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THE VALUE OF WAITING TO INVEST

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ABSTRACT

This paper studies the optimal timing of investment in an irreversible project where the benefits from the project and the investment cost follow continuous-time stochastic processes. The optimal time to invest and an explicit formula for the value of the option to invest are derived. The rule "invest if benefits exceed costs" does not properly account for the option value of waiting. Simulations show that this option value can be significant, and that for surprisingly reasonable parameter values it may be optimal to wait until benefits are twice the investment cost. Finally, we perform comparative static analysis on the valuation formula and on the rule for when to invest.

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I. INTRODUCTION

Suppose that the government is planning to build a canal through Everglades National Park. What is the appropriate way to perform a cost-benefit analysis? Clearly, one calculates the benefit from building the canal, and computes the direct cost of constructing it. An additional cost is the foregone benefit of the park as a recreational area. It would be incorrect, however, to simply compare these costs and benefits and then undertake to build the canal if benefits exceed costs.

The decision to build is essentially irreversible; the ecology of the Everglades will have been irreparably damaged. The decision to defer building is, however, reversible. This asymmetry, when properly taken into account, leads to a rule which says build the canal only if benefits exceed costs by a certain positive amount.

This point has been recognized by Krutilla (1967) and others (e.g., Henry (1974) and Greenley, Walsh and Young (1981)) and is also implicit in most investment models. The investment rule in the original Jorgenson (1963) formulation relies on the complete reversibility of investment; the more sophisticated adjustment-cost models lead to lower capital stocks, because it is recognized that investment cannot in the future be costlessly and instantaneously undone.

Although this point is known, it is often not dealt with.[1] The correct calculation involves comparing the value of investing today with the (present) value of the option of investing at all possible times in the future.[2] This is a comparison of mutually exclusive alternatives.

In this paper, we explicitly calculate a formula for the value of the option to invest in an irreversible project and study its properties. The model has applicability to a wide range of problems in both the public and private sectors; examples are discussed in Section II. In Section III we solve the valuation problem for three cases: where the present value of benefits from the project (were it undertaken today) follows geometric Brownian motion, where the present value of both benefits and the investment cost follow such a process, and where the present value of benefits almost always follows a Wiener process, but can jump discretely to zero, at that point making the option to invest worthless. In every case we assume that the option is infinitely-lived.

The first of our cases is formally identical to the problem of valuing an infinitely-lived call option on a dividend-paying stock. This correspondence is not surprising, as a stock option gives its owner the right to pay a fixed cost to (irreversibly) invest in a stock. This problem was solved for stock options by Samuelson and McKean (1970). What may be surprising is that the two models have different interpretations, and behave differently in response to parameter changes. In effect, the sensible ceteris paribus assumptions for the option to invest are different than those for a stock option.

It may be objected that the case of an infinitely-lived option to invest is uninteresting, since many real-life investment opportunities expire or become valueless at some point. We deal with this by allowing the present value of the benefit from undertaking the project to have an average downward drift, or by allowing the present value to jump to zero. In the latter case, the option eventually becomes valueless, but at an unknown date. The important omission in our model is the case where the option to invest expires at a known date in the future. A finitely-lived patent, for example, would in effect give the holder an option to invest with a known expiration date, and would be worth less than an infinitely-lived patent. It is typically not possible to solve analytically for the option value in this case. The omission is presumably less important in cases where the present value of benefits from the project is expected to decline at a rapid rate.

Our principal results, discussed in Section IV, are:

1) The rule: "invest now if the net present value of investing exceeds zero" is only valid if the variance of the present value of future benefits is zero or if the expected rate of growth of the present value is minus infinity. For surprisingly reasonable parameter values, it can be optimal to defer investing until the present value of the benefits from a project is <u>double</u> the investment cost.

2) In a world with risk-neutral investors, an increase in the variability of the present value of benefits from the project increases both the value of the investment opportunity and the amount by which the present value of benefits must exceed the investment cost for it to be optimal to invest immediately. Increases in the risk-free rate of interest have the opposite effect. The introduction of risk-averse investors (using, for example, the Capital Asset Pricing Model) however, can reverse these results, as is discussed in Section V.

3) If the present value of future benefits can discretely jump to zero, an increase in the probability of the jump has the same effect as increasing the risk-free rate of interest.

II. THE INVESTMENT PROBLEM

We study the investment decision of a firm which is considering the following investment opportunity: at any time t (up to a possible expiration date T), the firm[3] can pay a fixed cost, F_t, in order to install an investment project, where future net cash flows conditional on undertaking the project have a present value V_t. We emphasize that V_t is a present value and not the cash flow itself. It represents the appropriately discounted expected cash flows, given the information avaiable at time t. For the firm, V_t represents the market value of a claim on the stream of net cash flows that arise from installing the investment project at time t. The fixed cost, F_t, can be thought of as known with certainty, or as stochastic. The installation of capacity is irreversible, in that the capacity can only be used for this specific project.

The present value of future net cash flows is stochastic. In the simplest form of our model, this present value follows geometric Brownian motion of the form.

(1a) $$\frac{dV}{V} = \alpha_v dt + \sigma_v dz_v$$

where z_v is a standard Wiener process, with an expected value of zero. Thus the firm knows the present value of future net cash flows if it installs the project today. It is not sure, however, how new information will affect the present value if the capacity is installed in the future.[4] We also consider the possibility that at some (random) time in the future, the present value of net cash flows drops at once to zero.[5] Finally, we admit the possiblity that the cost of installation, F_t, is random. In that case, we assume that F_t follows

(1b) $$\frac{dF}{F} = \alpha_f dt + \sigma_f dz_f$$

In all of these cases the geometric Brownian motion assumption is crucial for the derivation of the formulas below.

