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ON SOME PRICE ADJUSTMENT SCHEMES

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The paper compares a stochastic approximation price adjustment equation with three Bayesian pricing schemes, of which two have one-period criterion functions and the third has a multi-period criterion function including a variable for a desired terminal stock level. The stochastic approximation price adjustment scheme is shown to be the same with two myopic Bayesian pricing schemes asymptotically with probability one. The Bayesian price adjustment equation for a multiperiod criterion function, under static price expectation assumption, is shown to be similar to one period price adjustment equation with probability one except for the presence of a stock level adjustment term.

1. INTRODUCTION

Consider an organized market dealing with a single commodity where trading takes place out of equilibrium. We suppose that prices are set either by a marketeer (for example, a trading specialist) or by a market authority (for example, in a centralized economy). Prices are set by such an economic agent in the face of unknown or imperfectly known market response.

We assume that the excess demand for the commodity in response to price p is modeled by $x(p) = f(p;\theta) + \xi$ where $f(p;\theta)$ is a known function of p with unknown parameter θ and where ξ is noise.[1] For example, the economic agent is assumed to know that $f(p;\theta)$ is linear in p, $f(p;\theta) = -\alpha p + \beta$, where the parameter vector $\theta = (\alpha, \beta)$ is unknown except for the fact that they are positive $\alpha, \beta > 0$.

We can investigate the pricing policy of the economic agent either by assuming that the economic agent has his subjective estimate of $x(p)$, in other words, subjective estimate of θ and employing the Bayesian approach; or by treating θ as an unknown constant vector and employing a price adjustment algorithm which is of the stochastic approximation type or other programming algorithm such as the stochastic gradient method, [1]. The Bayesian viewpoint is used in [3] to formulate the pricing policy.

In this paper, we first discuss the stochastic approximation adjustment in Section 2. In this scheme, $p_{t+1} - p_t$ is set equal to $a_t x(p_t)$ where the adjustment gain a_t approaches zero as $o(1/t)$ as $t \to \infty$.

We then compare it with a scheme in which the marketeer sets the price which, in his estimate, clears the market, i.e., he sets the price which clears his subjective estimate of the market excess demand. Since his estimate of θ changes with time, the equation for updating his estimate of θ implies a certain price adjustment scheme.

We show the relation of this equation with the stochastic approximation one. This is carried out in Section 3.1.

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[1] We assume a finite variance for noise.

In Section 3.2, the price is set to minimize the conditional expectation of $x(p_t)^2$. We then compare the resulting price scheme with that in Section 2. These two pricing schemes are therefore one-period or myopic Bayesian schemes.

In Section 4 we use a multi-period criterion function in the excess demands and the desired terminal stock level. We ask in what way the pricing scheme which results from an approximate optimization of the multi-period criterion function is related to that of Section 2 and show that except for the presence of a term to adjust the stock level, it behaves the same as the stochastic approximation scheme for large t.

2. Price Adjustment by Stochastic Approximation

In [3], we discussed the pricing policy which minimizes the expected multi-period cost, conditioned on the past observation. When we specialize it to a one-period policy where p is taken to be such that $E(x(p)|\mathscr{H}_t) = 0$, then the price at period t is adjusted (see Section 3.1 and Appendix 1 for the derivation) by

(1) $$p_{t+1} = p_t + k_t x_t, t = 1, 2, \ldots$$

where the adjustment gain is approximately given by

(2) $$k_t \simeq (1 + (p_t - \hat{p}_t)^2/s_t^2)/(1 + t)\alpha_t$$

where

(3) $$\hat{p}_t = \frac{1}{t} \sum_{s=1}^{t} p_s$$

(4) $$s_t^2 = \frac{1}{t} \sum_{s=1}^{t} (p_s - \hat{p}_t)^2$$

$$\alpha_t = E(\alpha|\mathscr{H}_t)$$

and where $\mathscr{H}_t$ is the agent's knowledge at time t

$$\mathscr{H}_t = \{\mathscr{H}_{t-1}, x(p_{t-1}), p_{t-1}\}$$

$$\mathscr{H}_0 = \{\text{the agent's } a\ priori \text{ knowledge on } \theta\},$$

so that α_t is the posterior estimate of α at period t.

