FORMULATION AND ESTIMATION OF DYNAMIC FACTOR DEMAND EQUATIONS UNDER NON-STATIC EXPECTATIONS: A FINITE HORIZON MODEL

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Under Non-Static Expectations: A Finite Horizon Model

Abstract

This paper proposes a discrete model of investment behavior that
incorporates general nonstatic expectations with a general cost of
adjustment technology. The combination of these two features usually
leads to a set of highly nonlinear first order conditions for the
optimal input plan; the expectational variables work in addition as
shift parameters. Consequently, an explicit analytic solution for
derived factor demand is in general difficult if not impossible to
obtain. Simplifying assumptions on the technology and/or the form of
the expectational process are therefore typically made in the literature.
In this paper we develop an algorithm for the estimation of flexible
forms of derived factor demand equations within the above general
setting. By solving the first order conditions numerically at each
iteration step this algorithm avoids the need for an explicit analytic
solution. In particular we consider a model with a finite planning
horizon. The relationship between the optimal input plans of the finite
and infinite planning horizon model is explored. Due to the discrete
setting of the model the forward looking behavior of investment is
brought out very clearly. As a byproduct a consistent framework for
the use of anticipation data on planned investment is developed.

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1. Introduction[1]

In recent years a large body of literature has developed on
dynamic factor demand models explicitly incorporating costs of
adjustment. In those studies the planning horizon of the firm is
typically taken to be infinite and an explicit analytic solution of the
first order conditions describing the optimal input path is determined.
However, for general technologies and general expectational processes
an explicit analytic solution is very difficult if not impossible to
obtain. This is caused by nonlinearities and the working of the
expectational variables as shift parameters. Consequently, factor
demand models derived from a cost of adjustment technology usually
fall into one of the following two groups. The first group allows for
general technologies but specifies static expectations.[2] An explicit
analytic solution is in general obtained by linearizing the first order
conditions around the steady state solution. Unfortunately, this
approach can empirically handle only one quasi-fixed factor unless
restrictions leading to a diagonal accelerator matrix are imposed.
This, however, rules out a priori interaction of the adjustment
processes of different factors. The other group of models relaxes
the assumption of static expectations but assumes rather simple
technologies or specific expectational processes.[3]

Against this background we construct here a model of investment
behavior that incorporates general nonstatic expectations with a
general cost of adjustment technology. In particular, we consider a
firm with a finite planning horizon and motivate this formulation by
planning costs.[4] For comparison, we relate the optimal input path of

the finite horizon model to that of the infinite horizon model. We then develop an algorithm for the estimation of flexible forms of derived factor demand equations within the above general settings. This algorithm avoids the need for an explicit analytic solution of the first order conditions by solving those conditions numerically at each iteration step of the estimation procedure. We note that this approach allows for more than one quasi-fixed factor without additional assumptions on the underlying technology.

Rather than deriving the factor demand equations from a continuous model and then deducing a discrete version of these equations for empirical purposes, we start out immediately with a discrete form of the model.[5] This discrete analysis makes especially obvious the interaction between future (planned), present and past factor inputs. The forward looking behavior of investment is brought out very clearly. As a byproduct, a consistent framework for the use of anticipatory data on planned investment is provided.

The paper is organized as follows: In Section 2 the general theoretical model is specified and the first order conditions for the optimal input path are derived. The algorithm for the estimation of that model is presented in Section 3. Concluding remarks are given in Section 4 followed by a technical appendix.

case of a fixed (but shifting) planning horizon of T + 1 periods--an
endogenous determination of the firm's planning horizon is left for
future research.[12]

The firm's objective at the beginning of each period t is to
maximize the expected value of the discounted stream of net cash flows
over the next T + 1 periods plus the discounted value of its stocks at
the end of the planning period.[13] In determining the value of its
terminal stocks the firm may engage itself in more or less complex
methods.[14] Consistent with the above remarks on planning costs we
consider a firm that computes the value of its terminal stocks as the
present value of the discounted stream of net cash flows beyond the
actual planning horizon by making the simplifying assumption of a
constant firm size and static expectations past the end of the planning
period. Under this assumption net cash flows beyond the planning period
are constant over time and the value of the terminal stocks can readily
be calculated.

Denote the expectations operator conditional on the firm's
information set at the beginning of period t with E_t and expected prices
and discount rates as, respectively, $p_{t,\tau} = E_t p_{t+\tau}$ and $r_{t,\tau} = E_t r_{t+\tau}$.
Using a "first order certainty equivalence" formulation the firm is then,
at the beginning of each period t, confronted with the following
optimization problem:[15]

$$
(2.3a) \qquad \max_{\{x_{t+\tau}\}_{\tau=0}^{T}} V_{t,T} = \sum_{\tau=0}^{T} R(x_{t+\tau}, x_{t+\tau-1}; p_{t,\tau}) \prod_{i=1}^{\tau} (1 + r_{t,i})^{-1}
$$

$$
+ S(x_{t+T}; p_{t,T}) \prod_{i=1}^{T} (1 + r_{t,i})^{-1}
$$

$$(2.3b) \qquad S(x_{t+T};P_{t,T}) = \sum_{\tau=T+1}^{\infty} R(x_{t+T},x_{t+T};P_{t,T})(1 + r_{t,T})^{-\tau}$$

$$= R(x_{t+T},x_{t+T};P_{t,T})/r_{t,T}$$

subject to the initial input vector x_{t-1}. (Recall that $x_{t+\tau} = x_{t+T}$, $P_{t,\tau} = P_{t,T}$ and $r_{t,\tau} = r_{t,T}$ for $\tau \geq T$).[16] We shall refer to the optimal input vectors to the above finite $T + 1$ period problem as the T-optimal input vectors. The T-optimal input vector planned at the beginning of period t for period $t + \tau$ will be denoted as $x_{t,\tau}^{T}$. Note that the terminal stocks are endogenous to the above optimization problem.

In general the firm will revise its plans every period in response to revisions in expectations and hence only implement the initial $\tau = 0$ portion of its respective plans. Therefore, in general, $x_t = x_{t,0}^{T}$ and $x_{t+\tau} \neq x_{t,\tau}^{T}$ for $\tau \geq 1$ where $x_{t+\tau}$ stands for the actual input vector in period $t + \tau$.

The firm's optimization problem (2.3) is of finite dimension. The following theorem concerning necessary and sufficient conditions hence follows immediately from standard results on the optimization of concave functions.[17]

Theorem 1. Given the above set of assumptions holds. Suppose all components of the sequence of planned input vectors $\{x_{t,\tau}^{T}\}_{\tau=0}^{T}$ lie in the interior of the input space D. Then the following first order conditions are necessary and sufficient for this sequence to maximize the present value $V_{t,T}$ given in (2.3):

2. The Theoretical Model

In this section we specify a general dynamic model of factor demand. The model consists of the following elements. First, the firm's technology is described by a general production function with internal costs of adjustment. Second, the firm's objective function and conditions for the optimal input path under a finite planning horizon are specified; the relationship between input plans under a finite and infinite planning horizon are explored. Finally, a general expectational model is specified. We thus provide a general dynamic model that incorporates expectations and a general cost of adjustment technology in a consistent analytic framework.

