_s = \tau_L = .25$

II: Neither rule binding, $\tau_s = .5, \tau_L .25$

III: Deduction limit binding, offset rule irrelevant, $\tau_s = \tau_L = 0$

IV: Bank or dealer at margin, $\tau_s = \tau_L = .5$

Timing option is always zero under tax scenario III.

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