Equation (1) with the differential adjustment parameter (gain) k_t is quite close to the adjustment scheme of the Robbins–Monro stochastic approximation [10].

We therefore consider a price adjustment equation

(5) $$p_{t+1} = p_t + a_t x_t$$

where $x_t = x(p_t) = -\alpha p_t + \beta + \xi_t$, and where a_t is specified below. We assume $\{\xi_t\}$ is a sequence of independently and identically distributed random variables with mean 0 and a finite variance σ^2.

Fact 1 (Chung)

The prices generated by (5) converge to β/α in probability as $t \to \infty$ for any sequence of adjustment gain such that $ta_t \to 1/\alpha, t \to \infty$.

Proof. This was established by Chung [6].

We can also show that p_t generated by (5) converges in mean square. Define the variance

$$v_t = E(p_t - \bar{p}_t)^2$$

where

$$\bar{p}_t = E(p_t).$$

We use the symbol $\sim$ to indicate the order of magnitude relation.

Fact 2 (Hodges–Lehmann)

For the adjustment parameter $a_t = c/t$, we have the order of magnitude relations for $2\alpha c > 1$, with a constant c,

$$E(p_{t+1}) \sim E(p_1)/t^{\alpha c} + c(1 - t^{-\alpha c})\beta/\alpha, \tag{6}$$

$$v_{t+1} \sim v_1/t^{2\alpha c} + \sigma^2/a^2 t \cdot (\alpha c)^2/(2c\alpha - 1).$$

Equation (6) remains valid for any other choice of a_t such that $ta_t \to c, 2\alpha c > 1$, as $t \to \infty$.

The Proof is due to Hodges–Lehmann [8].

The convergence with probability one also obtains for the price adjustment equation (5).

Fact 3

With $ta_t \to 1/\alpha$, the prices generated by (5) converge to β/α with probability one. $\hat{p}_t$ of (3) in conjunction (5) also converges with probability one.

Proof. With $a_t = 1/\alpha t$, p_t generated by (5) can be written as

$$p_{t+1} = \beta/\alpha + \eta_{t+1}$$

where

$$\eta_{t+1} = \frac{1}{t\alpha} \sum_{s=1}^{t} \xi_s.$$

Define $\zeta_t = \sum_{s=1}^{t} \xi_s/s, t = 1, \ldots$. It is easy to verify that $\{\zeta_t\}$ is a martingale and $\sup_t E|\zeta_t| < \infty$. Thus ζ_t converges to a finite limit with probability one (Chung [7]). By the Kronecker's lemma (Chung [7]), the convergence of ζ_t (with probability one) implies that $\eta_{t+1} \to 0$ (with probability one), as $t \to \infty$.

$\hat{p}_t$ of (3) is given as $\hat{p}_t = \beta/\alpha + \sum_{s=1}^{t} \eta_s/t$. Since $\sum_{u=1}^{t} \eta_u/u$ is also a martingale and $\sup_t E(\sum_{u=1}^{t} \eta_u/u)^2 < \infty$, $\sum_{s=1}^{t} \eta_s/t \to 0$, with probability one, hence $\hat{p}_t \to \beta/\alpha$ with probability one.

3. Relation with One Period Bayesian Pricing Policy

We discuss two pricing policies related to Bayesian policies involving x_t alone; i.e., we consider one-stage optimization in this section. Multiperiod pricing scheme is considered later in Section 4.

3.1. Consider a pricing policy whereby the agent sets p_t to make the expected excess demand $E(x_t|\mathscr{H}_t)$ zero. We have

$$E(x_t|\mathscr{H}_t) = p'\hat{\theta}_t$$

where

$$E(\theta|\mathscr{H}_t) = \hat{\theta}_t = (-\alpha_t, \beta_t)'.$$

From

$$(1) \qquad 0 = E(x_t|\mathscr{H}_t)$$

one has

$$(1)' \qquad p_t = \beta_t/\alpha_t, \qquad t = 0, 1, \ldots.$$

See Appendix 1 for the computation of $\hat{\theta}_t$.

