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TESTING THE AUTOCORRELATION STRUCTURE OF DISTURBANCES IN ORDINARY LEAST SQUARES AND INSTRUMENTAL VARIABLES REGRESSIONS

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ABSTRACT

This paper derives the asymptotic distribution for a vector of sample autocorrelations of regression residuals from a quite general linear model. The asymptotic distribution forms the basis for a test of the null hypothesis that the regression error follows a moving average of order $q \geq 0$ against the general alternative that autocorrelations of the regression error are non-zero at lags greater than q. By allowing for endogenous, predetermined and/or exogenous regressors, for estimation by either ordinary least squares or a number of instrumental variables techniques, for the case $q>0$, and for a conditionally heteroscedastic error term, the test described here is applicable in a variety of situations where such popular tests as the Box-Pierce (1970) test, Durbin's (1970) h test, and Godfrey's (1978b) Lagrange multiplier test are not applicable. The finite sample properties of the test are examined in Monte Carlo simulations where, with a sample sizes of 50 and 100 observations, the test appears to be quite reliable.

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I. Introduction

This paper derives the asymptotic distribution of a vector of sample autocorrelations of regression residuals from a quite general linear regression model. The model is allowed to have a regression error that is a moving average of order $q \geq 0$ with possibly conditionally heteroscedastic innovations; to have strictly exogenous, predetermined, and/or endogenous regressors; and to be estimated by a variety of Generalized Method of Moments estimators, such as ordinary least squares, two-stage least squares, or two-step two-stage least squares.[1]

One important use of the distribution derived here is to form the basis for a simple test of the null hypothesis that the regression error is a moving average of known order $q \geq 0$ against the general alternative that autocorrelations of the regression error are non-zero at lags greater than q. The test - denoted the ℓ test - is thus general enough to test the null hypothesis that the regression error has no serial correlation ($q=0$) or the null hypothesis that serial correlation in the regression error exists, but dies out at a known finite lag ($q>0$). This paper both describes how to implement the ℓ test and uses Monte Carlo simulations to evaluate its performance in finite samples.

The ℓ test is especially attractive because it can be used in at least three, frequently-encountered situations where such popular tests as the Box-Pierce (1970) test, Durbin's (1970) h test, and the Lagrange multiplier tests described by Godfrey (1978b) either are not applicable or are costly to compute.

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[1] See Hansen (1982) for a description of Generalized Method of Moments estimators. Cumby, Huizinga, and Obstfeld (1983) describe the two-step two-stage least squares estimator.

The first situation is when the regression contains endogenous variables. The three popular tests listed above are not valid when the regression has been estimated by instrumental variables, and the Box-Pierce test is further restricted to having only lagged dependent variables.[2] In contrast, the ℓ test can be used with not only with ordinary least squares but also with a wide class of instrumental variables estimators.

A second situation is when q>0, which arises in studies of asset returns over holding periods which differ from the observation interval and in studies where time aggregated data are used.[3] In this situation, existing tests that investigate the serial correlation of the regression error require estimating the parameters of the moving average error process, and therefore necessitate nonlinear estimation.[4] In contrast, the ℓ test described and analyzed in this paper avoids the use of nonlinear estimation because it is

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[2] Godfrey (1978a) describes a test that is valid with some instrumental variables estimators, but the test is not valid in the presence of conditionally heteroscedastic errors or with instrumental variables estimators such as two-step two-stage least squares. The test, like Durbin's h test, is also restricted to testing the significance of the first autocorrelation of the regression error.

[3] See, for example, work on returns in the foreign exchange market by Hansen and Hodrick (1980), the study of real interest rates by Huizinga and Mishkin (1984), the investigation of stock returns by Fama and French (1988), and work on the term structure of interest rates by Mishkin (1990). Hall (1988), Hansen and Singleton (1988), and Christiano, Eichenbaum, and Marshall (1987) address the issue of time aggregated data.

[4] This is true of the Box-Pierce test, the likelihood ratio test, and, as discussed in Godfrey (1978c), the Lagrange multiplier test. It is also true of a GMM approach that jointly estimates the parameters of primary interest and the residual autocorrelations. A procedure that would not require full maximum likelihood estimation of the moving average parameters is to implement a $C(\alpha)$ test. Such a test would be asymptotically equivalent to the likelihood ratio and Lagrange multiplier tests and would only require that the derivatives of the likelihood function be evaluated at initial consistent estimates. See Godfrey (1989) pp. 27-28.

based solely on the sample autocorrelations of regression residuals and a consistent measure of their asymptotic covariance matrix. The ℓ test thus reflects a desire for simplicity, and for ensuring that regression diagnostics do not become more costly or more difficult to compute than the original regression.

The third situation is conditional heteroscedasticity of the error term, a situation that is frequently detected in empirical studies. Monte Carlo simulations presented in this paper indicate that the presence of conditional heteroscedasticity may seriously undermine tests for serial correlation of regression errors that ignore its presence. The ℓ test can be used with either conditionally heteroscedastic or homoscedastic errors.

The outline of the paper is as follows. In section II, we derive the asymptotic distribution of the sample autocorrelations at lags q+1 to q+s of regression residuals from a model where the regression errors are a q^{th} order moving average with possibly conditionally heteroscedastic innovations. The regression is assumed to be estimated by instrumental variables, with instruments that are predetermined, but not necessarily strictly exogenous. We note how the distribution simplifies when the regression errors are conditionally homoscedastic and when all regressors are predetermined or strictly exogenous variables so that ordinary least squares is appropriate. Based on this asymptotic distribution of the sample autocorrelations of regression residuals, a test of the hypothesis that the true regression errors are a q^{th} order moving average process is presented in section III. Monte Carlo results presented in section IV illustrate how well the asymptotic distribution theory works in finite samples. Section V contains summary remarks.

II. Distribution of Sample Autocorrelations of Regression Residuals

The regression equation to be considered in this paper is

$$(1) \qquad y_t = X_t\delta + \varepsilon_t \qquad t = 1,\ldots,T$$

where y_t and ε_t are scalar random variables, X_t is a 1xk vector of the k scalar random variables $X_{1,t}$, $X_{2,t}$, ... $X_{k,t}$, and δ is a kx1 vector of unknown parameters. The vector of regressors, X_t, may include jointly endogenous variables (those contemporaneously correlated with ε_t), predetermined variables (those uncorrelated with ε_{t+j} for j ≥0 but are correlated with ε_{t-j} for some j >0), or strictly exogenous variables (those uncorrelated with ε_{t+j} for all j.).

The regression errors ε_t are assumed to have mean zero and satisfy two other conditions. First, though they are allowed to be conditionally heteroscedastic, they are assumed to be unconditionally homoscedastic. Second, for a known q≥0, their autocorrelations at all lags greater than q are required to be zero.

It is also assumed that there exists a 1xh vector of instrumental variables Z_t, comprised of h≥k scalar random variables $Z_{1,t}$, $Z_{2,t}$,..., $Z_{h,t}$, each of which are uncorrelated with ε_t. Z_t is required to be predetermined, but not necessarily strictly exogenous, with respect to ε_t. These assumptions are summarized by,

$$(2) \qquad E(\varepsilon_t) = 0, \qquad E(\varepsilon_t^2) = \sigma_\varepsilon^2,$$

$$(3) \qquad E(\varepsilon_t\varepsilon_{t-n})/\sigma_\varepsilon^2 = \rho_n,$$

and

(4) $E(\varepsilon_t | Z_t, Z_{t-1}, \ldots, \varepsilon_{t-q-1}, \varepsilon_{t-q-2}, \ldots) = 0.$

Furthermore, the hxh matrix

(5) $\Omega = \lim_{T\to\infty} (1/T)\ E(Z'\varepsilon\varepsilon'Z)$

is assumed to exist and be of full rank.

It is assumed that kx1 parameter vector δ in equation (1) has been estimated using a root-t consistent estimator of the form,

(6) $d = (X'Z\ A_T^{-1}\ Z'X)^{-1}\ X'Z\ A_T^{-1}\ Z'y,$

for some observable matrix A_T. This formulation is general enough to include ordinary least squares, $A_T = (X'X/T)$ and Z=X, two-stage least squares, $A_T = (Z'Z/T)$, and two-step two-stage least squares, A_T is a consistent estimate of Ω. The asymptotic covariance matrix for the estimator d is denoted V_d, with

(7) $V_d = D\ \Omega\ D'$

and the kxh matrix D given by,

(8) $D = \text{plim}\ T\ (X'Z\ A_T^{-1}\ Z'X)^{-1}\ X'Z\ A_T^{-1}.$

The objective of this section of the paper is to derive, within the framework of the model described by equations (1) - (8), the asymptotic covariance matrix of the sample autocorrelations of the regression residuals, $\hat{\varepsilon}_t = y_t - X_t d$. In the following section we show how a consistent estimate of this covariance matrix can be used to test the hypothesis that the sx1 vector $\rho = (\rho_{q+1}, \ldots, \rho_{q+s})' = 0$.

