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SPECTRAL BASED TESTING OF THE MARTINGALE HYPOTHESIS

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ABSTRACT

This paper proposes a method of testing whether a time series is a martingale. The procedure develops an asymptotic theory for the shape of the spectral distribution function of the first differences. Under the null hypothesis, this shape should be a diagonal line. Several tests are developed which determine whether the deviation of the sample spectral distribution function from a diagonal line, when treated as an element of a function space, is too erratic to be attributable to sampling error. These tests are consistent against all moving average alternatives. The testing procedure possesses the additional advantage that it eliminates discretion in choosing a particular  $H_1$  by the researcher and therefore guards against data mining. The tests may further be adjusted to analyze subsets of frequencies in isolation, which can enhance power against particular alternatives. Application of the test to stock prices finds some evidence against the random walk theory.

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## *Introduction*

The martingale hypothesis has a long and distinguished history in economic theory. Perhaps the oldest example is the random walk theory of stock prices. The efficiency of the stock market in aggregating information has frequently been equated with conditions requiring that stock price changes be unpredictable. Intuitively, arbitragers will eliminate any predictability in excess holding returns which could be detected empirically.

More recently, dynamic equilibrium approaches to macroeconomics have imposed martingale restrictions on numerous time series of interest. Hall [1978] is a fundamental paper which demonstrates circumstances where the marginal utility of consumption is a martingale. Similar results have been obtained for tax rates by Barro [1981] in the context of the optimal allocation of deadweight efficiency losses generated by distortionary funding of government spending.

This paper seeks to provide a general framework for testing whether a time series can be described as a martingale. The testing framework will unite various approaches which have become popular in applied work. Unification of these various procedures will permit explicit discussion of test power and the role of the researcher's priors in accepting or rejecting the null.

Several procedures for testing the martingale hypothesis are currently popular. One procedure, employed in the original Hall formulation, is to examine whether the time series under question follows an AR(1) with the lagged dependent variable coefficient equalling one. This procedure is essentially equivalent to exploring the properties of some of the elements of the autocorrelation function of the first differences of the data.

An alternative approach, explored in detail by Cochrane [1988], Lo and MacKinlay [1988] and Poterba and Summers [1987], examines the variance of the martingale difference (under the null)  $x_t$  versus the variance of  $\sum_{i=0}^{k-1} x_{t-i}$ . Under the null, the variance of the latter should equal  $k$  times the variance of the former. Some evidence exists that this test possesses excellent power properties relative to conventional autocorrelation tests, particularly in uncovering long run mean reversion.

A limiting form of the variance ratio test will be subsumed in our more general testing framework. In fact, the martingale null permits construction of an infinity of distinct tests, each of which is consistent against some class of alternatives. One advantage of our approach is that it tests all second moment implications of the martingale hypothesis. The testing framework we analyze avoids the need for the researcher to possess prior information on the alternative hypothesis. From a classical perspective, this feature is valuable as no uniformly most powerful test exists. From a Bayesian perspective, the test corresponds to a diffuse prior over all fixed martingale deviations. Our testing framework avoids the problem that a particular rejection may correspond to the maximum deviation within a large class of tests of the null.

The testing framework developed here analyzes the properties of the shape of the spectral distribution function estimates of the martingale difference sequence under  $H_0$ . All testable implications may be summarized in the statement that, under  $H_0$ , the spectral distribution function is a diagonal line. An asymptotic theory is developed to measure deviations of different sample spectral distribution estimates from a diagonal. The type of test we propose is thus analogous to a goodness-of-fit test for a probability distribution function.

There is a long standing literature on using the spectral distribution shape to test various hypotheses. Bartlett [1955] is an early

source of the idea that the *sup* of the cumulated normalized periodogram of white noise will converge to a Kolmogorov-Smirnov statistic. Grenander and Rosenblatt [1953, 1957] rigorously obtained quite similar results in the case where the variance of the time series is known. Their methodology represents the basis of our asymptotic theory. In the context of identically and normally distributed innovations, Durbin [1967] has obtained finite sampling results for one of our tests, the Kolmogorov-Smirnov statistic.

Our results extend this literature, which has concentrated on a single test statistic, to the more general question of analyzing the spectral distribution deviations from a diagonal as a problem of weak convergence in a random function space. A general asymptotic theory for the spectral distribution function will permit the construction of many test statistics of the martingale hypothesis. These tests possess different size and power properties. In addition, the robustness of the asymptotics to many forms of data heterogeneity will be established. The previous literature required either i.i.d or normal innovations.

Section 1 of the paper derives an asymptotic theory for the periodogram-based spectral distribution function as well as an array of martingale tests. Section 2 shows that this asymptotic theory carries over to a broad class of window estimators. Section 3 discusses spectral shape tests in the context of alternative approaches to uncovering deviations from the martingale null. Section 4 applies the tests to stock prices. The tests provide some evidence against the random walk hypothesis, confirming recent work of Lo and MacKinlay [1988] and Poterba and Summers [1987]. Section 5 contains summary and conclusions. A Technical Appendix follows which contains all proofs.

### *1. Spectral distribution function estimates and hypothesis testing*

Consider the time series  $x_t$ . The null hypothesis of interest is that  $x_t$  is a martingale difference sequence. With respect to the projections onto the Hilbert space generated by the history of  $x_t$ , this is equivalent to the statement that the autocovariance function of  $x_t$ ,  $\sigma_x(j)$ , is identically equal to zero at all leads and lags. As will be seen, the analysis will focus on the autocorrelation function  $\rho_x(j)$ . In order to develop an asymptotic distribution theory, it is necessary to place some restrictions on the properties of the martingale differences. These requirements are summarized in

*Definition 1.1.  $H_0$ : Null Hypothesis.*

*The following properties hold for  $x_t$*

i.  $E(x_t | \mathfrak{F}_{t-1}) = \mu$  where  $\mathfrak{F}_j$  is the  $\sigma$ -algebra generated by  $x_k$  for  $k \leq j$ .

ii.  $E(x_j^2) = \sigma^2$ .

iii.  $\lim_{T \rightarrow \infty} T^{-1} \sum_{j=1}^T E(x_j^2 | \mathfrak{F}_{j-1}) = \sigma^2 > 0$  almost surely.

iv. There exists a random variable  $W$  with  $E(W^4) < \infty$  such that  $P(|x_j| > u) \leq cP(|W| > u)$  for some  $0 < c < \infty$  and all  $j$ , all  $u \geq 0$ .

v.  $E(x_j^2 x_{j-r} x_{j-s}) = \kappa(r, s)$  finite and uniformly bounded  $\forall j, r \geq 1, s \geq 1$ .

vi.  $\lim_{T \rightarrow \infty} T^{-1} \sum_{j=1+\min(r,s)}^T x_{j-r} x_{j-s} E(x_j^2 | \mathfrak{F}_{j-1}) = \kappa(r, s)$  almost surely.

vii.  $E(x_j^8)$  is uniformly bounded  $\forall j$ .