Equation (1)′ is the policy such that $-\alpha_t p_t + \beta_t = 0$ for all $t = 0, 1, \ldots$. In other words, this pricing policy is the certainty equivalent policy of getting zero excess demand.

In case of Bayesian estimate updates, the convergence with probability one is established by the martingale convergence theorem, since $\{E(\theta|\mathscr{B}_t)\}$ with $\mathscr{B}_t\uparrow$ is a martingale, where $\mathscr{B}_t$ is the σ-algebra generated by $\mathscr{H}_t$. See Chung [pp. 312–331, 7].

Fact 4

The Bayesian pricing policies generate p_t such that $p_t \to \beta/\alpha$ a.s.

The conditional expectations are rather difficult to compute, except for several well known probability distribution functions.

Even though the marketeer knows that (1)′ is the price that clears the estimated excess demand, he may be therefore interested in a suboptimal pricing scheme which is easier to implement and which has *asymptotically* the same behavior as the optimal one given by (1)′.

The pricing equation of (4) given below is one of such schemes. It's relation to the optimal one is clearly seen by comparing (3) with (4).

From (1)′, we see that $p_{t+1} = \beta_{t+1}/\alpha_{t+1}$, where $\hat{\theta}_{t+1}$ is related to $\hat{\theta}_t$ by (2).

$$(2)^2 \qquad \hat{\theta}_{t+1} = (I - K_{t+1})\left[\hat{\theta}_t + \frac{\Lambda_t}{\sigma^2}\begin{pmatrix} p_t \\ 1 \end{pmatrix} x_t\right], \qquad K_{t+1} = \frac{\dfrac{\Lambda_t}{\sigma^2}\begin{pmatrix} p_t^2 & p_t \\ p_t & 1 \end{pmatrix}}{1 + (p_t, 1)\dfrac{\Lambda_t}{\sigma^2}\begin{pmatrix} p_t \\ 1 \end{pmatrix}}$$

$$\sigma^2\Lambda_t^{-1} = \sigma^2\Lambda_{t-1}^{-1} + \begin{pmatrix} p_{t-1}^2, p_{t-1} \\ p_{t-1}, 1 \end{pmatrix}$$

When $\hat{\theta}_{t+1}$ is expressed in terms of $\hat{\theta}_t$ and x_t, it is seen that the one period price-adjustment equation generating prices in the Bayesian case is approximately

[2] This is the same set of equations obtained by the Bayesian rule for independent Gaussian noises. In this section we merely consider this set as given. See [3].

equal to that of the stochastic approximation when some small terms are neglected. We state it as Fact 5.

Fact 5

The Bayesian one-period price adjustment equation generates p_t to clear the estimated excess demand and is recursively obtained by

$$p_{t+1} - p_t = k_t x_t \tag{3}$$

where k_t is given by

$$k_t = \frac{1}{(t+1)\alpha_t} + \frac{(p_t - \hat{p}_t)^2}{(t+1)\alpha_t s_t^2} + o\left(\frac{(p_t - \hat{p}_t)^2}{(t+1)\alpha_t s_t^2}\right).$$

See Appendix 1 for the derivation.

Motivated by this similarity in the price adjustment equations, we consider the convergence behavior of the price adjustment equation

$$p_{t+1} = p_t + a_t x_t,$$

where

$$a_t = \frac{1}{(t+1)\alpha_t} + \frac{(p_t - \hat{p}_t)^2}{(t+1)\alpha_t s_t^2}, \tag{4}$$

with

$$s_t^2 = \frac{1}{t}\sum_{u=1}^{t}(p_u - \hat{p}_t)^2. \tag{5}$$

where $\hat{\theta}_t = \begin{pmatrix} -\alpha_t \\ \beta_t \end{pmatrix}$ is generated by (2).

Note that the price adjustment gain (4) is k_t up to the term $o((p_t - \hat{p}_t)^2/(t+1)\alpha_t s_t^2))$, and that the first term of a_t in (4) is the same as the stochastic approximation price adjustment gain.