Let $\hat{r} = [\hat{r}_{q+1}, \hat{r}_{q+2}, \ldots, \hat{r}_{q+s}]'$ and

$$(9) \qquad \hat{r}_n = \frac{\sum_{t=n+1}^{T} \hat{\varepsilon}_t \hat{\varepsilon}_{t-n}}{\sum_{t=1}^{T} \hat{\varepsilon}_t^2} .$$

By the mean value theorem,

$$(10) \qquad \sqrt{T}\,\hat{r} = \sqrt{T}\,r + \frac{\partial r}{\partial \delta} \sqrt{T}\,(d-\delta),$$

where the sx1 vector $r = [r_{q+1}, r_{q+2}, \ldots, r_{q+s}]'$,

$$(11) \qquad r_n = \frac{\sum_{t=n+1}^{T} \varepsilon_t \varepsilon_{t-n}}{\sum_{t=1}^{T} \varepsilon_t^2} ,$$

and the jth row of the sxk matrix $\partial r/\partial \delta$ is evaluated at d_j^*, which lies between d and δ. Equation (10) shows that the asymptotic covariance matrix of $\hat{r}$ (the vector of sample autocorrelations of the regression residuals) can be derived as the asymptotic covariance matrix for the sum of r (the vector of sample autocorrelations of the true disturbances) and $\partial r/\partial \delta$ $(d-\delta)$. Only when $\partial r/\partial \delta$ can safely be ignored will the sampling variation in the estimation of δ not affect the sampling variation in the estimation of ρ.

Let the sxk matrix B have i,j^{th} element,

$$(12) \qquad B(i,j) = - [\, E(\varepsilon_{t-q-i} X_{j,t}) + E(\varepsilon_t X_{j,t-q-i}) \,] \,/\, E(\varepsilon_t^2)$$

We show in the appendix that $B = \text{plim}\ \partial r/\partial \delta$ and thus that BV_dB' is the asymptotic covariance matrix of $\partial r/\partial \delta$ $(d-\delta)$. In most models, the implication of equation (4) that $E(\varepsilon_t | \varepsilon_{t-q-1}, \varepsilon_{t-q-2}, \ldots) = 0$ will be sufficient to ensure $X_{j,t-q-1}$ is predetermined with respect to ε_t and thus that the second term of the sum in equation (12) is zero.

To complete the notation, let $\xi_{i,t} = \varepsilon_t \varepsilon_{t-q-i}$ for $i=1,\ldots,s$, $\omega_{j,t} = \varepsilon_t Z_{j,t}$ for $j=1,\ldots,h$, the ij^{th} element of the sxs matrix V_r be given by

$$(13) \qquad V_r(i,j) = \sigma_\varepsilon^{-4} \sum_{n=-q}^{q} E(\xi_{i,t}\xi_{j,t-n}).$$

and the ij^{th} element of the sxh matrix C be given by

$$(14) \qquad C(i,j) = \sigma_\varepsilon^{-2} \sum_{n=-q}^{q} E(\xi_{i,t}\omega_{j,t-n}).$$

In the appendix we show that V_r is the asymptotic covariance matrix of r and that the asymptotic covariance matrix of r with $\partial r/\partial \delta$ $(d-\delta)$ is BDC'. Proposition 1 combines these findings in giving the key result of this section.[5]

Proposition 1. **Given equations (1) through (14) and the regularity conditions stated in the appendix,** $\sqrt{T}\, \hat{r} \overset{A}{\sim} N(0,\, V_{\hat{r}})$, **where** $V_{\hat{r}} = V_r + BV_dB' + CD'B' + BDC'$.

Proposition 1 states that, in general, having to estimate the residuals will affect the asymptotic distribution of their sample autocorrelations. The following special cases of the general model provide further insight into proposition 1 and help clarify the relationship between tests based on the asymptotic distribution of $\hat{r}$ and tests of residual autocorrelation proposed elsewhere in the literature.

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[5] The proof of proposition 1 can be found in the appendix.

Case (i): Strictly Exogenous Regressors.

Since $B = 0$ when the regressors are strictly exogenous, $V_r^{\wedge} = V_r$ and one can safely ignore the fact that the true residuals are unavailable.

Case (ii): Conditionally Homoscedastic Residuals.

We show in the appendix that when the residuals are conditionally homoscedastic, V_r and C can be rewritten as

$$(15) \qquad V_r(i,j) = \sum_{n=-q}^{q} \rho_{n-i+j}\rho_n$$

and

$$(16) \qquad C(i,j) = \sum_{n=-q}^{q} \rho_n \, E(\varepsilon_{t-q-i} Z_{j,t-n}).$$

The well known result that the sample autocorrelations of a serially uncorrelated series are independent and asymptotically normal with variance 1/T follows from (15) with q=0. When q>0 the sample autocorrelations are not independent and, though asymptotically normal, do not have variance 1/T.

Case (iii): Conditionally Homoscedastic Residuals,

Predetermined Regressors, and q=0.

When the regressors are predetermined, ordinary least squares yields consistent estimates of δ, we can set $Z = X$, $A_T = X'X/T$, and the second term of B will be zero. Combining this with the assumption of conditional homoscedasticity (so that equation (16) is valid) and q=0 (so that $\rho_n = 0$ for $n \neq 0$) yields $C = -\sigma_\varepsilon^2 B$. Furthermore, $V_d = \sigma_\varepsilon^2 \operatorname{plim}(X'X/T)^{-1} = \sigma_\varepsilon^2 D$ so that $BDC' = -BV_dB'$. Finally, it follows from equation (15) that in this case $V_r = I$, and thus $V_r^{\wedge} = I - BV_dB'$.

Unlike the case of strictly exogenous regressors, when regressors are merely predetermined one cannot safely ignore the use of regression residuals rather than the true disturbances in estimating autocorrelations. The expression $V_{\hat{r}} = I - BV_dB'$ can be used to derive the well-known Durbin's (1970) h-test. Durbin (1970) considers testing whether the autocorrelation of the error term at lag one is zero in a model with lagged dependent variables and strictly exogenous variables as regressors. In this case B will contain all zeros except a single value of minus one in the position corresponding to the dependent variable lagged once. Using V_{d1} to denote the estimated variance of the coefficient on this variable, the asymptotic variance of the first autocorrelation of the regression residuals is seen to be $1/T - V_{d1}$, which matches the formula given by Durbin (1970).[6]

Case (iv): Conditionally Homoscedastic Residuals,

Only Lagged Dependent Variables, and q=0.

When the regression error is conditionally homoscedastic, X_t contains only k lagged values of y_t, and $q = 0$, we have a special case of the model considered by Box and Pierce (1970), who propose testing the hypothesis of zero correlation in the regression error by comparing $Q_s = \hat{r}'\hat{r}$ to the critical value of a chi-squared random variable with s-k degrees of freedom. Understanding the logic behind the Box-Pierce test and why the test in general fails when regressors other than lagged dependent variables are present becomes quite simple using the result from case (iii) that $V_{\hat{r}} = I - BV_dB'$.

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[6] Godfrey (1978b) also considers the case of lagged endogenous and/or strictly exogenous regressors, conditionally homoscedastic errors and q=0. Among other things, he extends Durbin (1970) by showing that the asymptotic covariance matrix for a vector of sample autocorrelations of regression residuals is $I - BV_dB'$, the formula derived above.

Specifically, it can be shown that when X_t contains only lagged values of y_t, V_d approaches $(B'B)^{-1}$ as s increases. It follows that as s increases, $\hat{V}_r$ approaches $I - B(B'B)^{-1}B'$, an idempotent matrix of rank s-k. Hence, for both large s and large T, Q_s will be approximately distributed as a chi-square with s-k degrees of freedom.[7,8] If, however, X_t contains any variables other than lagged dependent variables, V_d will not in general approach $(B'B)^{-1}$ and it is unlikely, though not impossible, that $I - BV_dB'$ will be an idempotent matrix.

III. Testing Residual Autocorrelations Equal to Zero

The results presented in section II can be used to develop a Wald test of the null hypothesis that the regression error in equation (1) is uncorrelated with itself at lags q+1 through q+s.[9] Proposition 2 presents this result.

Proposition 2. Let $\hat{V}_r$, $\hat{B}$, $\hat{C}$, $\hat{D}$, and $\hat{V}_d$ be consistent estimates of V_r, B, C, D, and V_d. Then, given the conditions of Proposition 1,

$$\ell_{q,s} = T\,\hat{r}'[\hat{V}_r + \hat{B}\hat{V}_d\hat{B}' + \hat{C}\hat{D}'\hat{B}' + \hat{B}\hat{D}\hat{C}']^{-1}\,\hat{r} \overset{A}{\sim} \chi^2(s)$$

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[7] If W is an nx1 random normal vector with mean 0 and nxn covariance matrix V whose trace is nonzero, then W'W is distributed as a chi-square random variable with n-m degrees of freedom if and only if V is idempotent and has rank n-m. See Johnson and Kotz (1970), pages 177-178.

[8] Ljung (1986) investigates how large s must be before the Q_s statistic approaches the chi-square distribution. She finds that in samples of 50 or 100 observations, s ≥ 10 is sufficient for all AR(1) models examined and that s ≥ 2 is sufficient for AR(1) models with the autoregressive paramter below .9.

[9] In many instances, instrumental variables are chosen as lagged endogenous variables so that rejecting the null hypothesis may call into question the validity of equation (4). In such cases it may be preferable to think of the null hypothesis being tested as a joint hypothesis concerning the serial correlation of the residuals and the validity of the instruments. Viewed in this way, the test described in this paper becomes an alternative to the J-statistic proposed in Hansen (1982).