The problem we study here is the timing of the installation of the capacity when the firm has the option of delaying installation. If the capacity were installed today, the net gain from undertaking the project would be its net present value $V_0 - F_0$. By delaying, the firm forgoes the rents on installed capacity. However, this cost is offset by a gain from waiting. By not (irreversibly) exercising the investment option, the firm retains the right to gain from favorable movements in $V - F$, yet it is protected from unfavorable movements because it also retains the option to forego the investment if it turns out that $V < F$. The irreversibility of the investment gives value to waiting. If the investment cost could always be recovered for certain, then waiting would have no value. It is optimal to invest when the cost of the foregone rents from delaying the investment exceed this gain from waiting.

There are at least four situations which this model can represent. We discuss each in turn, developing the first case, the franchise monopolist, in most detail.

A) Franchise Monopoly

A franchise monopolist[6] has an investment opportunity such that once he installs his capacity, he is protected from competition. This protection may arise from a patent or a trade secret. To be concrete, consider a project which produces a commodity, using a Cobb-Douglas production function

$$(2) \qquad Q_t = \overline{K}^{\alpha} L_t^{\beta}$$

where $\overline{K}$ is the fixed level of capital, and Q_t and L_t are quantity produced and labor employed at time t. The firm faces an inverse demand curve given by

(3) $$P_t = \theta_t Q_t^{-\frac{1}{\eta}}$$

where P_t is the price of the commodity at time t, η is the price elasticity of demand, and θ_t is a demand shift parameter following the stochastic process

(4) $$\frac{d\theta}{\theta} = \alpha_\theta dt + \sigma_\theta dz_\theta$$

At each point in time after the capacity installation profits are given by $\pi_t = P_t Q_t - \bar{w} L_t$, and labor usage is chosen to

(5) $$\text{Max } \pi_t = \underset{L_t}{\text{Max}} \; \theta_t \bar{K}^c L_t^d - \bar{w} L_t = B\theta_t^\gamma$$

where $c = \alpha(1 - \frac{1}{\eta})$, $d = \beta(1 - \frac{1}{\eta})$, $\bar{w}$ is the (fixed) wage, $\gamma = \frac{1}{1-d}$ and $B = \bar{K}^{\frac{c}{1-d}} \bar{w}^{\frac{-d}{1-d}} (d^{\frac{d}{1-d}} - d^{\frac{1}{1-d}})$. If r^* is the appropriate discount rate for profits, it is possible to show that when production continues indefinitely, the present value of expected maximized profits is

(6) $$V(\theta_0) = \frac{B\theta_0^\gamma}{r^* - \gamma\alpha_\theta - \frac{1}{2}\gamma(\gamma-1)\sigma_\theta^2}$$

Using Ito's lemma, it is easy to show that the present value of cash flows given by (6) follows the process (1), with $\sigma_v = \gamma\sigma_\theta$ and $\alpha_v = \gamma\alpha_\theta + \frac{1}{2}\sigma_\theta^2\gamma(\gamma-1)$. Recall that α_θ is the expected secular rate of growth in the demand price, while σ_θ is the standard deviation of that growth rate. Table I shows the relationship between the two parameters affecting cash flow α_θ and σ_θ, and the parameters for the present value of profits, α_v and σ_v.

The calculations use the formulas developed above.

Finally, the cost of capacity installation, F_t, may be stochastic, as it may depend upon other variables such as factor prices, which are themselves stochastic.

(Table I Here)

B) Competitive Industry with Stochastic Entry

A firm in an industry where entry is expected in the future will attempt to capture temporary rents. The investment opportunity consists of a project whose future net cash flows have a positive present value now, but which tend on average toward zero as time goes on because of lagged entry. In equation (1), this is represented by $\alpha_v < 0$. The stochastic component may arise because of stochastic entry and stochastic demand for the commodity being produced by the project. This kind of structure is present when a firm is in a strong competitive position, because other firms need time to "gear up" to enter. The advantage only results in temporary rents because other firms eventually compete away profits.

C) Unprotected Innovator

An unprotected innovator also tries to capture temporary rents. His investment opportunity consists of a new commodity that can be easily copied after a lag. Therefore V_t represents the present discounted value of net cash flows before entrants compete away profits. This differs from the previous case because α_v need not be less than zero. The demand for the commodity may increase over time, inducing an increase in temporary rents.

Many examples of this kind of project occur in the high-technology industries. When a firm introduces a new product, it realizes that others will copy it using "reverse engineering" techniques. As the others enter, profits disappear.

These industries also provide examples of how V_t might at some point drop to zero. While the unprotected innovator is waiting to introduce his commodity (or after he has done so), a new, more sophisticated, or cheaper version might be introduced by another innovator, rendering the former's product useless.

D) Cost-Benefit Analysis

This kind of analysis is also useful in certain types of cost-benefit analysis. Policy-makers may face an investment opportunity where V_t represents the present value of future benefits if the investment is undertaken at time t and F_t represents the value of those resources forgone by undertaking the project at that time.

As an example, consider the Everglades canal discussed in the introduction. If the canal is built at time t, then V_t represents the present value of future net benefits from the canal.[7] F_t represents the present value of recreational opportunities lost by building the canal plus the construction cost. In this context it may make sense to have F_t be stochastic, as the value of recreational opportunities may change over time.

E. Optimal Scrapping of a Project

By reinterpreting F as the value of the project and V as the scrap value, this same analysis can be used to study the optimal scrapping decision.

III. INVESTMENT TIMING AND THE VALUE OF WAITING

In this section we solve the problem of the optimal timing of the installation of an irreversible investment project. We derive an optimal decision rule and the value of the investment opportunity. We begin with the case of V_t following (1) with a fixed and known F_t, and then consider a stochastic F_t and the possibility that V_t may suddenly fall to zero.