The first condition is, of course, the null hypothesis of interest. Notice that a nonzero mean for  $x_t$  is permitted. This extends the tests to the case where the time series of interest is a random walk with drift. Conditions *ii.* through *vi.* place restrictions on the admissible degree of heterogeneity in the process. Condition *vii.* is required for characterizing the behavior of the periodogram but is not necessary for developing the asymptotics of the individual autocorrelations. The condition is necessary to prove tightness of the sample spectral distribution function estimates in a random function space. With the exception of *vii.*, these conditions are essentially as restrictive as the requirements imposed on innovations in order to develop functional central limit theorem arguments. (See Phillips [1987].)

The complete null hypothesis contains the weakest conditions presently known in the time series literature for developing limit laws and central limit theorems for sample autocorrelations. Hannan and Heyde [1972] prove

*Theorem 1.1. Asymptotic properties of sample autocorrelations.*

Let

$$\bar{x} = T^{-1} \sum_{j=1}^T x_j \tag{1.1}$$

$$\hat{\rho}_x(i) = \frac{T^{-1} \sum_{j=1}^{T-i} (x_j - \bar{x})(x_{j+i} - \bar{x})}{T^{-1} \sum_{k=1}^T (x_k - \bar{x})^2} \tag{1.2}$$

for all  $i \geq 1$ . If  $H_0$  holds, then

i.  $\hat{\rho}_x(i) \Rightarrow_w 0$ .

ii.  $T^{1/2} \hat{\rho} \Rightarrow_w N(0, I)$ ,  $I = k\pi k$  identity matrix, for any  $k$ -length vector  $\hat{\rho}$  of distinct autocorrelations. ( $\Rightarrow_w$  denotes weak convergence.)

In the frequency domain, all testable implications of the martingale null are summarized by the requirements placed on the shape of the spectral density

$$f_x(\omega) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \sigma_x(j) e^{-ij\omega} = \frac{\sigma_x(0)}{2\pi} \quad (1.3)$$

Under the null hypothesis, the spectral density is a rectangle. Equivalently, the spectral distribution function is a diagonal line.

$$F_x(\lambda) = \int_{-\pi}^{\lambda} f_x(\omega) = \frac{\sigma_x(0)}{2\pi} (\lambda + \pi) \quad (1.4)$$

The analysis of spectral shape means that the asymptotic theory will center on the convergence of random functions which estimate the complete spectral density, or equivalently the spectral distribution function. The random spectral density and distribution estimates throughout this paper, when normalized, are all (almost surely) elements of  $C[0,1]$ , the space of continuous functions defined on the interval  $[0,1]$ , endowed with the *sup* metric.

### *Periodogram estimates*

The computation of the asymptotic properties of spectral shape estimates will initially concentrate on the periodogram estimate of the spectral density



$$I_T(\omega) = \frac{1}{2\pi} \sum_{j=-(T-1)}^{T-1} \hat{\sigma}_x(j) e^{-ij\omega} \quad (1.5)$$

The deviations of the periodogram from the white noise spectral density

$$\gamma(\omega) = \frac{1}{2\pi} \left( \sum_{j=-(T-1)}^{T-1} \hat{\sigma}_x(j) e^{-ij\omega} - \sigma_x(0) \right) \quad (1.6)$$

would appear a natural object for measuring deviations from the null. However, the periodogram deviations are not directly interpretable since the periodogram will not, of course, converge pointwise to a rectangle due to the inconsistency of the individual frequency estimates. An alternative approach, though, will render this function useful. The cumulated deviations will, under the null hypothesis, converge to zero due to the law of large numbers introduced by the averaging of the individual frequency estimates through the integration of  $\gamma(\omega)$ . This insight is the basis of the seminal work of Grenander and Rosenblatt [1953,1957] on the asymptotics of the cumulated periodogram.

Grenander and Rosenblatt recognized that the *sup* of the cumulated periodogram will, with certain restrictions placed on the moments of the time series, converge to the *sup* of a Brownian motion. To see the heuristic argument, observe that the deviations of the sample spectral distribution function generated by the periodogram defined over  $\lambda \in [0, \pi]$  will equal

$$\Gamma_T(\lambda) = \int_0^\lambda \left( I_T(\omega) - \frac{\sigma_x(0)}{2\pi} \right) d\omega \quad (1.7)$$

which may be expressed as the sum

$$\frac{1}{2\pi} (\hat{\sigma}_x(0) - \sigma_x(0)) \lambda + \frac{1}{\pi} \sum_{j=1}^{T-1} \hat{\sigma}_x(j) \frac{\sin j\lambda}{j} \quad (1.8)$$