Proposition 2 states that if V_r, B, C, D, and V_d can be estimated consistently, then the $\ell_{q,s}$ statistic will be asymptotically distributed as a chi-square random variable with s degrees of freedom.[10] In the remaining part of this section we discuss how consistent estimates of V_r, B, C, D, and V_d can be formed.

Define the (h+s)x1 vector η_t by

$$(17) \qquad \eta_t = (\omega_{1,t}, \ldots, \omega_{h,t}, \xi_{1,t}, \ldots, \xi_{s,t})'.$$

Then the (h+s)x(h+s) spectral density matrix at frequency zero of η_t is,

$$(18) \qquad \Psi = \begin{bmatrix} \Omega & \sigma_\varepsilon^2 C' \\ \sigma_\varepsilon^2 C & V_r \sigma_\varepsilon^4 \end{bmatrix}.$$

Next, define the (2s)x(s+h) matrix Φ by

$$(19) \qquad \Phi = \begin{bmatrix} BD & 0 \\ 0 & \sigma_\varepsilon^{-2} I \end{bmatrix},$$

so that

$$(20) \qquad \Phi \Psi \Phi' = \begin{bmatrix} BV_dB' & BDC' \\ CD'B' & V_r \end{bmatrix}.$$

[10] Godfrey (1978b) considers a model with lagged endogenous and strictly exogenous regressors, conditionally homoscedastic errors and q=0. He shows that using $\hat{r}$ to test $\rho = 0$ is equivalent to the Lagrange multiplier test of the null hypothesis that the error term is serially uncorrelated against the alternatives that the error is MA(s) or AR(s) for s>0. Hence, in some models, the test described in proposition two is equivalent to a likelihood ratio test. However, Godfrey (1978c) shows that in the same model but with q>0, computation of the Lagrange multiplier test of the null hypothesis that the error term is MA(q) against the alternatives that the error is MA(q+s) or AR(q+s) requires that the moving average parameters be estimated. In this model the test described in proposition two may not possess all the desirable properties of a Lagrange multiplier or likelihood ratio test, but will be less computationally burdensome than those tests.

It follows from equation (20) that a consistent estimate of the asymptotic covariance matrix of $\hat{r}$ can be obtained from consistent estimates of Φ and Ψ. It also follows that if the consistent estimate of Ψ is positive definite, the resulting $\ell_{q,s}$ will be positive.

Consistently estimating Φ is straightforward. Let $\hat{E}$ be the Txs matrix,

$$(21) \qquad \hat{E}(i,j) = \begin{cases} \hat{\varepsilon}_{i-j-q} & \text{for } i-j-q > 0 \\ 0 & \text{otherwise} \end{cases}$$

so that the j^{th} column of $\hat{E}$ is the vector of regression residuals lagged q+j times. Then,

$$(22) \qquad \hat{\sigma}_{\varepsilon}^2 = (1/T) \sum_{t=1}^{T} \hat{\varepsilon}_t^2$$

$$(23) \qquad \hat{B} = -(\hat{E}'X/T)\hat{\sigma}_{\varepsilon}^2,$$

and

$$(24) \qquad \hat{D} = T\ (X'Z\ A_T^{-1}\ Z'X)^{-1}\ X'Z\ A_T^{-1},$$

are consistent estimates of σ_{ε}^2, B, and D respectively.

Consistently estimating Ψ is also straightforward. Let the (s+h)x1 vector $\hat{\eta}_t$ be given by,

$$(25) \qquad \hat{\eta}_t = (\hat{\varepsilon}_t Z_{1,t}, \ldots, \hat{\varepsilon}_t Z_{h,t}, \hat{\varepsilon}_t \hat{\varepsilon}_{t-q-1}, \ldots, \hat{\varepsilon}_t \hat{\varepsilon}_{t-q-s})'$$

and the (s+h)x(s+h) matrix R_n be given by,

$$(26) \qquad R_n = (1/T) \sum_{t=n+1}^{T} \hat{\eta}_t \hat{\eta}_{t-n}'.$$

Then, as described in Anderson (1971), there are a variety of (N+1)x1 weighting vectors $w^N = (w_0^N, \ldots, w_N^N)'$ such that the (s+h)x(s+h) matrix

$$(27) \qquad \hat{\Psi} = \sum_{n=-N}^{N} w^N_{|n|} R_n$$

is a consistent estimate of Ψ. Not all choices of w^N that give a consistent estimate of Ψ will also give a positive definite estimate, however.

Equations (15) and (16) in the previous section showed how the matrices V_r and C could be simplified in the case of homoscedastic errors.[11] With conditionally homoscedastic errors, the sxs matrix

$$(28) \qquad \hat{V}_r(i,j) = \sum_{n=-q}^{q} \hat{r}_{n-i+j}\hat{r}_n$$

is a consistent estimate of V_r, where $\hat{r}_j = 0$ for $|j| > q$. A consistent estimate of C is given by the sxh matrix

$$(29) \qquad \hat{C} = \hat{E}'\hat{V}_\varepsilon Z/T\hat{\sigma}^2_\varepsilon \text{ , where } \hat{V}_\varepsilon(i,j) = \hat{r}_{|i-j|}\hat{\sigma}^2_\varepsilon \text{ ,}$$

is an estimate of the covariance matrix of the error term.

While the analysis of this paper centers on the asymptotic distribution of simple autocorrelations, the results are also relevant for the asymptotic distribution of partial autocorrelations of regression residuals. Regressing $\hat{\varepsilon}_t$ on $\hat{\varepsilon}_{t-q-1}, \ldots, \hat{\varepsilon}_{t-q-s}$, yields the estimated coefficient vector $b = \hat{F}\hat{r}$, where $\hat{F} = (\hat{E}'\hat{E})^{-1}\hat{\varepsilon}'\hat{\varepsilon}$ converges in probability to a sxs matrix F. As a result, b converges in distribution to a Normal random variable with mean zero and covariance matrix $V_b = F\hat{V}_r F'$ when the null hypothesis is true, and the

[11] McLeod (1978) derives the asymptotic distribution of residual autocorrelations from univariate ARMA models with homoscedastic errors and, as we do here, suggests using a consistent estimate of the asymptotic covariance matrix to form a Wald test as an alternative to the Box-Pierce test. Breusch and Godfrey (1981) describe unpublished work by Sargan (1976) that suggests a test that is equivalent to the $\ell_{q,s}$ test when q=0 and the residuals are conditionally homoscedastic.

standard Wald statistic for testing b=0 will be numerically identical to the ℓ-statistic described in Proposition 2.[12] In the special case of q=0, F is an identity matrix so that even though b will not equal $\hat{r}$ in finite samples, one can replace $\hat{r}$ with b in proposition 2 and obtain a vaild test.

IV. Monte Carlo Experiments

In this section we present the results of Monte Carlo simulations that examine the finite sample distribution of the $\ell_{q,s}$ statistic in six models, with six specifications of the error term in each model. The models differ primarily in terms of whether the regressors are endogenous, predetermined, or strictly exogenous, though there is some variation in the number of regressors across models. The first four models, described in Tables 1 - 4, involve only predetermined or strictly exogenous variables. As a result, we use ordinary least squares to compute the parameter estimates. Model five, described in Table 6, is an overidentified, simultaneous equations model and is estimated by two-step two-stage least squares with the reduced form used to determine the choice of instruments. Model 6, described in Table 7, is a rational distributed lag model in autoregressive form and is also estimated by two-step two-stage least squares.[13] For all models, only the first equation is estimated and the second term of B is set to zero.

In each model we consider three specifications of a serially uncorrelated

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[12] Since the estimated covariance matrix for b reported by standard regression packages will not in general be a consistent estimate of V_b, testing b=0 with the typical F-test reported by these packages is not an asymptotically valid procedure.

[13] Both models five and six were estimated by two-stage least squares as well as two-step two-stage least squares. The performance of the ℓ-tests was the same for both estimation procedures, and thus we report only the results for the two-step two-stage procedure.

error (q=0) and three specifications of an error that follows a second-order moving average (q=2).[14] The first specification uses errors that are conditionally homoscedastic and normally distributed. The other two specifications use errors that are conditionally heteroscedastic. The two models of conditional heteroscedasticity used are an ARCH process and an Exponential ARCH process, both with innovations that are conditionally normally distributed.[15]

Each Monte Carlo experiment consists of 5,000 replications. Since every model we consider contains regressors that follow autoregressive processes, we set the initial values of these random variables to zero and generate 300 observations. Only the last 50 and 100 observations are used in the experiments. This should eliminate any impact of the initial conditions on the results. On each replication $\ell_{q,1}$, $\ell_{q,3}$, $\ell_{q,6}$ and $\ell_{q,12}$ are computed, corresponding to the hypotheses that one, three, six and twelve autocorrelations are equal to zero.

For each model and error specification, we use only one set of parameters. A prime concern in choosing parameter values for the models was to get roots close to the unit circle. Not only are roots close to unity frequent in real world data sets, but we suspect that such roots will present the greatest

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[14] Model 6 is the sole exception since estimating the rational distributed lag model in its autoregressive form induces a moving average error.

