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INVESTMENT, CAPACITY, AND UNCERTAINTY: A PUTTY-CLAY APPROACH

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Investment, Capacity, and Uncertainty: A Putty-Clay Approach
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ABSTRACT

We embed the microeconomic decisions associated with investment under uncertainty, capacity utilization, and machine replacement in a general equilibrium model based on putty-clay technology. In the presence of irreversible factor proportions, a mean-preserving spread in the productivity of investment raises aggregate investment, productivity, and output. Increases in uncertainty have important dynamic implications, causing sustained increases in investment and hours and a medium-term expansion in the growth rate of labor productivity.

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1 Introduction

In real business cycle models that emphasize the role of embodied technological change, technological booms are driven by increases in the mean level of productivity of investment projects. In this paper, we investigate the macroeconomic consequences of changes in the variance of productivity of investment projects. In recent papers, Campbell, Lettau, Malkiel and Xu (2001) and Goyal and Santa-Clara (2003) demonstrate that the idiosyncratic component of stock returns, a measure of uncertainty in the return on investment, is subject to large and highly persistent shifts over time. However, there has been little study of the effects of such shifts on the aggregate economy. We find that in the putty-clay model permanent shifts in the distribution of returns have first-order effects on aggregate investment, hours, and productivity. The dynamics resulting from such shifts differ sharply from those obtained with putty-putty capital or from shocks to the mean level of technology.

In an environment where capital is vintage specific and hence irreversible, increases in idiosyncratic uncertainty provide productivity benefits through the optimal allocation of variable factors such as labor across project outcomes that are embodied in fixed factors, such as capital. The size of the productivity gains and the transition dynamics associated with such productivity gains depend on the underlying production structure. If production is putty-putty, as in the benchmark vintage capital model introduced by Solow (1960), an increase in the variance of project returns has macroeconomic consequences that are isomorphic to an increase in the mean level of project returns. The productivity gains associated with the increase in variance occur quickly, and the subsequent boom in output, hours, and investment is relatively short-lived.

With putty-clay capital the dynamic response to a change in the variance of project outcomes differs significantly from that of a change to the mean level of technology. An increase in the variance of project returns extends the economic life of existing capital relative to new capital and produces a substantial delay in the arrival of productivity gains. A permanent increase in the variance of project returns generates a hump-shaped response of the *growth rate* of labor productivity with the peak response occurring ten to fifteen years later; in contrast, the peak response to a permanent increase in the level of technology occurs on impact. Overall, these results suggest a new mechanism whereby variation in the second moment of project returns can have first-order effects on aggregate quantities.

The Cobb-Douglas production technology has long been a popular choice in macro-economic research owing to its analytical convenience and empirically supported long-run balanced-growth properties. However, the malleability of capital inherent to the Cobb-Douglas framework severely restricts the analysis of issues related to irreversible investment, including investment under uncertainty, capacity utilization, and capital obsolescence and replacement. In particular, a key assumption needed to achieve aggregation with Cobb-Douglas technology and heterogeneous capital goods is that labor be equally flexible in the short and the long run. This assumption implies that, absent modifications such as costs of operating capital, all capital goods are used in production and that the short-run elasticity of output with respect to labor equals the long-run labor share of income.

Putty-clay technology, originally introduced by Johansen (1959), provides an alternative description of production and capital accumulation that breaks the tight restrictions on short-run production possibilities imposed by Cobb-Douglas technology and provides a natural framework for examining issues related to irreversible investment. With putty-clay capital, the ex ante production technology allows substitution between capital and labor, but once the capital good is installed, the technology is Leontief, with productivity determined by the embodied level of vintage technology and the ex post fixed choice of capital intensity.[2] An impediment to the adoption of the putty-clay framework has been the analytical difficulty associated with a model in which one must keep track of all existing vintages of capital. While recent research has made significant progress in incorporating aspects of putty- clay technology while preserving analytical tractability, these efforts have not provided a full treatment of issues related to irreversible investment and capacity utilization; see, for example, Cooley, Hansen and Prescott (1995), and Atkeson and Kehoe (1999).

In Gilchrist and Williams (2000), we develop a general equilibrium model

[2]Putty-clay capital was studied intensely during the 1960s, but received relatively little attention again until the 1990s. See Gilchrist and Williams (2000) for further references.

with putty-clay technology in which aggregate relationships are explicitly derived from the microeconomic decisions of investment, capacity utilization, and machine replacement. In that paper, we investigated the effects of shifts in disembodied and embodied technology on the economy. This model has already proven to be useful in a number of other applications (Gilchrist and Williams (2001), Wei (2003)). In this paper, we extend this model to incorporate time variation in the variability of project returns. We then examine the steady-state and dynamic relationships between uncertainty, productivity and investment.

Our analysis contributes to the large theoretical literature that identifies channels through which uncertainty may influence investment. Bernanke (1983) and Pindyck (1991) stress the negative influence that uncertainty has in a model where there exists an "option value" to waiting to invest. Because the uncertainty considered in this paper is resolved only after investment decisions are made, our work is more directly related to the work of Hartman (1972) and Abel (1983). These authors emphasize the positive effect that increased uncertainty may have on firm-level investment because expected profits increase with uncertainty. In our framework, increased uncertainty raises expected profits but reduces the expected marginal return to capital, causing a reduction in capital intensity at the microeconomic level. As a result, an increase in idiosyncratic uncertainty reduces investment at the project level but raises investment in the aggregate. The former result is broadly consistent with the empirical evidence of a negative relationship between investment and uncertainty at the firm level (Leahy and Whited 1996).

The remainder of this paper is divided as follows: Section 2 describes the model and equilibrium conditions. Section 3 shows the equilibrium determination of utilization rates and provides closed-form expressions for aggregate economic variables as functions of the equilibrium utilization rate. Section 4 considers the general-equilibrium implications of increased uncertainty on investment, output and productivity. Section 5 concludes.

2 The Model

In this section, we describe the putty-clay model and derive the equilibrium conditions. The underlying ex ante production technology is assumed to be Cobb-Douglas with constant returns to scale, but for capital goods in place, production possibilities take the Leontief form: there is no ex post substitutability of capital and labor at the microeconomic level. In addition to aggregate technological change, we allow for the existence of idiosyncratic uncertainty regarding the productivity of investment projects, the variance of which may change over time. To characterize the equilibrium allocation, we first discuss the optimization problem at the project level and then describe aggregation from the project level to the aggregate allocation.[3]

2.1 The Investment Decision

Each period, a set of new investment "projects" becomes available. Constant returns to scale implies an indeterminacy of scale at the level of projects, so without loss of generality, we normalize all projects to employ one unit of labor at full capacity. We refer to these projects as "machines." Capital goods require one period for initial installation and then are productive for $1 \leq M \leq \infty$ periods. The productive efficiency of machine i initiated at time t is affected by a random idiosyncratic productivity term. In addition, we assume all machines, regardless of their relative efficiency, fail at an exogenously given rate that varies with the age of the machine. In summary, capital goods are heterogeneous and are characterized by three attributes: vintage (age and level of aggregate embodied technology), capital intensity, and the realized value of the idiosyncratic productivity term.

The productivity of each machine, initiated at time t, differs according to the log-normally distributed random variable, $\theta(i)_t$, where

$$\ln \theta(i)_t \sim N(\ln \theta_t - \frac{1}{2}\sigma_t^2, \sigma_t^2).$$

The aggregate index θ_t measures the mean level of embodied technology of vintage t investment goods, and σ_t^2 is the variance of the idiosyncratic shock

[3]This section extends the analysis in Gilchrist and Williams (2000) by incorporating time-variation in the degree of idiosyncratic uncertainty.

for capital goods installed of that vintage.[4] The mean correction term $-\frac{1}{2}\sigma_t^2$ implies that $E(\theta(i)_t|\theta_t) = \theta_t$. We assume θ_t follows a nonstochastic trend growth process with gross growth rate $(1+g)^{1-\alpha}$.

The idiosyncratic shock to individual machines is not observed until after the investment decisions are made. We also assume that after the revelation of the idiosyncratic shock, further investments in existing machines are not possible. Subject to the constraint that labor employed, $L(i)_{t+j}$, is nonnegative and less than or equal to unity (capacity), final goods output produced in period $t+j$ by machine i of vintage t is

$$Y(i)_{t+j} = \theta(i)_t k(i)_t^\alpha L(i)_{t+j},$$

where $k(i)_t$ is the capital-labor ratio chosen at the time of installation. Denote the labor productivity of a machine by

$$X(i)_t \equiv \theta(i)_t k(i)_t^\alpha.$$

The only variable cost to operating a machine is the wage rate, W_t. Idle machines incur no variable costs and have the same capital costs as operating machines. Given the Leontief structure of production, these assumptions imply a cutoff value for the minimum efficiency level of machines used in production: those with productivity $X(i)_t \geq W_t$ are run at capacity, while those less productive are left idle. Owing to trend productivity growth and relatively long-lived capital, the mean labor productivity of the most recent vintage is substantially higher than that of all other existing machines. Obsolescence through embodied technical change implies that older vintages have lower average utilization rates than newer vintages.

