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ASSET PRICING WITH A FACTOR ARCH COVARIANCE STRUCTURE: EMPIRICAL ESTIMATES FOR TREASURY BILLS

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This structure is applied to monthly treasury bills from two to twelve months maturity and the value weighted NYSE returns index. The bills appear to have a single factor in the variance process and this factor is influenced or "caused in variance" by the stock returns.

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Asset pricing relations are developed for a vector of assets with a time varying covariance structure. Assuming that the eigenvectors are constant but the eigenvalues changing, both the Capital Asset Pricing Model and the Arbitrage Pricing Theory suggest the same testable implication: the time varying part of risk premia are proportional to the time varying eigenvalues. Specifying the eigenvalues as general ARCH processes, the model is a multivariate Factor ARCH model. Univariate portfolios corresponding to the eigenvectors will have (time varying) risk premia proportional to their own (time varying) variance and can be estimated using the GARCH-M model.

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I. INTRODUCTION

Much recent empirical literature has observed that asset returns typically exhibit time varying variances. This has been shown for exchange rates by Hsieh(1985), McCurdy and Morgan(1986), Domowitz and Hakkio(1984), Diebold and Nerlove(1988), and Engle and Bollerslev(1986) among others. It has also been observed by Engle, Lilien and Robins(1987) and Bollerslev, Engle and Wooldridge(1988) for short and long term interest rates and by Schwert, French and Stambaugh(1986), Chou(1986)and Bollerslav, Engle and Wooldridge (1988) for equities.

In the presence of such well established empirical regularity, the empirical tests of asset pricing theories must be reappraised. The general approach pioneered by Fama and MacBeth(1973) estimates the risk premia as if the covariance structure were constant and therefore may reject the CAPM even if it is valid in a dynamic context. Models of the factor structure of returns used in testing the APT, likewise are based upon estimates of the unconditional covariance matrix of returns which may or may not have the same factor structure as the time varying covariance matrix[1]. Inference about the number of factors is surely invalid in this context. Furthermore, the risk premia are potentially time varying so estimates of their mean will be inefficient or biased depending upon the context.

Several papers have attempted to estimate time varying risk premia in the presence of time varying variances. Engle, Lilien and Robins(1987) introduce the ARCH-M model to model returns as a function of own variances for interest rates, and Domowitz and Hakkio(1984) do the same for exchange rates. Bollerslev, Engle and Wooldridge(1988) extend this to a three asset system and apply a trivariate version of the CAPM.

In this paper the implications for the CAPM and APT of a time varying factor structure in the covariance matrix of returns are explored. Fortuitiously, both models suggest the same testable restrictions and both allow the model to be efficiently estimated without requiring the universe of stochastic returns. This argument is

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[1]Unlike the CAPM, the APT works well in an unconditional context. That is, an unconditional factor structure determines (in part) the behavior of unconditional risk premia. See Stambaugh (1983) and Rothschild (1986). However, a conditional (or time varying) factor structure also determines the way in which risk premia change over time. Empirical examination of this relationship requires estimates of the conditional (or time varying) factor structure.

presented in Section II. In section III, the econometric specification is developed which turns out to be the Generalized Factor ARCH structure of Engle(1987). Section IV. presents empirical estimates for the short end of the term structure and its interaction with equity markets. Section V ventures some conclusions.

II. Asset Pricing with Time Varying Covariances

Let $y_t \in \mathbb{R}^N$ be a vector of asset returns; that is y_{it} is the (random) return on an investment of $1 in asset i in period t. Denote the conditional mean vector and covariance matrix of y_t as μ_t and H_t respectively. If there is a riskless asset which earns rate of return ρ_t then the risk preimum of asset i is $\mu_{it} - \rho_t$. The goal of many asset pricing theories is to use the conditional covariance matrix, H_t, to explain risk premia. At each point in time the matrix H_t can be written in terms of its spectral decomposition:

$$H_t = \sum_{k=1}^{N} f_{kt} f_{kt}' h_{kt}$$

where f_{kt} are orthogonal eigenvectors and h_{kt} are non-negative eigenvalues.[2] To restrict the potential time series behavior of H_t we now assume that the eigenvectors are time invariant.[3] This major simplifying assumption on the behavior of the covariance matrix is

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[2]The f_{kt} are normalized such that $f_{kt}'\iota = 1$, where ι is a vector of one's. This particular normalization is chosen because, as we shall show later, the f_{kt} are related to the weights of particular portfolios that represent "factors".