2.1. The Firm's Technology

Consider a firm producing a single output good, y_t, from the cost of adjustment technology

$$(2.1) \qquad y_t = F(x_{t-1}, \Delta x_t, \theta) , \qquad \Delta x_t = x_t - x_{t-1} ,$$

where x_t stands for the s × 1 vector of end of period t stock of the factor inputs and θ denotes the vector of unrestricted technology parameters. The internal costs in terms of foregone output caused by changes in the stocks of input factors are represented by the Δx_t term.[6]

The generalized production function $F(\cdot)$ is taken to be twice continuously differentiable in all arguments. Let c_1 and c_2 be two s × 1 vectors of positive finite (possibly very large) constants, then $F(\cdot)$ is further assumed to be strictly concave in both vector arguments x_t and Δx_t on the entire admissible input space $D = \{(x_{t-1}, \Delta x_t):$

$0 \leq x_{t-1} \leq c_1$, $0 \leq \Delta x_t \leq c_2$}.[7] Also, output is bounded on the input space D by $0 \leq F(\cdot) \leq c_3$ where c_3 is some finite positive constant and $F(0,0) = 0$.

2.2. The Firm's Finite Horizon Objective Function and the Optimal Input Path

Consider a firm that is a price taker in both the output and the input market. Let $\hat{p}_t$ denote the price of the output good, w_t the $s \times 1$ vector of service prices, q_t the $s \times 1$ vector of prices for newly purchased investment goods, and r_t the nominal discount factor. Consider the stacked vectors $p_t = [\hat{p}_t, w'_t, q'_t]'$ and $v_t = [p'_t, r_t]'$ and assume that v_t lies in a compact subset of the R_+^{2s+2} with $\hat{p}_t, r_t > 0$. Further, let δ denote the $s \times s$ matrix of constant depreciation rates; then the net revenue function of the firm in period t is given by[8]

$$(2.2) \qquad R(x_t, x_{t-1}; p_t) = \hat{p}_t F(x_{t-1}, \Delta x_t) - w'_t x_{t-1} - q'_t(\Delta x_t + \delta x_{t-1}).$$

Note that under the above assumptions and for a given price vector p_t the firm's net revenue function is strictly concave in x_t and x_{t-1}.[9]

The planning horizon of the firm is assumed to be finite.[10] This formulation is motivated on the one hand by the existence of planning costs. Extending the planning horizon typically increases information requirements and leads at the same time to a more involved optimization problem. Both aspects are typically connected with higher costs. On the other hand, extending the planning horizon will generally increase the expected value of the discounted net revenue stream. This suggests that there is some finite optimal planning horizon.[11] We consider the

(2.4a) $\quad G(x^T_{t,\tau+1}, x^T_{t,\tau}, x^T_{t,\tau-1}; P_{t,\tau+1}, P_{t,\tau}, r_{t,\tau+1}, \theta)$

$$= R_1(x^T_{t,\tau}, x^T_{t,\tau-1}; P_{t,\tau}) + R_2(x^T_{t,\tau+1}, x^T_{t,\tau}; P_{t,\tau+1})/(1 + r_{t,\tau+1})$$

$$= 0 , \qquad \tau = 0, \ldots, T - 1 ,$$

(2.4b) $\quad G_*(x^T_{t,T}, x^T_{t,T-1}; P_{t,T}, r_{t,T}, \theta)$

$$= R_1(x^T_{t,T}, x^T_{t,T-1}; P_{t,T}) + [R_1(x^T_{t,T}, x^T_{t,T}; P_{t,T})$$

$$+ R_2(x^T_{t,T}, x^T_{t,T}; P_{t,T})]/r_{t,T}$$

$$= 0 ,$$

where $R_i(\cdot)$ stands for the derivative of $R(\cdot)$ with respect to the i-th argument vector (i = 1,2). Furthermore, this optimal input sequence is unique.

The above first order conditions can be interpreted in familiar terms. Substituting the definition of the net revenue function (2.2) and dropping superscripts T the first order conditions (2.4a) can (after some trivial rearrangements) be written as:

$$[\hat{p}_{t,\tau+1} F_1(x_{t,\tau}, \Delta x_{t,\tau+1}) - \hat{p}_{t,\tau+1} F_2(x_{t,\tau}, \Delta x_{t,\tau+1}) + q_{t,\tau+1}]/(1 + r_{t,\tau+1}) =$$

$$[-\hat{p}_{t,\tau} F_2(x_{t,\tau-1}, \Delta x_{t,\tau}) + q_{t,\tau} + (\delta q_{t,\tau+1} + \omega_{t,\tau+1})/(1 + r_{t,\tau+1})],$$

$$\Delta x_{t,\tau} = x_{t,\tau} - x_{t,\tau-1} ,$$

where $F_i(\cdot)$ stands again for the derivative of $F(\cdot)$ with respect to the i-th argument vector (i = 1,2). The L.H.S. of the above equation represents the discounted marginal gain in period $t + \tau + 1$ from investment in period $t + \tau$, including the marginal gain from not having to invest in period $t + \tau + 1$. The R.H.S. represents the discounted marginal costs in period $t + \tau$ and $t + \tau + 1$ from investment in period $t + \tau$. The first order condition (2.4b) has a completely analogous interpretation.

2.3. The Relationship Between the Optimal Input Plans of the Finite and an Infinite Horizon Model

The finite horizon model is related to the infinite horizon model. Suppose the firm has an infinite planning horizon; then its optimization problem can be stated as

$$(2.5) \qquad \max_{\{x_{t+\tau}\}_{\tau=0}^{\infty}} V_t = \sum_{\tau=0}^{\infty} R(x_{t+\tau}, x_{t+\tau-1}; P_{t,\tau}) \prod_{i=1}^{\tau} (1 + r_{t,\tau})^{-1} .$$

The following theorem gives necessary and sufficient conditions for a maximum. The proof is of standard kind in the dynamic programming literature and hence omitted here.[18]

Theorem 2. Given the above set of assumptions holds. Suppose all components of the sequence of planned input vectors $\{x_{t,\tau}\}_{\tau=0}^{\infty}$ lie in the interior of the input space D. Then the following first order conditions are necessary and sufficient for this sequence to maximize the present value V_t given in (2.5):

(2.6a)
$$G(x_{t,\tau+1}, x_{t,\tau}, x_{t,\tau-1}; P_{t,\tau+1}, P_{t,\tau}, r_{t,\tau+1})$$

$$= R_1(x_{t,\tau}, x_{t,\tau-1}; P_{t,\tau}) + R_2(x_{t,\tau+1}, x_{t,\tau}; P_{t,\tau+1})/(1 + r_{t,\tau+1})$$

$$= 0 , \qquad \tau = 0,1,\ldots ,$$

(2.6b)
$$\lim_{\tau \to \infty} R_1(x_{t,\tau}, x_{t,\tau-1}; P_{t,\tau}) \prod_{i=1}^{\tau} (1 + r_{t,\tau})^{-1} = 0 .$$

Furthermore, this optimal input sequence is unique.

The first order conditions (2.6a) have the same structure as the first order conditions (2.4a) and hence the same interpretation. Equation (2.6b) represents the usual transversality condition.