To derive the equilibrium allocation of labor, capital intensity, and investment, we begin by analyzing the investment and utilization decision for a single machine. Define the time t discount rate for time $t+j$ income by $\tilde{R}_{t,t+j} \equiv \prod_{s=1}^{j} R_{t+s}^{-1}$, where R_{t+s} is the one period gross interest rate at time

[4] The assumption of log-normally distributed idiosyncratic productivity, as in Campbell (1998), facilitates the analysis of aggregate relationships while preserving the putty-clay characteristics of the microeconomic structure. In particular, there exists a well-defined aggregate production function with a short-run elasticity of output with respect to labor strictly less than that of the Cobb-Douglas alternative. This result is in contrast to that of Houthakker (1953), who finds that a Leontief microeconomic structure aggregates to a Cobb-Douglas production function if the distribution of idiosyncratic uncertainty is Pareto.

$t+s$. At the machine level, capital intensity is chosen to maximize the present discounted value of profits to the machine

$$\max_{k(i)_t,\{L(i)_{t+j}\}_{j=1}^{M}} E\Big\{-k(i)_t + \sum_{j=1}^{M} \tilde{R}_{t,t+j}(1-\delta_j)(X(i)_t - W_{t+j})L(i)_{t+j}\Big\}, \quad (1)$$

$$\text{s.t.} \quad X(i)_t = \theta(i)_t k(i)_t^{\alpha},$$
$$0 \leq L(i)_{t+j} \leq 1, \;\; j = 1, \ldots, M,$$
$$0 < k(i)_t < \infty,$$

where δ_j is the probability a machine has failed exogenously by j periods and expectations are taken over the time t idiosyncratic shock, $\theta(i)_t$.

Because investment projects are identical ex ante, the optimal choice of the capital-labor ratio is equal across all machines in a vintage; that is, $k(i)_t = k_t, \forall i$. Denote the mean productivity of vintage t capital by $X_t = \theta_t k_t^{\alpha}$. Given the log-normal distribution for $\theta(i)_t$, the expected labor requirement at time t for a machine built in period s is given by

$$\Pr(X(i)_s > W_t | W_t, \theta_t) = 1 - \Phi(z_t^s),$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal and

$$z_t^s \equiv \frac{1}{\sigma_s}\Big(\ln W_t - \ln X_s + \frac{1}{2}\sigma_s^2\Big).$$

Letting $F(X(i)_s)$ denote the cumulative distribution function of $X(i)_s$, we can similarly compute expected output to be

$$\int_{W_t}^{\infty} X(i)_s dF(X(i)_s) = (1 - \Phi(z_t^s - \sigma_s))X_s,$$

where the expression on the right-hand side follows from the formula for the expectation of a truncated log-normal random variable.[5] Capacity utilization of vintage s capital at time t—the ratio of actual output produced from the capital of a given vintage to the level of output that could be produced when capital is fully utilized—equals $(1 - \Phi(z_t^s - \sigma_s))$.

[5] If $\ln(\mu) \sim N(\zeta, \sigma^2)$, then $E(\mu|\mu > \chi) = \frac{(1-\Phi(\gamma-\sigma))}{(1-\Phi(\gamma))}E(\mu)$ where $\gamma = (\ln(\chi) - \zeta)/\sigma$ (Johnson, Kotz and Balakrishnan 1994). This implies that $\int_{\chi}^{\infty} \mu dF(\mu) = E(\mu|\mu > \chi)\Pr(\mu > \chi) = (1-\Phi(\gamma-\sigma))E(\mu)$, where $F()$ denotes the cumulative distribution function of μ.

Expected net income in period t from a vintage s machine, π_t^s, conditional on W_t, is given by

$$\pi_t^s = (1-\delta_{t-s})\Big((1-\Phi(z_t^s-\sigma_s))X_s - (1-\Phi(z_t^s))W_t\Big).$$

Substituting this expression for net income into equation 1 eliminates the future choices of labor from the investment problem. The remaining choice variable is k_t.

The choice of k_t has a direct effect on profitability through its effect on the expected value of output X_t.[6] The first-order condition for an interior solution for k_t is given by

$$k_t = \alpha E_t \sum_{j=1}^{M} \tilde{R}_{t,t+j}(1-\delta_j)\left(1-\Phi(z_{t+j}^t-\sigma_t)\right)X_t. \quad (2)$$

New machines are put into place until the value of a new machine (the present discounted value of net income) is equal to the cost of a machine, k_t,

$$k_t = E_t \sum_{j=1}^{M} \tilde{R}_{t,t+j}(1-\delta_j)\left((1-\Phi(z_{t+j}^t-\sigma_t))X_t - (1-\Phi(z_{t+j}^t))W_{t+j}\right). \quad (3)$$

This is the free-entry or zero-profit condition. The first term on the right-hand side of equation 3 equals the expected present discounted value of output adjusted for the probability that the machine's idiosyncratic productivity draw is too low to profitably operate the machine in period $t+j$. The second term equals the expected present value of the wage bill, adjusted for the probability of such a shutdown. Equations 2 and 3 jointly imply that, in equilibrium, the expected present value of the wage bill equals $(1-\alpha)$ times the expected present value of revenue.

[6] For any given realization of θ_{it}, a higher choice of k_t raises the probability that a machine will be utilized in the future. This increase in utilization raises both expected future output and expected future wage payments. Because the marginal machine earns zero quasi rents, this effect has no marginal effect on profitability, however; that is,

$$\frac{\partial \pi_t^s}{\partial z_t^s} = \frac{1}{\sigma_s}\phi(z_t^s-\sigma_s)X_s - \frac{1}{\sigma_s}\phi(z_t^s)W_t = 0,$$

where $\phi(\cdot)$ denotes the probability density function for a standard normal random variable.

2.2 Aggregation

Total labor employment, L_t, is equal to the sum of employment from all existing vintages of capital

$$L_t = \sum_{j=1}^{M}(1 - \Phi(z_t^{t-j}))(1 - \delta_j)Q_{t-j}, \tag{4}$$

where Q_{t-j} is the quantity of new machines started in period $t-j$. Aggregate final output, Y_t, is

$$Y_t = \sum_{j=1}^{M}(1 - \Phi(z_t^{t-j} - \sigma_{t-j}))(1 - \delta_j)Q_{t-j}X_{t-j}. \tag{5}$$

In the absence of government spending or other uses of output, aggregate consumption, C_t, satisfies

$$C_t = Y_t - k_t Q_t, \tag{6}$$

where $k_t Q_t$ is gross investment in new machines.

2.3 Preferences

To close the model, we posit that the representative household maximizes the present value of household utility, where preferences are described by

$$\frac{1}{1-\gamma} E_t \sum_{t=0}^{\infty} \beta^t N_t \Big(\frac{C_t (N_t - L_t)^{\psi}}{N_t} \Big)^{1-\gamma}, \tag{7}$$

where $\beta \in (0,1), \psi > 0$, $\gamma^{-1} > 0$ is the intertemporal elasticity of substitution, and $N_t = N_0(1+n)^t$ is the size of the household, the growth rate of which is assumed to be exogenous and constant.[7] Households optimize over these preferences subject to the standard intertemporal budget constraint. We assume that claims on the profits streams of individual machines are traded; in equilibrium, households own a diversified portfolio of all such claims.

The first-order condition with respect to consumption is given by

$$U_{c,t} = \beta E_t R_{t,t+1} U_{c,t+1}, \tag{8}$$

[7]Note that we assume that per capita utility is weighted by the size of the household. The following analysis would be nearly unchanged if instead we assumed that preferences were not weighted by the size of the household.

where $U_{c,t+s}$ denotes the marginal utility of consumption. The first-order condition with respect to leisure/work is given by

$$U_{c,t}W_t + U_{L,t} = 0, \tag{9}$$

where $U_{L,t}$ denotes the marginal utility associated with an incremental increase in work (decrease in leisure).

3 The Steady State

Our goal in this paper is to analyze the effect of changes in project uncertainty on output, productivity, and capacity both in the long-run and along the transition path. To determine the long-run effect of an increase in uncertainty, we first characterize the steady-state conditions of the economy and show how the steady-state conditions depend on the level of idiosyncratic uncertainty. In analyzing the effects of changes in uncertainty, we first consider the firm's employment and output decisions in partial equilibrium and then proceed to consider the investment and capacity decisions in a general equilibrium analysis.

We begin with the case of no technological or population growth. This case is analytically tractable and allows us to prove the uniqueness of the steady-state equilibrium utilization rate and to derive closed-form solutions for the steady-state values of all variables as functions of this utilization rate. The case of positive growth is taken up in later in the section. For the positive growth case, we provide a set of sufficient conditions for there to exist a unique non-stochastic balanced growth path.

3.1 The Zero-Growth Economy

The zero-growth economy is obtained by setting $g = n = 0$. For further simplicity, we assume $M = \infty$ and $\delta_j = 1 - (1-\delta)^{j-1}$ for some depreciation rate $\delta > 0$. Letting lower case letters denote steady-state per capita quantities and suppressing time subscripts, we define $z = (1/\sigma)[\ln(w) - \ln(\theta k^{\alpha}) + \frac{1}{2}\sigma^2]$. Equation 4 then implies that steady-state labor equals the steady-state capital utilization rate times the total stock of machines, q/δ ,

$$l = (1 - \Phi(z))(q/\delta), \tag{10}$$

while equation 5 implies that steady-state output is equal to the steady-state capacity utilization rate times potential output

$$y = (1 - \Phi(z - \sigma))\theta k^{\alpha}(q/\delta). \tag{11}$$

For a given capital-labor ratio and stock of machines, labor and output are increasing in the rate of utilization.