A) F_t Fixed with no Jumps in V_t

Suppose initially that V_t is random, but that F_t is fixed at F. That is, there is a known cost of investing, but the (present) value of the benefits in the future is uncertain. (As we discussed above, the value of the benefits from investing today is known with certainty.) One can think of the investment timing problem as a standard first-passage problem in the theory of stochastic processes.[8] That is, there is a boundary which is a function of time alone, such that investment is undertaken the first time that V_t passes the boundary. This boundary may be found by solving recursively backward.

Suppose, for example, that the investment opportunity expires at T,[9] and that it is currently time 0. It is obvious that if we reach T and have not already undertaken investment, then it will be optimal to do so provided that $V_T > F$. Thus, $C_T^* = F$ constitutes a boundary at T, at which the investment opportunity is undertaken. In a similar way, working backwards, for any t it is possible to derive a C_t^* such that, if the investment opportunity is still unexercised at t, then undertaking it will be optimal if $V_t \geq C_t^*$, and not if the inequality is reversed. Using this dynamic programming approach, C_t^* (for every t) is chosen to maximize the value of the option <u>given</u> that it is still unexercised. From the recursive structure of this problem, it is clear that the boundary schedule chosen in this way, $\{C_t^*\}_0^T$, will maximize the current value of the investment opportunity. Thus the optimal decision rule involves deriving a boundary schedule C_t^*, $t \in [0,T]$, such that as long as $V_t < C_t^*$ the firm will defer the investment. When $V_t = C_t^*$ the firm invests and the net present value of the project is then $C_t^* - F$.

For an arbitrary boundary $\{C_t\}_0^T$, the value of this opportunity is the expected present value of the payoff:

$$X(T) = E_0\{e^{-\rho t'}[C_{t'} - F]\}$$

where t' is the date at which V first reaches this boundary C. The expectation is taken over the first passage times t' and ρ is the appropriate discount rate (which for now can be thought of as the risk-free rate.) We derive ρ in Section IV.[10] Let $G(\{C_s\}_0^t, V_0, t)$ be the probability that the first passage of V across this arbitrary boundary C occurs at or before time t. Then the optimal boundary is found by solving the problem

$$X^*(T) = \underset{\{C_t\}}{\text{Max}} \int_0^T e^{-\rho t}[C_t - F]g(\{C_s\}_0^T, V_0, t)dt \tag{8}$$

where $g(\cdot)$ is the density function associated with $G(\cdot)$. In general, the first passage density $g(\cdot)$ is complicated because it is calculated <u>conditional</u> on V having not already reached C.[11]

In the special case where the investment opportunity is infinitely lived, it is possible to solve (8) explicitly. When $T = \infty$, it is possible to remove calender time from the problem, from which it follows that C cannot depend on t, hence $C_t^* = C^*$ for all t.[12] Cox and Miller (1965) present a solution technique for this problem.[13] Because $C^* - F$ is constant, (8) reduces to

(9) $$\max_{C}\ [C - F]E_0\{e^{-\rho t'}\}$$

The expectation is calculated for an arbitrary boundary C by Cox and Miller, who show that[14]

(10) $$E_0\{e^{-\rho t'}\} = \left(\frac{V_0}{C}\right)^{\varepsilon}$$

where ε is the solution to the quadratic equation

(11) $$\rho = \frac{1}{2}\sigma_v^2\varepsilon^2 + (\alpha_v - \frac{1}{2}\sigma_v^2)\varepsilon$$

Having found the expectation in (9) for an arbitrary boundary, we now find the boundary which maximizes the value of the investment opportunity. Using (10), the solution to (9) is

(12) $$X^*(\infty) = [C^* - F]\left(\frac{V_0}{C^*}\right)^{\varepsilon}$$

where

(13) $$C^* = F\left(\frac{\varepsilon}{\varepsilon-1}\right)$$

is the optimal boundary.[15] Notice that for the solution to be well-defined (i.e., for $\varepsilon > 1$), it is necessary that $\alpha_v < \rho$. Otherwise the investment will always be undertaken immediately.[16]

The significance of the option to invest at some point in the future can be seen in Table II. This table presents the value of an investment option (from (12)) which has a zero net present value (V = F) if the investment is undertaken today. For example, if $\sigma_v^2 = .02$, $r = .02$ and $\alpha_v = 0$ (implying no expected increase in V), then the investment project is still worth 25 cents for every dollar that the project installation costs. Even if the present value of benefits decreases at an annual rate of 5 per cent ($\alpha_v = -.05$), the investment opportunity is still worth 6.3 cents for every dollar that the project installation costs. As the table shows and as we will verify later in general, increasing σ_v^2 will increase the value of the investment opportunity.

(Table II Here)

The case where T is finite has not been solved analytically (Samuelson (1970)). The general solution procedure in such cases involves using a discrete approximation to the continuous-time problem and applying a dynamic-programming argument to obtain numerical approximations to the solution (cf., Ingersoll (1976)). Brock, Rothschild and Stiglitz (1982) have attacked with great generality the problem of investment timing when there is no cost of investing (F = 0). In order to obtain non-trivial results without an investment cost, their model has an α_v which is a declining function of V.

B) F_t Stochastic with no Jumps in V_t

Now we consider consider the same problem in A) above, except that F_t is also random and follows the stochastic process (1b). As before, the problem is formulated as a first passage problem, but now there are two plausible ways to characterize the solution: 1) invest when the difference V - F reaches a barrier, or 2) invest when the ratio V/F reaches a barrier. Fortunately, the economics of the situation help us choose between these. It is sensible that doubling the size of the project *ceteris paribus* should double its value.

This implies that the value of the project should be homogeneous of degree one in V and F (Merton (1973)), so that the second characterization is the correct one. If the first characterization were adopted, the barrier could be reached simply by increasing the scale of the project sufficiently, so that the value could not be homogeneous in V and F.

