TECHNICAL WORKING PAPER SERIES

A CES INDIRECT PRODUCTION FUNCTION

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Technical Working Paper No. 188

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
October 1995

I thank the C.V. Starr Center for Applied Economics at New York University for technical and financial help, and Jess Benhabib, Jayasri Dutta, Jordi Gali, and Ken Wolpin for helpful comments. This paper is part of NBER's research program in Productivity. Any opinions expressed are those of the author and not those of the National Bureau of Economic Research.

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ABSTRACT

This paper derives an indirect production function that is, in a special case, of a constant elasticity of substitution form. This is not a contribution to the theory of aggregation generally. Instead it is a microfoundation for a specific but popular production function -- the CES -- that helps us express the important concept of the elasticity of substitution in terms of more primitive, and more intuitive concepts of the returns to scale. The paper presents a simple lemma, and then shows that several and diverse applications have a common logical structure: the production function often used in growth theory, the utility function when there is household production, human capital theory, and the concept of the aggregate technology shock.

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A CES Indirect Production Function

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October 14, 1995.

1. Introduction

In this paper I derive an indirect production function that is, in a special case, of a constant elasticity of substitution form. This is not a contribution to the theory of aggregation generally. Instead it is a microfoundation for a specific production function -- the CES -- that helps us express the important concept of the elasticity of substitution in terms of more primitive, and (to me) more intuitive concept of the returns to scale. The main result is given in section 2, along with some remarks. Section 3 then discusses some applications.

2. The main result

Let $i \in [0, A]$ be an index, where A could be infinity. Let $\mathbf{h} = (h_i)$ and $\mathbf{x} = (x_i)$ be nonnegative vectors with individual elements $h_i \in R_+$ and $x_i \in X$ indexed by i, and let (f^i) be a vector of real-valued functions defined on X.

(1)
$$Y(H, \mathbf{x}) = \max_{(h_i)_0^\infty} \left\{ \int_0^A h_i^{\alpha} f^i(x_i)\, di \right\},$$

subject to the resource constraint

(P)

(2)
$$\int_0^A h_i\, di \leq H.$$

In words, the problem **(P)** is one of maximizing the sum, over the index i, of the index-specific

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"outputs" $h_i^{\alpha} f^i(x_i)$, with $\mathbf{x}$ given, and with $\mathbf{h}$ constrained by (2). Implicit in the constraint is the assumption that a unit of H can be allocated to at most one location i. The result is:

Lemma: If $\alpha < 1$,

$$(3) \qquad Y(H, \mathbf{x}) = H^{\alpha}\left(\int_0^A f^i(x_i)^{1/(1-\alpha)} di\right)^{1-\alpha}.$$

Proof: Form the Lagrangean

$$L = \int_0^A h_i^{\alpha} f^i(x_i)\, di + \lambda\left[H - \int_0^A h_i\, di\right]$$

The Lagrangean is differentiable, so that at an interior solution (dictated by α being less than 1) the derivative with respect to the control, $\mathbf{h}$, must be zero. So, we differentiate under the integral sign with respect to h_i to get the first-order necessary conditions for a maximum: $\alpha h_i^{\alpha-1} f^i(x_i) = \lambda$, so that

$$(4) \qquad h_i = (\alpha/\lambda)^{1/(1-\alpha)} f^i(x_i)^{1/(1-\alpha)}$$

Integrating both sides of (4) over $i \in [0, A]$, and assuming that at the optimal $\mathbf{h}$, the constraint (2) binds with equality, we get

$$(5) \qquad (\alpha/\lambda)^{1/(1-\alpha)} = H\left[\left(\int_0^A f^i(x_i)^{1/(1-\alpha)} di\right)\right]^{-1}.$$

Substituting this into (4) gives

$$(6) \qquad h_i = H f^i(x_i)^{1/(1-\alpha)}\left[\left(\int_0^A f^i(x_i)^{1/(1-\alpha)} di\right)\right]^{-1}.$$

Substituting for h_i into the maximand and observing that

$$1 + \frac{\alpha}{(1-\alpha)} = \frac{1}{(1-\alpha)}$$

implies the assertion of the lemma. ■

3. Remarks

3.1 Relation to the literature: The lemma says nothing new about aggregation generally -- it is a special case of an old result about when one can aggregate and express a production of many capital inputs in terms of a capital aggregate. Solow (1955-6) showed that this can be done in general if and only if the marginal rate of substitution between the different types of capital be independent of the aggregate amount of labor. This condition, also known as the Leontieff condition, requires that

$$\frac{\partial Y/\partial x_i}{\partial Y/\partial x_j} \quad \text{be independent of } H,$$

and evidently it is met in the case at hand. My perusal of this literature failed to uncover the result in the lemma above; the closest thing to it is in eq. (29) of Whitaker (1968), which easily could have been shown to imply the result -- but wasn't. The modern literature on the vintage capital model [e.g., Benhabib and Rusticchini (1993), and Cooley *et al* (1995)] also overlooks the result.