It is shown in Appendix 2 that the second term of a_t is at most $o(1/t)$, a.s.

Proposition 1

The price adjustment equation (3), with the adjustment gain given by (4) and (5), converges to α/β a.s. if $f_t = o(t^{-1})$, a.s., where f_t is defined by (2), Appendix 2.

Proof. Let $r_t = p_t - \beta/\alpha$. r_t obeys the difference equation

$$r_{t+1} = (1 - \alpha a_t) r_t + a_t \xi_t \tag{6}$$

where from (1) and (2) of Appendix 2 we see that $1 - \alpha a_t = t(1 - f_t)/(t+1)$ which is less than 1 a.s.

From (6), denoting by $\mathscr{B}_t$ σ-algebra generated by ξ_s, $s < t$, we have

$$E(r_{t+1}^2 | \mathscr{B}_t) = (1 - \alpha a_t)^2 r_t^2 + a_t^2 \sigma^2 \leq r_t^2 + a_t^2 \sigma^2.$$

Therefore, if $\Sigma \overline{a_t^2} < \infty$, where $a_t = (1 + tf_t)/(t + 1)$, then by Cor. 1 of [11], r_t^2 converges a.s., hence r_t also converges a.s. to a finite random variable.

We show in Appendix 2 that $f_t \le q_t^2/\sum^t q_u^2$ where q_t^2 is bounded a.s. and $\Sigma q_t^2 \to \infty$ a.s. Then from Dini's theorem (p. 125 of [15]), $\Sigma f_t^2 < \infty$ a.s. Thus, $\Sigma a_t^2 < \infty$ follows if $\Sigma f_t/t < \infty$ a.s. This convergence obtains for any $f_t = o(t^{-\delta})$, $\delta > 0$.

The a.s. convergence to zero follows from Fact 1 if $f_t = o(t^{-1})$, a.s. See Claim, Appendix 2.

Remark. See Proposition 1 of Appendix 2 for a proof of convergence of $\hat{r}_t$ to zero.

3.2. Suppose now that the agent wants to set p_t to minimize $E(x_t^2|\mathscr{H}_t)$ as close to zero as possible, rather than setting $E(x_t|\mathscr{H}_t)$ equal to zero.

This seemingly trivial modification from Section 1 introduces some complications, as we will see. Let $\tilde{p}_t = (p_t, 1)'$.

We have

$$E(x_t^2|\mathscr{H}_t) = (\tilde{p}_t'\hat{\theta}_t)^2 + \sigma^2 + \tilde{p}_t'\Lambda_t\tilde{p}_t.$$

Thus, the agent chooses p_t given by

$$p_t = \frac{\alpha_t\beta_t - \sigma^2\lambda_{2t}}{\alpha_t^2 + \sigma^2\lambda_{1t}} \tag{7}$$

where from (A.3) of Appendix 1, $\lambda_{1t} = 1/ts_t^2 + r_{1t}$ and $\lambda_{2t} = -\hat{p}_t/ts_t^2 + r_{2t}$, where r_{1t} and r_{2t} are $o(1/ts_t^2)$ with probability one. Substituting these into (7) we obtain

$$p_t = \beta_t/\alpha_t + \frac{\sigma^2}{ts_t^2\alpha_t^2}(\hat{p}_{t-1} - \beta_t/\alpha_t) + \gamma_t, \tag{7$'$}$$

where $\gamma_t = 0(1/ts_t^2)$ (with probability one).

Unlike the certainty equivalent pricing policy, this price given by (7)$'$ takes into account uncertainties (estimation error covariance) of the parameter θ. The second term represents this correction.