It is possible to use the same method as before to derive the optimal decision rule and value of the investment opportunity. Because the optimal rule is to invest when V_t/F_t reaches a barrier Δ^*, the expected present value of the payoff is

$$(14) \qquad E_0\{F_{t'}[\Delta^*-1]e^{-\rho t'}\} = [\Delta^*-1]E_0\{F_{t'}e^{-\rho t'}\}$$

where the expectation is taken over the joint density of F_t and the first-passage times for V_t/F_t. Fortunately, it is not necessary to derive the joint density for F_t and t' in order to evaluate (14). The derivation of (14) is involved, however, so it is relegated to the Appendix. From the Appendix, the value of the opportunity is

$$(15) \qquad (\Delta^*-1)F_0\left(\frac{V_0/F_0}{\Delta^*}\right)^{\varepsilon'}$$

where ε' is the solution to the quadratic

$$(16) \qquad \rho = \frac{1}{2}\varepsilon'(\varepsilon'-1)\sigma^2 + \varepsilon'\alpha_v + (1-\varepsilon')\alpha_f$$

$\sigma^2 = \sigma_v^2 + \sigma_f^2 - 2\sigma_{vf}$, σ_{vf} is the instantaneous covariance of the rates of increase of V and F, $\Delta^* = \varepsilon'/(\varepsilon'-1)$. The solution is:

(16') $$\varepsilon' = \sqrt{\left(\frac{\alpha_v - \alpha_f}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2(\rho - \alpha_f)}{\sigma^2}} + \left(\frac{1}{2} - \frac{\alpha_v - \alpha_f}{\sigma^2}\right)$$

It is easy to show that (15) reduces to (10) if F_t is constant. Notice also that when V is fixed and F follows (16), then (15) represents the value of the option to scrap a project, where the value of the project is F and the scrap value is fixed at V. This requires setting $\alpha_v = 0$ and $\sigma^2 = \sigma_v^2$. Merton (1973) obtains the same formula for the value of a perpetual put on a stock. A stochastic V then would represent a stochastic scrap value.

C) F_t Fixed and Jumps in V_t

Once again we assume that the investment cost F is fixed, but now there is a positive probability that the present value of net future cash flows, V_t can take a discrete jump to zero. If this happens the investment opportunity becomes worthless. Thus the stochastic process for V_t is a mixed Poisson-Wiener process of the form

(17) $$\frac{dV}{V} = \alpha_v dt + \sigma_v dz_v + dq$$

where

$$dq = \begin{cases} -1 \text{ with probability } \lambda dt \\ 0 \text{ with probability } 1 - \lambda dt \end{cases}$$

The occurence of the Poisson event induces the process to stop, since zero is a natural absorbing barrier for a geometric Brownian motion process.

Calculating the value of the investment opportunity and the optimal decision rule is made easier by noticing that when the Poisson event occurs, it is as if the investment opportunity expires, since its value becomes zero. Thus, calculating the value of the investment opportunity when V_t can jump to zero is just like calculating the value of an investment opportunity with an uncertain expiration date. The opportunity expires just when V_t falls to zero. The value in this case is easily calculated thanks to a result in Merton (1971).

The value of the investment opportunity conditional on its expiration at time T ($X^*(T)$) is given by (8). The distribution of first occurence times for a Poisson event with parameter λ is exponential. Thus, if the Poisson event is uncorrelated with the first passage time for V, then for the risk-neutral investor, the expected present value of the payoff from the investment opportunity with uncertain expiration date is

$$X^* = \int_0^\infty \lambda e^{-\lambda T} X^*(T)\,dT \tag{18}$$

Following Merton, this may be integrated by parts to give

$$X^* = \max_{\{C_t\}} \int_0^\infty e^{-(\lambda+\rho)t}[C_t - F]\,g(\{C_s\}_0^t, V_0, t)\,dt \tag{19}$$

But this is exactly the problem we solved above for the fixed investment cost with no Poisson jump. The discount rate ρ has been replaced by $\rho + \lambda$.[17] The formula is therefore the same as (12) with the discount rate adjustment. When we consider risk-averse investors, it will be necessary to assume that the jump risk is uncorrelated with both V_t and other systematic sources of uncertainty in the economy.

IV. RESULTS AND DISCUSSION

In this section we discuss some implications of the foregoing. For simplicity, we treat only the case of a fixed investment cost with no jumps in V. For the most part, we will also focus only on the case when investors are risk-neutral, though we will mention the risk-averse case.

A) NPV > 0 Rules

It is commonly asserted that an investment should be undertaken if its net present value exceeds zero. This is correct, except when choosing among mutually exclusive projects. In that case, one chooses the project with the greatest net present value. Undertaking an investment today and undertaking it tomorrow are mutually exclusive actions. The model we have presented simply provides a way to choose among the mutally exclusive alternatives of investing today or waiting.[18]

In general, for the firm facing an infinitely-lived investment opportunity, it will not be optimal to invest unless V exceeds F by some positive amount. From (12), it can be seen that a firm will be willing to invest at V = F only when $\sigma_v^2 = 0$ or $\alpha_v = -\infty$.[19]

How important is this effect? Table III displays values of C^*/F computed from (11) and (13), for the risk-neutral case. A rule of thumb is the following: when $\alpha_v = 0$ and $r = \sigma_v^2$, then $C^*/F = 2$. Thus, if V has a zero expected rate of change in value, the risk-free rate is .02, and the variance of the rate of change of V is a not unreasonable .02,[20] then V will have to be twice the investment cost before it is optimal to invest today. The size of this barrier is lowered if r increases or if σ_v^2 or α_v decreases.

Even if V is expected to decline by 25% over the following year, it can be optimal, for reasonable variances, to defer investing until V exceeds the cost of investing by as much as 20%. Clearly the option value of waiting can be important.