Equations 10 and 11 provide an implicit relationship between labor and output that may be interpreted as the short-run production function for this economy (holding k and q fixed). Taking the ratio of partial derivatives, $\frac{\partial y}{\partial z} / \frac{\partial l}{\partial z}$, we obtain an expression for the marginal product of labor (MPL),

$$MPL = \frac{\phi(z - \sigma)}{\phi(z)}\theta k^{\alpha}. \tag{12}$$

Taking second derivatives, we obtain $\frac{\partial MPL}{\partial l} = -\sigma MPL$, which implies strict concavity of the short-run production function.

Combining equations 10 , 11, and 12, the elasticity of output with respect to labor may be expressed as

$$\frac{\partial \ln y}{\partial \ln l} = \frac{h(z - \sigma)}{h(z)} \tag{13}$$

where $h(x) \equiv \phi(x)/(1 - \Phi(x))$, the hazard rate for the standard normal. This ratio plays a key role in determining the equilibrium rate of capital utilization.

Combining equations 2, 10, and 11 and solving for the capital-labor ratio per machine yields

$$k = \left(\frac{\alpha}{r + \delta}(1 - \Phi(z - \sigma))\theta\right)^{\frac{1}{1-\alpha}}, \tag{14}$$

where $r = \beta^{-1} - 1$ is the steady-state equilibrium real interest rate. Except for the adjustment for capacity utilization $(1 - \Phi(z - \sigma))$, this is the standard expression for the steady-state capital-labor ratio in a zero-growth economy. The adjustment factor implies that the optimal capital-labor ratio for new machines is increasing in the capacity utilization rate.

In equilibrium, the wage rate equals the marginal product of labor or, equivalently, the efficiency level of the marginal machine. The first-order condition for the labor-leisure decision, equation 9, and the aggregate resource constraint, equation 6, yield

$$w = \psi \frac{c}{1-l}, \tag{15}$$

and

$$c = y - kq. \tag{16}$$

To close the model and solve for the equilibrium rate of capital utilization, we first express the zero-profit condition as a monotonic function of z. In steady state, per-period profits (net of capital expenditures) are given by

$$\Pi = (1 - \Phi(z - \sigma))\theta k^{\alpha} - (1 - \Phi(z))w - (r + \delta)k, \tag{17}$$

where $(r + \delta)k$ equals per-period capital expenditures. Using equation 14, we may alternatively express per-period capital expenditures by $\alpha(1 - \Phi(z - \sigma))\theta k^{\alpha}$. Net profits may then be written

$$\Pi = (1 - \Phi(z))[(1 - \alpha)\frac{y}{l} - w].$$

The free-entry condition requires that expected net profits equal zero, so that, in equilibrium, labor's share of output equals the wage bill: $(1-\alpha)y = wl$, just as in the neoclassical vintage model with Cobb-Douglas production. In the vintage model, this equality is achieved by allocating more labor to high-efficiency machines and less to low-efficiency machines so that the marginal product of labor is equal across machines. Each factor (labor and capital) is paid its share of output so that net profits are zero. In the putty-clay model, marginal products are not equalized across individual machines. Instead, a worker employed on a highly efficient machine is more productive than one employed on a low efficiency machine. Free entry of new machines then determines the utilization rate consistent with zero equilibrium net profits.

To see the link between free entry and utilization, we use the equilibrium condition that the wage rate equals the productivity of the marginal machine to obtain $\frac{\partial \ln y}{\partial \ln l} = \frac{wl}{y}$. Equation 13 combined with the free-entry condition then determines the steady-state value of z and thereby the steady-state capital utilization rate $(1 - \Phi(z))$. We state this result in the following proposition (proofs of all propositions appear in the appendix).

Proposition 1 *For the zero-growth economy, there exists a unique equilibrium value of z that satisfies:*

$$1-\alpha=\frac{h(z-\sigma)}{h(z)}, \tag{18}$$

where $h(x)=\phi(x)/(1-\Phi(x))$ is the hazard rate for the standard normal.

To complete the description of the model, we combine equations 11, 14, 15, and 16 and use the free entry condition to solve for steady-state labor

$$l=\frac{(1-\alpha)}{1-\alpha+\psi(1-\alpha\delta/(r+\delta))}. \tag{19}$$

Note that steady-state labor is independent of z and σ, the degree of idiosyncratic uncertainty. As in the case of the standard Solow vintage model, a mean-preserving spread to idiosyncratic productivity acts like an aggregate disembodied productivity shock with respect to the labor allocation decision and thus has no effect on steady-state labor. (The Solow vintage capital model is described in the appendix.) Equilibrium values for all remaining aggregate variables are then computed from z and l.

An implication of Proposition 1 is that the steady-state capacity utilization rate is decreasing in σ.[8] The degree of idiosyncratic uncertainty and the resulting distribution of unused capacity determine the short-run response of the economy to shocks. In the very short run, during which the distribution of machines is fixed, an expansion of output is achieved through the utilization of marginal machines, and the sensitivity of the marginal product of labor to increases in output depends on the density of machines available on the margin. Letting $(1/w)$ denote the inverse of the real wage, we can obtain the short-run elasticity of supply by taking logs and then, using the definition of z, differentiating equation 11 with respect to w to obtain:

$$-\frac{\partial \ln y}{\partial \ln(1/w)}=\frac{h(z-\sigma)}{\sigma}.$$

The hazard rate $h(z-\sigma)$ measures machine efficiency for machines in use, relative to overall machine efficiency. Because the degree of idiosyncratic uncertainty determines the equilibrium utilization rate, it also influences the

[8]See the proof of Proposition 4.

slope of the aggregate supply curve. For a low level of σ, the equilibrium capacity utilization rate is high, implying that the marginal machine is in the upper tail of the productivity distribution. In this region of the distribution, machine quality falls off rapidly and marginal cost rises sharply as output expands. Thus, a low degree of idiosyncratic uncertainty implies a low elasticity of supply and a steep supply curve.

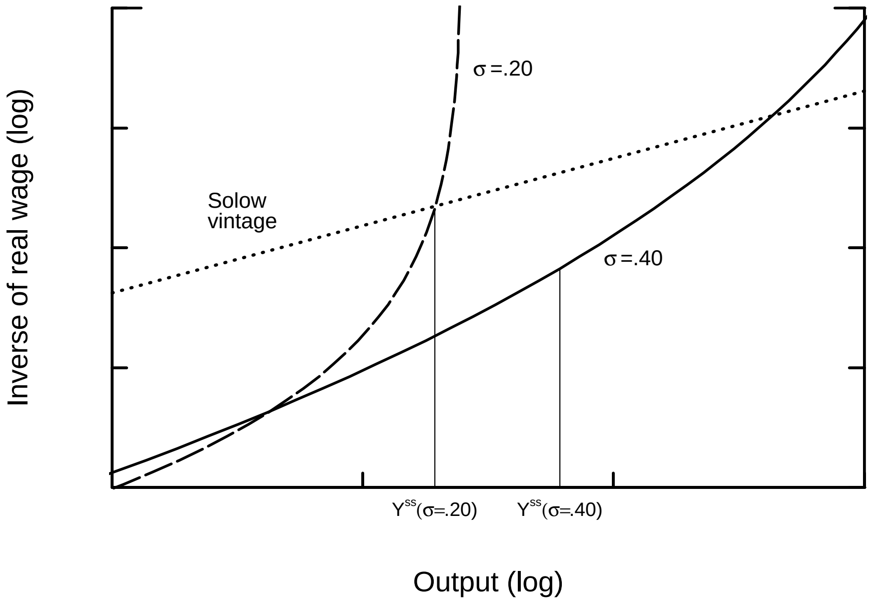
Figure 1 plots the log of the inverse of the wage against the log of output for the Solow vintage model and two choices of σ for the putty-clay model. In each case, the stock of capital goods is held fixed at its steady-state level, and each curve is computed by varying the amount of labor input. In log-terms, the slope of the supply curve is equal to the inverse of the elasticity of supply. In the Solow vintage model, the elasticity of supply is constant and equal to $\frac{1-\alpha}{\alpha}$, and therefore the supply curve is log-linear. In the putty-clay model, the elasticity of supply, $\frac{h(z-\sigma)}{\sigma}$, is decreasing in utilization, implying that the slope of the supply curve increases as output increases. For any given σ, the slope of the supply curve is increasing at an increasing rate as lower and lower quality machines are brought on line.

At the steady-state equilibrium, the slope of the short-run aggregate supply curve is negatively related to the degree of idiosyncratic uncertainty and is always greater than the slope implied by the Solow vintage model. These results are summarized in the following proposition.

Proposition 2 *For the zero-growth putty-clay economy, the slope of the aggregate supply curve holding capital fixed,* $\frac{\partial \ln(1/w)}{\partial \ln y} = \frac{\sigma}{h(z-\sigma)}$*, is increasing and convex in* $\ln y$*. Evaluated at the steady-state equilibrium, the slope of the aggregate supply curve is decreasing in the level of idiosyncratic uncertainty and is bounded below by* $\frac{\alpha}{1-\alpha}$.