It is helpful to renormalize this expression by mapping  $\lambda$  onto  $\pi t$ ,  $t \in [0,1]$ , and multiplying the entire function by  $\sqrt{2}$ . As  $T \rightarrow \infty$ , a nondegenerate asymptotic distribution will exist for

$$W_T(t) = \frac{1}{\sqrt{2}} T^{1/2} (\hat{\sigma}_x(0) - \sigma_x(0))t + \frac{\sqrt{2}}{\pi} \sum_{j=1}^{T-1} T^{1/2} \hat{\sigma}_x(j) \frac{\sin j\pi t}{j} \quad (1.9)$$

if this expression can be arbitrarily well approximated by

$$\frac{1}{\sqrt{2}} T^{1/2} (\hat{\sigma}_x(0) - \sigma_x(0))t + \frac{\sqrt{2}}{\pi} \sum_{j=1}^k T^{1/2} \hat{\sigma}_x(j) \frac{\sin j\pi t}{j} \quad (1.10)$$

for large  $k$  and if an asymptotic theory is developed for the individual autocovariances. Convergence of the normalized deviations of the spectral distribution function to a process proportional to Brownian motion follows from three arguments: 1) A sequence of *i.i.d.*  $N(0,1)$  random variables,  $\{\epsilon_j\}$ , may be employed to construct a dense approximation of Brownian motion through the transformation

$$\epsilon_0 t + \frac{\sqrt{2}}{\pi} \sum_{j=1}^k \epsilon_j \frac{\sin j\pi t}{j} \Rightarrow_w B(t) \quad t \in [0,1] \quad (1.11)$$

(where convergence occurs as  $k \rightarrow \infty$ .)

2) The sample spectral distribution function and the above construction of Brownian motion may be arbitrarily well approximated by a finite number of terms, and 3) The normalized autocovariances,  $T^{1/2}(\hat{\sigma}_x(j) - \sigma_x(j))$  (which equal  $T^{1/2} \hat{\sigma}_x(j)$  for  $j \neq 0$ ), converge to uncorrelated normal random variables under the null hypothesis.

There are, however, several problems with the Grenander and Rosenblatt formulation from the perspective of econometric implementation. Grenander and Rosenblatt developed only a limited testing framework as they proved convergence of a function (the *sup*) of the cumulated periodogram rather than convergence of the cumulated

periodogram considered as a random function. They thus derived the behavior of a particular mapping of a sequence of elements of  $C[0,1]$  to  $\mathfrak{R}$ . Second, these authors derived their results under the assumption of stationarity. It is well established that many time series which are predicted by theory to be martingales, such as stock prices, exhibit considerable heteroskedasticity. Third, these authors only considered the case where the null hypothesis concerning  $f_x(\omega)$  is completely specified *a priori*. The martingale hypothesis, however, does not restrict the variance of the process. The null says nothing about the particular values of the spectral distribution function, only that the shape is a straight line. Therefore, tests of the spectral distribution need to be normalized to eliminate dependence on the population variance. This modification has the additional implication that unlike previous work, our asymptotic theory does not depend upon the fourth cumulant of the data, which affects the sampling properties of the variance of  $x$ .

Alternatively, we develop an asymptotic theory for the sample spectral distribution function as an element of a random function space. This permits many different aspects of the null hypothesis to be examined. The asymptotic theory in turn permits the construction of a comprehensive array of specification tests based upon the spectral distribution function. These additional tests are quite important from the perspective of finite sample size and power (See Bernard [1989] and Durlauf [1990] for more details.) Further, this asymptotic theory relies only on the properties of the sample autocorrelation function. As a result, the theory is robust to many forms of heteroskedasticity and is unaffected by nuisance parameters. This feature contrasts our work with Durbin [1967, 1969] who developed some results by assuming the data were both stationary and normal.

In order to develop a general theory of martingale testing, we follow the insights of Grenander and Rosenblatt and consider the

periodogram-based estimate of deviations of the spectral distribution function from its theoretical shape when the periodogram is normalized by the sample variance.

$$\begin{aligned}
 U_T(t) &= \sqrt{2} T^{1/2} \int_0^{\pi t} \left( \frac{I_T(\omega)}{\hat{\sigma}_x(0)} - \frac{1}{2\pi} \right) d\omega ; \quad t \in [0,1] \\
 &= \frac{\sqrt{2}}{\pi} \sum_{j=1}^{T-1} T^{1/2} \hat{\rho}_x(j) \frac{\sin j\pi t}{j} ; \quad t \in [0,1]
 \end{aligned} \tag{1.12}$$

These cumulated deviations will provide the basis for hypothesis testing, in that the normalized deviations will converge only under the null hypothesis. The hypothesis tests will not be subject to the difficulties outlined above. By normalizing by the sample variance, the limiting distribution is determined by the asymptotics of the normalized (by  $T^{1/2}$ ) sample autocorrelations. This renders the asymptotics robust to substantial data heterogeneity. In addition, the shape of the normalized spectral distribution is completely characterized for the null hypothesis that the time series is a martingale difference.

In normalizing the periodogram by the sample variance, the cumulated deviations are forced to sum to zero. In fact, the normalized spectral distribution function relates to the original spectral distribution function through

$$U_T(t) = \frac{W_T(t) - tW_T(1)}{\hat{\sigma}_x(0)} \tag{1.13}$$

This transformation ties down the limiting Brownian motion discussed earlier. This idea is formalized in

*Theorem 1.2. Spectral shape asymptotics.*

*If  $x_t$  fulfills  $H_0$ , then*

$$U_T(t) \Rightarrow_w U(t) \quad \text{on } t \in [0,1]$$

where  $U(t)$  is the Brownian bridge on  $t \in [0,1]$ .

Testing the martingale hypothesis requires determining whether the cumulated deviations of  $U_T(t)$  are too large to be attributable to sampling error. In fact, several statistics are available which map the random function into a scalar random variable. The Continuous Mapping Theorem (CMT) immediately implies