[15] Engle (1982) describes ARCH models and Nelson (1990) describes Exponential ARCH models. The ARCH parameter is chosen so that the fourth moment of the regression error will exist, as is required when obtaining the asymptotic distribution of the residual autocorrelations. Diebold (1986) presents conditions for the existence of the moments of ARCH processes. An Exponential ARCH has two advantages over the simple ARCH. First, it allows us to determine the persistence of shocks to the conditional variance with the coefficient on the lagged log variance. Second, when the innovations are normally distributed, all moments of the regression error exist.

challenge to acceptable behavior of the test statistics in finite samples.

For all error specifications, both conditionally homoscedastic and conditionally heteroscedastic, the $\ell_{q,s}$ statistics we compute are based on heteroscedastic-consistent estimates. There are two reasons for this. First, in practice the econometrician is unlikely to know a priori whether a given data set is conditionally heteroscedastic or not. Therefore, having a test that works well on both conditionally homoscedastic and conditionally heteroscedastic data is desirable. Second, when the errors are not serially correlated, the heteroscedasticity-consistent estimate of Ψ is guaranteed to be positive definite so that the resulting $\ell_{q,s}$ statistic is positive.[16]

When the regression error is serially uncorrelated, Ψ is estimated according to equation (27) with N=0.[17] Three estimators are used to compute Ψ when the error is a second-order moving average. The first uses (27) with the "Gaussian" weights $w_i^N = \exp(-i^2/2N^2)$ and N=2.[18] If this fails to yield a positive definite estimate of Ψ, N is sucessively reduced by one until a positive definite estimate is obtained.

The other two estimators will necessarily produce positive definite estimates of Ψ. The first is a modified Bartlett estimator (Anderson (1971) and Newey and West (1987)), which uses equation (27) with $w_i^N = (N-i+1)/(N+1)$ and N=5. The second of the positive definite estimators is the VAR estimator

[16] It is of course possible that using an estimate of Ψ which is not positive definite may lead to ℓ statistics which are sometimes negative, but nonetheless have a distribution which closely matches the chi-square distribution in the crucial right-hand tail region. Our experience, however, suggests that estimates of Ψ that are not positive definite yield ℓ statistics with distributions quite far from the chi-square distribution.

[17] In all cases where N = 0, $w_0^N = 1$.

[18] See Brillinger (1975), p55.

proposed by Cumby, Huizinga, and Obstfeld (1983), which estimates Ψ by fitting $\hat{\eta}$ to a second order vector autoregression that is then inverted to obtain the moving average representation of $\hat{\eta}$. The moving average representation is truncated at lag two and used to compute the spectral density of η.[19]

In computing the $\ell_{q,s}$ statistics for the least squares residuals, we make the small-sample adjustment to the sample autocorrelations suggested by Ljung and Box (1978). That is, $\hat{r}_n$ of equation (9) is replaced by $[(T+2)/(T-n)]^{.5}\ \hat{r}_n$ when computing the test statistic. For residuals from the instrumental variables regressions, $\hat{r}_n$ of equation (9) is replaced by $T/(T-n)\ \hat{r}_n$.

Tables 1 - 4 contain the results of the Monte Carlo experiments for the four models estimated by ordinary least squares. Each entry in the table provides the percent of the 5000 replications that exceeds the five percent critical value for a $\chi^2(s)$ random variable. In general, the frequency of rejection is very close to five percent. The similarity between the results with samples of 50 and 100 are striking. With both sample sizes, most of the rejection frequencies fall between 3.5% and 6.5%.[20]

There are three exceptions to the generally favorable performance of the

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[19] Other procedures for estimating Ψ were also investigated. One procedure set N=2 and used weights of unity in equation (27). This estimate of Ψ is not guaranteed to be positive definite and the frequency of not positive definite estimates was very high. More importantly, the resulting ℓ statistics falsely rejected the null hypothesis far too often. A Parzen estimator (Parzen (1961)), like the Modified Bartlett estimator is guaranteed to be positive definite but performed very poorly, yielding unacceptably small rejection frequencies when the number of autocorrelations tested is large. We also experimented with diferent values of N for the modified Bartlett procedure. These other values of N led to ℓ statistics with poorer properties than those we report.

[20] We also carried out Monte Carlo experiments using estimated partial autocorrelation and $\hat{V}_r$ when q=0. The performance of the tests based on partial autocorrelations was substantially worse than the performance of the ℓs tests.

test statistic. First, and most striking, is the exceptionally low rejection percentages that are obtained using the modified Bartlett estimator with s>1. In many instances, no rejections were found when twelve autocorrelations were tested. Second, while the rejection percentages obtained with both the Gaussian and VAR estimators of Ψ are close to the values predicted by the asymptotic theory, there is a tendency for the test using the VAR estimator to reject too frequently when s=12 and when the an ARCH process is used to generate the errors.[21] Finally, when the regressors are strictly exogenous (models II and IV), there is a tendency for the test to reject too frequently when s=1, regardless of how Ψ is estimated.

Table 5 presents the rejection frequencies obtained when the Q-test suggested by Box and Pierce (1970) (as modified by Ljung and Box (1978)) is applied to the same residuals used in the tests in Tables 1 - 4. The statistics are,

$$(32) \qquad Q_s = \sum_{n=1+j}^{s+j} [(T+2)/(T-n)]\, \hat{r}_n^{\,2}$$

with j=0 for the serially uncorrelated errors and j=2 for the second order moving average errors.

It should be emphasized that the Q test is valid in only two cases. First, in the univariate autoregressive model with serially uncorrelated,

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[21] We did not use the VAR estimator with a sample of 50 observations as fitting the vector autoregressive representation of η would involve estimating a number of parameters that is large relative to the sample, especially with s=12. The Monte Carlo experiments were also performed with an ARCH parameter of 0.9. As the fourth moment of η does not exist with this value of the parameter, it is not surprising that the performance of the test statistic deteriorates somewhat. The deterioration is substantial with the VAR estimator but small otherwise.

homoscedastic errors (a case originally considered by Box and Pierce), where Q_s is approximately distributed as $\chi^2(s-1)$.[22] Second, since use of regression residuals is equivalent to use of the true regression errors when all regressors are strictly exogenous, the Q_s statistic in model 2 with serially uncorrelated, homoscedastic errors will be distributed as $\chi^2(s)$. We report the results of using Q_s in the other cases in order to demonstrate the dangers of applying the statistic in inappropriate circumstances.[23]

The results reported in Table 5 indicate that when the errors are serially uncorrelated and the data are homoscedastic, the Q_s test performs reasonably well. While the Q_s test rejects slightly too often and the $\ell_{q,s}$ test exhibits rejection frequencies closer to five percent, the overall behavior of the two tests is comparable. The performance of the Q_s test deteriorates somewhat when the data are heteroscedastic, especially in model 4.[24] When the errors are MA(2) and residual autocorrelations past lag two are tested, the performance of the Q_s statistic falls substantially. These experiments indicate that the Q_s test is likely to be wildly misleading when used to test a null hypothesis other than that all residual autocorrelations are zero.

Tables 6 and 7 contain the results from the Monte Carlo experiments for the two models estimated with instrumental variables. As is the case in

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[22] Since the Q_s test is distributed as $\chi^2(s-1)$, we do not compute the Q_s test for s=1.

[23] Since the Q test is not valid in most of the models, it is not clear which critical value to choose. We chose to use the critical value from the chi-square distribution with s-1 degrees of freedom for models I, III and IV since each included one lagged dependent variable. We chose to use the critical value from the chi-square distribution with s degrees of freedom for model II because it is the correct choice with serially uncorrelated, homoscedastic errors.

[24] The performance of the Q_s test when an ARCH parameter of 0.9 is used is markedly worse than the results reported in Table 5.

Tables 1 - 4, there is little difference between the performance of the test in samples of 50 and 100 observations. The frequency of rejection when the modified Bartlett estimator are again extremely low. Excluding a few cases where only one autocorrelation is tested, the rejection frequencies for the simultaneous system (model 5), are generally close to five percent. As is the case with tests using ordinary least squares residuals, the tests sometimes rejects too frequently when only one autocorrelation is tested. There appears to be a tendency for the test to reject too seldom in the rational distributed lag model (model 6), with the smallest rejection frequencies generally occurring at s=12.

In closing this section we return to the point made earlier that the Gaussian estimator of Ψ that we use is not guaranteed to be positive definite in finite samples. Table 8 provides evidence on the frequency with which failure to obtain a positive definite estimate occurs in the data sets generated for models 1 and 6. The results for models 2 through 5 are very similar to those for model 1. The evidence clearly shows that the likelihood of obtaining an estimate of Ψ that is not positive definite increases with N (the number of autocovariance matrices summed), increases with s (the number of autocorrelations being tested equal to zero), and increases when heteroscedasticity is introduced, although less so when the EGARCH model is used than when the ARCH model is used.[25]

[25] Monte Carlo experiments were also carried out with a sample of 200 observations using the Gaussian estimator. The resulting estimates of Ψ are positive definite much more frequently than is the case with a sample of 100 observations.

V. Concluding Remarks

In this paper we have derived the asymptotic distribution of a vector of autocorrelations of regression residuals from a quite general linear model. The model is allowed to have a true regression error that is either conditionally heteroscedastic or conditionally homoscedastic and is either a moving average of order $q>0$ or serially uncorrelated. The model can have a mix of strictly exogenous and predetermined regressors, so that ordinary least squares is used for estimation, or a mix of strictly exogenous, predetermined, and endogenous regressors, so that an instrumental variables procedure is used. In this latter case, the instruments need only be predetermined and not strictly exogenous.