[3]Under this assumption, $E(H_t) = \Sigma_{k=1}^{N} f_k f_k' \sigma_k^2$, where $\sigma_k^2 = E(h_{kt})$. Thus, the f_k are also eigenvectors of the unconditional expectation of H_t.

statistically convenient and, we would argue, quite reasonable. If there is a single risky characteristics, then our restriction implies that the **relative** riskiness of assets remains constant over time while total riskiness varies. This observation will be developed below. The eigenvalues, h_{kt}, can be decomposed into a non-stochastic, time invariant component ω_k, and a time varying component λ_{kt}, so that:

$$h_{kt} = \omega_k + \lambda_{kt}, \quad \omega_k, \lambda_{kt} \geq 0 \text{ for all } k, t.$$

Finally, we suppose that only $K \leq N$ of the λ_{kt} are ever non-zero. Then we obtain the decomposition of H_t:

(1.a) $$H_t = \Omega + V_t$$

(1.b) $$\Omega = \sum_{k=1}^{N} f_k f_k' \omega_k$$

(1.c) $$V_t = \sum_{k=1}^{K} f_k f_k' \lambda_{kt}$$

Equation (1.a) states that the variance covariance matrix can be decomposed into a constant part Ω and a time varying part V_t. Both Ω and V_t are symmetric, positive semidefinite matrices. Equation (1.b) is the spectral decomposition of Ω and (1.c) is the same for V_t. The matrix V_t is of rank K which may be considerably less than N. If K is small, then (1.c) implies that the dynamic structure of the variance covariance matrix is determined by the dynamic behavior of a few parameters, the eigenvalues of V_t, which can have a rich stochastic structure. The techniques developed for estimating dynamic process of variances can be used to analyze their behavior. Indeed as we shall see the λ_{kt} are variances of the returns on particular portfolios and the f_k are corresponding vectors of portfolio weights.

In this paper we will (i) analyze the stochastic structure of the

λ_{kt} and (ii) use the results of this analysis to estimate asset pricing equations which are variants of

$$\mu_{it} - \rho_t = \alpha_i + \sum_k \beta_{ik}\lambda_{kt} . \tag{2}$$

Equation (2) is suggested by the specification (1), some strong additional assumptions and either of the two leading theories of asset pricing, the capital asset pricing model (CAPM) or the arbitrage pricing theory (APT).

A. The CAPM

We consider first the CAPM. Suppose there is a riskless asset whch earns the rate of return ρ_t[4]. The CAPM pricing equation may be written

$$\mu_t - \rho_t \iota = \delta_t H_t w_t \tag{3}$$

where ι is a vector of ones, δ_t is the market price of risk and w_t is a normalized vector of portfolio weights which is proportional to the market portfolio. Assume that the market price of risk and the composition of the market portfolio are constant[5]. That is

$$\delta_t = \delta \text{ and } w_t = w. \tag{4}$$

Then,

$$\mu_t - \rho_t \iota = \delta\Omega w + \delta V_t w = \delta\Omega w + \delta\sum_k (f_k' w) f_k \lambda_{kt} .$$

which is of the form (2).

The careful reader will have noted that the assumptions in (4) are not likely to be consistent with the CAPM. That is, if the CAPM really

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[4]This derivation assumes the existence of a riskless asset. All the formulae have analogs if there is not a riskless asset.

[5]These are the strong additional assumptions mentioned above.

determines prices of securities, the market portfolio w is determined by the stochastic realization of asset prices, which is governed, in part, by the variance-covariance matrix of asset returns. As a result it is difficult to write down a model in which the CAPM holds (the market portfolio is mean variance efficient) and the market shares are constant. This does not mean that our basic equation (2) is inconsistent with rational behavior on the part of investors. As the work which Harrison and Kreps (1979) initiated on Martingale measure has made clear any stochastic specification of security prices which does not permit arbitrage opportunities is consistent with rational behavior (this is the stochastic counterpart of Sonnenschein's work on the, meager, implications of utility maximization for the behavior of excess demand functions[6]). Thus our equation is suggested by the CAPM but can be derived exactly only by approximating the market portfolio by the fixed weight portfolio most highly correlated with the market. In most cases this should be a good approximation; in any case the equation is an interesting hypothsis for empirical examination.

Consider now the returns to two assets which have only one risk characteristic, namely factor loadings on factor k. For these assets, the risk premia are given by

$$\mu_{it} - \rho_t = \delta(f_k{}'w)f_{ik}(\omega_k + \lambda_{kt}).$$

The ratio of the risk premia for any such assets is proportional only to the factor loadings and is therefore time invariant if and only if the eigenvectors are time invariant.

Note that for individual portfolios formed using eigenvectors as

[6]See Schafer and Sonnenschein (1982) for a survey of this literature.

weights, the relation between (changes in) risk and return is particularly simple. Let $\tilde{f}_{kt}$ be the scalar random return on the portfolio with weights f_k. Then

$$E(\tilde{f}_{kt}) - \rho_t = f_k'\mu_t - \rho_t$$
$$= \delta f_k'\Omega w + \delta(f_k'w)\lambda_{kt}$$

and

$$\mathrm{Var}(\tilde{f}_{kt}) = f_k'\Omega f_k + \lambda_{kt}$$

Thus, the time varying part of the risk premia of $\tilde{f}_{kt}$ is proportional to the time varying part of its variance, which is simply λ_{kt}.