*Corollary 1.1. Asymptotic behavior of spectral shape tests.*

Under the null hypothesis, for any  $\Omega$ , a closed Borel subset of  $[0,1]$

$$\text{i. } \int_{\Omega} \frac{U_T(t)^2}{t(1-t)} dt \Rightarrow_w \int_{\Omega} \frac{U(t)^2}{t(1-t)} dt$$

$$AD_T = \int_0^1 \frac{U_T(t)^2}{t(1-t)} dt \Rightarrow_w \int_0^1 \frac{U(t)^2}{t(1-t)} dt \triangleq \text{Anderson-Darling statistic.}$$

$$\text{ii. } \int_{\Omega} U_T(t)^2 dt \Rightarrow_w \int_{\Omega} U(t)^2 dt$$

$$CVM_T = \int_0^1 U_T(t)^2 dt \Rightarrow_w \int_0^1 U(t)^2 dt \triangleq \text{Cramér-von Mises statistic.}$$

$$\text{iii. } \sup_{t \in \Omega} |U_T(t)| \Rightarrow_w \sup_{t \in \Omega} |U(t)|$$

$$KS_T = \sup_{t \in [0,1]} |U_T(t)| \Rightarrow_w \sup_{t \in [0,1]} |U(t)| \triangleq \text{Kolmogorov-Smirnov statistic.}$$

$$\text{iv. } \sup_{[s \in \Omega_i; t \in \Omega_j]} |U_T(t) - U_T(s)| \Rightarrow_w \sup_{[s \in \Omega_i; t \in \Omega_j]} |U(t) - U(s)|$$

$$K_T = \sup_{[0 \leq s, t \leq 1]} |U_T(t) - U_T(s)| \Rightarrow_w \sup_{[0 \leq s, t \leq 1]} |U(t) - U(s)| \triangleq \text{Kuiper statistic.}$$

Subsequent discussion will focus on the  $AD_T$ ,  $CVM_T$ ,  $KS_T$ , and  $K_T$  statistics, as the various significance levels of their asymptotic distributions are tabulated. (See Shorack and Wellner [1987].) Tests of the null using any of these statistics will possess asymptotic power one against any stationary non-white noise alternative. This occurs due to the  $T^{1/2}$  term blowing up the deviations.

*Corollary 1.2. Consistency of spectral shape tests.*

$AD_T$ ,  $CVM_T$ ,  $KS_T$ ,  $K_T$ , all diverge if  $x_t$  is any other MA process fulfilling requirements ii. to vii. of the null hypothesis.

*Frequency interval analysis*

An alternative testing framework for the martingale hypothesis may be developed by considering point estimates of the spectral distribution function. This approach may be appropriate when the researcher has some prior information on the location of the alternative. For example, if a researcher believes that the alternative to the martingale model is long run mean reversion, maximizing test power might dictate an examination of the behavior of the low frequencies.

More generally, one can formulate the spectral implications of a martingale difference through the Cramér representation of a random variable

$$x_t = \int_0^\pi \cos(\omega t) du(\omega) + \int_0^\pi \sin(\omega t) dv(\omega) \quad (1.14)$$

where at each fixed  $\omega$ ,  $du(\omega)$  and  $dv(\omega)$  are zero mean orthogonal random variables. This orthogonal decomposition in turn identifies a measure of how different intervals of frequencies contribute to the total variance of a time series. The percentage contribution of interval  $[\lambda_1, \lambda_2]$  to total variance is

$$\frac{2 \int_{\lambda_1}^{\lambda_2} f_x(\omega) d\omega}{\sigma_x(0)} \quad (1.15)$$

Under the null,  $\frac{2 \int_{\pi s}^{\pi t} f(\omega) d\omega}{\sigma_x(0)} = t-s$ .

Specific alternatives will place precise restrictions on this ratio. For example, mean reversion implies that the percentage contribution of  $[0, \lambda]$  is small when  $\lambda$  is small. Testing whether the power within an interval is consistent with white noise can be achieved in a straightforward application of Theorem 1.2. Noting that  $U_T(t) - U_T(s)$  corresponds to deviations of spectral power over  $[\pi s, \pi t]$  relative to the null,

*Corollary 1.3. Asymptotic behavior of point estimates of the spectral distribution function.*

Under  $H_0$ , for  $s \leq t$ ,  $U_T(t) - U_T(s) \Rightarrow_w N(0, (t-s) - (t-s)^2)$ . Under any alternative such that  $\frac{2 \int_{\pi s}^{\pi t} f(\omega) d\omega}{\sigma_x(0)} \neq t-s$ ,  $U_T(t) - U_T(s)$  diverges.

## 2. Window estimates of the spectral density

In empirical work, the periodogram is normally not directly analyzed due to the inconsistency of the individual frequency estimates. Spectral windows are typically applied to smooth the periodogram so as to generate consistency at a countable set of frequencies. These smoothed, normalized spectral estimates take the form

$$h_T(\omega) = \int_{-\pi}^{\pi} \frac{I_T(\theta)}{\hat{\sigma}_x(0)} W_T(\omega - \theta) d\theta \quad (2.1)$$

where the window  $W_T(\theta)$  equals

$$W_T(\theta) \triangleq \frac{1}{2\pi} \sum_{j=-(T-1)}^{T-1} \delta_T(j) e^{-ij\theta} \quad (2.2)$$

Since some martingale tests are interpretable as window estimators at the zero frequency, it is useful to understand the asymptotic properties of window-based spectral distribution function estimates.

We work with windows that fulfill

*Definition 2.1. Admissible spectral window.*

*The sequence of weights  $\delta_T(j)$  fulfills*

*i.  $\delta_T(j)$  is uniformly bounded in  $j$  and  $T$ .*

*ii.  $\lim_{T \rightarrow \infty} \delta_T(j) \Rightarrow 1$  for fixed  $j$ .*

These requirements are weaker than those necessary to prove that individual frequency estimates are consistent. The relevant issue for hypothesis testing concerns the interpretation of spectral shape tests which employ a smoothed, normalized estimator,



$$U_T^W(t) = \sqrt{2} T^{1/2} \int_0^{\pi t} \left( h_T(\omega) - \frac{1}{2\pi} \right) d\omega \quad (2.3)$$

Despite the pointwise inconsistency of the periodogram, the employment of a consistent window estimate in its place will have no effect on the asymptotics of the cumulated spectral shape. This was originally recognized by Parzen [1957]. Intuitively, this occurs because the cumulated sample spectral density estimates already average over individual frequencies so as to produce estimates of proportions of the total variance of the time series. This idea may be extended to show that standard windows used in the construction of the sample spectral distribution function asymptotically wash out.