3.2 The equal allocations case: When f does not depend on i, and when $x_i = x$ for all i, h_i is allocated equally over the different i, so that $h_i = H/A$, and

$$\text{(7)} \qquad Y(H, \mathbf{x}) = H^{\alpha} A^{1-\alpha} f(x),$$

emerges either from **(1)** or from **(3)**. This case is often analyzed in aggregative models.

3.3 h_i can be a composite input: The restriction that h_i be a scalar can sometimes be relaxed. Suppose that the index-specific output is not $h_i{}^{\alpha} f^i(x_i)$, but

$$y_i = h_{1,i}{}^{\alpha_1} h_{2,i}{}^{\alpha_2} f^i(x_i).$$

Suppose that the endowments are now H_1 and H_2, respectively. As long as α_1 and α_2 do not depend on i, their optimal allocation over i will satisfy the equation

$$h_{1,i} = \frac{H_1}{H_2} h_{2,i}$$

and the composite input h is then defined by the equation

$$h_i^{\alpha} = (H_1/H_2)^{\alpha_1} \; h_{2,i}^{\alpha_1+\alpha_2}.$$

so that we have $\alpha = \alpha_1 + \alpha_2$.

3.4 <u>No restrictions on X</u>: There is no restriction on the domain of the f^i, only that they be real-valued. This means that very complicated objects can be aggregated.

3.5 <u>Corner solutions for h</u>: With $\alpha < 1$, it is not optimal to set $h_i = 0$ unless $f^i = 0$. But if $h_i = 0$ for $i \in A^* \subset [0, A]$, say, then the lemma is still true, but with the domain of integration in (3) being the complement of the set A^*. On the other hand, the lemma fails at the "other" corner: if there are increasing returns to scale at some locations so that one or a few locations i swallows up the entire endowment of H, the expression in (3) is false.

3.6 <u>Different grades of capital goods</u>: We can write $f^i(x_i) = (1 + g)^i x_i$ so that each successive x_i is of higher quality than the previous one. The lemma still applies.

4. Returns to scale and the elasticity of substitution

Suppose that each individual process has the production function $\theta_i h_i^{\alpha} x_i^{\gamma}$. The returns to scale at the individual level are $\alpha + \gamma$, and the same is true at the aggregate level in (3) when A is taken as fixed. The parameters of the micro production function determine the elasticity of substitution of the x's in (3), which is

$$\sigma = \frac{(1 - \alpha)}{(1 - \alpha - \gamma)}.$$

So, as $\alpha + \gamma \rightarrow 1$ and we approach constant returns to scale at each individual location, σ approaches infinity.

If the x_i's are not given, but can be allocated (based on a second resource constraint, say $\int x_i \, di \leq K$, as long as $\alpha + \gamma < 1$, there is a preference for variety, and each location gets some resources, h_i, and x_i. But as $\alpha + \gamma \rightarrow 1$ and we approach constant returns to scale at each location, the preference for variety and the gains to the division of labor (i.e., of H) disappear. As returns become constant, the most efficient location i absorbs all the resources, aggregate output tends to

$$Y \quad \rightarrow \quad \left\{ \mathrm{Max}_{i \in [0, A]} \, \theta_i \right\} H^{\alpha} K^{(1-\alpha)}.$$

5. Applications

The lemma is useful in a variety of applications:

5.1 **Growth theory**: Here h_i would be efficiency units of labor applied to the type-i capital goods. If there are x_i such goods, the output produced by such goods would be $h_i^{*} f^{i}(x_i)$. This is the output of the i'th production process, and aggregate output is just the sum of the individual outputs. If, in addition, we postulate that $f^{i}(x_i) = x_i^{\gamma}$, say, then

$$\textbf{(8)} \qquad Y(H, \mathbf{x}) = H^{\alpha} \left(\int_0^A x_i^{\gamma/(1-\alpha)} \, di \right)^{1-\alpha}.$$

A couple of points are noteworthy:

5.1A <u>The equal allocations case</u>: If in the above equation $x_i = x$ for all i,

$$Y(H, \mathbf{x}) = H^{\alpha} A^{1-\alpha} x^{\gamma}.$$

5.1B <u>The x_i must be interdependent</u>: Because a higher $f^{i}(x_i)$ means that location i will draw some resources (H) away from other locations, the x_i can not be independent. By this I mean that this microfoundation can <u>not</u> produce an indirect production function of the form:

$$Y = H^{a} \int x_i^{b} \, di$$

used, among others, by Romer (1990), and Jones (1995). In such formulation, the marginal product of x_i does not depend on x_j, and this is not possible in the present framework. Benhabib, Perli and Xie's formulation is consistent with **(8)**.