Consider now the familiar Roll critique of empirical tests of the CAPM. Suppose we could observe an additional set of assets N+1,...,N* which together with the original set of assets comprises the universe of assets. Denoting the returns to the first set by y_{1t} and the second by y_{2t} the covariance matrix becomes H_t^* given by

$$H_t^* = \begin{bmatrix} \Omega_{11} + V_t & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}$$

where it is simply assumed that the covariance matrix of the new asset returns is time invariant. With $w'^* = (w_1'^*, w_2'^*)$ as the new set of portfolio shares, the vector of risk premia implied by the CAPM is again given by

$$\mu_t - \rho_t\iota = \delta\, H_t^*\, w^*$$

so that the risk premia to the first set of assets are

$$\mu_{1t} - \rho_t\iota = \delta\, \Omega_{11}w_1^* + \delta\Omega_{12}w_2^* + \delta V_t w_1^*.$$

The difference between this formula and that derived above, is simply in the intercept term. Without prior knowledge of the Ω's, w's or δ, the parameters of the intercept are not identified so the estimation can proceed without observation of y_2. Thus by assuming that the unobserved asset returns satisfy the constant covariance assumptions traditionally made for all assets in empirical tests of the CAPM, the Roll critique can be avoided. This seems an appropriate assumption for some of the most problematic assets such as human capital, housing and consumer durables.

B. The APT

The derivation of (2) from the APT is somewhat less clean. Let us review the conventional APT. Suppose there are many assets (i.e. N is large) and that the constant variance-covariance matrix H has a factor structure

(5) $$H = BB' + D$$

where B is N by K and D is diagonal[7]. Then the APT states that

(6) $$\mu_i - \rho = \sum_k b_{ik} \tau_k + e_i$$

where τ_k is the factor risk premium and e_i is a small pricing error.

The sense in which pricing errors are small is sometimes a source of confusion. The APT was deisgned to capture an intuitively plausible notion. If the stochastic structure of most asset returns is determined by a few common factors and idiosyncratic risk, then idiosyncratic risk should be diverisifiiable; in equilibrium investors will only be rewarded for bearing undiversifiable factor risk. It follows then that an asset's risk premia will be determined by the asset's correlation with factors. In cruder terms, the risk premia of many assets will be

[7]Weaker conditions on D will suffice. See Chamberlain and Rothschild (1983) and Reisman (1988).

well explained by a few things. In a sense the APT is a generalization of the CAPM which states that an asset's risk premium is completely determined by its covariance with the market. Ross, who developed the theory, demonstrated the correctness of this intuition by showing that if the number of assets was infinite ($N = \infty$) the sum of the pricing errors was finite ($\sum_i e_i^2 < \infty$) so that pricing errors were in aggregate a negligible fraction of risk premia.

This way of stating the APT has led to the belief that for the theory to be useful N must be large. This is not the case. The APT can be quite useful if N is not large but K is small (relative to N) and if pricing errors (i.e. the part of risk premia which is not a reward for bearing factor risk) are small. If the number of assets is not large no theory guarantees that pricing errors will be relatively small. However it is an empirical possibility and one which can be investigated. The situation with small N is no different from the situation with large N. When N is large, factor risk will only explain risk premia if there is a factor structure. That is, if there is a decomposition like equation (5)[8], then equation (6) will hold and pricing errors will be relatively small. Whether or not there is a factor structure when the number of assets is large is an empirical and not a theoretical matter. No deductive theory guarentees that asset returns will have a factor structure. We think it is likely that there will be a factor structure

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[8]The precise statement is that there must be an approximate factor structure. For a precise definition see Chamberlain and Rothschild (1983). The basic idea is that for every collection of N assets, the variance-covariance matrix can be decomposed into two positive semidefinite matrices. One is of rank K and the other has (for all N) bounded eigen values.

because we believe that there are many more assets than there are important influences on asset returns. In exactly the same way, whether or not factor risk premia explain asset prices well when the number of assets is not large is an empirical not a theoretical question[9]. In this paper we explore the possibility that dynamic pricing errors are small, and that changes in risk premia are well explained by changes in factor variances. This is the basis of the APT version of our basic estimating equation (2).

Returning to our development of the APT let us write equation (6) as

(7) $$\mu_i - \rho \approx \sum_k f_{ik}\tau_k\lambda_k$$

where $f_k = b_k/(b_k'\iota)$ is a normalized eigenvector of B. We call τ_k the price of the k^{th} factor risk because if $\tilde{f}_k$ is the random return from a portfolio with weights f_k then

(8) $$E(\tilde{f}_k) - \rho \approx \tau_k \lambda_k \quad \text{and} \quad Var(\tilde{f}_k) \approx \lambda_k$$

so that $\dfrac{E(\tilde{f}_k - \rho)}{Var(\tilde{f}_k)} \approx \tau_k$[10].