Proposition 2 implies that the short-run response of the economy to shocks to technology depends on σ, the degree of idiosyncratic uncertainty. For low levels of σ, production is relatively inflexible in the short-run and marginal costs rise rapidly as output expands. Owing to such sharply rising marginal costs, an increase in θ, the mean level of technology, leads to only a modest increase in output, hours, and investment in the short-run. Over time, as more investment occurs, capacity expands and the economy can increase output

Figure 1: Aggregate Supply Curve



Notes: The solid line shows the short-run relationship between the inverse of the real wage and output for the putty-clay model with $\sigma = 0.4$. The dashed line shows the same for the economy with $\sigma = 0.2$. The dotted line shows this relationship for the Solow vintage model. These calculations are based on the steady-state values of k, q, and θ.

at lower cost. Transition dynamics are more prolonged for low σ economies than high σ economies. As documented in Gilchrist and Williams (2000), for sufficiently low levels of σ, the model is capable of generating hump-shaped output dynamics in response to persistent shocks to technology.[9]

3.2 The Balanced Growth Economy

Generalizing the results obtained above for the case of positive growth is complicated by the fact that the economy can no longer be summarized by a single vintage of capital and associated hazard rate, but instead depends on a set of hazard rates that vary across vintages of capital. Nonetheless, we are able to obtain conditions under which a unique balanced growth path exists. We also obtain similar results regarding how the shape of the short-run supply curve

[9] Campbell and Fisher (2000) provide an alternative analysis of the relationship between idiosyncratic uncertainty and aggregate dynamics.

depends on σ, the degree of idiosyncratic uncertainty.

Along the balanced growth path, per capita output, consumption, and investment grow at rate g, and labor and labor capacity grow at rate n. We use lower case letters to indicate steady-state values of variables, normalized by appropriate time trends, and $\tilde{k}$ to indicate the normalized steady-state capital-labor ratio. We define the growth-adjusted discount rate $\tilde{\beta} \equiv \beta(1+g)^{-\gamma}$. Let z denote the difference between the average efficiency of the leading edge technology and the current wage rate in steady state, $z \equiv (\ln w - \ln x + \frac{1}{2}\sigma^2)/\sigma$, and let $z(i)$ denote the difference between the average efficiency of vintage i and the current wage rate $z(i) \equiv z + (i/\sigma)\ln(1+g)$.

On the balanced growth path, the normalized levels of output, consumption, labor, and the wage rate are given by

$$y = qx \sum_{j=1}^{M} ((1+g)(1+n))^{-j}(1-\delta_j)\Big(1-\Phi(z(j)-\sigma)\Big), \tag{20}$$

$$c = y - \tilde{k}q, \tag{21}$$

$$l = q\sum_{j=1}^{M}(1+n)^{-j}(1-\delta_j)\Big(1-\Phi(z(j))\Big). \tag{22}$$

$$w = \frac{(1+g)^{-j}x\phi(z(j)-\sigma)}{\phi(z(j))}, j = 1, \ldots, M. \tag{23}$$

Note that y/q, c/q, l/q, and w depend only on the values of $\tilde{k}$ (directly and indirectly through $x = \tilde{k}^{\alpha}$) and z. The first-order condition for $\tilde{k}$ and the zero-profit condition yield two equations in z and $\tilde{k}$

$$\tilde{k} = \alpha\sum_{j=1}^{M}\tilde{\beta}^j(1-\delta_j)\Big\{\Big(1-\Phi(z(j)-\sigma)\Big)x\Big\}, \tag{24}$$

$$\tilde{k} = \sum_{j=1}^{M}\tilde{\beta}^j(1-\delta_j)\Big\{\Big(1-\Phi(z(j)-\sigma)\Big)x - (1+g)^j\Big(1-\Phi(z(j))\Big)w\Big\}. \tag{25}$$

By combining these last three equations, we obtain the balanced growth equilibrium condition for z.

As in the zero-growth economy, an equilibrium value of z is determined by setting utilization rates so that a weighted average of vintage labor shares equals $1-\alpha$. In the case of positive growth, however, these weights are not fixed constants as in the zero-growth case, but instead depend on z. As a result, with positive growth one cannot rule out a priori the existence of multiple steady-state values of z without additional assumptions, as stated in the following proposition.

Proposition 3 *Let*

$$v(z(j)) = \frac{\tilde{\beta}^j(1-\delta_j)(1-\Phi(z(j)-\sigma))}{\sum_{i=1}^{M}\tilde{\beta}^i(1-\delta_i)(1-\Phi(z(i)-\sigma))}, \tag{26}$$

define a set of weights such that $\sum_{j=1}^{M} v(z(j)) = 1$, *then there exists at least one steady-state value of* z *that satisfies*

$$(1-\alpha) = \sum_{j=1}^{M} v(z(j))\frac{h(z(j)-\sigma)}{h(z(j))}. \tag{27}$$

A sufficient condition for uniqueness of the equilibrium is that the sum $\sum_{j=1}^{M}\tilde{\beta}^j(1-\delta_j)(1-\Phi(z(j)-\sigma))$ *be log-concave in* z.

Note that the possibility of multiple balanced growth equilibria exists only in the case of nonzero trend technological growth. This potential for multiple steady states distinguishes this model from its putty-putty counterpart. Nonetheless, numerical analysis of the model suggests that multiple equilibria occur only in "unusual" regions of the parameter space, for example, when the trend growth rate of technology is extremely large and the value of α lies in a limited range. Further discussion of these issues appears in the appendix. In the following analysis, the parameterized version of the model possesses a unique steady state.

4 Reallocation Benefits, Investment, and Uncertainty

In this section, we consider the general equilibrium effects of a permanent increase in uncertainty. We first analyze the effect of an increase in σ on the long-run properties of the model. We then characterize the transition dynamics as the economy moves from a less flexible, low idiosyncratic uncertainty economy to a more flexible, high idiosyncratic uncertainty economy. This exercise is of interest for theoretical reasons – in our model, a mean preserving spread to project outcomes has first-order effects on both transition dynamics and steady-state levels. It is also of interest for empirical reasons – recent evidence shows that the idiosyncratic variance associated with the return to capital at the firm level has doubled over the postwar period and explains a large fraction of the total volatility of stock market returns (Campbell et al. 2001).

4.1 The steady-state effect of an increase in uncertainty

We start by considering the effect of an increase in σ on the steady-state of the putty-clay model. To obtain analytical results we focus on the zero-growth economy. As in the Solow vintage model, an increase in idiosyncratic uncertainty increases productivity by allowing labor to be reallocated from low productivity to high productivity projects. In the putty-clay model, this reallocation is limited by the Leontief nature of production however.[10] A complete characterization of the effect of an increase in σ on the steady-state equilibrium is summarized in the following proposition.

Proposition 4 *For the zero-growth economy, $\frac{dz}{d\sigma} > 1$, the steady-state capital-labor ratio per machine, k, and capacity utilization are strictly decreasing in σ. Output, consumption, total investment, and the wage rate are increasing in σ, with elasticity $\frac{d\ln y}{d\ln\sigma} = \sigma h(z)$. By comparison, in the Solow model, $\frac{d\ln y}{d\ln\sigma} = \frac{1}{\alpha} > \sigma h(z)$.*

The long-run elasticity of output with respect to σ in the Solow model can be deduced directly from equation 29 by computing the equivalent variation in θ implied by variation in σ, and noting that the long-run elasticity of output with respect to θ is unity.

In the case of strictly positive growth, we still obtain the result that $\frac{dz}{d\sigma} > 1$ and $\frac{dk}{d\sigma} < 0$. With growth, changes in σ influence the effective depreciation rate. As a result, labor is not independent of σ and the aggregate effects are

[10] In the Solow vintage model, project-level capital expenditures are irreversibly tied to a specific realization of idiosyncratic productivity $\theta(i)_t$ but labor can be costlessly reallocated across projects after the realization occurs. A mean-preserving spread causes a reallocation of labor from low productivity to high productivity machines, equalizing the marginal product of labor across machines. This reallocation increases productivity in proportion to σ and raises the return to capital, causing investment and output to increase.

difficult to characterize analytically. Numerical calculations indicate that the results in proposition 4 generalize to the case of positive growth for relevant values of σ.

Proposition 4 implies that an increase in idiosyncratic uncertainty *reduces* investment at the project level but *increases* aggregate investment. Project managers pay k and in effect buy an option to produce in the future. The option is exercised (production occurs) if the ex post realization of revenues exceeds wage costs.[11] Ceteris paribus, an increase in uncertainty raises the value of the option and increases expected profits. Although expected profits per machine increase with σ, the partial equilibrium effect (holding wages fixed) on k is ambiguous and depends on the utilization rate.[12] In general equilibrium, higher profits induce new entry. New entry drives up the wage rate, thereby reducing utilization and eroding profits. The increase in the wage rate causes a reduction in k. The additional investment that occurs through the extensive margin more than offsets the reduction in investment that occurs through the intensive margin, and aggregate investment unambiguously rises in response to an increase in idiosyncratic uncertainty.

The productivity gains associated with an increase in σ depend on the extent to which the economy can reallocate labor from low productivity to high productivity projects. The elasticity of output with respect to σ captures the benefits from this reallocation of labor. In steady state, $\ln(X(i_t) = \ln(\theta(i)_t k^\alpha)$ is normally distributed with variance σ^2. The standard formula for a truncated normal implies

$$\frac{d\ln y}{d\ln\sigma} = \sigma h(z) = E(\ln(X(i)_t)|X(i)_t > W) - E\ln(X(i)_t). \quad (28)$$

[11]Pindyck (1988) also considers the option value associated with machine shutdown. In his framework, holding constant the option value associated with waiting to invest, an increase in uncertainty raises investment. This result contrasts with that described below.