5.2. **Consumption theory**: Since Spence (1976) and Dixit and Stiglitz (1977), the CES functional form has been used for utility functions. The justification is usually on grounds of simplicity. But the lemma implies that these Dixit-Stiglitz preferences emerge quite naturally if one adopts Becker's (1965) approach to consumption theory. In Becker's model, utility is a function of "commodities". Commodities are produced with goods and with time, and so one can derive an indirect utility function as a function of goods alone, assuming that time is allocated optimally. Our approach can yield a CES form for this indirect utility function.

To see this, let x_i be the quantity of good i, and let there be as many commodities as goods[2] and let Z_i be the amount of the i'th commodity. Write utility as

(9) $$U = \int_0^A \theta_i Z_i \, di,$$

and suppose that

(10) $$Z_i = \mu_i h_i^{\alpha} x_i^{\gamma}$$

where h_i is assumed to be time, and H is the individual's time endowment. If the individual can occupy herself with just one good at a time, then (2) is the appropriate constraint. Then the expression on (3) is the correct indirect utility, with $f^i(x_i) = \theta_i \mu_i x_i^{\gamma}$.

One can generalize the constraint on h_i and allow the use of multiple goods at a time. If a unit of time can simultaneously be devoted to the "enjoyment" of η goods, then ηH should be inserted in place of H on the right-hand sides of (2) and (3), where the latter would then read

(10) $$Y(H, x) = (\eta H)^{\alpha} \left(\int_0^A (\theta_i \mu_i x_i^{\gamma})^{1/(1-\alpha)} \, di \right)^{1-\alpha}.$$

5.3 **Different types of human capital**: In sections 5.1 and 5.2, the x_i denoted consumption goods and/or capital goods of different types, but the cooperating factor was homogeneous. Now we turn things around, and assume that the x_i denote different types of human capital, or different amounts of human ability that, for one reason or another, can not bed

[2] This is assumed for ease of exposition. One could in fact allow each commodity to be produced with several goods, and with time.

aggregated into a single index. In the next two examples, some other homogeneous factor is allocated across different types of human capital.

5.3A Akerlof (1970) presents a model in which A would denote the number of workers in the economy, while x_i would stand for the i'th worker's ability. The output produced by a worker, $h_i \cdot f(x_i)$ depends only on his or her ability, and on the amount of capital, h_i, that the worker works with. Physical capital is homogeneous, and its economy-wide supply is H units. A unit of capital can be assigned to at most one worker, and so the constraint (2) holds. Akerlof's equilibrium coincides with the aggregate output-maximizing solution, and so (6) represents the equilibrium allocation of capital to worker i (but without the i superscript on f), and aggregate output is given by (3).[3]

5.3B Lucas (1978) presents a model in which a homogeneous aggregate composed of raw labor and physical capital is allocated to managers of different types (each manager employs the same capital/labor ratio so they can be aggregated into a homogeneous, composite input, as described in section 3.3). So $f^i(x^i)$ is the ability of the i'th manager, and there is a measure A of managers; H stands for the economywide equilibrium quantity of the raw labor-physical capital aggregate, and the output of each manager is $h \cdot f^i(x^i)$. Each manager has a limited span of control, so that $\alpha < 1$. There are no external effects in the model, and aggregate output is just the sum of the outputs produced by the managers, as in (1), and so the lemma applies, with aggregate output again given by (3), as in Akerlof's example.

5.3C Lucas (1993) presents a model in which each i denotes a different good, or a different production process. There are different grades of goods (as in section 3.6), and x_i denotes the level of experience on producing good i. The current allocation of labor would be h_i and A would denote the most advanced good produced to date. If the allocation of labor were such that current output was maximized, the lemma would apply.

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[3] The main point of Akerlof's paper is, however, that some low-ability individuals may not, in equilibrium, get any capital, because the marginal product of h_i remains bounded as h_i goes to zero, which can not happen if $\alpha < 1$. See section 3.5 for how this would affects things.

3.D. **Aggregation of shocks**: The real business cycle approach has for the most part dealt with aggregate shocks to technology. But in fact, there are shocks to sectors, to firms, to plants. So we can think of $f^i(x) = x_i$ as a technology shock to location i, determining the productivity of the inputs allocated to that location. The inputs are homogeneous, and h_i is the amount allocated to location i. Then if aggregate output is the sum of the outputs produced at the individual locations, the lemma applies, and we can write aggregate output as a function of the aggregate factor endowment H and the aggregate shock Z,

(12) $$Y = H^{\alpha} Z,$$

where the aggregate shock is given by

(14) $$Z = \left(\int_0^A x_i^{1/(1-\alpha)} di\right)^{1-\alpha}.$$

Curiously, since $x^{1/(1-\alpha)}$ is convex, a mean-preserving spread of the local shocks raises Y!

6. Conclusion

Since the CES form is so popular, it is useful to interpret its parameters in terms of primitives. This note has presented a simple lemma, and shown that several and diverse applications have a common logical structure.

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