The simplest way to turn the APT into a dynamic story which justifies (2) is to assume that

(9) $$H_t = V_t + \Omega = B_t B_t' + D$$

where $B_t = (\sqrt{\lambda_{1t}}\, b_1, \ldots, \sqrt{\lambda_{Kt}}\, b_K)$ and D (= Ω) is diagonal. The λ_{kt} are the eigenvalues of $B_t B_t' = V_t$. They change over time. The b_k

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[9] For an instructive estimate of the size of pricing errors in finite economies see Dybvig (1983).

[10] This could be made more precise by letting $\tilde{f}_k$ be a random variable which has only factor risk. See Admati and Pfleiderer (1985) and Chamberlain (1983).

are column vectors of B_t and eigenvectors of both H_t and V_t. As in our discussion of the CAPM, the eigenvectors of V_t are constant while the eigenvalues are changing. The meaning is slightly different; here the eigenvectors of V_t are also vectors of factor loadings.

If (9) holds, then the basic APT equation can be written

$$\mu_{it} - \rho_t \approx \sum_k f_{ik} \gamma_k \lambda_{kt} \tag{10}$$

which the reader will recognize as a version of our basic estimating equation (2).

The assumptions needed to derive (10) are overly strong. Suppose that

$$H_t = \Omega + B_t B_t' \tag{11}$$

where Ω, the constant part of the variance covariance matrix, can be decomposed further into factors and non-factor risk. That is

$$\Omega = CC' + D \tag{12}$$

where CC' is of rank $K^* < N$ and D is diagonal. Again the APT implies that

$$\mu_{it} - \rho_t \approx \alpha_i + \sum_k f_{ik} \gamma_k \lambda_{kt} \tag{13}$$

where the constant α_i includes both pricing errors (the e_i from equation (6)) and the effect of constant factors on prices.

Both derivations state that changes in risk premia can be attributed to changes in the variance-covariance matrix of returns. If the variance-covariance matrix has a particularly simple dynamic structure, (if its eigenvectors are constant while its eigenvalues are changing) then it is straightforward to use time series techniques to analyze the effects of changing variances on risk premia. This approach contrasts with most empirical work on asset pricing. By and large this work ignores the dynamic structure of the variance covariance matrix and uses cross section techniques to estimate constant risk premia. In terms of our basic equation (2), the concern is with the estimation of the α_i. Our approach ignores the restrictions which theory places on these intercept terms and uses time series techniques to estimate the way in which a changing variance covariance matrices affects changing risk premia. Our discussion of the Roll Critique on page 7 and 8 above is an example of this approach.

While the time series approach allows us to focus attention on the time varying components of the risk premia and examine a small number of assets, it could be that the constant part of risk premia are of much more importance and interest than the variable part. Whether or not this is so is an empirical question. We do note that if changing second moments influence risk premia significantly then our approach can be combined with more traditional cross sectional approaches to estimate both the constants and the and the time varying terms in (2). If variance covariance matrices change in the way we specify and if risk premia are systematically related to variance-covariance matrices, then our techniques will provide demonstrably superior estimates than will traditional techniques.

III. Econometric Specification

Let y_t^* be the vector of asset excess retunrs. (i.e. $y_t^* = y_t - \rho_t \iota$) With the factor structure described in (1), the asset pricing equation (2), and the additional assumption on conditional normality, the density of y_t^* given , F_{t-1}, the sigma field of all past information, is given by

$$(14) \qquad y_t^*|F_{t-1} \sim N(\mu_t^*, H_t)$$

$$\mu_{it}^* = \mu_{it} - \rho_t = \alpha_i + \sum_{k=1}^{K} \beta_{ik}\lambda_{kt} \quad , \ i = 1, \ldots, N$$

$$H_t = \Omega + \sum_{k=1}^{K} f_k f_k' \lambda_{kt}$$

To complete the specification, an expression for λ_{kt} is necessary which makes these variances measureable with respect to F_{t-1}. The conditional variance of the random portfolio excess return, $\tilde{f}_{kt}^*$, given by $f_k' y_t^*$, is defined to be h_{kt} which can be expressed as

$$h_{kt} = f_k' \Omega f_k + \lambda_{kt}.$$

A natural but possibly restrictive assumption is therefore that portfolio k has a univariate ARCH or GARCH representation.

Definition: A portfolio has a univariate variance representation if it can be expressed as a GARCH(p,q) process conditional on the full multivariate information set.