[12]In the putty-clay model, the relationship between k and profits—given by equation 17—combines both the standard concave relationship owing to diminishing returns to capital and the effect of k on expected utilization rates. The expected marginal product of capital (i.e. the derivative of gross profits with respect to k) is given by $MPK = \alpha(1-\Phi(z-\sigma))k^{\alpha-1}$. The derivative of the marginal product of capital with respect to σ, holding wages fixed, equals $\frac{\partial \text{MPK}}{\partial\sigma} = \phi(z-\sigma)\alpha k^{\alpha-1}\frac{z}{\sigma}$. For $z < 0$, that is, at steady-state capital utilization rates exceeding 50%, a mean-preserving spread reduces capacity utilization and the optimal choice of k.

Thus the size of the reallocation benefits depends on the difference between the average efficiency of machines in use relative to the mean-efficiency of all machines.[13]

This result has important implications for the transition dynamics described below. A permanent increase in σ causes the real wage to rise. In the initial periods following the rise in σ, the real wage reflects the existing distribution of capital however and is low relative to the new steady-state. Low real wages imply only limited reallocation benefits in the short-run. Over time, as real wages rise, so do the reallocation benefits associated with the increase in σ. Thus, relative to an increase in the mean-level of productivity, increases in the variance of productivity are expected to have delayed effects on output, hours and investment.

4.2 The Dynamic Effects of Increases in Uncertainty

We now consider the transition dynamics associated with an increase in idiosyncratic uncertainty. Using numerical simulations of the log-linearized model, we examine the effect of a one-time permanent increase in idiosyncratic uncertainty for the growth economy described above.

We calibrate our model to match standard long-run properties of the post-war U.S. data, including average capacity utilization. We set $\alpha = 0.3, \beta = 0.98, \delta = 0.08, \gamma = 1$ and $\psi = 3$. In steady state, the capital utilization rate of capital goods that are i periods old equals $(1 - \Phi(z + (i/\sigma)\ln(1 + g)))$. From this formula we see that the two key determinants of the vintage utilization schedule are the long-run growth rate of embodied technology and the degree of idiosyncratic uncertainty.

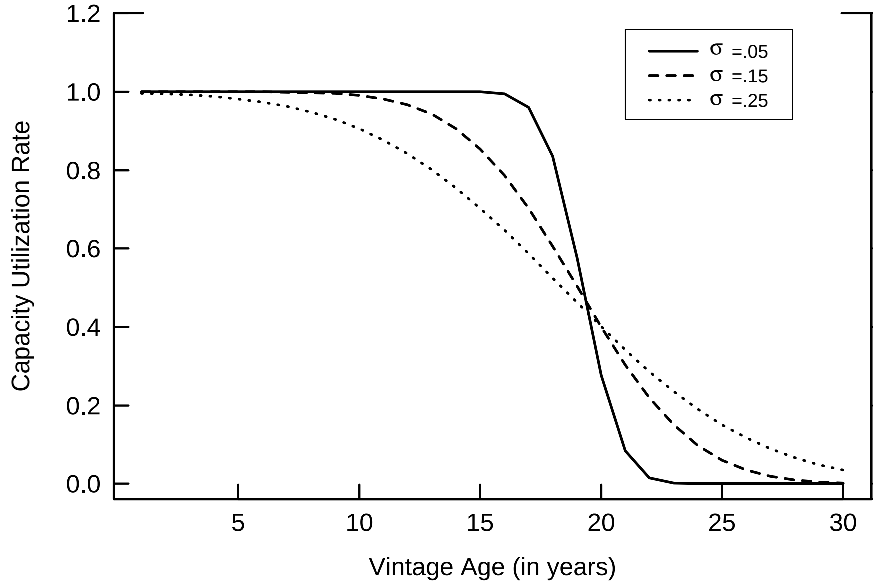
The primary effect of positive trend productivity growth on machine replacement is to shorten the useful life of capital goods. Owing to more rapid growth in real wages, an increase in g speeds up the process of machine replacement, shifting the utilization schedule forward in time. The degree of idiosyncratic uncertainty, on the other hand, mainly affects the shape of the

[13] For $x \sim N(\mu, \sigma_x)$,

$$E(x|x > \omega) = E(x) + \sigma_x h\left(\frac{\mu - \omega}{\sigma_x}\right)$$

where $h(z) = \phi(z)/(1 - \Phi(z))$.

Figure 2: Steady-State Capacity Utilization



Notes: The figure plots the steady-state capacity utilization for a given vintage rate as a function of the age of capital in the vintage for economies with different parameterizations of σ.

utilization schedule. In the case of low idiosyncratic uncertainty, the depreciation schedule is close to that of the "one-hoss shay" whereby machines of any given vintage have essentially the same economic lifespan. For high σ, the depreciation schedule resembles exponential decay.[14] Figure 2 shows the steady-state capital utilization rates for different values σ.[15] In order to match the 82 percent average capacity utilization rate of the manufacturing sector of the U.S. economy over the period 1960-2000, we choose $\sigma = 0.2$.

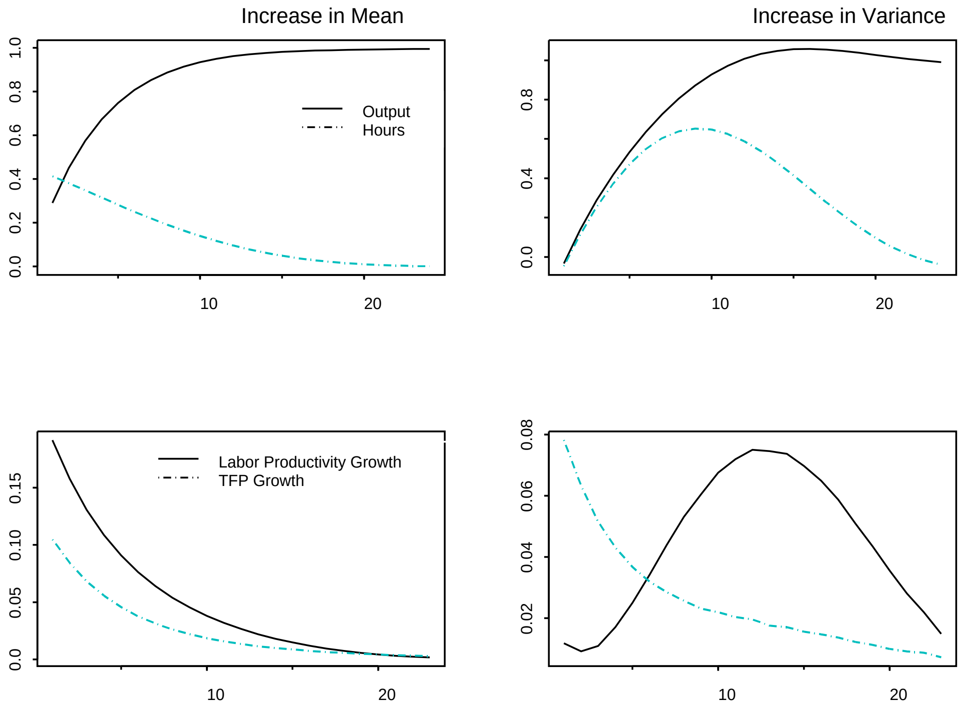
Figures 3 and 4 compare the dynamic responses from an increase in σ to that of an increase in the mean level of embodied technology, θ.[16] In each

[14]The pattern of scrapping relates to the potential presence of "replacement echoes" of the type studied by Boucekkine, Germain and Licandro (1997), where an initial investment surge leads to recurring spikes in investment as successive vintages are retired. In the context of our putty-clay model, pronounced replacement echoes occur only in the absence of mechanisms that lead to the smoothing of capital goods replacement over time. Specifically, necessary conditions for replacement echoes to exist in our putty-clay model are a low degree of idiosyncratic uncertainty and a high intertemporal elasticity of consumption.

[15]For the examples shown in figure 2, we assume there is no "exogenous" capital depreciation ($\delta = 0$).

[16]Total factor productivity (TFP) is constructed using the standard Solow formula:

Figure 3: Dynamic Responses of Hours, Output, and Productivity



Notes: The left-hand panels show the impulse responses to an unanticipated permanent one percent increase in the mean level of productivity, θ. The right-hand panels show the impulse responses to a permanent increase in the idiosyncratic variance of productivity, σ^2. The scale of the increase in σ is chosen so that the long-run increase in output is the same (one percent) as in the shock to the mean level of productivity. All results are shown as percentage point deviations from steady-state; time periods correspond to years.

case, we calibrate the size of the shock to produce a 1 percent increase in the steady-state level of output. In the case of a shock to σ, this corresponds to an increase in σ of about 0.04, which implies a 20 percent increase in the standard deviation of returns. Such a 20 percent increase in volatility is conservative relative to the range of low frequency movements in idiosyncratic volatility documented by Campbell et al. We focus on a permanent increase in σ based on the finding by Campbell et al. (2001) that the idiosyncratic component of stock returns exhibits near unit root behavior.