It is interesting to note that our finite horizon model can be interpreted as a constrained infinite horizon model. Suppose $P_{t,\tau} = P_{t,T}$ and $r_{t,\tau} = r_{t,T}$ for $\tau \geq T$. Then upon substitution of (2.3b) into (2.3a) it is readily seen that we can formulate this finite horizon optimization problem equivalently as

(2.7)
$$\max_{\{x_{t+\tau}\}_{\tau=0}^{\infty}} V_t \quad \text{subject to the constraints } x_{t,\tau} = x_{t,T}$$
$$\text{for } \tau \geq T.$$

We introduce the following notation. With $X(t) = \{x_{t,\tau}\}_{\tau=0}^{\infty}$ we denote the sequence of optimal input plan vectors corresponding to the infinite horizon model (2.5); similarly, we denote the sequence of optimal input plans corresponding to the finite $T + 1$ period planning horizon model (2.3) with $X_T(t) = \{x_{t,\tau}^T\}_{\tau=0}^{\infty}$ where $x_{t,\tau}^T = x_{t,T}^T$ for $\tau \geq T$. In Appendix A we prove the following theorem relating T-optimal input plans to those of the infinite horizon model.

Theorem 3. Given the above set of assumptions holds. Assume that the firm adopts static expectations over the entire future and that the smallest eigenvalue of the negative Hessian of the net revenue function is bounded from below by some arbitrary small number, say $\rho > 0$.[19] Then

$$(2.8) \qquad \lim_{T \to \infty} X_T(t) = X(t)$$

in the sense that $\lim_{T \to \infty} x_{t,\tau}^T = x_{t,\tau}$ for all $\tau = 0,1,\ldots < \infty$.[20] That is, for every finite $\tau \geq 0$ and any arbitrary small number $\varepsilon > 0$ there exists an index $T_0(\varepsilon,\tau)$ such that $\left| x_{t,\tau}^T - x_{t,\tau} \right| < \varepsilon$ for all $T \geq T_0$. Furthermore, if $\tau' < \tau$ then also $\left| x_{t,\tau'}^T - x_{t,\tau'} \right| < \varepsilon$ for all $T \geq T_0$.

The first half of the above theorem shows that under certain sufficient conditions the sequence of T-optimal input plans converges as $T \to \infty$ to the optimal input plan of the infinite horizon model. The second half of the theorem implies furthermore that, loosely speaking, the speed of convergence is higher in the initial than in the later periods of the plan. This last result is of interest since (as remarked above) in each period t the firm realizes only the initial $\tau = 0$ portion of its plan.

2.4. The Expectational Model

We return to the finite horizon model. As a consequence of the presence of adjustment costs the investment decisions cease to be myopic as is revealed by the first order conditions (2.4). Each element of the T-optimal input plan depends on the entire stream of expectational values $\{v_{t,\tau}\}_{\tau=0}^T$.

We allow for expectational processes of the general form

$$(2.9) \qquad v_{t,\tau} = \phi_\tau(v_{t-1},v_{t-2},\ldots,\omega_{t-1},\omega_{t-2},\ldots,\theta_*) \;, \quad \tau = 0,\ldots,T,$$

where ω denotes the vector of exogenous variables entering the expectation formation process and θ_* is the vector of unconstrained expectational parameters.[21] We assume that $\phi_\tau(\cdot)$ is twice continuously differentiable in all arguments.

To add more meaning to the above abstract formulation of the expectational model we give two illustrative examples:

(i) Rational expectations. Let v_t be generated by the following linear model

$$Bv_t + Av_{t,0} + C\omega_t = \eta_t$$

where B, A and C are real matrices and η_t is assumed to be i.i.d. with zero mean and nonsingular covariance matrix. Suppose the exogenous variables ω_t follow the ARMA(p,q) process

$$a(L)\omega_t = b(L)\varepsilon_t$$

with $a(L) = I - a_1 L - \ldots - a_p L^p$ and $b(L) = I + b_1 L + \ldots + b_q L^q$, where the b_i's and c_i's are real matrices and ε_t is white noise.[22] Then if expectations are formed rationally

$$(2.10) \qquad v_{t,\tau} = -(B + A)^{-1} C\omega_{t,\tau} \;, \qquad \tau = 0,1,\ldots \;,$$

with $\omega_{t,\tau} = \sum_{j=1}^{\tau} c_j \omega_{t,\tau-j} + \sum_{j=\tau+1}^{\infty} c_j \omega_{t+\tau-j}$, where $c(L) = b(L)^{-1}a(L) = I - c_1 L - c_2 L^2 - \ldots$. It is readily seen that the above rational expectations model represents a special case of (2.9).

(ii) Expectations as weighted averages. Now let expectations be formulated by the following simple weighting scheme (or their logarithmic analog)

$$(2.11) \qquad \dot{v}_{t,\tau} = \sum_{j=1}^{\tau} \alpha_j \dot{v}_{t,\tau-j} + \sum_{j=\tau+1}^{k} \alpha_j \dot{v}_{t+\tau-j} \ , \qquad \tau = 0,1,\dots \ ,$$

where dots denote relative differences. Such weighting schemes are frequently used in applied econometric work and are seen to also represent special cases of (2.9).[23]

The above two examples should demonstrate that the expectational model (2.9) can handle a wide range of expectational processes. Clearly, static and adaptive expectations or mixed forms of those expectations are also special cases of this model.

3. The Estimation Technique

In the following an empirical specification of the factor demand equations corresponding to the finite horizon model is given. We show how these equations can be estimated without requiring an explicit analytic solution of the first order conditions. As a byproduct we also develop a rigorous framework for the use of anticipatory data on factor inputs. For notational convenience superscripts T used to denote the T-optimal input vectors are henceforth dropped.

3.1. The Empirical Model

We assume that the T-optimal input path lies in the interior of the input space for all parameter vectors θ of the parameter space. Theorem 1 then implies that (for given initial and expectational values) there exists a single-valued mapping from the parameter space into the solution space:[24]

$$(3.1) \qquad x_{t,\tau} = \rho_\tau(\theta; z_t) \ , \qquad \tau = 0, \ldots, T \ ,$$

where $z_t = [x'_{t-1}, v'_{t,0}, \ldots, v'_{t,T}]'$. Theoretically the firm realizes in each period t the initial portion of its respective input plans, i.e., $x_{t,0} = x_t$. Hence by substituting the above explicit expression for one-period-ahead plans into the initial $\tau = 0$ first order condition of the T-optimal input plan (2.4) we get

$$(3.2) \qquad f(x_t, z_t, \theta) = G[\rho_1(\theta, z_t), x_t, x_{t-1}, P_{t,1}, P_{t,0}, r_{t,1}, \theta] = 0 \ .$$

In case of a linear quadratic production function and static expectations the above system will reduce to the familiar flexible accelerator model

as the planning horizon extends towards infinity (see Prucha and Nadiri (1981) for details). Our theoretical factor demand model is an abstraction from reality. Therefore, the above first order relationship will not hold exactly when confronted with empirical data. We hence specify the following stochastic relationship

$$(3.3) \qquad f(x_t, z_t, \theta) = \epsilon_t , \qquad 1, \ldots, n ,$$

where ϵ_t denotes the $s \times 1$ vector of disturbances. For ease of presentation we assume for the time being that the expectational variables are observable.[25] Then in the above (in general) implicit nonlinear simultaneous equation system, x_t represents the vector of endogenous, and z_t the vector of predetermined variables. We assume that typical modeling assumptions are satisfied.[26] Computer programs for the estimation of such systems are readily available.[27]