(Table III Here)

B) Comparative Statics

In this section, we will discuss the effect on X^*, the option value, and C^*, of changes in σ_v^2, r and α_v. The effect on on the value of the investment opportunity of induced changes in C^* will be ignored, which is permissible by the envelope theorem, since C^* was chosen so as to maximize the value of the opportunity. We will also ignore the effect of parametric changes in V, although from equation (6) it is obvious that changes in these parameters will in general change V, and hence X^*.

i) Variance

It is possible, though tedious, to show that $\partial X/\partial \varepsilon < 0$ and $\partial X/\partial \sigma_v^2 < 0$, so that an increase in variance, holding V fixed, will raise the value of the option to invest. This occurs because increasing the variance of changes in V will increase the chance that V will have either large positive or large negative deviations from its expected path. The investor is not hurt any more by the large negative surprises than he would be by small negative surprises because in either case there is no need to invest; the investor does benefit from the large positive surprises, however. The result that value increases with variance is a standard property of options.[21]

An increase in ε also increases C^* since the owner of the investment opportunity can take advantage of large positive deviations of V from its expected path, which are now more likely.

ii) Risk-Free Rate

An increase in the risk-free rate raises ε and thus lowers the value of the investment opportunity. A change in r obviously leaves the first-passage distribution unaffected, so it does not change the expected time to investment. However, the present value of any particular passage to the boundary is lowered by the increase in the discount rate. Hence, the value of the option is lower.

This result should be compared to the standard option-pricing result that an increase in the risk-free rate raises the value of the option. In standard option-pricing models, it is implicitly assumed that the rate of return on the stock (which is analogous to V) rises with the increase in the risk-free rate. Put another way, the dividend rate on the stock is held fixed when the risk-free rate increases. In our model, $\delta_v = r - \alpha_v$ could be held fixed with an increase in r only if α_v also rose. For our purposes, this seems less interesting than allowing the dividend to change when the risk-free rate changes.

Because ε increases, C^*/F will fall. This is because the cost of waiting (foregone rents) has gone up.

iii) Expected Rate of Change

An increase in α_v will obviously raise the value of the claim on the investment opportunity.

C^*/F also increases, because ε decreases. Intuitively, C^*/F increases so that the owner can take advantage of the increase in the expected future value of V.

V. VALUATION BY RISK-AVERSE INVESTORS

Up to this point, we have taken the rate at which future payoffs are discounted, ρ, as given. In a world of risk-neutral investors ρ would equal r, the risk-free rate of interest. In this section we derive the appropriate formula in a world with risk-averse investors. The technique we use is to show that ρ--which is the equilibrium expected rate of return on the investment opportunity--must be a weighted average of the equilibrium expected rates of return on assets with the same risk as V and F.

A) V and F Stochastic

Consider the formula for the value of the investment opportunity when the investment cost is stochastic. From (16) this may be written

$$X^*(\infty) = (\Delta^*-1)\Delta^{*^{-\varepsilon'}} F^{1-\varepsilon'} V^{\varepsilon'} \tag{20}$$

X^* is of course also the equilibrium price of a claim on the investment opportunity. The rate of return on such a claim can be derived by taking an Ito expansion of X^*:

$$\frac{dX^*}{X^*} = (1-\varepsilon')\frac{dF}{F} + \varepsilon'\frac{dV}{V} + (\varepsilon'-1)\varepsilon'(\tfrac{1}{2}\sigma_f^2 + \tfrac{1}{2}\sigma_v^2 - \sigma_{vf})dt$$

$$= [(1-\varepsilon')\alpha_f + \varepsilon'\alpha_v + (\varepsilon'-1)\varepsilon'\tfrac{1}{2}\sigma^2]dt + (1-\varepsilon')\sigma_f dz_f + \varepsilon'\sigma_v dz_v \tag{21}$$

The unanticipated component of the return on X^* is $(1-\varepsilon')\sigma_f dz_f + \varepsilon'\sigma_v dz_v$, which is a weighted average of the unanticipated components in the rates of change of V and F. Therefore, if $\hat{\alpha}_x$ is the equilibrium expected return on the claim to the investment opportunity, then

$$\hat{\alpha}_x = (1-\varepsilon')\hat{\alpha}_f + \varepsilon'\hat{\alpha}_v \tag{22}$$

where $\hat{\alpha}_f$ and $\hat{\alpha}_v$ are the required expected rates of return on assets with the same stochastic components as F and V.

If we equate the required expected rate of return (22) with the actual expected rate of return on X^* in (21), we get the following quadratic equation in ε' :

$$\hat{\alpha}_x = (1-\varepsilon')\hat{\alpha}_f + \varepsilon'\hat{\alpha}_v = (1-\varepsilon')\alpha_f + \varepsilon'\alpha_v + \frac{1}{2}(\varepsilon'-1)\varepsilon'\sigma^2 \tag{23}$$

Notice that this quadratic equation is exactly that which generates ε' in (15), with $\rho = \hat{\alpha}_x$. Thus we have defined what ρ must be in equilibrium.

(23) has the solution

$$\varepsilon' = \sqrt{\left(\frac{\delta_f - \delta_v}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{\delta_f}{\sigma^2}} + \left(\frac{1}{2} - \frac{\delta_f - \delta_v}{\sigma^2}\right) \tag{24}$$

where $\delta_f = \hat{\alpha}_f - \alpha_f$ and $\delta_v = \hat{\alpha}_v - \alpha_v$. When investors are risk-neutral, $r = \hat{\alpha}_f = \hat{\alpha}_v$ and (24) reduces to (16) with $\rho = r$.

This formula may be converted to the case where the investment cost is known and constant by setting $\alpha_f = 0$, $\hat{\alpha}_f = r$, $\delta_f = r$ and $\sigma^2 = \sigma_v^2$.