$\Delta \ln TFP_t = \Delta \ln(Y_t) - \alpha \Delta \ln(K_t) - (1-\alpha)\Delta \ln(N_t)$, where capital is measured using the perpetual inventory method $K_t = (1-\delta)K_{t-1} + I_t$.

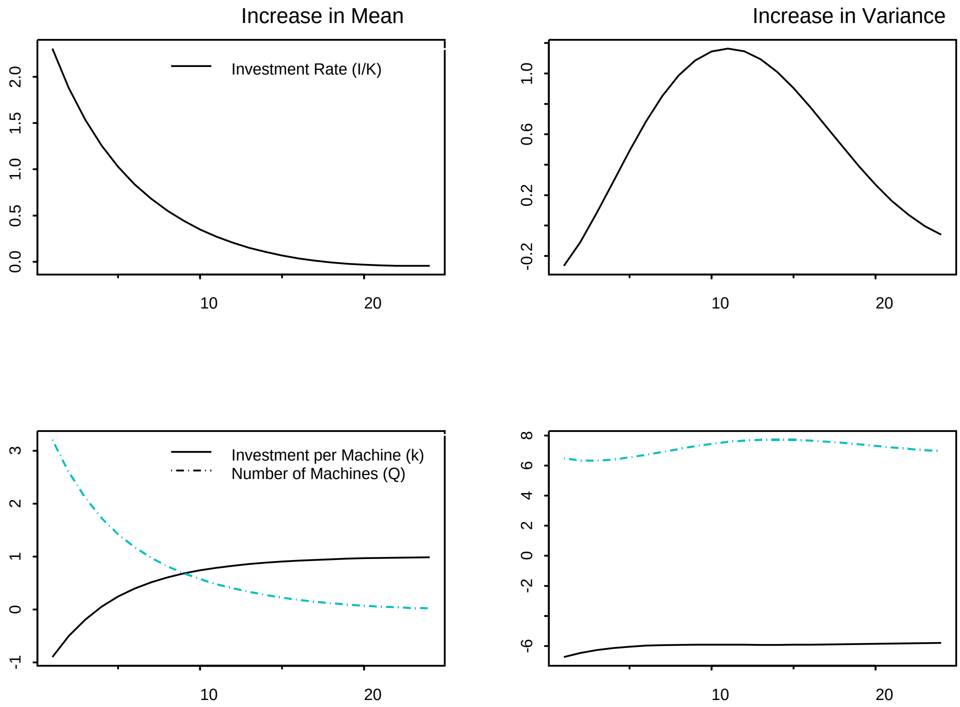
The dynamic response of the economy to an increase in the mean level of technology embodied in capital is discussed in Gilchrist and Williams (2000). An increase in θ represents a reduction in the cost, in terms of foregone consumption, of new capital goods. Owing to the associated increase in investment demand, investment spending, labor, and output all increase upon impact of the shock. The economy continues to display high levels of investment (relative to output) and high levels of employment for a number of years after the shock. Output rises, while the investment rate (I/K) and employment fall monotonically along the transition path. Productivity growth – both labor and total-factor productivity – is highest at the onset of the shock, when the newer, more productive capital is small relative to the existing stock of old capital. Consequently, the growth rates of both labor and total-factor productivity fall monotonically over time. In percentage terms, the productivity gains associated with the new investment fall as the existing capital stock embodies a larger fraction of new capital relative to old capital.

The transition dynamics associated with an increase in the idiosyncratic variance differ dramatically from those associated with an increase in the mean level of embodied technology. While the main transition dynamics of an increase in θ occur in the first five years after the shock, an increase in σ produces a transition dynamic whose peak effect occurs ten to fifteen years after the initial innovation. In the long run, an increase in variance raises output, labor productivity, and real wages, and causes an increase in aggregate investment. In the short run, an increase in σ has little effect on output, employment, or investment. Over time, output and employment rise, with employment peaking nearly ten years after the shock. The investment rate (I/K) peaks even later. Although total-factor productivity growth is highest at the onset of the shock, the growth rate of labor productivity rises for ten years. Overall, the increase in the variance of idiosyncratic returns produces a medium-run boom in labor hours, investment, and labor-productivity growth.[17]

An increase in the variance of project outcomes yields a substantial delay

[17] While Gilchrist and Williams (2000) document the fact that the putty-clay model with low sigma produces a hump-shaped response of labor hours to permanent increases in the mean level of embodied technology, neither the standard real business-cycle model nor the putty-clay model produces a hump-shaped response of labor productivity growth to permanent increases in either embodied or disembodied technology.

Figure 4: Investment Dynamics



Notes: See Figure 3.

in the investment response and productivity gains. This delay occurs for two reasons. First, the productivity gains associated with an increase in variance depend on the extent of reallocation. As discussed above, the extent of reallocation is tied to the real wage. Prior to the increase, the current real wage reflects the existing capital stock whose distribution is determined by the initially low level of σ. At the onset of the shock, new investment is small relative to existing capital, and the increase in variance has very little impact on the real wage. Because wages are low relative to the new steady state, utilization rates on new machines are relatively high, and there is very little reallocation of labor within the new vintages. Over time, as the existing capital stock reflects the higher level of σ, real wages rise and the economy experiences a higher rate of machine shutdown and reallocation. Because reallocation benefits are slow to arrive, there is little incentive to increase aggregate investment in the short run. The pace of investment reflects the benefits to reallocation. As a result, the economy displays strong co-movement among the investment-output ratio, labor hours, and labor productivity growth.

The second reason for the delay is that an increase in the variance of project outcomes for new machines extends the effective life of existing capital and slows down the rate at which new machines are introduced to the economy. An increase in variance implies a lower level of capital intensity for new machines relative to old machines. As figure 4 shows, capital intensity of new machines falls immediately to near its new long-run steady-state value. The drop in machine intensity is offset by a surge in investment at the extensive margin, resulting in a relatively stable investment-to-output ratio. However, existing machines are more capital intensive than is optimal relative to the new steady state. The relatively high capital intensity of old machines implies a higher mean efficiency level for old machines relative to new machines and a lower probability of shutdown for existing capital than would have occurred with no change in variance. As the scrappage rate of existing machines falls, the economy has less incentive to invest in new machines in the short-run. Over time, old machines are eventually scrapped, and investment picks up.

5 Conclusion

In this paper, we investigate the macroeconomic consequences of changes in idiosyncratic uncertainty of project returns in a putty-clay model of capital accumulation. The model that we develop provides a set of microeconomic foundations for the analysis of investment under uncertainty, capacity utilization, and machine retirement in a general equilibrium framework. Aggregation over heterogeneous capital goods results in a well-defined aggregate production function that preserves the putty-clay microeconomic structure. The aggregate production function takes an intermediate form between that of Cobb-Douglas and that of Leontief, depending on the degree of idiosyncratic uncertainty.

In this environment, an increase in idiosyncratic uncertainty reduces investment at the project level but raises aggregate labor productivity and investment. Relative to an increase in the mean level of technology, an increase in idiosyncratic variance also has important implications for transition dynamics. In the putty-clay model, an increase in variance results in a pronounced expansion in output, hours and investment, whose combined effect produces a sustained increase in trend labor productivity growth over a ten-to-fifteen year period.

The long-lasting expansion following an increase in variance is a result of two opposing forces – a decrease in the desired capital-to-labor ratio at the machine level and an increase in the desired number of machines in the aggregate. Because existing machines have high capital-to-labor ratios relative to new machines, the rate of economic depreciation on existing machines falls and the overall rate of investment is diminished relative to an expansion triggered by an increase in mean productivity. More generally, the putty-clay model implies that expansionary forces that reduce desired capital-to-labor ratios at the machine level have long lasting transition dynamics. In the putty-clay model, the desired capital-to-labor ratio at the machine level is a decreasing function of the rate of technological change. We therefore expect that transition dynamics to a new steady-state growth rate are much slower in the putty-clay model relative to the more standard putty-putty alternative. By the same logic, our results also suggest that permanent reductions in labor income taxes imply substantially slower dynamic adjustment in the putty-clay model. We leave a full exploration of these issues to future research.

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Appendix

Results regarding the hazard rate of the standard normal distribution:

In the following, let $h(x)$ denote the hazard rate for the standard normal distribution, $h(x) \equiv \phi(x)/(1-\Phi(x))$. From the definition of the hazard rate, we know $h(x) = E(y|y > x),\ y \sim N(0,1)$, which implies that $h(x) > 0$ and $h(x) > x$, for all x.