Now suppose we have observations on the firm's input plans one period ahead.[28] Then the substitution of the explicit analytic solution (3.1) for $x_{t,1}$ is unnecessary. All variables entering the initial $\tau = 0$ first order condition are observable. We can hence estimate this relationship directly and our empirical factor demand model will be of the form

$$(3.4) \qquad g(x_t, x_{t,1}, z_t, \theta) = G[x_{t,1}, x_t, x_{t-1}, P_{t,0}, P_{t,1}, r_{t,1}, \theta] = \epsilon_t ,$$

$$t = 1, \ldots, n .$$

In this model both $x_{t,1}$ and z_t represent the predetermined variables. That plans one period ahead should be viewed as predetermined is obvious

from the explicit analytic solution (3.1). The merit of this formulation

of the empirical model is that it provides a consistent framework to

incorporate anticipatory data into factor demand systems. Note also that

this formulation brings out the forward-looking character of the

investment decision very clearly. Furthermore, estimating (3.4) rather

than (3.3) should yield more efficient estimates. This stems from the

fact that in the latter case $x_{t,1}$ is modeled as a function of the unknown

parameter vector and is hence subject to estimation errors.[29]

3.2. An Iterative Estimation Routine Based on a Numerical

Solution of the First Order Conditions

Consider again the case where anticipatory data are not available.

For general technologies the first order conditions will be nonlinear

in both variables and parameters. In case of nonstatic expectations

the expectational variables will in addition serve as shift paramaters.

As a consequence, practically an explicit analytic solution of the first

order conditions may often not be available.[30] In such a case we are not

able to estimate the model in its compact form (3.3). In the following

we hence show how to estimate the model in the "unsubstituted" form

$$(3.5a) \qquad g(x_t, x_{t,1}, z_t, \theta) = \varepsilon_t.$$

using the implicit definition of the unobserved vector of one-period-

ahead plans $x_{t,1}$ through the system of first order conditions (2.4).

We denote this system more compactly as

$$(3.5b) \qquad h(x_{t,0}, x_{t,1}, \ldots, x_{t,T}, z_t, \theta) = 0.\text{[31]}$$

The basic ideas how to estimate the parameters of our factor

demand model from (3.5) are first explained by means of a simple

"ad hoc" iteration scheme. Thereafter we present an iteration scheme

with proven convergence properties. The discussion is given in terms

of the full information maximum likelihood (FIML) estimator. The

results also apply, with minor modifications, to other estimation

procedures.

Suppose we have some estimate of the parameter vector θ, say $\theta^{(i)}$.[32]

Using this estimate we can then calculate the corresponding value for

the planned-one-period-ahead input vector, say $x_{t,1}^{(i)}$, by solving the

implicit system of first order conditions (3.5b) numerically--thus avoiding

the need for an explicit analytic solution.[33] Once we have obtained

numerical values for $x_{t,1}^{(i)}$ we use them as "observations" on the planned-

one-period-ahead input vector.

In this sense all variables of model (3.5a) are then "observed."

We may now apply the FIML routine (or any other estimation routine) to

this model and thus obtain a new vector of parameter estimates, say

$\theta^{(i+1)}$. We repeat this iteration scheme until there is no significant

change between two subsequent sets of parameter values. (The FIML

routine itself is an iterative procedure. Hence the above iteration

scheme may be modified such that at the estimation stage we only make a

few subiterations.)[34] As can readily be seen from the subsequent

discussion the values of the likelihood functions of model (3.3)

evaluated at $\theta^{(i)}$ and that of model (3.5a) evaluated at $\theta^{(i)}$ and $x_{t,1}^{(i)}$

are identical.[35] Hence the maximum of the likelihood function and

consequently the ML estimates of model (3.3) and that of (3.5a) subject

to the constraint (3.5b) will be identical.[36] The above ad hoc iteration scheme will stop at the maximum of the likelihood; it has, however, no proven convergence properties. The appealing part of this scheme is that it can be executed with existing econometric software.

For general application an iteration scheme with proven convergence properties is desirable. By construction systems (3.3) and (3.5) are just different representations of the same model. Hence in developing such an iteration scheme it seems natural to start out with an algorithm that could be used to estimate our model in its compact form (3.3) and then modify this algorithm such that it can handle the estimation of our model in its "unsubstituted" form (3.5). The estimation scheme developed below takes the FIML algorithm put forward by Berndt et al. (1974) as a starting point.[37]

The concentrated log likelihood function corresponding to model (3.3) is

$$(3.6) \qquad L(\theta) = \text{const} + (1/n)\sum_{t=1}^{n} \ln \det[J_t(\theta)] - (1/2) \ln \det[S(\theta)]$$

where $S = (1/n)\sum_{t=1}^{n} f_t f_t'$ and $J_t = (\partial/\partial x_t)f_t$ with $f_t = f(x_t, z_t, \theta)$.[38] The gradient of the log likelihood function is given by $\partial L/\partial \theta' = p - q$, where

$$p(\theta) = \sum_{t=1}^{n} p_t, \qquad p_t = \frac{\partial \text{vec}(J_t)}{\partial \theta'} \text{vec}(J_t'^{-1})$$

(3.7)

$$q(\theta) = \sum_{t=1}^{n} q_t, \qquad q_t = \frac{\partial f_t}{\partial \theta'} [\sum_{t=1}^{n} f_t f_t']^{-1} f_t.$$

We further need the sample covariance matrix of the gradient multiplied by n^2

(3.8) $R(\theta) = n\sum_{t=1}^{n}[p_t - q_t][p_t - q_t]'$.

Berndt et al. (1974) then showed that the following iteration scheme

(3.9) $\theta^{(i+1)} = \theta^{(i)} + \lambda^{(i)}[R(\theta^{(i)})]^{-1}[p(\theta^{(i)}) - q(\theta^{(i)})]$

converges to a stationary point, given the stepwidth $\lambda^{(i)}$ is computed

in a certain reasonable way.[39] For starting values reasonable close

to the maximum of the likelihood function this iteration scheme will

hence converge to that maximum. The matrix $(1/n)R^{-1}$ is furthermore a

consistent estimator of the variance-covariance matrix of the maximum

likelihood estimator of θ.

    For given values of $x_{.,1} = [x'_{1,1},\ldots,x'_{n,1}]'$ the concentrated log

likelihood function corresponding to model (3.5a) is

(3.10) $L_*(\theta, x_{.,1}) = \text{const} + (1/n)\sum_{t=1}^{n} \ln \det[J_{*t}(\theta, x_{t,1})]$

$$- (1/2)\ln \det[S_*(\theta, x_{.,1})]$$

where $S_* = (1/n)\sum_{t=1}^{n}g_t g'_t$ and $J_{*t} = (\partial/\partial x_t)g_t$ with $g_t = g(x_t, x_{t,1}, z_t, \theta)$.