B) Jumps in V

For the Poisson case of Section IIIC, we can use Ito's lemma for Poisson processes (see Merton (1971)) to calculate the equilibrium required expected rate of return on a claim on the option to invest. Assume that F is fixed and that the Poisson event is independent of dV. The Ito derivative of (12) is then

$$\frac{dX^*}{X^*} = \varepsilon\frac{dV}{V} + \varepsilon(\varepsilon-1)\frac{1}{2}\sigma_v^2 dt - \varepsilon\lambda dt \tag{25}$$

where again λ is the Poisson parameter. Thus

$$E(\frac{dX^*}{X^*}) = \varepsilon(\alpha_v - \lambda) + \varepsilon(\varepsilon-1)\frac{1}{2}\sigma_v^2 dt \tag{26}$$

Equating this expected rate of return to the required rate of return $\hat{\alpha}_v$, we find that

$$\varepsilon = \sqrt{(\frac{r - \delta_v^*}{\sigma_v^2} - \frac{1}{2})^2 + \frac{2r}{\sigma_v^2}} + (\frac{1}{2} - \frac{r - \delta_v^*}{\sigma_v^2}) \tag{27}$$

where $\delta_v^* = \hat{\alpha}_v + \lambda - \alpha_v$. Note that this is identical to (24) with $\delta_f = r$ (because the investment cost is fixed) and δ_v replaced by δ_v^*.

C) Relation to Option-Pricing Formulas

In the known investment cost case, this risk-adjusted formula is equivalent to the formula derived by Samuelson and McKean for valuing an American call option on a stock, where V_t is the price of the stock, F is the exercise price, δ_v is the proportional dividend rate and the time to maturity is infinite. This is not surprising, because a perpetual investment opportunity of the type we discuss is simply an American option to purchase the present discounted value of future net cash flows (the "stock") by paying the investment cost (the "exercise price").

The value of the discounted stream of net cash flows is increasing at an expected rate α_v, which is less than $\hat{\alpha}_v$, the rate at which the price of a financial asset with the same financial risk would be expected to grow under the ICAPM. Therefore, the expected "capital gain" on V_t is the same as the expected capital gain on this financial asset, if the asset also pays a proportional dividend at a rate $\delta_v = \hat{\alpha}_v - \alpha_v$. (The total rate of return on the asset is thus $\alpha_v + \delta_v = \hat{\alpha}_v$.) Similarly, the stochastic investment cost formula would also be the formula for an American call option on a dividend-paying stock when the exercise price is stochastic.

It can be shown that the investment opportunity will always be undertaken immediately if $\delta_v < 0$, and will never be undertaken if $\delta_v = 0$. Similarly, a perpetual American call option on a non-dividend paying stock will never be exercised.

D) Changes in Comparative Statics Results Due to Risk Aversion

Most of the comparative static calculations in Section IV are severely complicated once $\hat{\alpha}_v$ is determined by an asset pricing model, and is no longer the risk-free rate. Suppose that asset rates of return are set according to the Capital Asset Pricing Model. An increase in variance will—in addition to the direct effect on ε—increase $\hat{\alpha}_v$ if the change in V is positively correlated with the return on the market (i.e., if V has a positive beta). This increase in $\hat{\alpha}_v$, holding α_v fixed, will raise the required expected rate of return on the investment opportunity and thus will lower X^*, offsetting the beneficial direct effect of an increase in variance. If V is highly enough correlated with the market, an increase in variance can lower X^*. For negative beta projects, the reverse is true—the increase in variance lowers $\hat{\alpha}_v$, thus reinforcing the effect of the variance increase and raising X^*.

The effects of changes in the risk-free rate can also be different because a change in the risk-free rate may change the risk premium on the market, $\alpha_m - r$. The net effect on X^* will be different for positive and negative beta projects, and will depend on the direction of change in the risk-premium.

VI. CONCLUSION

This paper has discussed the problem of the optimal timing of investment when the benefit from (and possibly the cost of) investing is a random variable. The general conclusion is that it is almost always optimal to defer investing until the present value of the project's cash flows exceeds the cost of investing by some positive amount. This amount can be surprisingly large even for moderate parameter choices. We show in a risk-neutral world, for example, that if the variance of the rate of change of the project's value equals the risk free rate, and the project value has a zero expected rate of change, then it will always be optimal to defer taking the project until the present value of the cash flows is <u>double</u> the investment cost.

The basic insights appear to be applicable in a variety of areas, from the analysis of environmental issues, to questions in industrial organization and investment theory.

Appendix: Derivation of $E_0\{F_{t'} e^{-\rho t'}\}$

The purpose of this Appendix is to derive the expectation on the right hand side of (14). We argued in Section IIIB that it will be optimal for the firm to invest when $D_t = V_t/F_t$ reaches a barrier Δ^*. The first step in our derivation is to characterize the joint probability density function of F_t <u>and</u> the first passage time of D_t across Δ^*. Denote this density function as $g(F_t, t; F_0, D_0, \Delta^*)$. Then, using arguments similar to those in Cox and Miller (pp. 208-211, 246-247), this density must satisfy the Kolmogorov backward equation

(A1) $$g_t = \tfrac{1}{2}g_{dd}\sigma_d^2 D^2 + \tfrac{1}{2}g_{ff}\sigma_f^2 F^2 + g_{fd}\sigma_{fd}FD + g_d\alpha_d D + g_f\alpha_f F$$

where $g_d = \frac{\partial g}{\partial D_0}$, $g_f = \frac{\partial g}{\partial F_0}$, etc., but $g_t = \frac{\partial g}{\partial t}$.