Result 1: *$h(x)$ is monotonically increasing in x, with $\lim_{x\to-\infty} h'(x) = 0$ and $\lim_{x\to+\infty} h'(x) = 1$.*
Proof: *Taking the derivative of $h(x)$, we have $h'(x) = h(x)(h(x) - x) > 0$, where the inequality follows directly from the definition of the hazard rate of the standard normal. To establish the lower limit of $h'(x)$, first note that $\lim_{x\to-\infty} h(x) = 0$. Then, $\lim_{x\to-\infty} h'(x) = -\lim_{x\to-\infty} xh(x) = -\lim_{x\to-\infty} x\phi(x) = 0$, where the final equality results from applying l'Hopital's rule. To establish the upper limit, note that application of l'Hopital's rule yields $\lim_{x\to+\infty} h'(x) = \lim_{x\to+\infty} h(x)/x$. Applying l'Hopital's rule yields $\lim_{x\to+\infty} \frac{h(x)}{x} = \lim_{x\to+\infty}(1 + \frac{1}{x^2}) = 1$, which establishes the result.*

Result 2: *$h(x)$ is log-concave, that is, $ln(h(x))$ is strictly concave in x.*[18]
Proof: *To prove log-concavity, we need to show that $\frac{\partial \ln(h(x))}{\partial x} = \frac{h'(x)}{h(x)}$ is decreasing in x, which is true if $h'(x) < 1$. Consider $h''(x)$*

$$h''(x) = h(x)[(h(x) - x)^2 + (h'(x) - 1)]$$

which is strictly positive if $h'(x) \geq 1$. Suppose $h'(x^*) \geq 1$ for some x^*. Then, $h'(x)$ is increasing at x^*, implying $h'(x) > 1$ and $h''(x) > 0$ for all $x > x^*$, a result which contradicts $\lim_{x\to+\infty} h'(x) = 1$, established in Result 1. Alternatively, it is straightforward to show that for the standard normal distribution Var($y|y > x$) $= 1 - h'(x)$, which implies $h'(x) < 1$ for all x.

Result 3: *$h(x)$ is strictly convex in x.*
Proof: *Let $g(x) = [(h(x) - x)^2 + (h'(x) - 1)]$; then, $h''(x) > 0$ iff $g(x) > 0$. Given the limiting results established above, it is straightforward to obtain*

[18] *Bagnoli and Bergstom (1989) provide some results on properties of log-concave distribution functions, including a proof that the reliability function $1 - \Phi(x)$ is log-concave. We require, however, that the hazard rate itself be log-concave.*

$\lim_{x\to-\infty} g(x) = \infty$ *and* $\lim_{z\to+\infty} g(x) = 0$. *Now, consider* $g'(x)$

$$g'(x) = 2(h(x) - x)(h'(x) - 1) + h(x)g(x)$$

which is strictly negative if $g(x) \leq 0$. *Suppose* $g(x^*) \leq 0$ *for some* x^*, *implying that* $g'(x^*) < 0$. *This then implies that* $g(x) < 0$ *and* $g'(x) < 0$ *for all* $x > x^*$, *a result which contradicts* $\lim_{x\to+\infty} g(x) = 0$.

Result 4: *For a given constant* $c > 0$, $\frac{h(x-c)}{h(x)}$ *is monotonically increasing in* x *with* $\lim_{x\to-\infty} \frac{h(x-c)}{h(x)} = 0$ *and* $\lim_{x\to+\infty} \frac{h(x-c)}{h(x)} = 1$.
Proof: *To prove that* $h(x-c)/h(x)$ *is monotonically increasing in* x, *we compute*

$$\frac{\partial(h(x-c)/h(x))}{\partial x} = \frac{h(x-c)}{h(x)}\Big\{h(x-c) - (x-c) - (h(x) - x)\Big\}.$$

which is positive if the term in brackets is positive. We therefore need to show that $h(y) - y > h(x) - x$ *for* $y < x$ *which is true if* $h'(x) < 1$, *that is, if* $h(x)$ *is log-concave, which is proven in Result 2 above. To show the lower limit, we note that* $\frac{h(x-c)}{h(x)} = \left(e^{(2xc-c^2)/2}\right)\left(\frac{1-\Phi(x)}{1-\Phi(x-c)}\right)$ *and take limits. To establish the upper limit, we use the mean value theorem to obtain* $h(x) = h(x-c) + ch'(x^*)$ *for* $x - c < x^* < x$. *We then use* $x^* < x$ *and* $h''(x) > 0$ *to obtain the bounds* $1 > \frac{h(x-c)}{h(x)} > 1 - \frac{h'(x)}{h(x)}c$. *Result 1 implies* $\lim_{x\to+\infty} \frac{h'(x)}{h(x)} = 0$, *which establishes the result.*

Result 5: *For a given constant* $c > 0$, $c\,(h(x-c) - (x-c))\,(h(x) - x) > h(x-c) - h(x) + c$.
Proof: *Let*

$$f(x) = c[\omega(x-c)\omega(x)] + [\omega(x) - \omega(x-c)],$$

where $\omega(x) = h(x) - x > 0$. *Taking limits we obtain* $\lim_{x\to-\infty} \omega(x) = \infty$, $\lim_{x\to+\infty} \omega(x) = 0$, *implying* $\lim_{x\to-\infty} f(x) = \infty$ *and* $\lim_{x\to+\infty} f(x) = 0$. *Taking derivatives, we have* $\omega'(x) = h'(x) - 1 < 0$ *and* $\omega''(x) = h''(x) > 0$. *Since* $\omega(x)$ *is decreasing and strictly convex in* z, *we have* $\omega'(x) < \omega'(x-c)$ *and*

$$f'(x) = c[\omega'(x-c)\omega(x) + \omega(x-c)\omega'(x)] + [\omega'(x) - \omega'(x-c)] < 0.$$

Given that $\lim_{x\to-\infty} f(x) = \infty$ *and* $\lim_{x\to+\infty} f(x) = 0$, $f'(x) < 0$ *implies* $f(x) > 0$ *for all* x.

Proof of proposition 1: *Result 4 implies that for any given* $\sigma > 0$, $h(z-\sigma)/h(z)$ *is monotonically increasing with* $\lim_{z\to-\infty}\frac{h(z-\sigma)}{h(z)} = 0$ *and* $\lim_{z\to+\infty}\frac{h(z-\sigma)}{h(z)} = 1$. *Hence, there is a single value of* z *that satisfies equation 18.*

Proof of proposition 2: *Let* $\eta = \frac{\partial \ln y}{\partial \ln(1/w)} = \frac{h(z-\sigma)}{\sigma}$ *denote the elasticity of supply. From Result 1, we know that* $h(z)$ *is log-concave which implies* $h(z-\sigma)-(z-\sigma) > h(z)-z$ *and* $h'(z-\sigma) < 1$ *. Using* $h'(z) = h(z)(h(z)-z)$, *and taking partial derivatives, we have:* $\frac{\partial\eta}{\partial \ln y} = -\left(\frac{h(z-\sigma)-(z-\sigma)}{\sigma}\right) < 0$ *and* $\frac{\partial^2\eta}{\partial(\ln y)^2} = \frac{(h'(z-\sigma)-1)}{\sigma h(z-\sigma)} < 0$ *implying that the slope of the supply curve,* η^{-1} *, is increasing and convex in ln(y). To show* η^{-1} *is decreasing in* σ, *we note that in equilibrium* $\eta = \frac{(1-\alpha)h(z)}{\sigma}$ *. Taking derivatives, we obtain*

$$\frac{d\eta}{d\sigma} = (\frac{h'(z)}{h(z)}\frac{dz}{d\sigma} - \frac{1}{\sigma})\eta$$

and totally differentiating equation 18 we obtain $\frac{dz}{d\sigma} = \frac{h(z-\sigma)-(z-\sigma)}{h(z-\sigma)-h(z)+\sigma} > 1$ *where the inequality again follows from log-concavity of* $h(z)$ *. Combining these expressions we have*

$$\frac{d\eta}{d\sigma} = \left[\frac{(h(z-\sigma)-(z-\sigma))\,(h(z)-z)}{h(z-\sigma)-h(z)+\sigma} - \frac{1}{\sigma}\right]\eta.$$

Result 5 relies on convexity of $h(z)$ *to show that the term in brackets is strictly positive for any* $\sigma > 0$ *. This establishes that* $\frac{d\eta}{d\sigma} > 0$ *, and the slope of the supply curve is strictly decreasing in* σ *at the steady-state equilibrium. To establish the lower bound for the slope of the supply curve, in equilibrium, we note that* $h(z-\sigma) = (1-\alpha)h(z)$ *and* $h(z-\sigma)-(z-\sigma) > h(z)-z$ *implies* $\alpha h(z) < \sigma$ *and therefore* $\eta = \frac{(1-\alpha)h(z)}{\sigma} < \frac{1-\alpha}{\alpha}$ *.*

Proof of proposition 4: *Equation 18 defines* z *as an implicit function of* σ *with* $\frac{dz}{d\sigma} > 1$ *following immediately from the increasing hazard property of the standard normal distribution (see the proof of proposition 3). Differentiating the capital-labor ratio* k *with respect to* σ, *and using equation 18 we obtain* $\frac{d\ln k}{d\sigma} = -h(z)(\frac{dz}{d\sigma} - 1) < 0$. *Because steady-state labor is independent of* σ, *the flow of new machines is proportional to the inverse of the capital utilization rate. Thus, an increase in* σ *leads to a fall in the steady-state capital utilization rate proportional to the increase in the number of new machines:*

$\frac{d\ln q}{d\sigma} = h(z)\frac{dz}{d\sigma} > 0$. *Combining* $\frac{d\ln k}{d\sigma}$ *with* $\frac{d\ln q}{d\sigma}$*, we obtain the result that investment kq is increasing in* σ*:* $\frac{d\ln(kq)}{d\sigma} = h(z) > 0$*. Equations 11 and 14 imply that output is linear in investment:* $y = \frac{1-\beta(1-\delta)}{\alpha\beta}kq$ *so that* $\frac{d\ln y}{d\sigma} = h(z) > 0$*. Thus, output and investment rise by the identical* $h(z)$ *percent in response to a unit increase in* σ*, and the investment share of output is invariant to the degree of idiosyncratic uncertainty. Finally the result that, at the equilibrium, the selasiticity* $\sigma h(z) < \frac{1}{\alpha}$ *is established in the proof of proposition 2.*