*Theorem 2.1 Asymptotic equivalence of window generated and periodogram generated spectral shape tests.*

*Under  $H_0$ , for any window fulfilling Definition 2.1*

$$U_T^W(t) \Rightarrow_w U(t).$$

*Corollary 2.1. Asymptotic behavior of window based spectral shape tests.*

*For any window fulfilling Definition 2.1, all asymptotic results in Corollaries 1.1 to 1.3 will still hold if  $U_T^W(t)$  replaces  $U_T(t)$ .*

This asymptotic equivalence does not, of course, imply that the window choice is irrelevant in finite samples. Window choice can in particular affect finite sample test size.

### *3. Relationship to other tests*

Recent work on the testing of the martingale null has focused on the question of the behavior of the variance of long differences in various time series. Cochrane [1988] is an early source of this methodology. This approach centers on the variance ratio

$$\Lambda(k) = \frac{\text{Var}(\sum_{i=0}^{k-1} x_{t-i})}{k \text{Var}(x_t)}. \quad (3.1)$$

Under  $H_0$ ,  $\Lambda(k)=1$ .

As recognized by Campbell and Mankiw [1988], since the sample variance ratio equals

$$\Lambda_T(k) = \sum_{j=-k}^k \left(\frac{k-|j|}{k}\right) \hat{\rho}_x(j) \quad (3.2)$$

the statistic is proportional to the Bartlett estimate of the zero frequency. Variance ratio tests, as  $k \rightarrow \infty$ , therefore examine the rectangular null at the zero frequency.

The spectral shape tests may therefore be contrasted with the variance ratio methodology in terms of the way in which the alternative hypothesis is formulated by the researcher. The various spectral shape tests examine how on average the entire range of frequencies  $[0, \pi]$  deviate from the  $H_0$  population values. This computation is appropriate when the researcher has a diffuse prior over the entire class of alternatives. In the time domain, this is analogous to testing the entire autocorrelation function to see whether the percentage of statistically significant autocorrelations is too great to be attributable to sampling variation.

The variance ratio tests, alternatively, focus on the zero frequency

in isolation and correspond to a researcher's prior that the high frequency deviations of the alternative from the null are small relative to the low frequency deviations. This test will naturally not be consistent against all alternatives, such as  $x_t = \epsilon_t + \epsilon_{t-1} - \epsilon_{t-2}$ . The test can only be justified if the researcher's prior on  $H_1$  is concentrated on the low frequencies.

The complex relationship between the variance ratio test and spectral shape tests may be seen in the window based Cramér-von Mises statistic

$$CVM_T^W = 2T \int_0^1 \left( \int_0^{\pi t} (h_T(\omega) - \frac{1}{2\pi}) d\omega \right)^2 dt \quad (3.3)$$

For the Bartlett window,  $h_T(0) = \Lambda_T(0)$ . The *CVM* test thus employs information in the very low frequencies. The mapping of individual frequencies into the spectral shape tests is, however, quite nonlinear. This occurs because the zero frequency affects all terms in the integrand.

In comparing the tests, note that there is some information in all frequencies for processes with long run mean reversion. For example, if  $x_t = \epsilon_t - \epsilon_{t-100}$ , then the frequencies 0 and  $\pi$  both provide equal information as  $f_x(\pi) = f_x(0) = 0$ . Even with a prior belief that the values of the low order autocovariances are zero, this does not imply that high order frequencies do not contain useful information. General spectral shape tests will exploit information at high frequencies. Ignoring the information available at these frequencies can only be justified by a very specialized prior.

It is also possible to modify the spectral shape tests so as to maximize power against long run mean reversion. By choosing  $\Omega = [0, \lambda]$  for  $\lambda$  near zero, Corollaries 1.1 and 2.1 allow inferences to concentrate on low frequency deviations in a way similar to the variance ratio tests. Further, examination of point estimates as suggested in Corollary 1.3

permits the identification of whether low frequencies exhibit a deficiency of power relative to the random walk null.

#### *4. Application to stock prices*

This section presents an application of the spectral shape tests to some time series of excess holding returns on stock portfolios. Recent authors, most notably Lo and MacKinlay [1988] and Poterba and Summers [1987] have challenged the conventional view that stock price returns are unpredictable-i.e. do not form a martingale difference sequence. These authors, using variance ratio tests, have concluded that stock prices exhibit some mean reversion.

As Section 3 has suggested, the spectral shape tests may be interpreted as searching over all frequencies of the spectral density for martingale difference violations whereas the variance bounds tests may be interpreted as examining the zero frequency in isolation. The Lo-MacKinlay and Poterba-Summers papers are therefore a natural place to explore the importance of the researcher's beliefs concerning the location of alternatives in affecting the outcome of a hypothesis search. We concentrate on periodogram estimates. Different Bartlett window estimates generated similar results.

Lo and MacKinlay explored the weekly fluctuations for two CRSP NYSE-AMEX aggregate portfolios-one weighted by value, the other equally weighted across all stocks reported on the exchange. The returns on these portfolios are based on the one week changes in closing Wednesday prices. Following these authors, if the exchange was closed on a Wednesday, the Thursday price was employed. If the exchange was also closed on Thursday, the previous Tuesday price was used. If the

Table 1

Spectral Shape Tests of Deviations of Weekly Excess Returns From White Noise

Periodogram-based Tests  
Lo and MacKinlay Data Set

	$AD_T$	$CVM_T$	$KS_T$	$K_T$
Value Weighted Portfolio				
Entire Sample	6.05**	0.86**	1.42*	1.53
First Half	8.31**	1.33**	2.07**	2.13**
Second Half	1.32	0.17	0.74	0.91
Equal Weighted Portfolio				
Entire Sample	90.3**	11.4**	4.85**	4.89**
First Half	45.4**	5.88**	3.63**	3.64**
Second Half	42.6**	5.30**	3.51**	3.64**
Asymptotic Critical Values				
5%	2.49	0.46	1.36	1.75
1%	3.86	0.74	1.64	2.01
Finite Sample Critical Values ( $T=1000$ )				
5%	3.14	0.49	1.16	1.60

\* denotes significant at asymptotic 5% level

\*\* denotes significant at asymptotic 1% level

exchange was closed on all three days, the value was treated as missing. The data consist of 1216 observations running from September 6, 1962 to December 26, 1985. The sample is also divided in half to see whether the properties of the returns are stable.