This assumption says that, conditional on all the returns data, only the particular linear combination of surprises defining the portfolio are relevant for forecasting the variance of the portfolio. Thus for $\epsilon = y-\mu$,

$$f'y|\ F_{t-1} \sim D(\ f'\mu_t, h_t)$$

$$h_t = \omega + \sum_{i=1}^{p} \theta_i (f'\epsilon_{t-i})^2 + \sum_{i=1}^{q} \varphi_i h_{t-i}$$

which for the GARCH(1,1) model becomes:

$$(15) \qquad h_{kt} = \omega_k + \theta_k^2\ \eta_{kt-1}^2 + \phi_k^2\ h_{kt-1}$$

where $\eta_{kt} = f_k' \epsilon_t$, the surprise in the k^{th} portfolio. Under this assumption the multivariate covariance matrix can be written as

$$(16) \qquad H_t = \Omega + \sum_{k=1}^{K} \{\theta_k^2 f_k f_k'(f_k'\epsilon_{t-1})^2 + \phi_k^2 f_k'H_{t-1}f_k\}.$$

This is an example of the Factor GARCH structure introduced by Engle(1987). It is also an example of the general positive definite structure proposed for multivariate ARCH models in Baba et al(1986)

$$(17) \qquad H_t = \Omega + \sum_{k=1}^{K}\sum_{i=1}^{p} \{A_{ki}\epsilon_{t-i}\epsilon_{t-i}'A_{ki}' + B_{ki}H_{t-i}B_{ki}'\}.$$

Under the assumption that A and B are symmetric rank one matrices with common eigenvectors for each k, equation (16) can be derived.

A more general model than (16) can be achieved by relaxing the univariate portfolio assumption. Let

$$(18) \qquad h_{kt} = \omega_k + \sum_{j=1}^{K} \{\theta_{jk}^2\, \eta_{jt-1}^2 + \phi_{jk}^2\, h_{jt-1}\}.$$

If $\theta_{kj}=\phi_{kj}=0$ for $k\neq j$ then the univariate portfolio assumption holds. If not, the information in one portfolio is useful in predicting the variance of another. In this case, using the terminology of Granger, Robins and Engle(1985), there is "**causality in variance**" from one portfolio to another. In particular if $\theta_{jk}\neq 0$ then there is causality in variance from portfolio j to portfolio k.

Substituting (18) into (14) gives a model of the form (17) but with

$$A_{k1} = \sum_{j=1}^{K} \theta_{jk}\, f_k f_j', \qquad B_{k1} = \sum_{j=1}^{K} \phi_{jk}\, f_k f_j' .$$

This model is still a special case of (17) since A and B will have rank one. Portfolios will still have no time varying covariance but will have causality in variance.

For these specifications as well as more general ones, the portfolios will all have risk premia derived from (2) or (14) which depend only upon their own variance and possibly an intercept. Therefore under the univariate portfolio representation assumption, the appropriate model for the portfolio is simply the ARCH-M model as used by Engle Lilien and Robins(1987) as well as Domowitz and Hakkio(1984).

The estimation and testing of this model is rather straight forward if it is assumed that the f's are known. The portfolios are then constructed and tested for causality in variance from other information sources using the standard GARCH-M likelihood function. When the variance process of the portfolio is correctly formulated, it is then possible to test the risk premium. Violations of the assumption that only the own variance enters the risk premium in an otherwise correct model, indicate failure of the asset pricing theory or the assumption of constant eigenvectors. If the variance process is not correctly specified however, then rejection of the risk premium formulation may only be reflecting the inadequacy of the variance model which includes less than all of the relevant information. Finally, the individual returns can be examined to determine whether the variance is a linear combination of the variances of the factors, and whether the risk premium is a linear combination of the variances of the factors.

IV. Application to the Pricing of Treasury Bills

To investigate whether the specification (14) ,(15) and (16) is useful in modelling the dynamic behavior of asset risk premia, we try applying it to the pricing of the short end of the term structure.

Data

The data are one month returns of two to twelve months treasury bills from August 1964 to November 1985 obtained from the CRSP Government Bond Tape. This is the updated version of the data set used by Fama (1984). A companion set of value weighted equity returns for NYSE-AMSE from the same sample period is taken from the CRSP Index Tape.

From this dataset, the monthly **excess** returns of the two to twelve months T-bills and the stock market portfolio are constructed by subtracting from their monthly returns the one month T-bill rate under the assumption that it represents a riskless return. Figure 1 presents the plots of these excess returns for 3,6,9 and 12 month maturities. Clearly they move together with common periods of high volatility suggesting the plausibility of the Factor ARCH model.

The unconditional covariance matrix of these excess returns is nearly singular as shown by the principle component analysis in Table 1. The largest eigenvalue represents 92% of the total variance and the first two are 99.67% of the total variance. It is natural therefore to focus attention on the first two factors of the data set. The factor loadings of these factors are also tabulated and these make it clear that the first factor loads primarily on the stock index and is therefore called Factor S, and the second eigenvalue loads primarily on the bills and is consequently called Factor B.

In order to implement the Factor ARCH model described in the previous sections of the paper, it is necessary to identify the factors. While it is in principle possible to estimate these jointly as a maximum likelihood procedure, this is an approach left for further research.

In this paper we explore two methods for identifying the factors. In the first we assume that the eigenvectors of $E(H_t)$ are also eigenvectors of the unconditional covariance matirx of excess returns.[11] We also assume that the eigenvectors of $E(H_t)$ with the largest eigenvalues correspond to the eigenvectors of V_t. Under these assumptions the two portfolios can be constructed using the weights given in Table I and are called respectively Factor S and Factor B.