Recall that the explicit analytic solution for $x_{t,1}$ given in (3.1)

does not depend on x_t. It is then evident from (3.2) and (3.4) that

$L(\theta) = L_*[\theta, \rho_1(\theta, z_1), \ldots, \rho_1(\theta, z_n)]$. Consequently, the gradient of L

can also be expressed as $\partial L/\partial \theta' = \partial L_*/\partial \theta' + \sum_{t=1}^{n}(\partial x_{t,1}/\partial \theta')(\partial L_*/\partial x'_{t,1}) =$

$= p_* - q_*$, where

$$p_*(\theta, x_{.,1}) = \sum_{t=1}^{n} p_{*t} \ , \quad p_{*t} = [\frac{\partial \text{vec}(J_{*t})}{\partial \theta'} + \frac{\partial x_{t,1}}{\partial \theta'} \frac{\partial \text{vec}(J_{*t})}{\partial x'_{t,1}}] \text{vec}(J'^{-1}_{*t})$$

(3.11)

$$q_*(\theta, x_{.,1}) = \sum_{t=1}^{n} q_{*t} \ , \quad q_{*t} = [\frac{\partial q_t}{\partial \theta'} + \frac{\partial x_{t,1}}{\partial \theta'} \frac{\partial q_t}{\partial x'_{t,1}}][\sum_{t=1}^{n} g_t g'_t]^{-1} g_t \ .$$

The derivatives of $x_{t,1}$ with respect to θ can be calculated from

(3.12) $\quad [\frac{\partial x_{t,0}}{\partial \theta'} \ , \ \frac{\partial x_{t,1}}{\partial \theta'} \ , \ \ldots \ , \ \frac{\partial x_{t,T}}{\partial \theta'}]' = [\frac{\partial h}{\partial x_{t,0}} \ , \ \ldots \ , \ \frac{\partial h}{\partial x_{t,T}}]^{-1} \frac{\partial h}{\partial \theta}$

and can hence be expressed in the form $\partial x_{t,1}/\partial \theta = \psi(x_{t,0}., \ldots,.x_{t,T}, \theta)$.
Expression (3.12) follows from differentiating the first order
conditions (3.5b) implicitly.[40] We define

(3.13) $\quad R_*(\theta, x_{.,1}) = n \sum_{t=1}^{n} [p_* - q_*][p_* - q_*]' \ .$

Now denote the solution values of the first order condition corresponding
to $\theta^{(i)}$ as $x_{t,\tau}^{(i)}$; that is, $x_{t,\tau}^{(i)} = \rho_\tau(\theta^{(i)}, z_t)$ or, equivalently,

(3.14) $\quad h(x_{t,0}^{(i)}, x_{t,1}^{(i)}, \ldots, x_{t,T}^{(i)}, \theta^{(i)}) = 0 \ .$

Define further $\partial x_{t,1}^{(i)}/\partial \theta = \psi(x_{t,0}^{(i)}, \ldots, x_{t,T}^{(i)}, \theta^{(i)})$; then clearly

$$p(\theta^{(i)}) = p_*(\theta^{(i)}, x_{.,1}^{(i)}) \ , \qquad\qquad q(\theta^{(i)}) = q_*(\theta^{(i)}, x_{.,1}^{(i)}) \ ,$$

$$R(\theta^{(i)}) = R_*(\theta^{(i)}, x_{.,1}^{(i)}) \ .$$

This implies that the iterating equation (3.9) can be expressed as

$$(3.15) \qquad \theta^{(i+1)} = \theta^{(i)} + \lambda^{(i)}[R_*(\theta^{(i)}, x^{(i)}_{\cdot,1})]^{-1}[P_*(\theta^{(i)}, x^{(i)}_{\cdot,1})$$

$$- q_*(\theta^{(i)}, x^{(i)}_{\cdot,1})] \ .$$

Consequently, the iteration step (3.9) can be equivalently performed by first solving (3.14) for the plan values $x^{(i)}_{t,0}, \ldots, x^{(i)}_{t,T}$ $(t = 1, \ldots, n)$ corresponding to $\theta^{(i)}$ and then using those values to calculate $\theta^{(i+1)}$ from (3.15). Note that (3.14) and (3.15) are expressed solely in terms of elements of the "unsubstituted" model (3.5). Hence this iteration scheme is directly applicable to that form of the model while yielding identical values as the iteration scheme (3.9) operating on the "substituted" model (3.3). Knowledge of an explicit analytic solution is therefore not required. (Note also that setting $\partial x_{t,1}/\partial\theta = 0$ in all expressions entering (3.15) will yield a special case of the "ad hoc" iteration scheme discussed above, where only one subiteration is made at the estimation stage.)

So far the idealizing assumption that observation on the expectational variables $\{v_{t,\tau}\}^T_{\tau=0}$ are available was maintained. We now drop this assumption. If the expectational model is very simple, that model may be substituted directly into the factor demand equations (3.3) or (3.5). The technology parameter θ and the expectational parameters θ_* can then be estimated jointly from the so obtained system without further complications. This approach can e.g. be taken if expectations are modeled as weighted averages as shown in (2.11).

The general form of the expectational model is given by (2.9). If this model represents some forecasting rule from a stochastic model the expectational parameters can, in a first step, be estimated separately from this stochastic model. They can then be used to generate a series of estimates on the expectational variables, $\{\hat{v}_{t,\tau}\}_{\tau=0}^{T}$. Treating those variables as data we can then, in a second step, apply the above discussed methods to estimate the investment model.[41] However, as is common in such two step procedures, the variance of the estimators of the technology parameters is underestimated. To avoid this problem the factor demand model and the expectational model may be estimated simultaneously using iteration scheme (3.14) and (3.15). In this case the objective function would be composed of the elements of the factor demand and expectational model and, in addition to the values for planned inputs, new expectational series would have to be generated at each iteration step. Separate estimation of the expectational and the factor demand model should provide good starting values for this joint procedure.

We finally note that although the estimation procedure (3.14) and (3.15) was introduced within the particular context of a factor demand model its usefulness is not limited to this context. Rather, the procedure can be applied to any econometric model that can be written in the form (3.5a) where $x_{t,1}$ is some unobserved vector implicitly defined by a system of equations of the form (3.5b).

4. Conclusion and Summary

In this paper we have formulated a dynamic model of factor demand that incorporates nonstatic expectations and costs of adjustment in a consistent framework; an algorithm for the estimation of that model has been developed. Several contributions are of interest:

1. The model is fairly general in that there are no restrictions imposed on the form of the underlying production function, the nature of the expectation process, or the characterization of the adjustment costs and the number of quasi-fixed factors. Achieving this degree of generality has been rather elusive in modeling dynamic input behavior.

2. Further, this general model is formulated with a finite planning horizon and its relation to the optimal input path of the infinite horizon model is established.

3. Another feature of this model is that it brings out the forward-looking feature of investment behavior and provides a consistent framework for integrating anticipation on planned investment with actual investment.

4. The estimation procedure is based on an iteration scheme that solves the first order conditions numerically; thus the need for an explicit analytic solution is avoided.

A potential extension of the model is to incorporate the utilization rates of factor inputs as decision variables and thereby capture some of the variation in input decisions that are presently left out of the model. Further, the model could readily be extended to incorporate various tax parameters. Since the expectation process is very general, the model lends itself to an analysis of dynamic input

vectors x_{t-1} and Δx_t. Then the net revenue function $R(x_t, x_{t-1})$ defined in (2.2) is strictly concave in the argument vectors x_t and x_{t-1}.