The Economy with Growth

Proof of proposition 3: *Let* $\Psi(z) \equiv \sum_{j=1}^{M} \tilde{\beta}^j(1-\delta_j)(1-\Phi(z(j)-\sigma))\frac{h(z(j)-\sigma)}{h(z(j))}$*, and* $\Gamma(z) \equiv \sum_{j=1}^{M} \tilde{\beta}^j(1-\delta_j)(1-\Phi(z(j)-\sigma))$*. Then* $\frac{\Psi(z)}{\Gamma(z)} = \sum_{j=1}^{M} v(z(j))\frac{h(z(j)-\sigma)}{h(z(j))}$ *and the balanced growth equilibrium condition may be written*

$$(1-\alpha) = \frac{\Psi(z)}{\Gamma(z)}.$$

Following the proof of result 2, it is straightforward to show that $\lim_{z\to-\infty} \frac{\Psi(z)}{\Gamma(z)} = 0$ *and* $\lim_{z\to+\infty} \frac{\Psi(z)}{\Gamma(z)} = 1$*. Thus, by continuity of* $\frac{\Psi(z)}{\Gamma(z)}$*, there exists at least one value of z that satisfies the equilibrium condition.*

To prove that log-concavity of $\Gamma(z)$ *implies uniqueness of the equilibrium we show that* $\frac{\partial^2 \ln\Gamma(z)}{\partial z^2} < 0$ *implies* $\frac{\Psi(z)}{\Gamma(z)}$ *is monotonically increasing in z. Taking derivatives and using the facts that* $\Psi'(z) = \sigma\Psi(z) + \Gamma'(z)$ *and* $\Gamma'(x) = -\sum_{i=1}^{M} \tilde{\beta}^j(1-\delta_j)\phi(z(j)-\sigma)$*, we obtain*

$$\frac{\partial\frac{\Psi(z)}{\Gamma(z)}}{\partial z} = \sigma\frac{\Psi(z)}{\Gamma(z)} + \left(1-\frac{\Psi(z)}{\Gamma(z)}\right)\frac{\partial\ln\Gamma(z)}{\partial z}.$$

Taking second derivatives we obtain

$$\frac{\partial^2\frac{\Psi(z)}{\Gamma(z)}}{\partial z^2} = \left(\sigma - \frac{\Gamma'(z)}{\Gamma(z)}\right)\frac{\partial\frac{\Psi(z)}{\Gamma(z)}}{\partial z} + \left(1-\frac{\Psi(z)}{\Gamma(z)}\right)\frac{\partial^2\ln\Gamma(z)}{\partial z^2}$$

If $\frac{\partial^2\ln\Gamma(z)}{\partial z^2} < 0$ *then the second term in this expression is negative. Given* $\Gamma'(x) < 0$ *we have* $\left(\sigma - \frac{\Gamma'(z)}{\Gamma(z)}\right) > 0$*, so that* $\frac{\partial\frac{\Psi(z)}{\Gamma(z)}}{\partial z} < 0$ *implies that the first term is also negative. Now suppose* $\frac{\partial\frac{\Psi(z^*)}{\Gamma(z^*)}}{\partial z} < 0$ *for some* z^**. Then we have*

$\frac{\partial^2 \frac{\Psi(z^*)}{\Gamma(z^*)}}{\partial z^2} < 0$, *implying that* $0 < \frac{\Psi(z^*)}{\Gamma(z^*)} < 1$ *and* $\frac{\Psi(z)}{\Gamma(z)}$ *is strictly decreasing on* (z^*, ∞) *, a result that contradicts* $\lim_{z \to +\infty} \frac{\Psi(z)}{\Gamma(z)} = 1$.

Uniqueness of equilibrium depends on log-concavity of $\Gamma(z)$. In the remainder of this section of the appendix, we provide some analysis of the conditions needed to guarantee the log-concavity of $\Gamma(z) \equiv \sum_{j=1}^{M} \tilde{\beta}^j (1-\delta_j)(1-\Phi(z(j)-\sigma))$. We also consider what set of parameter values may lead to multiple equilibria.

To begin, consider the function $\Gamma''(z)\Gamma(z) - \Gamma'(z)^2$ which, if negative, guarantees log-concavity of $\Gamma(z)$ and hence uniqueness of the equilibrium. We know that the reliability function $(1-\Phi(x))$ is log-concave and $\Gamma(z)$ is the weighted sum of such functions, which, while not sufficient to guarantee log-concavity, suggests that it may be difficult to produce circumstances under which it does not obtain. Let $\omega_j = \tilde{\beta}^j(1-\delta_j)$ and $z_j = z(j) - \sigma$. After some manipulation we obtain

$$\begin{aligned}
\Gamma''(z)\Gamma(z) - \Gamma'(z)^2 &= \sum_{j=1}^{M}\sum_{k=1}^{M} \omega_j \omega_k \phi(z_j)[(z_j)(1-\Phi(z_k) - \phi(z_k)] \\
&= \sum_{j=1}^{M}\sum_{k=1}^{M} \omega_j \omega_k \phi(z_j)\phi(z_k)\left(\frac{z_j}{h(z_k)} - 1\right) \\
&= \sum_{j=1}^{M}\sum_{k=1}^{M} \omega_j \omega_k \phi(z_j)\phi(z_k)\left(\frac{z_k}{h(z_k)} - 1\right) \\
&\quad + \sum_{j=1}^{M}\sum_{k=1}^{M} \omega_j \omega_k \phi(z_j)\phi(z_k)\left(\frac{z_j - z_k}{h(z_k)}\right).
\end{aligned}$$

The first term in this expression is clearly negative. The second term may be positive if for some j, k we have large productivity differentials between vintages j and k.

Because z_j is linearly increasing in machine age, a large productivity differential is likely to occur when vintage j is substantially older and less productive than vintage k. In this case, however, the contribution of this term to the sum is relatively small owing to discounting, both explicitly through the term $\tilde{\beta}^j(1-\delta_j)$ and implicitly through a low value of $\phi(z_j)$. Furthermore, for any positive term $(z_j - z_k)/h(z_k) > 0$ there is an equally weighted negative term $(z_k - z_j)/h(z_j) < 0$. This suggests that only under extreme parameterizations can we have large productivity differentials that yield positive values of any magnitude for

$$\omega_j \omega_k \phi(z_j)\phi(z_k) \left(\frac{(z_j - z_k)}{h(z_k)} + \frac{(z_k - z_j)}{h(z_j)} \right),$$

the weighted sum of these components. In turn, these positive values must be large enough to offset the negative sum in the first component of $\Gamma''(z)\Gamma(z) - \Gamma'(z)^2$.

Note that log-concavity of $\Gamma(z)$ is a sufficient, not necessary, condition for a unique steady-state value of z. Indeed, it is not necessary that $\Psi(z)/\Gamma(z)$ be monotonically increasing, as long as it crosses $(1-\alpha)$ only once. Numerical experiments suggest that multiple equilibria only occur when both the trend productivity growth rate is exorbitantly high, so that productivity differentials across vintages are large, and when discounting through the real interest rate and depreciation is very low. For example, we obtain multiple equilibria in the model when $\sigma = 0.1$, $\delta_j = 0, j = 1, \ldots, M$, $\gamma = 0.1$, $g = 0.6$ (60 percent per annum), and $\beta = 0.999$. These parameter values imply that $z_2 - z_1 > 5$. Relatively small adjustments in parameter values result in the number of equilibria collapsing to one. We have found no evidence of multiple equilibria using more conventional parameterizations that would typically characterize the capital accumulation process in a general equilibrium model calibrated based on empirical moments of industrialized economies.

The Solow Vintage Model

By relaxing the restriction that ex post capital-labor ratios are fixed, the model described above collapses to the putty-putty vintage capital model initially introduced by Solow (1960), modified to allow for time-varying idiosyncratic uncertainty. For the Solow vintage model, define the capital aggregator, K_t, by

$$K_t \equiv \sum_{j=1}^{M} \left(\theta_{t-j} e^{\frac{(1-\alpha)\sigma_{t-j}^2}{2\alpha}} \right)^{1/\alpha} (1-\delta_j) I_{t-j}, \tag{29}$$

where I_t denotes gross aggregate capital investment in period t. The two terms that multiply investment flows measure the level of embodied technology at the time of installation of the capital good, θ, and the scale correction that results from aggregating across machines with differing levels of idiosyncratic productivity, subject to the marginal product of labor being equal across all machines. Aggregate production in period t is given by

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}. \tag{30}$$

If we assume that $\delta_j = 1 - (1-\delta)^{j-1}$ and $M = \infty$, we obtain the following capital accumulation equation:

$$K_t = (1-\delta)K_{t-1} + \left(\theta_{t-1} e^{\frac{(1-\alpha)\sigma_{t-1}^2}{2\alpha}}\right)^{1/\alpha} I_{t-1}.$$

Note that in this economy with putty-putty capital, a mean-preserving spread to idiosyncratic productivity is equivalent to an increase in embodied productivity at the aggregate level. Both of these factors enter the model through the capital accumulation equation and are equivalent to a reduction in the economic cost of new capital goods.