We then use this asymptotic distribution to propose a Wald test of the hypothesis that the regression error follows a moving average of order q by testing that the autocorrelations of the residuals at lags $q+1$ through $q+s$ are jointly zero. The finite sample properties of the test are examined in Monte Carlo simulations, using six different models and a variety of specifications for the regression errors. With sample sizes of 50 and 100 observations and s ranging from one to twelve the test is quite reliable. The probability of type one error is in general very close to the level predicted by asymptotic theory.

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Appendix

This appendix provides a proof of the main proposition in the text. Let y_t, ε_t, $X_{1,t}$, ..., $X_{k,t}$, $Z_{1,t}$, ..., $Z_{h,t}$ be scalar random variables on which we have observations for t=1,...,T. Define X_t and Z_t to be the 1xk and 1xh vectors $(X_{1,t}, \ldots, X_{k,t})$ and $(Z_{1,t}, \ldots, Z_{h,t})$, and define y, X, and Z to be the Tx1, Txk, and Txh matrices $(y_1, \ldots, y_T)'$, $(X_1', \ldots, X_T')'$, and $(Z_1', \ldots, Z_T')'$. Define $\eta_t = (\omega_{1,t}, \ldots, \omega_{h,t}, \xi_{1,t}, \ldots, \xi_{s,t})'$ for $\omega_{j,t} = \varepsilon_t Z_{j,t}$ and $\xi_{i,t} = \varepsilon_t \varepsilon_{t-q-i}$, and let A_T be an observable hxh matrix. We assume that for a known constant q and unknown kx1 vector of constants δ,

(A1) $\{ X_t, Z_t, \varepsilon_t \}$ is wide sense stationary and ergodic,

(A2) $y_t = X_t\delta + \varepsilon_t$,

(A3) $E(\varepsilon_t) = 0$,

(A4) $E(\varepsilon_t | Z_t, Z_{t-1}, \ldots, \varepsilon_{t-q-1}, \varepsilon_{t-q-2}, \ldots) = 0$.

(A5) $E(\eta_t \eta_{t-i}')$ is finite for i=0,...,q,

(A6) $\Psi = E(\eta_t\eta_{t-q}') + E(\eta_t\eta_{t-q+1}') + \ldots + E(\eta_t\eta_{t+q-1}') + E(\eta_t\eta_{t+q}')$ is positive definite,

(A7) (1/T) plim X'Z exists and has rank k,

and

(A8) plim A_T = A exists and is nonsingular.

Define $E(\varepsilon_t^2) = \sigma_\varepsilon^2$, $E(\varepsilon_t\varepsilon_{t-n})/\sigma_\varepsilon^2 = \rho_n$, r_n to be the sample autocorrelation of ε_t at lag n, the sx1 vector $r = (r_{q+1}, r_{q+2}, \ldots, r_{q+s})'$, the kx1 vector $d = (X'Z A_T^{-1} Z'X)^{-1} X'Z A_T^{-1} Z'y$, $\hat{r}_n$ to be the sample autocorrelation of $\hat{\varepsilon}_t = y_t - X_t d$, at lag n, and the sx1 vector $\hat{r} = (\hat{r}_{q+1}, \hat{r}_{q+2}, \ldots, \hat{r}_{q+s})$. We also define C to be the sxh matrix that has σ_ε^{-2} times the sum from n equals

-q to q of $E(\xi_{i,t}\omega_{j,t-n})$ as its ij^{th} element, V_r to be the sxs matrix that has σ_ε^{-4} times the sum from n equals -q to q of $E(\xi_{i,t}\xi_{j,t-n})$ as its ij^{th} element, E to be the Txs matrix that has q+j zeros followed by ε_t, t=1,...,T-q-j as its jth column, and B to be the sxk matrix that has $-[E(\varepsilon_{t-q-i}X_{j,t}) + E(X_{j,t-q-i}\varepsilon_{,t})]/\sigma_\varepsilon^2$ as its ij^{th} element.

Lemma A1: Given the assumptions (A1) - (A8) and the definitions stated above, d is a consistent estimate of δ and $\sqrt{T}\,(d-\delta) \overset{A}{\sim} N(0,V_d)$ where $V_d = D\,\Omega\,D'$, $D = \text{plim}\ T\,(X'Z\,A_T^{-1}\,Z'X)^{-1}\,X'Z\,A_T^{-1}$ and $\Omega = E(\omega_t\omega'_{t-q}) + E(\omega_t\omega'_{t-q+1}) + \ldots + E(\omega_t\omega'_{t+q-1}) + E(\omega_t\omega'_{t+q})$.

Proof: The proof can be found in Cumby, Huizinga and Obstfeld (1983).

Lemma A2: Given the assumptions (A1) - (A8) and the definitions stated above, then plim $\partial r/\partial\delta = B$.

Proof: $\partial r/\partial\delta$ has as its ij^{th} element

$$(\partial r_i/\partial\delta_j)\big|_{\delta=d_i^*} = \frac{\partial}{\partial\delta_j}\frac{\sum e_t e_{t-q-i}}{\sum e_t^2}, \quad \text{where } e_t = y_t - X_t d_i^*$$

and d_i^* lies between d and δ. Differentiating, we obtain,

$$-\frac{\sum[X_{j,t}e_{t-q-i} + X_{j,t-q-i}e_t]}{\sum e_t^2} + 2\,\frac{\sum e_t e_{t-q-i}\sum X_{j,t}e_t}{[\sum e_t^2]^2}.$$

Therefore, using (A1), the fact that d is a consistent estimate of δ (Lemma 1), and the fact that d_i^* lies between d and δ, we get

$$\text{plim} \left. \frac{\partial r_i}{\partial \delta_j} \right|_{\delta=d_i^*} = - \frac{E(X_{j,t}\varepsilon_{t-q-i})}{\sigma_\varepsilon^2} - \frac{E(X_{j,t-q-i}\varepsilon_t)}{\sigma_\varepsilon^2} + 2\rho_{q+i} \frac{E(X_{j,t}\varepsilon_t)}{\sigma_\varepsilon^2}$$

Since $\rho_j=0$ for $j>q$, the third term in this sum is zero and the lemma is proved.

The proof of Proposition 1 is now straightforward.

Proposition 1: Given the assumptions (A1) - (A8) and the definitions above, $\sqrt{T}\,\hat{r} \overset{A}{\sim} N(0,V_{\hat{r}})$ with $V_{\hat{r}} = V_r + BV_dB' + BDC' + CD'B'$.

Proof: By the mean-value theorem,

$$\sqrt{T}\,\hat{r} = \sqrt{T}\,r + \sqrt{T}\,\frac{\partial r}{\partial \delta}(d - \delta),$$

where the ith row of $\partial r/\partial\delta$ is evaluated at d_i^*, which lies between d and δ. Stacking the terms on the right-hand side of this expression and substituting the definitions of d and E gives,

$$\sqrt{T}\begin{bmatrix} \partial r/\partial\delta\ (d-\delta) \\ r \end{bmatrix} = \begin{bmatrix} \partial r/\partial\delta\ T\ (X'ZA_T^{-1}Z'X)^{-1}X'ZA_T^{-1} & 0 \\ 0 & (\varepsilon'\varepsilon/T)^{-1}\ I \end{bmatrix}\begin{bmatrix} Z'\varepsilon/\sqrt{T} \\ E'\varepsilon/\sqrt{T} \end{bmatrix}$$

where I is an sxs identity matrix. By Lemma 2, (A7) and (A8),

$$\text{plim} \begin{bmatrix} \partial r/\partial\delta\ T\ (X'ZA_T^{-1}Z'X)^{-1}\ X'ZA_T^{-1} & 0 \\ 0 & (\varepsilon'\varepsilon/T)^{-1}\ I \end{bmatrix} = \begin{bmatrix} BD & 0 \\ 0 & \sigma_\varepsilon^{-2}\ I \end{bmatrix} = \Phi$$

and by a central limit theorem in Hannan (1973), (see Hansen (1982)),

$$\begin{bmatrix} Z'\varepsilon/\sqrt{T} \\ E'\varepsilon/\sqrt{T} \end{bmatrix} \overset{A}{\sim} N(0,\Psi)$$

for

$$\Psi = \begin{bmatrix} \Omega & C'\sigma_\varepsilon^2 \\ C\,\sigma_\varepsilon^2 & V_r\sigma_\varepsilon^4 \end{bmatrix}.$$

Thus,

$$\sqrt{T} \begin{bmatrix} \partial r/\partial\delta\ (d-\delta) \\ r \end{bmatrix} \overset{A}{\sim} N(0, \Phi\ \Psi\ \Phi')$$

where

$$\Phi\ \Psi\ \Phi' = \begin{bmatrix} BV_dB' & CD'B' \\ BDC' & V_r \end{bmatrix}.$$

Since $\sqrt{T}\ \hat{r}$ is the sum of the two random vectors that are asymptotically normally distributed with covariance matrix $\Phi\ \Psi\ \Phi'$, it follows that $\sqrt{T}\ \hat{r}$ is asymptotically normally distributed with covariance matrix given by $BV_dB' + BDC' + CD'C' + V_r$ and the proof is completed.