<u>Proof</u>: Let R_{ij} and F_{ij} denote the matrix of second order derivatives of $R(\cdot,\cdot)$ and $F(\cdot,\cdot)$, respectively, with respect to the i-th and j-th argument vector $(i,j = 1,2)$. We then prove the strict concavity of $R(\cdot,\cdot)$ by showing that the Hessian matrix

$$\begin{bmatrix} R_{11} & R_{21} \\ R_{12} & R_{22} \end{bmatrix} = \hat{p}_t \begin{bmatrix} F_{22} & F_{12} - F_{22} \\ F_{21} - F_{22} & F_{11} - F_{12} - F_{21} + F_{22} \end{bmatrix}$$

is negative definite. This is the case if and only if

$$[\alpha',\beta'] \begin{bmatrix} R_{11} & R_{21} \\ R_{12} & R_{22} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \hat{p}_t [\beta',\alpha' - \beta'] \begin{bmatrix} F_{11} & F_{21} \\ F_{12} & F_{22} \end{bmatrix} \begin{bmatrix} \alpha \\ \alpha - \beta \end{bmatrix} < 0$$

for all nonzero vectors $[\alpha',\beta']' \in R^{2s}$. However, since $\hat{p}_t > 0$, the validity of the above inequality follows immediately from the negative definiteness of the Hessian of the production function.   Q.E.D.

<u>Lemma A.2</u>. The sequence of present values of the discounted net revenue

stream corresponding to the finite $(T+1)$-period planning horizon model converges to the present value of the discounted net revenue stream of the infinite horizon model as T tends to infinity: $\lim_{T\to\infty} V[X_T] = V[X]$.

<u>Proof</u>: Consider the input sequence $\hat{X}_T(t) = \{\hat{x}_{t,\tau}\}_{\tau=0}^{\infty}$ where $\hat{x}_{t,\tau} = x_{t,\tau}$ for $\tau = 0,\ldots,T$ and $\hat{x}_{t,\tau} = x_{t,T}$ for $\tau \geq T$. That is, the first $T + 1$ elements of this input sequence correspond to those of the optimal input

plan of the infinite horizon model while all subsequent elements are kept constant at the optimal input level for period t + T. The present value of the discounted stream of net revenues corresponding to this input sequence is

$$(A.2) \qquad V[\hat{X}_T] = \sum_{\tau=0}^{T} R(x_{t,\tau}, x_{t,\tau-1})\gamma^{\tau} + \frac{\gamma^{T+1}}{1-\gamma} R(x_{t,T}, x_{t,T}) \ .$$

By definition $X_T(t)$ is that input path that yields the maximum present values among all input paths for which input levels are kept constant from period t + T onwards; hence $V[\hat{X}_T] \leq V[X_T]$. Since the finite horizon problem may be viewed as an infinite horizon problem with an (additionally) restricted solution space stemming from the constraints $x_{t,\tau}^T = x_{t,T}^T$ for all $\tau > T$, we further have $V[X_T] \leq V[X]$. Combining the two inequalities gives $V[\hat{X}_T] \leq V[X_T] \leq V[X]$. Therefore, in order to prove the lemma it is sufficient to show that $\lim_{T\to\infty} V[\hat{X}_T] = V[X]$.

Our catalog of assumptions implies that the net revenue function is bounded on the input space, i.e., $k = \sup\{|R(\cdot,\cdot)|\} < \infty$. Hence

$$\left| V[\hat{X}_T] - V[X] \right| = \left| \sum_{\tau=T+1}^{\infty} [R(x_{t,T}, x_{t,T}) - R(x_{t,\tau}, x_{t,\tau-1})]\gamma^{\tau} \right| \leq 2k \frac{\gamma^{T+1}}{1-\gamma} \ .$$

Note that $0 < \gamma < 1$. Hence for every $\varepsilon > 0$ there exists an index T_0 such that $2k \frac{\gamma^{T_0+1}}{1-\gamma} < \varepsilon$ and hence $\left| V[\hat{X}_T] - V[X] \right| < \varepsilon$ for all $T \geq T_0$. This of course implies that $\lim_{T\to\infty} V[\hat{X}_T] = V[X]$. \hfill Q.E.D.

behavior under different nonstatic tax regimes. Finally, the model could be extended by endogenizing the length of the planning horizon of the firm. It would be important to know the determinants of the length of the planning horizon of the firm and whether the planning horizon of the firm is different with respect to different inputs.

Appendix A: Proof of Theorem 3

Throughout this appendix we maintain the assumptions underlying Theorem 3. Specific reference to the conditional character of the revenue function on the static expectational values will be suppressed in the following. Consider the optimal input plans for the finite horizon model, $X_T(t) = \{x_{t,\tau}^T\}_{\tau=0}^{\infty}$ where $x_{t,\tau}^T = x_{t,T}^T$ for all $\tau \geq T$, and that for the infinite horizon model $X(t) = \{x_{t,\tau}\}_{\tau=0}^{\infty}$. Note that the assumption of static expectations implies in particular that $r_{t,\tau} = r_{t,0}$ for all $\tau \geq 1$. Hence the present value functions of the finite and infinite horizon model can be written, respectively, as

$$(A.1a) \qquad V[X_T] = \sum_{\tau=0}^{T} R(x_{t,\tau}^T, x_{t,\tau-1}^T)\gamma^\tau + \frac{\gamma^{T+1}}{1-\gamma} R(x_{t,T}^T, x_{t,T}^T)$$

$$(A.1b) \qquad V[X] = \sum_{\tau=0}^{\infty} R(x_{t,\tau}, x_{t,\tau-1})\gamma^\tau$$

where $\gamma = (1 + r_{t,0})^{-1}$. The strategy of the proof is as follows. First we show that the sequence of present values corresponding to the finite $(T + 1)$-period planning horizon models converge to the present value of the infinite horizon model as T tends to infinity: $\lim_{T\to\infty} V[X_T] = V[X]$. We then use this result to prove the convergence of the optimal input plans, i.e., $\lim_{T\to\infty} X_T = X$ in the sense that

$$\lim_{T\to\infty} x_{t,\tau}^T = x_{t,\tau} \text{ for all } \tau \geq 0.[42] \text{ Several lemmas are needed.}$$

<u>Lemma A.1.</u> Suppose the output price $\hat{p}_t$ is positive. Further, let the production function $F(x_{t-1}, \Delta x_t)$ be strictly concave in the argument

Lemma A.3. The difference between the present values of the discounted net revenue stream corresponding to the finite $(T + 1)$-period planning horizon model and the present value of the discounted net revenue stream of the infinite horizon model can be expressed as: $V[X_T] - V[X] = \sum_{\tau=0}^{\infty} \phi_\tau^T \gamma^\tau$ where $\phi_\tau^T = (a_\tau^T)' A_\tau^T a_\tau^T$ with

$$(A.4) \qquad a_\tau^T = \begin{bmatrix} x_{t,\tau}^T - x_{t,\tau} \\ x_{t,\tau-1}^T - x_{t,\tau-1} \end{bmatrix}, \quad A_\tau^T = \begin{bmatrix} R_{11}(x_{t,\tau}^+, x_{t,\tau-1}^+) & R_{21}(x_{t,\tau}^+, x_{t,\tau-1}^+) \\ R_{12}(x_{t,\tau}^+, x_{t,\tau-1}^+) & R_{22}(x_{t,\tau}^+, x_{t,\tau-1}^+) \end{bmatrix}$$

and where $x_{t,\tau}^+$ is some point between $x_{t,\tau}^T$ and $x_{t,\tau}$ and where $x_{t,\tau-1}^+$ is some point between $x_{t,\tau-1}^T$ and $x_{t,\tau-1}$, respectively.