It is easy to show, by applying Ito's lemma to D = V/F and equating drift and stochastic terms, that

(A2) $$\alpha_d = \alpha_v - \alpha_f + \tfrac{1}{2}\sigma_f^2 - \sigma_{vf}$$

$$\sigma_d^2 = \sigma_v^2 + \sigma_f^2 - 2\sigma_{vf}$$

$$\sigma_{fd} = \sigma_{vf} - \sigma_f^2$$

Therefore (A1) can be rewritten as

(A1') $$g_t = \frac{1}{2}g_{dd}[\sigma_v^2 + \sigma_f^2 - 2\sigma_{vf}]D^2 + \frac{1}{2}g_{ff}\sigma_f^2F^2 + g_{fd}[\sigma_{vf} - \sigma_f^2]DF$$
$$+ g_d[\alpha_v - \alpha_f + \sigma_f^2 - \sigma_{vf}]D + g_f\alpha_f F$$

Now let

(A3) $$L = \int_0^\infty \int_0^\infty e^{-\rho t}Fg(F, t; F_0, D_0, \Delta^*)dtdF$$

where $L = E_0\{F_{t'}e^{-\rho t'}\}$ is the expectation on the right hand side of (14) in the text. We assume that L exists. (If it did not, then the option would have infinite value.) Notice that

(A4) $$L_{dd} = \int_0^\infty \int_0^\infty Fg_{dd}dtdF$$

$$L_d = \int_0^\infty \int_0^\infty Fg_d dtdF$$

and so forth for L_{ff}, L_{fd}, and L_f. It is permissible to exchange differentiation and integration in this way because we are taking derivatives in (A1) with respect to initial values D_0 and F_0, while the integration is over t and F_t, holding fixed D_0 and F_0. (This is why (A1) is called the backward equation.) It is also possible to show, using integration by parts, that

(A5) $$L_t = \rho L$$

Therefore, multiplying (A1') by $F_te^{-\rho t}$ and integrating over first passage times and F_t, we get the partial differential equation

(A6) $$\rho L = \frac{1}{2}L_{dd}[\sigma_v^2 + \sigma_f^2 - 2\sigma_{vf}]D^2 + \frac{1}{2}L_{ff}\sigma_f^2F^2 + L_{fd}[\sigma_{vf} - \sigma_f^2]DF$$
$$+ L_d[\alpha_v - \alpha_f + {}^2\sigma_f - \sigma_{vf}]D + L_f\alpha_f F$$

Now, assume (guess) that the form of L is

(A7) $$L = \Delta^{*^{-\varepsilon'}} FD^{\varepsilon'}$$

Then (A6) can be written as

(A8) $$\rho = \frac{1}{2}\varepsilon'(\varepsilon'-1)[\sigma_v^2 + \sigma_f^2 - 2\sigma_{vf}] + \varepsilon'[\sigma_{vf} - \sigma_f^2]$$
$$+ \varepsilon'[\alpha_v - \alpha_f + \sigma_f^2 - \sigma_{vf}] + \alpha_f$$

which can be rewritten as the quadratic

(A9) $$\rho = \frac{1}{2}\varepsilon'(\varepsilon'-1)\sigma^2 + \varepsilon'\alpha_v + (1-\varepsilon')\alpha_f$$

where $\sigma^2 = \sigma_v^2 + \sigma_f^2 - 2\sigma_{vf}$.

The solution to this quadratic is given by equation (16') in the text. Therefore, (A7) (with ε' given by (16')) solves the partial differential equation (A6).

We need only show that the solution satisfies the boundary conditions of the problem. The main boundary condition is that when the investment is undertaken, then L must equal the investment cost. Recall that L is the expectation of the present discounted value of the investment cost. Since it is known with certainty (when the investment is undertaken) that the investment cost is F_0 and that it will be paid immediately, the expectation equals F_0. It is easy to see from (A7) that when $D = \Delta^*$. then $L = F$. The second two boundary conditions are that when $D = 0$, $L = 0$ and that the limit as $F \to \infty$ of L is 0. Both of these conditions are met by (A7). Therefore, (A7) is the solution to the expectation on the right hand side of (14), so (15) is the value of the investment opportunity. Δ^* is simply the boundary that maximizes (15).

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FOOTNOTES

1. Several recent papers deal with irreversibility. Baldwin and Meyer (1979) discuss irreversibility when mutually exclusive investment opportunities arrive stochastically over time. Bernanke (1981) develops a model where waiting to invest is optimal pending the resolution of significant uncertainty. Bernanke also provides a useful discussion of previous papers dealing with irreversibility and their relation to financial option models and search theory models. Brock, Rothschild and Stiglitz (1982) study the general tree-cutting problem for a variety of stochastic processes.

2. Brealey and Myers (1981), for example, discuss the problem of optimal timing of investment in a certainty case, and they proceed in this way explicitly.

3. We set up the investment problem in the context of a firm. The same analysis applies for public-sector cost-benefit analyses.

4. A firm could realize V_t by installing the project and selling the rights to the net cash flows. We are assuming that it knows the market value of the claim on the net cash flows if it installs the capacity today, but it is uncertain about what this market value will be if it waits.

5. This event is modelled as Poisson process.

6. This is a term used by Brock, Miller, and Scheinkman (1981), who study a problem related to the one in this paper.

7. It is important to note that while the government cannot, if it wishes, immediately realize V_t by selling a claim on the future net benefits from the project, it knows V_t with certainty at time t. V_t represents the analysts' best guess of the gain in social welfare after installing the project today. It includes, as part of the discounting, an adjustment for the risk that the benefits from the project may turn out to be small in the future.

8. Cox and Miller (1965) provide a good introduction to first-passage problems.

9. While we have said earlier that we only treat the case in which $T = \infty$, we set up the problem for an arbitrary T. We do this both for expositional reasons and because it will prove useful in solving the case in which there may be jumps in V_t.

10. In general, in the case with risk-averse investors, there is no reason to expect ρ to be constant over time. We deal exclusively with the case $T = \infty$, however and ρ is then constant, as we will show.