The alternative approach is to construct portfolios with prespecified weights. Those selected for analysis are 1) equal weights on each of the bill returns with a zero on the stock index which is labeled EW bill, and 2) zero weights on all the bill returns with all the weight on the stock index which is therefore just the Value Weighted index labeled VW stock.

Assuming that there are two factors and that these have univariate variance GARCH(1,1) representations, the model to be estimated for the portfolios are implicit in (14) and (15). These are simply the GARCH-M models used by Domowitz and Hakkio(1984), Engle, Lilien and

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[11] The unconditional covariance matrix of excess returns is:

$$\Sigma = E(y_t^* - E(\mu_t^*))(y_t^* - E(\mu_t^*))' = E(H_t) + E(\mu_t^* - E(\mu_t^*))(\mu_t^* - E(\mu_t^*))'$$

In general, we expect the last term to be small because the variability of the risk premia should be a lot less than the variability of excess returns themselves. Thus, reasonable estimates of f_k may be taken from Σ.

Robins(1987).and French Schwert and Stambaugh(1987). The maximum likelihood estimates are presented in Table II.

The results appear consistent with the model. The two bill and two stock portfolios seem very similar in their behavior. The bill portfolios have highly significant risk premia while the stocks are only marginally significant. The GARCH efffects are very strong and are essentially integrated in each case. See Engle and Bollerslev(1986) and Engle(1987) for an analysis of the integrated ARCH models.

These models are now examined to determine whether the assumption that they have univariate variance representations is satisfied. We therefore check for causality in variance as parameterized in equation (18). A series of Lagrange Multiplier tests of the models in Table II are constructed and are reported in Table III.

The test results are rather striking. **The squared surprises from the stock market over the previous month are important in explaining the variance in the bills this month, while the past surprises in the bill market are not significant in forecasting equity variances.** There appears to be one way causality in variance. Further tests were carried out to determine whether the conditional variances of one market last period would enter the other variance equation. These tests were all insignificant as one tailed tests since the likelihood declined in the permissible portion of the parameter space.

The preferred models for the bill market require another term in the variance equation; (15) becomes (19).

$$(19) \qquad h_{B,t} = \theta_B^2\, \eta_{B,t-1}^2 + \varphi_B^2\, h_{B,t-1} + \theta_{SB}^2\, \eta_{S,t-1}^2$$

where B refers to FACTOR B and EW bill and S refers to FACTOR S and VW stock. The intercept has been dropped in (19) because the data does not support a postive value and a negative value is not permitted. The maximum likelihood estimates of these models are presented in Table IV. The results are better than the previous estimates in the sense that the likelihood is improved and the significance of the ARCH-M terms in the risk premia is even greater.

The theory implies that the portfolio risk premia will depend only on their own conditional variance. We therefore test whether the past squared surprise and conditional variance in one portfolio enter the mean equation of the other portfolio. This test checks whether other information available at the beginning of the period is useful in predicting returns. For the two bill portfolios, the likelihood ratio tests with two degrees of freedom are 5.76 and 5.80 which are marginally insignificant at the 5% level. For the stock portfolios, the test statistics are .10 and .02 which are highly insignificant.

Since our model implicitly implied the constancy of the conditional covariance between the two factor representing portfolios, we can perform an interesting diagnostic test for our specification by checking whether the cross product of the residuals for the two factor representing portfolios has autoregressive structure. The simplest way to do this is to regress the cross product of residuals on its lags and then look at the individual t-statistics and the TR^2 of the regression. These are essentially LM tests for time varying covariances. The results suggest the rejection of the constant conditional covariance assumption, particularly for the **a priori** weighted portfolios. This may be evidence that the dynamic structure is too restrictive or that we have not chosen the portfolios correctly. It is also possible that the test has incorrect size in this case.

The theory also predicts the behavior of the individual assets which make up the portfolios. The two factor ARCH model implies that both the mean and the variance of the returns should be a weighted average of the variances of the two factors. Because the factor weights are nearly zero for the stock factor, the effects of the stock variances should be small or zero.

We now examine models for 3,6,9 and 12 month T-bills. The results are presented in Table V. The stock variance is dropped from the variance equation in each case because likelihood was declining from the boundary of the parameter space. Although all the coefficients are positive, the significance is not great even for the bill variances. Very likely this is due to multicollinearity between the assets. As a benchmark for comparison, the GARCH(1,1)-M models are estimated for each asset and are presented in Table VI. The conditional variances are all significant or nearly so in the mean, yet the likelihood is below that of the Factor model in each case. The difference is generally quite large and gives substantial support to our formulation. It also reinforces the interpretation of the low t-statistics as due to multicollinearity. Rather than rely on a non-nested test between the two models, we now estimate the artificially nested model which has all terms. Non-negativity constraints are a particular problem here but the final results are given in Table VII. For TB-3, TB-6, TB-9 and TB-12, the two degree of freedom LR statistics for testing the two factor model against this more general model are 6.34, 6.14, 1.63 and 4.23

respectively. That is , for TB-3 and TB-6 the own variance effect is marginally significant, while for TB-9 and TB-12 it is not. This result is basically encouraging for our model formulation and presumably with slightly more parameter richness, even better diagnostic statistics could be obtained.