Proof: Expanding $R(x_{t,\tau}^T, x_{t,\tau-1}^T)$ around $(x_{t,\tau}, x_{t,\tau-1})$ in a second order Taylor series gives

$$(A.5) \qquad R(x_{t,\tau}^T, x_{t,\tau-1}^T) - R(x_{t,\tau}, x_{t,\tau-1}) = R_1(x_{t,\tau}, x_{t,\tau-1})(x_{t,\tau}^T - x_{t,\tau})$$
$$+ R_2(x_{t,\tau}, x_{t,\tau-1})(x_{t,\tau-1}^T - x_{t,\tau-1}) + \phi_\tau^T .$$

Making use of the definition of an infinite sum we can write

$$(A.6) \qquad V[X_T] - V[X] = \lim_{M \to \infty} \sum_{\tau=0}^{M} [R(x_{t,\tau}^T, x_{t,\tau-1}^T) - R(x_{t,\tau}, x_{t,\tau-1})] \gamma^\tau .$$

By employing the first order conditions (2.6a) and the Taylor series expansion (A.5) it is readily seen that

$$(A.7) \qquad \sum_{\tau=0}^{M} [R(x_{t,\tau}^T, x_{t,\tau-1}^T) - R(x_{t,\tau}, x_{t,\tau-1})] \gamma^\tau =$$
$$= \sum_{\tau=0}^{M} \phi_\tau^T \gamma^\tau + R_1(x_{t,M}, x_{t,M-1})(x_{t,M}^T - x_{t,M}) \gamma^M .$$

The boundedness of the admissible input vector and the transversality condition (2.6b) imply that the last term on the R.H.S. of the above equation converges to zero as $M \to \infty$. Therefore, substitution of (A.7) into (A.6) yields

$$V[X_T] - V[X] = \lim_{M \to \infty} \sum_{\tau=0}^{M} \phi_\tau^T \gamma^\tau = \sum_{\tau=0}^{\infty} \phi_\tau^T \gamma^\tau . \qquad \text{Q.E.D.}$$

Proof of Theorem 3: The strict concavity of the net revenue function implies that the Hessian matrix A_τ^T is negative definite. Clearly, $-A_\tau^T$ is positive definite; let Λ_τ^T be the diagonal matrix containing the (positive) eigenvalues of this matrix and U_τ^T the matrix of corresponding eigenvectors. Then $-A_\tau^T = (U_\tau^T)'\Lambda_\tau^T U_\tau^T$ where $(U_\tau^T)'U_\tau^T = I_{2s}$. Let $\lambda_{\tau,\min}^T$ be the smallest eigenvalue; then by assumption $\lambda_{\tau,\min}^T \geq \rho > 0$ for all $\tau \geq 0$ and $T \geq 0$. Further, let $a_{\tau,i}^T$ be the i-th component of the vector a_τ^T; it then follows as an immediate consequence of the above consideration that for all $i = 1,\ldots,2s$:

$$(A.8) \qquad |\phi_\tau^T| = |(a_\tau^T)'(U_\tau^T)'\Lambda_\tau^T U_\tau^T a_\tau^T| \geq \lambda_{\tau,\min}^T |a_\tau^{T'} a_\tau^T| \geq \rho(a_{\tau,i}^T)^2$$

Lemmas A.2 and A.3 imply that $\lim_{T \to \infty} V[X_T] - V[X] = \lim_{T \to \infty} [\sum_{\tau=0}^{\infty} \phi_\tau^T \gamma^\tau] = 0$.

That is, for every $\varepsilon > 0$ there exists an index T_0 such that $|\sum_{\tau=0}^{\infty} \phi_\tau^T \gamma^\tau| = $ $= \sum_{\tau=0}^{\infty} |\phi_\tau^T| \gamma^\tau < \varepsilon$ for all $T \geq T_0$. (The last equality follows since all ϕ_τ^T are negative.) This implies that $|\phi_\tau^T| \gamma^\tau < \varepsilon$ for all $T \geq T_0$ and finite $\tau \geq 0$. Making use of the inequality (A.8) implies further that

$$(A.9) \qquad (a_{\tau,i}^T)^2 \leq \rho^{-1} |\phi_\tau^T| < \varepsilon/(\rho\gamma^\tau)$$

for all $T \geq T_0$ and finite $\tau \geq 0$. Therefore, for every finite $\tau \geq 0$ and

any arbitrary small $\epsilon' > 0$ we can find an index $T_0(\tau, \epsilon')$ such that $\left| a_{\tau,i}^T \right|^2 < \epsilon'$ for all $T \geq T_0(\epsilon', \tau)$ by choosing $\epsilon = \epsilon' \rho \gamma^\tau$ in (A.9). This of course implies that $\lim_{T \to \infty} (a_{\tau,i}^T)^2 = 0$ and further $\lim_{T \to \infty} a_{\tau,i}^T = 0$. It then follows from the definition of the vector a_τ^T that $\lim_{T \to \infty} x_{t,\tau}^T = x_{t,\tau}$

for all finite $\tau \geq 0$. This proves the first part of Theorem 3. The second part of Theorem 3 regarding the speed of convergence is obvious from (A.9) since $0 < \gamma < 1$.

2. Most of these studies choose a continuous time framework. Examples are Eisner and Strotz (1963), Gould (1968), Lucas (1967), Mortensen (1973) and Treadway (1969, 1970, 1971, 1974). For empirical applications see Berndt, Fuss and Waverman (1977, 1979, 1980), Berndt, Morrison and Watkins (1981), Denny, Fuss and Waverman (1979a, 1979b, 1981), Morrison and Berndt (1981), Nadiri and Rosen (1969, 1973) and Schramm (1970).

3. See, e.g., Bitros and Kelejian (1976), Gould (1968), Lorie and Smith (1970), Norstrøm (1974), Taylor (1970), Tinsley (1970, 1971) for cost of adjustment models with nonstatic expectations in a continuous time framework. For such models in discrete time see Craine (1971, 1975), Hartman (1972, 1973), Kennan (1979), Lucas and Prescott (1971), Meese (1980), Nerlove (1967, 1972), Pindyck and Rotemberg (1982) and Sargent (1978, 1979).

4. The case of a infinite planning horizon is discussed in
Prucha and Nadiri (1981).

5. The optimal path of factor inputs derived from a continuous
time model is typically described by a set of second order differential
equations--see, e.g., Treadway (1974) and the references cited therein.
Transforming the set of differential equations into difference
equations involves judgment about the proper discrete dual of the first
and second order differential operator. This, however, is not always
a trivial task. For instance, the expressions for the flexible
accelerator derived from a continuous or discrete model are in general
not identical. See Prucha and Nadiri (1981) for further discussion
of this issue.