11. Note that we have assumed that there is a single stopping boundary. We have not proved that the upper limit to the stopping region is infinity. It is logically conceivable that if V_t were sufficiently greater than C_t^*, then it would pay to wait. Recall, however, that the value of waiting is basically derived from the downside protection afforded by the option <u>not</u> to invest. This downside protection becomes less relevant, <u>ceteris paribus</u>, the greater is V - F. Therefore, if the cost of waiting exceeds the value for some C_t^*,

then it should also exceed it for all $V > C_t^*$.

12. Merton (1973) makes this point in a discussion of option pricing.

13. McKean, in his Appendix to Samuelson (1970), also solves the same problem as the solution to a partial differential equation.

14. Cox and Miller (1965) solve the same problem where V follows arithmetic Brownian Motion. Our solution can be obtained from their solution by noticing that if V follows (1a), then ℓnV follows arithmetic Brownian Motion with drift $\alpha_v - \frac{1}{2}\sigma_v^2$. Making these substitutions in their formula (38) yields (10) above.

15. Merton (1973) shows that this condition is equivalent to Samuelson's "high contact" boundary condition.

16. It is straightforward to figure the expected length of time until the investment occurs. Notice that (10) is the moment generating function for the first passage density $g(\cdot)$. Taking the derivative of each side of (10) with respect to ρ and evaluating at $\rho = 0$ yields

$$E_0\{t'\} = \frac{-\ell n(\frac{V_0}{C^*})(\frac{V_0}{C^*})}{\alpha_v - \frac{1}{2}\sigma_v^2}$$

Note that since $V_0 < C^*$, $E_0\{t'\}$ is positive. Equation (14) is only valid if $\alpha_v > \frac{1}{2}\sigma_v^2$. If $\alpha_v < \frac{1}{2}\sigma_v^2$, then there is a positive probability that V_t will never reach <u>any</u> boundary set greater than V_0 and the expectation ceases to exist.

17. Merton (1976) first obtained this result, when he showed that the formula for a call option written on a stock for which there is a possibility of complete ruin, is obtained by replacing r with $r + \lambda$ in the Black-Scholes formula. Merton shows that the possibility of complete ruin for the stock makes a call option <u>more</u> valuable. In our case, the possibility of complete ruin makes the option to invest <u>less</u> valuable. We discuss this later.

18. We thank Alex Kane for pointing this out to us.

19. Financial theory (the CAPM) conveys the lesson that zero-beta assets can be treated the same as risk-free assets (e.g., both will earn the risk-free rate in equilibrium). Nevertheless, in this context, (as is true in all option pricing models) a truly risk-free asset ($\sigma_v^2 = 0$) differs from a risky zero-beta asset.

20. The annual standard deviation of the annual rate of return on the stock market is .2, which implies a σ_v^2 of .04.

21. It should be noted that this result is a consequence of the assumption that V follows geometric Brownian Motion with constant parameters. Brock, Rothschild and Stiglitz show that when the stochastic process for V has a lower absorbing barrier sufficiently close to the current value of V, then an increase in variance can lower the value of the option. With processes like (1), zero is a natural absorbing barrier, but one which is never reached in finite time.

TABLE I

Expected Growth Rates and Standard Deviations of Growth Rates of V for Different Demand Elasticities

		$\sigma_\theta = .05$		$\sigma_\theta = .25$	
η	α_θ	α_v	σ_v	α_v	σ_v
	.05	.1425	.1500	-.0375	.7500
∞	.00	-.0075	.1500	-.1875	.7500
	-.05	-.1575	.1500	-.3375	.7500
	-.25	-.7575	.1500	-.9375	.7500
	.05	.0500	.0500	.0500	.2500
1.0	.00	.0000	.0500	.0000	.2500
	-.05	-.0500	.0500	-.0500	.2500
	-.25	-.2500	.0500	-.2500	.2500
	.05	.0073	.0071	.0110	.0357
0.1	.00	.0002	.0071	.0038	.0357
	-.05	-.0070	.0071	-.0033	.0357
	-.25	-.0356	.0071	-.0319	.0357

*These calculations assume that the output elasticity of labor is $\beta = 2/3$.

TABLE II

Value of Investment Opportunity When V = F = 1

	r = .02			r = .05		
	α_v					
σ_v^2	.00	-.05	-.25	.00	-.05	-.25
.02	.250	.063	.014	.162	.059	.014
.04	.341	.113	.028	.225	.102	.028
.06	.404	.155	.041	.272	.138	.041
.08	.452	.190	.054	.309	.168	.053
.10	.491	.222	.066	.341	.194	.065
.30	.684	.419	.167	.523	.365	.162
.50	.763	.524	.243	.615	.463	.235
1.0	.848	.661	.374	.730	.599	.360

Note: Entries caluclated using (12) with

$$\varepsilon = \sqrt{\left(\frac{\alpha_v}{\sigma_v^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma_v^2}} + \left(\frac{1}{2} - \frac{\alpha_v}{\sigma_v^2}\right)$$

TABLE III

Values of C^*/F for Various Parameters

	r = .02			r = .05		
σ_v^2	$\alpha_v = 0$	$\alpha_v = -.05$	$\alpha_v = -.25$	$\alpha_v = 0$	$\alpha_v = -.05$	$\alpha_v = -.25$
.02	2.00	1.20	1.04	1.56	1.17	1.04
.04	2.62	1.36	1.08	1.86	1.32	1.08
.06	3.19	1.53	1.12	2.13	1.46	1.12
.08	3.73	1.69	1.16	2.38	1.58	1.16
.10	4.27	1.85	1.20	2.62	1.71	1.19
.30	9.39	3.34	1.77	4.79	2.82	1.56
.50	14.4	4.78	1.96	6.85	3.87	1.91
1.0	26.9	8.39	2.90	11.9	6.42	2.77

Note: Entries are calculated using equations (10) and (12) in the text.