V. Conclusion

The factor ARCH model is theoretically appealing and appears to be reasonably successful in modelling the short end of the term structure. Many caveats remain and the unanswered questions are innumerable. In particular, the difficulties with estimating the number of factors and the factor weights, and the numerical problems with the non-negativity constraints in the variance equations are prime directions for further investigation.

<u>Figure 1</u>

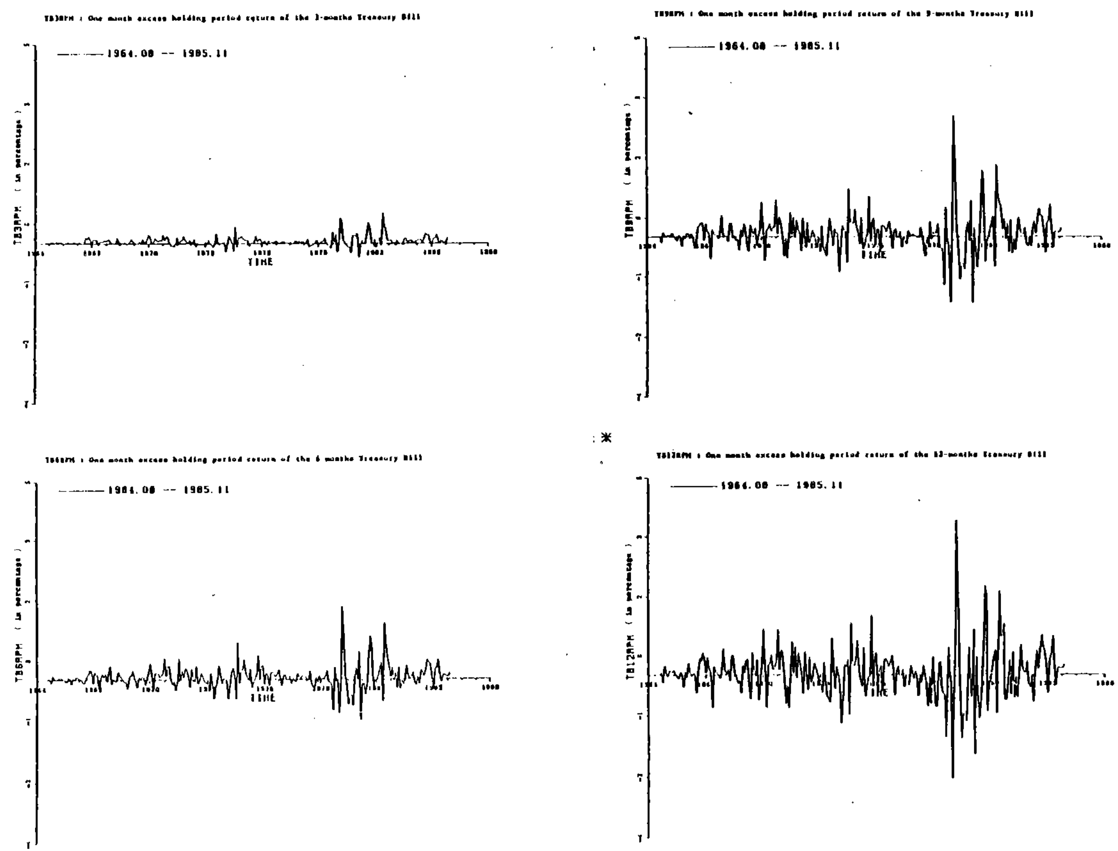
TB3RPM : One month excess holding period return of the 3-months Treasury Bill
1964.08 -- 1985.11
TB3RPM (in percentage)
TIME
TB9RPM : One month excess holding period return of the 9-months Treasury Bill
1964.08 -- 1985.11
TIME
TB6RPM : One month excess holding period return of the 6 months Treasury Bill
1964.08 -- 1985.11
TB6RPM (in percentage)
TIME
TB12RPM : One month excess holding period return of the 12-months Treasury Bill
1964.08 -- 1985.11
TB12RPM (in percentage)
TIME

TABLE I

EIGENVALUES AND EIGENVECTORS

Variables	Factor S	Factor B
TB2	-.0020	.0124
TB3	-.0044	.0275
TB4	-.0085	.0430
TB5	-.0121	.0610
TB6	-.0151	.0758
TB7	-.0173	.0880
TB8	-.0218	.1029
TB9	-.0262	.1218
TB10	-.0281	.1370
TB11	-.0276	.1493
TB12	-.0377	.1620
VWSTOCK	1.2008	.0195