6. The discussion of the various sources of adjustment costs
is for instance summarized in Soderstrom (1976); hence, this discussion
will not be repeated here. The assumption that output in period
t depends on flows from stocks at the end of period t - 1 implies that
investment in period t only becomes productive in period t + 1. This
assumption could easily be relaxed without changing the analysis in
any essential way by using some weighted average of the stocks at the
end of periods t - 1 and t. Similarly, net investment could be replaced
by gross investment to represent adjustment costs without essentially
affecting any of the subsequent result.

7. The strict concavity assumption in the second argument
vector of F(·) implies that we are considering only nonperfectly
variable factors. Perfectly variable factors could be incorporated
with only minor changes in the analysis.

8. For expositional reasons we neglect all tax parameters. They can, however, easily be incorporated.

9. For a proof of this result see Lemma 1 in Appendix A.

10. Most of the investment literature assumes an infinite planning horizon. We speculate that one of the reasons for the prevalence of this unrealistic assumption is that in the case of an infinite horizon model, no explicit specification of the value of the terminal stocks is needed. See Telser and Graves (1972, p. 1), and Arrow and Kurz (1970, p. 30), for further discussion.

11. See Elliasson (1976) for an empirical description of the planning behavior of approximately 80 U.S. and non-U.S. firms. The longest planning horizon encountered in this study was up to the year 2000. Furthermore, the details of the plan are reported to decrease with the length of the planning period. In particular, plans beyond a five-year period are typically found to be rather sketchy.

12. At the estimation stage we may compare estimation results obtained under various assumptions concerning the length of the planning horizon. We speculate that the choice of the proper planning horizon may be undertaken in a way similar to the choice of the maximum lag length in ARMA models.

13. Our subsequent results can readily be modified to a cost minimizing framework.

14. Extreme methods would be on the one hand to set the value of the terminal stocks equal to some fixed constant--in particular, equal to zero. This approach is taken by Schramm (1970). On the other hand, the firm may formulate expectations with respect to the exogenous

the familiar linear algebra problem of finding a solution to a fourth order polynomial. There also, in general, no explicit analytic solution is available.

31. For a discussion of linear explicit models containing unobserved variables see, e.g., Robinson (1974) and the literature cited therein.

32. Initial estimates may be obtained by replacing $x_{t,1}$ in (3.5a) by the actual input vector x_{t+1}. For the use of the actual input vector x_{t+1} in the specific context of an infinite horizon model with rational expectations see Pindyck and Rotemberg (1982).

33. Numerical solution algorithms for nonlinear implicit equations are readily available; e.g., TSP provides such a routine. For a general survey on solution algorithms see Ortega and Rheinboldt (1970).

34. Compare Fair and Taylor (1980) for a related iteration scheme in the context of a nonlinear rational expectations model.

35. In deriving the likelihood function of model (3.5), $x_{t,1}$ is treated as "observed." Explicit expressions for the likelihood functions are given in (3.6) and (3.10).

36. An alternative method of finding maximum likelihood estimates without the need for an explicit analytic solution would hence be to formulate the problem as a constrained optimization problem. Computer programs supporting this approach are available. These programs are, however, generally not carried by econometric packages.

37. Note, however, that the subsequent discussion applies as well to most other estimation algorithms.

38. We use the conventions on vector differentiation of Dhrymes (1978).

39. See Berndt et al. (1974) for details. The matrix $-R(\theta)$ converges to the Hessian of the log likelihood function as $n \to \infty$. Hence, the above iteration scheme is closely related to the Newton method.

40. Inspection of the first order conditions shows that the first matrix on the R.H.S. of (3.12) is block-tri-diagonal.

41. This two-step procedure is analogous to the one proposed by Wallis (1980) in the context of a rational expectation model in which the exogenous variables are generated by an ARMA process. Those exogenous variables correspond to our expectational variables, which are exogenous to the investment process.

42. In this paper convergence of a sequence of finite dimensional vectors is understood to be defined in terms of the "maximum absolute element vector norm." That is, consider the sequence of $s \times 1$ dimensional vectors $z^1, z^2, \ldots$ and the $s \times 1$ dimensional vector z. Let z^j_i be the i-th element of the vector z^j and z_i the i-th element of the vector z. Then if for every $\varepsilon > 0$ there exists an index J such that

$$\max_{i=1,\ldots,s} |z^j_i - z_i| < \varepsilon \text{ for all } j \geq J,$$ then we say the sequence of

vectors $z^1, z^2, \ldots$ converges to the vector z; in symbols: $\lim_{j \to \infty} z^j = z$.

variables over the infinite horizon and then take the maximized present value of the net revenue stream past the planning horizon as the value of its terminal stocks. This, of course, would lead us back to the infinite horizon model.

15. On the certainty equivalence principle see, e.g., Simon (1956), Theil (1957, 1964, 1965) and Malinvaud (1969). The method of certainty equivalence can be viewed as a limited information approach in that only the first moments of the distribution of $p_{t+\tau}$ and $r_{t+\tau}$, $\tau = 0,\ldots,T$, rather than the exact distribution must be known. For an interesting limited information approach based on the knowledge of the first and second moments see Bitros and Kelejian (1976).

16. For a related objective function in the case of a multiperiod consumption plan see Day (1969) and the literature references therein.

17. Sufficiency and uniqueness follow from the strict concavity of the net revenue function. For theorems on the optimization of concave functions see, e.g., Takayama (1974).

18. The proof can be obtained from the authors upon request. Sufficiency and uniqueness follow again from the strict concavity of the net revenue function.

19. The strict concavity of the production function implies the strict concavity of the net revenue function as shown in Lemma A.1 of Appendix A. This in turn implies that all eigenvalues of the negative Hessian of the net revenue function are positive. We assume here in addition that those eigenvalues are bounded from below.

20. For this paper, convergence of a sequence of vectors is defined in terms of the "maximum absolute element vector norm." See footnote 42 for more details.

21. To accommodate perfect foresight models we could also include future values of v into the argument list of the function $\phi_\tau(\cdot)$.

22. This model is considered in Wallis (1980). It contains, as special cases, various rational expectations models introduced in the literature. For the case where ω_t follows a pure AR process see also Revankar (1980).

23. See, e.g., Ando et al. (1974) and Coen and Hickman (1970).

24. Since the first order conditions are continuously differentiable in all arguments and their Jacobian is nonsingular the implicit function theorem implies further that this function is continuously differentiable at each and hence all admissible parameter values.

25. This assumption will be relaxed later on.

26. For the statistical theory on 2SLS, 3SLS and FIML estimation in implicit nonlinear simultaneous systems see Amemiya (1977), Gallant (1977), Gallant and Holly (1980) and Gallant and Jorgenson (1979). Since our model contains lagged endogenous variables we have to keep in mind the qualifications made in Gallant (1977, p. 73).

27. For instance, TSP allows in its FIML routine for the estimation of implicit nonlinear simultaneous equation systems.

28. Plan values for inputs are, e.g., reported in the annual McGraw-Hill capital expenditure surveys on a firm-by-firm basis. This body of data has been considered in Eisner (1978).

29. Clearly, this argument neglects the possibility of measurement errors.

30. This is not to say that a solution of the first order conditions does not exist. For clarification consider, for instance,

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