In the text, we discuss how the asymptotic distribution of $\sqrt{T}\ \hat{r}$ is affected when the assumption that ε_t is conditionally homoscedastic,

(A9) $E(\varepsilon_t\varepsilon_{t-n}|\ Z_t,\ Z_{t-1},\ \ldots,\ \varepsilon_{t-q-1},\ \varepsilon_{t-q-2},\ \ldots) = E(\varepsilon_t\varepsilon_{t-n}),\ 0 \le n \le q,$

is added to assumptions (A1) - (A8) above. In particular, equations (15) and (16) give forms of equations (13) and (14) which are claimed to be valid when this assumption is added. To verify that equation (15) is in fact correct, note that when (A9) holds and $-q \le n \le q$,

$$\sigma_\varepsilon^{-4} E(\xi_{i,t}\xi_{j,t-n}) = \sigma_\varepsilon^{-4} E(\varepsilon_t\varepsilon_{t-q-i}\varepsilon_{t-n}\varepsilon_{t-n-q-j})$$
$$= \sigma_\varepsilon^{-4} E(\varepsilon_{t-q-i}\varepsilon_{t-n-q-j} E(\varepsilon_t\varepsilon_{t-n}|\varepsilon_{t-q-i},\varepsilon_{t-n-q-j}))$$
$$= \sigma_\varepsilon^{-4} E(\varepsilon_{t-q-i}\varepsilon_{t-n-q-j})E(\varepsilon_t\varepsilon_{t-n}) = \rho_{n+j-i}\rho_n.$$

Equation (16) can be verified in a similar manner.

TABLE 1: EMPIRICAL 5% REJECTION PROBABILITIES FOR ℓ STATISTICS

Model 1 - Univariate Autoregression

$$y_t = .9\, y_{t-k} + e_t$$
$$e_t = u_t + a_1\, u_{t-1} + a_2\, u_{t-2}$$

ERROR TERM		COVARIANCE METHOD	REJECTION PERCENTAGES 100 OBSERVATIONS				REJECTION PERCENTAGES 50 OBSERVATIONS			
			ℓ.,1	ℓ.,3	ℓ.,6	ℓ.,12	ℓ.,1	ℓ.,3	ℓ.,6	ℓ.,12
HOMOSKEDASTIC	MA(0)		5.74	5.32	4.70	3.96	5.74	4.96	4.78	5.10
HOMOSKEDASTIC	MA(2)	GAUSSIAN	6.60	4.66	5.10	6.78	7.24	3.88	4.82	6.96
HOMOSKEDASTIC	MA(2)	BARTLETT	5.90	2.20	0.04	0.00	5.20	0.14	0.00	0.00
HOMOSKEDASTIC	MA(2)	VAR	6.24	5.64	5.04	8.74				
ARCH	MA(0)		5.46	4.36	3.84	3.94	5.58	4.40	3.88	5.04
ARCH	MA(2)	GAUSSIAN	5.80	3.64	5.12	6.06	6.24	2.80	4.60	6.94
ARCH	MA(2)	BARTLETT	4.60	1.34	0.00	0.00	4.44	0.06	0.00	0.00
ARCH	MA(2)	VAR	5.60	5.36	6.34	14.24				
EGARCH	MA(0)		5.00	5.00	3.74	4.14	5.38	4.56	4.44	4.82
EGARCH	MA(2)	GAUSSIAN	6.40	4.96	4.96	7.04	6.78	3.64	4.46	6.28
EGARCH	MA(2)	BARTLETT	5.98	2.24	0.02	0.00	4.94	0.10	0.00	0.00
EGARCH	MA(2)	VAR	6.28	6.14	5.16	9.60				

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Numbers given are the percent of ℓ statistics exceeding the 5% critical value for a ^{2}(s) random variable. With a true rejection probability of 5%, the 95% confidence interval for the estimates in this table is [4.4%,5.6%]. The disturbance term $u_t = h_t v_t$ where v_t~i.i.d. N(0,1). For the Homoskedastic error $h_t = 1$, for the ARCH error $h_t^2 = 1 + .4\, u_{t-1}^2$, and for the EGARCH error $\ln(h_t^2) = 1 + .7 \ln(h_{t-1}^2) + .2\, (|v_{t-1}| - (2/\pi)^{1/2})$. For the MA(0) error $a_1 = a_2 = 0$ and $\sigma_e^2/\sigma_y^2 = .19$. For the MA(2) error $a_1 = .9$, $a_2 = .81$, and $\sigma_e^2/\sigma_y^2 = .19$. In order to ensure a predetermined regressor, k=1 for the MA(0) error and k=3 for the MA(2) error. The Gaussian covariance method uses equation (27) in the text with $w_i^N = \exp(-i^2/2N^2)$ and N=2, unless this fails to yield a positive definite estimate. In this case, N is sucessively reduced by one until a positive definite estimate is obtained. The Bartlett covariance method uses equation (27) with $w_i^N = (N-i+1)/(N+1)$ and N=5. The VAR covariance method fits $\hat{\eta}$ (see equation (25) in the text) to a second order vector autoregregression, inverts the autoregression to obtain a moving average representation, truncates the moving average representation at lag two, and forms the estimated the spectral density of η from the parameters of this estimated moving average representation.

TABLE 2: EMPIRICAL 5% REJECTION PROBABILITIES FOR ℓ STATISTICS

Model 2 - Single Exogenous Regressor

$$y_t = x_t + e_t$$
$$e_t = u_{1,t} + a_1 u_{1,t-1} + a_2 u_{1,t-2}$$
$$x_t = .9 x_{t-1} + u_{2,t}$$

ERROR TERM		COVARIANCE METHOD	REJECTION PERCENTAGES 100 OBSERVATIONS $\ell_{.,1}$	$\ell_{.,3}$	$\ell_{.,6}$	$\ell_{.,12}$	REJECTION PERCENTAGES 50 OBSERVATIONS $\ell_{.,1}$	$\ell_{.,3}$	$\ell_{.,6}$	$\ell_{.,12}$
HOMOSKEDASTIC	MA(0)		4.82	5.00	4.88	4.28	5.54	5.58	4.72	4.70
HOMOSKEDASTIC	MA(2)	GAUSSIAN	7.64	5.74	5.82	6.36	9.28	4.64	5.58	5.16
HOMOSKEDASTIC	MA(2)	BARTLETT	7.20	2.18	0.08	0.00	7.14	0.12	0.00	0.00
HOMOSKEDASTIC	MA(2)	VAR	6.76	6.16	4.92	7.10				
ARCH	MA(0)		4.74	4.72	4.28	3.74	5.58	4.78	4.38	5.30
ARCH	MA(2)	GAUSSIAN	7.42	4.64	5.36	6.20	8.40	4.24	5.32	5.22
ARCH	MA(2)	BARTLETT	6.28	1.50	0.04	0.00	5.64	0.06	0.00	0.00
ARCH	MA(2)	VAR	6.78	5.92	7.02	13.28				
EGARCH	MA(0)		5.02	4.90	4.10	4.16	4.92	4.34	3.36	1.64
EGARCH	MA(2)	GAUSSIAN	7.82	5.16	6.30	6.34	9.44	4.40	5.56	5.34
EGARCH	MA(2)	BARTLETT	7.08	2.24	0.12	0.00	6.32	0.12	0.00	0.00
EGARCH	MA(2)	VAR	7.12	5.78	5.50	9.82				

Numbers given are the percent of ℓ statistics exceeding the 5% critical value for a $\chi^2(s)$ random variable. With a true rejection probability of 5%, the 95% confidence interval for the estimates in this table is [4.4%,5.6%]. The disturbance term $u_{i,t} = h_{i,t}v_{i,t}$ for i=1,2, where the 2x1 vector v_t~i.i.d. N(0,I). For the Homoskedastic error $h_{i,t} = 1$, for the ARCH error $h^2_{i,t} = 1 + .4\, u^2_{i,t-1}$, and for the EGARCH error $\ln(h^2_{i,t}) = 1 + .7 \ln(h^2_{i,t-1}) + .2\{|v_{i,t-1}| - (2/\pi)^{1/2}\}$. For the MA(0) error $a_1 = a_2 = 0$ and $\sigma^2_e/\sigma^2_y = .16$. For the MA(2) error $a_1 = .9$, $a_2 = .81$, and $\sigma^2_e/\sigma^2_y = .32$. The Gaussian covariance method uses equation (27) in the text with $w^N_i = \exp(-i^2/2N^2)$ and N=2, unless this fails to yield a positive definite estimate. In this case, N is sucessively reduced by one until a positive definite estimate is obtained. The Bartlett covariance method uses equation (27) with $w^N_i = (N-i+1)/(N+1)$ and N=5. The VAR covariance method fits $\hat{\eta}$ (see equation (25) in the text) to a second order vector autoregregression, inverts the autoregression to obtain a moving average representation, truncates the moving average representation at lag two, and forms the estimated the spectral density of η from the parameters of this estimated moving average representation.