Table 1 reports the various spectral shape statistics for the periodogram based estimates. These estimates provide reasonably strong evidence against the martingale difference null. Four basic conclusions may be drawn. 1) The null hypothesis is generally rejected for both portfolios over the entire sample. The one exception is the Kuiper statistic for the value weighted portfolio. 2) The rejections for the equal weighted portfolio are overwhelming and substantially stronger than for the value weighted portfolio. 3) The rejections for the value weighted portfolio disappear in the second half of the sample period. Interestingly, these results are quite consistent with Lo and MacKinlay. 4) The extremely high values of the Anderson-Darling statistic relative to the Cramér-von Mises statistic suggest that the low frequency components of the equal weighted portfolio returns are the source of the overall rejections.

Table 1 also lists the finite sample critical values generated by 4000 replications based upon *i.i.d.*  $N(0,1)$  errors. The finite sample critical values do affect inference slightly. The Kuiper statistic over the entire sample now rejects for the value weighted portfolio. Interestingly, the Cramér-von Mises statistic performs quite well in that the finite sample 5% confidence level is quite near its asymptotic counterpart. Durlauf [1990] confirms that for independent normal errors, the size of the *CVM* test is 5% for as few as 40 observations.

The differences between the behavior of value and equal weighted portfolio returns affects the sorts of economic interpretations suggested by violations of the null. For example, the equal weighted portfolio requires many more transactions across time than the value weighted

Table 2

Spectral Shape Tests Deviations of Monthly Excess Returns From White Noise

Periodogram-Based tests  
Poterba and Summers Data Set

	$AD_T$	$CVM_T$	$KS_T$	$K_T$
Value Weighted Portfolio	11.1**	1.86**	2.57**	2.75**
Equal Weighted Portfolio	5.37**	0.92**	2.10**	2.24**

\* denotes significant at asymptotic 5% level

\*\* denotes significant at asymptotic 1% level

portfolio. This occurs because all capital gains must be followed by portfolio adjustments to preserve relative weights. Mean reversion may thus be caused by transaction costs. Further, the equal weighted portfolio gives substantial weight to small firms in determining holding returns. Investors may be risk averse with respect to small firms facing possible bankruptcy, particularly over longer horizons.

Poterba and Summers examined the monthly returns on the CRSP-NYSE value weighted and equal weighted portfolios. The data run from January 1926 to December 1985. Table 2 explores the spectral behavior of aggregate excess holding returns over the entire sample.

The overall results for monthly holding returns are inconsistent with the null hypothesis. Over the entire sample, the periodogram estimates consistently reject the null for both portfolios for all four statistics. The results in Table 2 clearly show that the Poterba-Summers conclusions were not essentially affected by the particular alternative examined.

One should certainly not read the results of these empirical exercises as strongly demonstrating that excess holding returns are not white noise. The test statistics require the existence of eighth moments, which may not hold for the data. Pagan and Schwert [1989] provide some evidence that even second moments may not exist. The important conclusion from our results is evidence against the random walk theory can be deduced which does not depend on examination of a particular frequency.

The frequency domain methods have provided a number of interesting empirical results in other areas. Durlauf [1989,1990] uses spectral shape tests to argue that annual log per capita output can be well modelled as a random walk with drift. Bizer and Durlauf [1990], analyzing tax behavior, reject the null hypothesis that tax changes are unpredictable, casting some doubt on the optimal tax smoothing model



of Barro [1981].

## 5. *Summary and conclusions*

This paper has presented a method of using spectral distribution estimates to test whether a given time series is a martingale difference. The tests explore the cumulated deviations of the sample spectral distribution function from its theoretical shape—a diagonal line—under the null hypothesis. These cumulated deviations, when scaled, behave as a Brownian bridge. An asymptotic theory is developed for a broad class of martingale difference processes.

When applied to the entire spectral distribution function, the testing framework is consistent against all fixed alternatives. At the same time, each test simultaneously analyzes all frequency components of the data. As a result, the various test statistics correspond to a diffuse prior over the location of alternative hypotheses. These tests avoid data mining by embodying *all* implications of the null hypothesis. The tests can further be adjusted to explore different subsets of frequencies, which may be appropriate when a researcher has some prior on the nature of likely alternatives.

Application of the tests to weekly and monthly stock returns revealed some evidence against the null hypothesis that the holding returns are martingale differences. This result confirms recent research demonstrating that stock prices exhibit long run mean reversion. Violations of the random walk theory appear to be robust to a relatively diffuse formulation of a researcher's beliefs concerning the class of alternatives.

One extension of the techniques in this paper would consist of

multivariate generalizations of the test statistics. Most martingale formulations require that increments in a time series be unpredictable given all available information, not just the history of the series. In this way, the more powerful tests of theories such as the efficient markets hypothesis can be implemented.

## Technical Appendix

### Proof of Theorem 1.2

Let  $\epsilon_j, j=1, \dots$ , represent an infinite sequence of  $N(0,1)$  random variables. Standard arguments verify that

$$z(t) = \frac{\sqrt{2}}{\pi} \sum_{j=1}^{\infty} \epsilon_j \frac{\sin j\pi t}{j} \quad t \in [0,1] \quad (\text{A.1})$$

is a representation of a Brownian bridge. We now consider  $U_T(t)$

$$U_T(t) = \frac{\sqrt{2}}{\pi} \sum_{j=1}^{T-1} T^{1/2} \hat{\rho}_x(j) \frac{\sin j\pi t}{j} \quad (\text{A.2})$$

To prove weak convergence of  $U_T(t)$  to  $z(t)$ , we need to show that the probability measures  $\mu_T$  associated with the  $U_T(t)$  sequence converge to the probability measure associated with  $U(t)$ . From Billingsley Theorems 8.1 and 8.2, necessary and sufficient conditions for this convergence are *i.* For any  $\delta > 0$ ,  $\exists$  an  $\xi > 0$  such that  $P(|U_T(0)| > \delta) \leq \xi \quad \forall T \geq 1$ . *ii.* The joint distribution of  $\{U_T(t_1) \dots U_T(t_k)\}$  converges to the joint distribution of  $\{U(t_1) \dots U(t_k)\}$  for any finite sequence  $t_1 \dots t_k$ . *iii.* For every  $\gamma > 0$  and  $\eta > 0$ ,  $\exists$  an  $\epsilon \in (0,1)$  and  $\exists$  an integer  $T_0$  such that  $P(\sup_{|s-t| < \epsilon} |U_T(t) - U_T(s)| \geq \gamma) \leq \eta$  for all  $T \geq T_0$ .