	EIGENVALUES	TRACE PERCENTAGE
Factor S	19.67	92.23
Factor B	1.59	7.45
3	.03	.14
4	.02	.08
5	.01	.04

TABLE II

UNIVARIATE GARCH-M PORTFOLIO MODELS

	EW Bill	VW STOCK	FACTOR B	FACTOR S
MEAN EQ				
CONST	.017	-4.11	.0073	-4.9
	.92)	(-1.4)	(.29)	(-1.4)
h_t	.50	.23	.46	.19
	(2.9)	(1.5)	(3.0)	(1.5)
VARIANCE EQ				
CONST	.004	2.1	.0074	3.1
	(2.1)	(1.7)	(2.4)	(1.7)
η^2_{t-1}	.27	.049	.25	.049
	(5.1)	(1.7)	(4.2)	(1.7)
h_{t-1}	.72	.84	.73	.84
	(16)	(12)	(14)	(12)
lnL	-29.24	-719.61	-89.25	-766.05

TABLE III

LAGRANGE MULTIPLIER TESTS FOR CAUSALITY IN VARIANCE IN GARCH-M MODELS

VARIANCE EQ ADDITIONAL VARIABLES	EW BILL	FACTOR B	VW STOCK	FACTOR S
$\eta^2_{\text{EW BILL}, t-1}$	—	—	.40	—
$\eta^2_{\text{FACTOR B}, t-1}$	—	—	—	.27
$\eta^2_{\text{VW STOCK}, t-1}$	7.2*	—	—	—
$\eta^2_{\text{FACTOR S}, t-1}$	—	6.0*	—	—

TABLE IV

PORTFOLIO MODELS WITH CAUSALITY IN VARIANCE

	EW BILL	FACTOR B
MEAN EQ		
CONST	.0073	-.011
	(.39)	(-.45)
h_t	.65	.59)
	(3.4)	(3.4
VAR EQ		
η^2_{t-1}	.23	.21
	(4.8)	(3.9)
h_{t-1}	.71	.71
	(14)	(13)
$\eta^2_{\text{VW STOCK}, t-1}$	.00042	——
	(2.9)	
$\eta^2_{\text{FACTOR S}, t-1}$	——	.00054
		(3.1)
lnL	-23.76	-83.85

TABLE V

TWO FACTOR MODEL OF INDIVIDUAL ASSET EXCESS RETURNS

	TB3	TB6	TB9	TB12
MEAN EQ				
CONST	-.01 (-.29)	-.10 (-.86)	-.21 (-1.1)	-.17 (-.68)
$h_{FACTOR\ B,t}$	.18 (3.2)	.26 (1.8)	.39 (1.6)	.37 (1.0)
$h_{FACTOR\ S,t}$	.0014 (.71)	.0046 (.97)	.0081 (1.1)	.0061 (.60)
VAR EQ				
$h_{FACTOR\ B,t}$	.056 (15)	.44 (15)	1.2 (16)	2.4 (15)
lnL	255.42	15.71	-106.20	-192.12

TABLE VI

GARCH-M MODELS FOR INDIVIDUAL ASSETS

	TB3	TB6	TB9	TB12
MEAN EQ				
CONST	.032	.036	.021	.021
	(6.5)	(2.3)	(.81)	(.56)
h_t	1.7	.56	.39	.19
	(2.8)	(2.9)	(2.7)	(1.8)
VAR EQ				
CONST	.0002	.0053	.0094	.013
	(1.4)	(2.6)	(2.4)	(1.7)
η^2_{t-1}	.38	.45	.24	.21
	(4.7)	(5.6)	(5.8)	(3.9)
h_{t-1}	.70	.57	.73	.78
	(23)	(11)	(15)	(15)
lnL	244.67	10.10	-113.56	-196.0

TABLE VII

TWO FACTORS MODEL WITH ARCH(1) - M EFFECT FOR INDIVIDUAL ASSETS

	TB3	TB6	TB9	TB12
<u>MEAN EQ</u>				
CONSTANT	-0.021 (-0.50)	-0.14 (-1.3)	-0.30 (-1.7)	-0.34 (-1.5)
h_t	1.4 (0.67)	1.8 (1.4)	5.6 (0.41)	4.51 (0.80)
$h_{FACTOR\ B,t}$	0.61 (0.40)	-0.69 (-1.1)	-6.5 (-0.4)	-10. (-0.81)
$h_{FACTOR\ S,t}$	0.0017 (1.0)	0.0068 (1.5)	0.013 (1.7)	0.013 (1.4)
<u>VARIANCE EQ</u>				
ϵ^2_{t-1}	0.31 (2.3)	0.25 (2.1)	0.048 (0.41)	0.073 (0.82)
$h_{FACTOR\ B,t}$	0.050 (9.3)	0.36 (9.4)	1.1 (9.2)	2.2 (11.)
lnL	258.59	18.78	-105.39	-190.0

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