TABLE 3: EMPIRICAL 5% REJECTION PROBABILITIES FOR ℓ STATISTICS

Model 3 - Bivariate Autoregression

$$y_{1,t} = .5\, y_{1,t-k} + .4\, y_{2,t-k} + e_{1,t}$$
$$y_{2,t} = .4\, y_{1,t-k} + .5\, y_{2,t-k} + e_{2,t}$$
$$e_{1,t} = u_{1,t} + a_1\, u_{1,t-1} + a_2\, u_{1,t-2}$$
$$e_{2,t} = u_{2,t} + a_1\, u_{2,t-1} + a_2\, u_{2,t-2}$$

ERROR TERM		COVARIANCE METHOD	REJECTION PERCENTAGES 100 OBSERVATIONS				REJECTION PERCENTAGES 50 OBSERVATIONS			
			$\ell_{\cdot,1}$	$\ell_{\cdot,3}$	$\ell_{\cdot,6}$	$\ell_{\cdot,12}$	$\ell_{\cdot,1}$	$\ell_{\cdot,3}$	$\ell_{\cdot,6}$	$\ell_{\cdot,12}$
HOMOSKEDASTIC	MA(0)		5.38	5.44	5.10	3.82	5.72	5.58	4.92	5.10
HOMOSKEDASTIC	MA(2)	GAUSSIAN	7.28	4.54	5.06	4.52	7.98	4.20	3.68	5.48
HOMOSKEDASTIC	MA(2)	BARTLETT	6.90	2.06	0.14	0.00	6.22	0.22	0.00	0.00
HOMOSKEDASTIC	MA(2)	VAR	6.32	4.72	3.70	5.38				
ARCH	MA(0)		5.20	4.18	4.38	3.96	6.06	4.64	4.74	5.48
ARCH	MA(2)	GAUSSIAN	6.92	3.68	4.22	4.86	8.26	3.40	3.60	5.82
ARCH	MA(2)	BARTLETT	6.10	1.44	0.08	0.00	6.54	0.06	0.00	0.00
ARCH	MA(2)	VAR	6.20	4.28	4.90	10.46				
EGARCH	MA(0)		5.54	5.04	4.66	4.28	5.66	5.40	5.22	5.50
EGARCH	MA(2)	GAUSSIAN	7.24	4.68	4.86	5.16	7.80	3.56	3.66	5.48
EGARCH	MA(2)	BARTLETT	6.74	1.98	0.10	0.00	6.12	0.06	0.00	0.00
EGARCH	MA(2)	VAR	6.34	4.66	4.50	7.02				

Numbers given are the percent of ℓ statistics exceeding the 5% critical value for a $\chi^2(s)$ random variable. With a true rejection probability of 5%, the 95% confidence interval for the estimates in this table is [4.4%,5.6%]. The disturbance term $u_{i,t} = h_{i,t} v_{i,t}$ for i=1,2, where the 2x1 vector v_t~i.i.d. N(0,I). For the Homoskedastic error $h_{i,t} = 1$, for the ARCH error $h^2_{i,t} = 1 + .4\, u^2_{i,t-1}$, and for the EGARCH error $\ln(h^2_{i,t}) = 1 + .7\ln(h^2_{i,t-1}) + .2^2\{|v_{i,t-1}| - (2/\pi)^{1/2}\}$. For the MA(0) error $a_1 = a_2 = 0$ and $\sigma^2_e/\sigma^2_y = .32$. For the MA(2) error $a_1 = .9$, $a_2 = .81$, and $\sigma^2_e/\sigma^2_y = .32$. In order to ensure a predetermined regressor, k=1 for the MA(0) error and k=3 for the MA(2) error. The Gaussian covariance method uses equation (27) in the text with $w^N_i = \exp(-i^2/2N^2)$ and N=2, unless this fails to yield a positive definite estimate. In this case, N is sucessively reduced by one until a positive definite estimate is obtained. The Bartlett covariance method uses equation (27) with $w^N_i = (N-i+1)/(N+1)$ and N=5. The VAR covariance method fits $\hat{\eta}$ (see equation (25) in the text) to a second order vector autoregregression, inverts the autoregression to obtain a moving average representation, truncates the moving average representation at lag two, and forms the estimated the spectral density of η from the parameters of this estimated moving average representation.

TABLE 4: EMPIRICAL 5% REJECTION PROBABILITIES FOR ℓ STATISTICS

Model 4 - Transfer Function Model

$$y_t = .9\, y_{t-k} + x_t + e_t$$
$$e_t = u_{1,t} + a_1 u_{1,t-1} + a_2 u_{1,t-2}$$
$$x_t = .9\, x_{t-1} + u_{2,t}$$

ERROR TERM		COVARIANCE METHOD	REJECTION PERCENTAGES 100 OBSERVATIONS				REJECTION PERCENTAGES 50 OBSERVATIONS			
			$\ell_{.,1}$	$\ell_{.,3}$	$\ell_{.,6}$	$\ell_{.,12}$	$\ell_{.,1}$	$\ell_{.,3}$	$\ell_{.,6}$	$\ell_{.,12}$
HOMOSKEDASTIC	MA(0)		5.64	5.96	5.48	4.80	6.06	6.06	6.04	5.86
HOMOSKEDASTIC	MA(2)	GAUSSIAN	10.68	7.24	6.82	5.60	13.68	6.34	5.32	4.50
HOMOSKEDASTIC	MA(2)	BARTLETT	9.94	3.60	0.12	0.00	10.56	0.12	0.00	0.00
HOMOSKEDASTIC	MA(2)	VAR	9.54	7.66	5.62	6.56				
ARCH	MA(0)		5.58	4.54	4.82	4.32	5.96	5.66	5.20	6.10
ARCH	MA(2)	GAUSSIAN	10.12	6.60	6.60	4.88	12.00	5.78	4.60	4.32
ARCH	MA(2)	BARTLETT	9.00	2.38	0.08	0.00	8.78	0.06	0.00	0.00
ARCH	MA(2)	VAR	9.44	7.22	6.96	12.42				
EGARCH	MA(0)		5.52	5.28	5.22	5.04	6.56	5.72	5.50	6.04
EGARCH	MA(2)	GAUSSIAN	10.46	6.72	6.62	5.60	13.00	6.60	4.66	4.78
EGARCH	MA(2)	BARTLETT	9.82	3.10	0.10	0.00	9.88	0.06	0.00	0.00
EGARCH	MA(2)	VAR	9.32	7.04	6.06	9.38				

Numbers given are the percent of ℓ statistics exceeding the 5% critical value for a $\chi^2(s)$ random variable. With a true rejection probability of 5%, the 95% confidence interval for the estimates in this table is [4.4%,5.6%].The disturbance term $u_{i,t} = h_{i,t}v_{i,t}$ for i=1,2, where the 2x1 vector v_t~i.i.d. N(0,I). For the Homoskedastic error $h_{i,t} = 1$, for the ARCH error $h^2_{i,t} = 1 + .4\, u^2_{i,t-1}$, and for the EGARCH error $\ln(h^2_{i,t}) = 1 + .7 \ln(h^2_{i,t-1}) + .2\{|v_{i,t-1}| - (2/\pi)^{1/2}\}$. For the MA(0) error $a_1 = a_2 = 0$ and $\sigma^2_e/\sigma^2_y = .004$. For the MA(2) error $a_1 = .9$, $a_2 = .81$, and $\sigma^2_e/\sigma^2_y = .017$. In order to ensure a predetermined regressor, k=1 for the MA(0) error and k=3 for the MA(2) error. The Gaussian covariance method uses equation (27) in the text with $w^N_i = \exp(-i^2/2N^2)$ and N=2, unless this fails to yield a positive definite estimate. In this case, N is sucessively reduced by one until a positive definite estimate is obtained. The Bartlett covariance method uses equation (27) with $w^N_i = (N-i+1)/(N+1)$ and N=5. The VAR covariance method fits $\hat{\eta}$ (see equation (25) in the text) to a second order vector autoregregression, inverts the autoregression to obtain a moving average representation, truncates the moving average representation at lag two, and forms the estimated the spectral density of η from the parameters of this estimated moving average representation.

TABLE 5: EMPIRICAL 5% REJECTION PROBABILITIES FOR Q_s STATISTICS

MODEL 1 - UNIVARIATE AUTOREGRESSION:

ERROR TERM		s = 3	s = 6	s = 12
HOMOSKEDASTIC	MA(0)	7.52	5.64	6.08
HOMOSKEDASTIC	MA(2)	24.46	29.46	40.58
ARCH	MA(0)	13.84	9.80	8.40
ARCH	MA(2)	23.70	28.68	37.10
EGARCH	MA(0)	8.92	7.30	7.22
EGARCH	MA(2)	26.44	31.40	40.46

MODEL 2 - SINGLE EXOGENOUS REGRESSOR:

ERROR TERM		s = 3	s = 6	s = 12
HOMOSKEDASTIC	MA(0)	4.78	5.14	6.06
HOMOSKEDASTIC	MA(2)	25.40	34.32	45.12
ARCH	MA(0)	11.98	10.12	8.40
ARCH	MA(2)	25.96	34.18	43.78
EGARCH	MA(0)	7.02	6.72	6.96
EGARCH	MA(2)	27.38	35.38	46.94

MODEL 3 - BIVARIATE AUTOREGRESSION:

ERROR TERM		s = 3	s = 6	s = 12
HOMOSKEDASTIC	MA(0)	5.64	5.14	5.00
HOMOSKEDASTIC	MA(2)	17.20	26.42	39.78
ARCH	MA(0)	7.96	6.56	5.68
ARCH	MA(2)	17.12	26.64	38.02
EGARCH	MA(0)	6.44	6.18	6.28
EGARCH	MA(2)	17.90	27.72	40.78

MODEL 4 - TRANSFER FUNCTION:

ERROR TERM		s = 3	s = 6	s = 12
HOMOSKEDASTIC	MA(0)	11.56	9.20	8.50
HOMOSKEDASTIC	MA(2)	37.10	43.02	52.00
ARCH	MA(0)	22.34	16.18	13.24
ARCH	MA(2)	37.06	42.92	49.86
EGARCH	MA(0)	15.32	11.98	10.62
EGARCH	MA(2)	37.44	43.12	52.24

All tests are based on 5,000 replications and samples of 100 used in estimation. The numbers given are the percent of Q statistics that exceed the 5% critical value for a $\chi^2(s-1)$ random variable, except in model 2, where a $\chi^2(s)$ random variable is used. With a true rejection probability of 5%, the 95% confidence interval for the estimates in this table is [4.4%,5.6%].