Condition *i.* is trivial since by construction  $U_T(0) = 0 \quad \forall T$ .

In order to verify Conditions *ii.* and *iii.*, rewrite  $U_T(t)$  as

$$U_T(t) = \frac{\sqrt{2}}{\pi} \left( \sum_{j=1}^k T^{1/2} \hat{\rho}_x(j) \frac{\sin j\pi t}{j} + \sum_{j=k+1}^{T-1} T^{1/2} \hat{\rho}_x(j) \frac{\sin j\pi t}{j} \right) \quad (\text{A.3})$$

By the Hannan-Heyde CLT and the CMT, for any fixed  $k$

$$U_T^k(t) = \frac{\sqrt{2}}{\pi} \sum_{j=1}^k T^{1/2} \hat{\rho}_x(j) \frac{\sin j \pi t}{j} \Rightarrow_w U^k(t) = \frac{\sqrt{2}}{\pi} \sum_{j=1}^k \rho_{lim}(j) \frac{\sin j \pi t}{j} \quad (\text{A.4})$$

where  $T^{1/2} \hat{\rho}_x(j)$  converges to a sequence of *i.i.d.*  $N(0,1)$  random variables  $\rho_{lim}(j)$ . Convergence requirements *ii.* and *iii.* thus hold for the truncated estimate  $U_T^k(t)$ . Next, define

$$R_T^k(t) = \frac{\sqrt{2}}{\pi} \left| \sum_{j=k+1}^{T-1} T^{1/2} \hat{\rho}_x(j) \frac{\sin j \pi t}{j} \right| \quad (\text{A.5})$$

$$R^k(t) = \frac{\sqrt{2}}{\pi} \left| \sum_{j=k+1}^{\infty} \epsilon_j \frac{\sin j \pi t}{j} \right| \quad (\text{A.6})$$

where  $\epsilon_j$ 's are defined as above. Suppose for any  $\zeta > 0$  and  $\xi > 0$ , there exists a  $k$  and a  $T_0$  such that for all  $T \geq T_0$

$$P\left(\sup_{t \in [0,1]} R_T^k(t) \geq \zeta\right) \leq \xi \quad (\text{A.7})$$

We claim that if (A.7) is true, then the Theorem is verified. To see this, note that Condition *ii.* holds through application of the Cramér-Wold device to

$$\sum_{i=1}^r \lambda_i U_T(t_i) \quad (\text{A.8})$$

where  $t_1 \dots t_r$  are fixed. Convergence of (A.8) to  $\sum_{i=1}^r \lambda_i U(t_i)$  for  $U(t) = U^k(t) + R^k(t)$  requires that for any  $\nu > 0$  and  $\psi > 0$ ,

$$\limsup_{T \rightarrow \infty} P\left(\left| \sum_{i=1}^r \lambda_i U(t_i) - \sum_{i=1}^r \lambda_i U_T(t_i) \right| \geq \nu\right) \leq \psi \quad (\text{A.9})$$

This expression can be bounded by

$$\limsup_{T \rightarrow \infty} P\left(\left| \sum_{i=1}^r \lambda_i U(t_i) - \sum_{i=1}^r \lambda_i U_T(t_i) \right| \geq \nu\right) \leq$$

$$\begin{aligned}
& \limsup_{T \rightarrow \infty} \\
& P(|\sum_{i=1}^r \lambda_i U^k(t_i) - \sum_{i=1}^r \lambda_i U_T^k(t_i)| + |\sum_{i=1}^r \lambda_i R^k(t_i)| + |\sum_{i=1}^r \lambda_i R_T^k(t_i)| \geq \nu) \quad (\text{A.10})
\end{aligned}$$

By the convergence in (A.4), for any  $\nu, \psi > 0$

$$\limsup_{T \rightarrow \infty} P(|\sum_{i=1}^r \lambda_i U^k(t_i) - \sum_{i=1}^r \lambda_i U_T^k(t_i)| \geq \frac{\nu}{3}) \leq \frac{\psi}{3} \quad (\text{A.11})$$

By construction of the Brownian bridge, for any  $\nu, \psi > 0$

$$\limsup_{k \rightarrow \infty} P(|\sum_{i=1}^r \lambda_i R^k(t_i)| \geq \frac{\nu}{3}) \leq \frac{\psi}{3}. \quad (\text{A.12})$$

Finally, (A.7) implies that the third term in (A.10) greater than any  $\frac{\nu}{3}$  with probability less than  $\frac{\psi}{3}$  through suitable choice of  $k$  and sufficiently large  $T$ . Therefore, if (A.7) holds, then the sum of the three terms on the right hand side of (A.10) can be made smaller than any arbitrary  $\psi > 0$  with probability arbitrarily near 1, for sufficiently large  $k$  as  $T \rightarrow \infty$ . Since this holds for any weights  $\lambda_i$ , the standard Cramér-Wold argument ensures that all finite dimensional distributions converge.

Condition *iii*. follows from a version of the triangle inequality

$$\begin{aligned}
& P(\sup_{|s-t| < \epsilon} |U_T(t) - U_T(s)| \geq \gamma) \leq \\
& P(\sup_{|s-t| < \epsilon} |U_T^k(t) - U_T^k(s)| + |R_T^k(t) - R_T^k(s)| \geq \gamma) \quad (\text{A.13})
\end{aligned}$$

To see that this last expression can be made arbitrarily small for fixed  $k$ , consider the two components  $\sup_{|s-t| < \epsilon} |U_T^k(t) - U_T^k(s)|$  and  $\sup_{|s-t| < \epsilon} |R_T^k(t) - R_T^k(s)|$  separately. For all  $T$  greater than or equal to some  $T_1$ ,  $\sup_{|s-t| < \epsilon} |U_T^k(t) - U_T^k(s)|$  can be bounded below any  $\frac{\gamma}{2}$  with probability greater than  $1 - \frac{\eta}{2}$  since the sequence of probability measures associated

with  $U_T^k(t)$  is tight. For all  $T$  greater than or equal to some  $T_2$ , the  $\sup_{|s-t| < \epsilon} |R_T^k(t) - R_T^k(s)|$  is bounded below any  $\frac{\gamma}{2}$  with probability greater than  $1 - \frac{\eta}{2}$  if (A.7) holds. Therefore, choosing  $T_0$  as the maximum of  $T_1$  and  $T_2$  allows one to make this last probability smaller than any arbitrary  $\eta$ , implying that Condition *iii.*, tightness, holds if (A.7) is true.

Tedious algebra in Grenander and Rosenblatt [1957] pg. 189 shows that (A.7) holds whenever

$$E|\hat{\rho}_x(j)\hat{\rho}_x(r+j)\hat{\rho}_x(k)\hat{\rho}_x(r+k)| \quad (\text{A.14})$$

is  $O_p(T^{-2})$ , which holds whenever  $E(x_j^8)$  is uniformly bounded. This verifies the Theorem.

*Proof of Corollary 1.3*

The CMT ensures that  $U_T(t) - U_T(s) \Rightarrow_w U(t) - U(s)$ . Further,  $U(t) - U(s) = B(t) - B(s) - (t-s)B(1)$ , which is distributed as  $N(0, t-s - (t-s)^2)$ .

*Proof of Theorem 2.1*

The window estimator of the cumulated spectral deviations is

$$U_T^W(t) = \frac{\sqrt{2}}{\pi} \sum_{j=1}^{T-1} T^{1/2} \delta_T(j) \hat{\rho}_x(j) \frac{\sin j\pi t}{j} \quad (\text{A.15})$$

We analyse

$$\begin{aligned} U_T^W(t) - U_T(t) &= \frac{\sqrt{2}}{\pi} \sum_{j=1}^{T-1} T^{1/2} (\delta_T(j) - 1) \hat{\rho}_x(j) \frac{\sin j\pi t}{j} \\ &= \frac{\sqrt{2}}{\pi} \sum_{j=1}^k T^{1/2} (\delta_T(j) - 1) \hat{\rho}_x(j) \frac{\sin j\pi t}{j} + \frac{\sqrt{2}}{\pi} \sum_{j=k+1}^{T-1} T^{1/2} (\delta_T(j) - 1) \hat{\rho}_x(j) \frac{\sin j\pi t}{j} \end{aligned} \quad (\text{A.16})$$

We now show that (A.16) converges to zero, which combined with Theorem 1.2 verifies Theorem 2.1. Consider the first summation, for fixed  $k$ . By the Cauchy-Schwarz inequality

$$\begin{aligned} & \left| \frac{\sqrt{2}}{\pi} \sum_{j=1}^k T^{1/2} (\delta_T(j) - 1) \hat{\rho}_x(j) \frac{\sin j \pi t}{j} \right| \leq \\ & \frac{\sqrt{2}}{\pi} \left( \sum_{j=1}^k \left( T^{1/2} \hat{\rho}_x(j) \frac{\sin j \pi t}{j} \right)^2 \right)^{1/2} \left( \sum_{j=1}^k (\delta_T(j) - 1)^2 \right)^{1/2} \end{aligned} \quad (\text{A.17})$$

This upper bound converges to zero as  $T \rightarrow \infty$  since the first summation on the right of the inequality is an  $O_p(1)$  random variable and the second summation converges to zero since  $\delta_T(j) \Rightarrow 1$  for fixed  $j$ . This means that the first summation in (A.16) converges weakly to zero.

Define

$$R_T^{W,k}(t) = \frac{\sqrt{2}}{\pi} \left| \sum_{j=k+1}^{T-1} T^{1/2} (\delta_T(j) - 1) \hat{\rho}_x(j) \frac{\sin j \pi t}{j} \right| \quad (\text{A.18})$$

Following the proof of Theorem 1.2, Theorem 2.1 is verified if for any  $\zeta > 0$  and  $\xi > 0$ , there exists a  $k$  such that for  $T \geq T_0$ ,

$$P\left( \sup_{t \in \{0,1\}} R_T^{W,k}(t) \geq \zeta \right) \leq \xi \quad (\text{A.19})$$

Rewriting  $R_T^{W,k}(t)$  as

$$R_T^{W,k}(t) = \frac{\sqrt{2}}{\pi} \left| \sum_{j=k+1}^{T-1} T^{1/2} \delta_T(j) \hat{\rho}_x(j) \frac{\sin j \pi t}{j} - \frac{\pi}{\sqrt{2}} R_T^k(t) \right|, \quad (\text{A.20})$$

eq. (A.7) and the triangle inequality imply that proving (A.19) holds is equivalent to showing for any  $\zeta > 0$  and  $\xi > 0$ , there exists a  $k$  such that for  $T \geq T_0$

$$P\left(\sup_{t \in [0,1]} \frac{\sqrt{2}}{\pi} \left| \sum_{j=k+1}^{T-1} T^{1/2} \delta_T(j) \hat{\rho}_x(j) \frac{\sin j \pi t}{j} \right| \geq \frac{\xi}{2} \right) \leq \frac{\xi}{2} \quad (\text{A.21})$$

By analogy to the proof of Theorem 1.2, this last limit holds if

$$E|\delta_T(j)\delta_T(r+j)\delta_T(k)\delta_T(r+k)\hat{\rho}_x(j)\hat{\rho}_x(r+j)\hat{\rho}_x(k)\hat{\rho}_x(r+k)| \quad (\text{A.22})$$

is  $O_p(T^{-2})$ . Uniform boundedness of the window weights means that any conditions ensuring that (A.14) is  $O_p(T^{-2})$  also mean that the same holds for (A.22), which proves the result.



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