TABLE 6: EMPIRICAL 5% REJECTION PROBABILITIES FOR ℓ STATISTICS

Model 5 - Simultaneuos Equations System

$$y_{1,t} = .8\ y_{2,t} + .18\ y_{1,t-k} + .36\ x_{1,t} + e_{1,t}$$
$$y_{2,t} = .8\ y_{1,t} + .18\ y_{2,t-k} + .36\ x_{2,t} + e_{2,t}$$
$$e_{i,t} = u_{i,t} + a_1\ u_{i,t-1} + a_2\ u_{i,t-2} \qquad i=1,2$$
$$x_{i,t} = .9\ x_{i,t-1} + u_{i+2,t} \qquad i=1,2$$

ERROR TERM		COVARIANCE METHOD	REJECTION PERCENTAGES 100 OBSERVATIONS				REJECTION PERCENTAGES 50 OBSERVATIONS			
			$\ell_{..1}$	$\ell_{..3}$	$\ell_{..6}$	$\ell_{..12}$	$\ell_{..1}$	$\ell_{..3}$	$\ell_{..6}$	$\ell_{..12}$
HOMOSKEDASTIC	MA(0)		5.38	5.22	4.28	3.96	5.28	5.14	5.28	9.10
HOMOSKEDASTIC	MA(2)	GAUSSIAN	8.96	5.24	4.82	3.76	11.04	5.56	4.38	10.24
HOMOSKEDASTIC	MA(2)	BARTLETT	9.10	2.98	0.60	0.08	10.28	0.98	0.26	0.46
HOMOSKEDASTIC	MA(2)	VAR	7.82	4.68	3.36	5.00				
ARCH	MA(0)		5.28	4.28	3.66	4.02	5.04	4.00	4.50	8.78
ARCH	MA(2)	GAUSSIAN	9.24	4.96	4.56	4.14	9.74	4.62	4.32	11.60
ARCH	MA(2)	BARTLETT	9.30	2.86	0.44	0.10	8.56	0.82	0.40	0.50
ARCH	MA(2)	VAR	8.58	5.34	4.30	8.42				
EGARCH	MA(0)		5.62	4.98	4.34	3.76	4.96	4.78	4.72	8.96
EGARCH	MA(2)	GAUSSIAN	9.24	5.08	4.56	4.08	10.06	5.20	4.44	10.64
EGARCH	MA(2)	BARTLETT	9.56	2.68	0.48	0.04	9.44	0.88	0.16	0.58
EGARCH	MA(2)	VAR	8.22	4.54	3.24	6.08				

Numbers given are the percent of ℓ statistics exceeding the 5% critical value for a $\chi^2(s)$ random variable. With a true rejection probability of 5%, the 95% confidence interval for the estimates in this table is [4.4%,5.6%].The disturbance term $u_{i,t} = h_{i,t}v_{i,t}$ for i=1,4, where the 4x1 vector v_t~i.i.d. N(0,I). For the Homoskedastic error $h_{i,t} = 1$, for the ARCH error $h^2_{i,t} = 1 + .4\ u^2_{i,t-1}$, and for the EGARCH error $\ln(h^2_{i,t}) = 1 + .7\ \ln(h^2_{i,t-1}) + .2\ (|v_{i,t-1}| - (2/\pi)^{1/2})$. For the MA(0) error $a_1 = a_2 = 0$ and $\sigma^2_e/\sigma^2_y = .03$. For the MA(2) error $a_1 = .9$, $a_2 = .81$, and $\sigma^2_e/\sigma^2_y = .08$. In order to ensure a predetermined regressor, k=1 for the MA(0) error and k=3 for the MA(2) error. The Gaussian covariance method uses equation (27) in the text with $w^N_i = \exp(-i^2/2N^2)$ and N=2, unless this fails to yield a positive definite estimate. In this case, N is sucessively reduced by one until a positive definite estimate is obtained. The Bartlett covariance method uses equation (27) with $w^N_i = (N-i+1)/(N+1)$ and N=5. The VAR covariance method fits $\hat{\eta}$ (see equation (25) in the text) to a second order vector autoregregression, inverts the autoregression to obtain a moving average representation, truncates the moving average representation at lag two, and forms the estimated the spectral density of η from the parameters of this estimated moving average representation.

TABLE 7: EMPIRICAL 5% REJECTION PROBABILITIES FOR ℓ STATISTICS

Model 6 - Rational Distributed Lag

$$y_t = -.75\, y_{t-1} + .15\, y_{t-2} + x_t + u_{1,t} - .75\, u_{1,t-1} + .15\, u_{1,t-2}$$
$$x_t = .9\, x_{t-1} + u_{2,t}$$

ERROR TERM		COVARIANCE METHOD	REJECTION PERCENTAGES 100 OBSERVATIONS				REJECTION PERCENTAGES 50 OBSERVATIONS			
			$\ell_{..1}$	$\ell_{..3}$	$\ell_{..6}$	$\ell_{..12}$	$\ell_{..1}$	$\ell_{..3}$	$\ell_{..6}$	$\ell_{..12}$
HOMOSKEDASTIC	MA(2)	GAUSSIAN	5.26	3.76	2.80	2.46	5.26	3.60	3.16	6.92
HOMOSKEDASTIC	MA(2)	BARTLETT	4.48	1.56	0.14	0.00	3.16	0.16	0.00	0.00
HOMOSKEDASTIC	MA(2)	VAR	3.52	2.32	1.50	2.16				
ARCH	MA(2)	GAUSSIAN	4.80	3.74	2.26	2.10	4.36	2.88	2.62	7.66
ARCH	MA(2)	BARTLETT	3.80	1.12	0.06	0.00	2.12	0.10	0.00	0.08
ARCH	MA(2)	VAR	3.92	2.80	2.04	4.00				
EGARCH	MA(2)	GAUSSIAN	5.82	3.76	2.44	2.36	5.30	3.12	3.20	7.58
EGARCH	MA(2)	BARTLETT	4.58	1.08	0.04	0.00	3.36	0.18	0.00	0.06
EGARCH	MA(2)	VAR	4.10	2.50	1.68	2.46				

Numbers given are the percent of ℓ statistics exceeding the 5% critical value for a $\chi^2(s)$ random variable. With a true rejection probability of 5%, the 95% confidence interval for the estimates in this table is [4.4%,5.6%].The disturbance term $u_{i,t} = h_{i,t}v_{i,t}$ for i=1,2, where the 2x1 vector v_t~i.i.d. N(0,I). For the Homoskedastic error $h_{i,t} = 1$, for the ARCH error $h_{i,t}^2 = 1 + .4\, u_{i,t-1}^2$, and for the EGARCH error $\ln(h_{i,t}^2) = 1 + .7\ln(h_{i,t-1}^2) + .2(|v_{i,t-1}| - (2/\pi)^{1/2})$. $\sigma_e^2/\sigma_y^2 = .38$. The Gaussian covariance method uses equation (27) in the text with $w_i^N = \exp(-i^2/2N^2)$ and N=2, unless this fails to yield a positive definite estimate. In this case, N is sucessively reduced by one until a positive definite estimate is obtained. The Bartlett covariance method uses equation (27) with $w_i^N = (N-i+1)/(N+1)$ and N=5. The VAR covariance method fits $\hat{\eta}$ (see equation (25) in the text) to a second order vector autoregregression, inverts the autoregression to obtain a moving average representation, truncates the moving average representation at lag two, and forms the estimated the spectral density of η from the parameters of this estimated moving average representation.

Table 8: FREQUENCY OF ESTIMATED Ψ FOUND NOT POSITIVE DEFINITE

Model 1 - UNIVARIATE AUTOREGRESSION

ERROR TERM	N	s = 1	s = 3	s = 6	s = 12
HOMOSKEDASTIC	2	0	2	246	2584
	1	0	0	0	109
ARCH	2	0	15	395	2905
	1	0	0	9	217
EGARCH	2	0	1	352	2792
	1	0	0	1	202

Model 6 - RATIONAL DISTRIBUTED LAG

ERROR TERM	N	s = 1	s = 3	s = 6	s = 12
HOMOSKEDASTIC	2	93	268	641	3078
	1	0	0	0	1
ARCH	2	169	419	1034	3580
	1	0	2	3	19
EGARCH	2	122	332	824	3363
	1	0	0	0	3

The numbers given are the number of times, out of 5000 replications of samples of 100, that a not positive-definite estimate of Ψ is obtained. Estimates are formed with the Gaussian covariance method, which uses equation (27) in the text and $w_i^N = \exp(-i^2/2N^2)$. When $N = 0$, $w_0^N = 1$ and the estimate of Ψ is guaranteed to be positive definite.