From (7)$'$, p_{t+1} is given as

$$p_{t+1} = \frac{\beta_{t+1}}{\alpha_{t+1}} + \frac{\sigma^2}{(t+1)s_{t+1}^2\alpha_{t+1}^2}\left(\frac{\hat{p}_t - \beta_{t+1}}{\alpha_{t+1}}\right) + \gamma_{t+1},$$

where γ_t is some higher order terms in $1/ts_t^2$. We know from (A.6) of Appendix 1 that

$$\beta_{t+1}/\alpha_{t+1} = \beta_t/\alpha_t + k_t(x_t - \hat{x}_t).$$

Proposition 2

The one-period Bayesian price adjustment scheme which results from minimizing $E(x_t^2|\mathscr{H}_t)$ is the same as that of the certainty equivalent price adjustment equation up to $o(1/ts_t^2)$ a.s.

4. Open-Loop Feedback Policy and Other Policies which Incorporate Price Expectation Behavior

In this section we consider a criterion function involving more than one period. We show that the approximation under static price expectation to the resultant open-loop feedback optimal pricing policy gives rise to a price adjustment equation similar to (3.3). See [5], [9] for the discussion of open-loop feedback policy. See [3] for another approximation method.

Suppose a static price expectation holds. Then the price at time t, p_t, will be chosen to minimize the following expression:

$$E(J_t|\mathscr{H}_t)$$

where the criterion function is taken to be

$$J_t = (S_{T+1} - S^*)^2 + \lambda \sum_{u=t}^{T} x_u^2,$$

where S^* is the desired terminal stock and S_{T+1} is the actual terminal stock. Here we assume t and T are sufficiently large.

Using the relation $S_{T+1} = S_t - \sum_{u=t}^{T} x_u$, we express J_t as

$$J_t = (S_t - S^*)^2 - 2(S_t - S^*) \sum_{u=t}^{T} x_u + \sum_{v=t}^{T} \sum_{u=t}^{T} x_v x_u + \lambda \sum_{u=t}^{T} x_u^2.$$

From

$$\left.\begin{aligned} E(x_u|\mathscr{H}_t) &= \hat{\theta}_t'\tilde{p}_t \\ E(x_u^2|\mathscr{H}_t) &= (\hat{\theta}_t\tilde{p}_t)^2 + \sigma^2 + \tilde{p}_t'\Lambda_t\tilde{p}_t \end{aligned}\right\} \quad \text{for } u \geq t$$

and

$$E(x_v x_u|\mathscr{H}_t) = (\hat{\theta}_t'\tilde{p}_t)^2 + \sigma^2\delta_{vu} + \tilde{p}_t'\Lambda_t\tilde{p}_t, \qquad u, v \geq t,$$

we have

$$\begin{aligned} E(J_t|\mathscr{H}_t) = {} & (S_t - S^*)^2 - 2(S_t - S^*)[(T + 1 - t)(\hat{\theta}_t'\tilde{p}_t)] \\ & + (T + 1 - t)(T + 1 - t + \lambda)[(\hat{\theta}_t'\tilde{p}_t)^2 + \tilde{p}_t'\Lambda_t\tilde{p}_t] \\ & + (\lambda + 1)(T + 1 - t)\sigma^2. \end{aligned}$$

Hence p_t^* which minimizes the above is given by

$$p_t^* = \frac{\beta_t}{\alpha_t} \frac{1 - \sigma^2\lambda_{2t}/\beta_t\alpha_t}{1 + \sigma^2\lambda_{1t}/\alpha_t^2} - \frac{1}{\alpha_t} \frac{\Delta S_t}{1 + \sigma^2\lambda_{1t}/\alpha_t^2} \tag{6}$$

where

$$\Delta S_t = (S_t - S^*)/(T + 1 - t + \lambda).$$

Note that with λ sufficiently large, $\Delta S_t \approx 0$. Then (6) reduces to (3.7). Substituting (A.3b)–(A.3d) of Appendix 1 into (6), we can write p_t^* as

$$\begin{aligned} p_t^* &= \frac{\beta_t}{\alpha_t}\left(1 - \sigma^2 \frac{\lambda_{2t} + \lambda_{1t}\beta_t/\alpha_t}{\alpha_t\beta_t}\right) - \frac{\Delta S_t}{\alpha_t}\left(1 - \frac{\sigma^2\lambda_{1t}}{\alpha_t^2}\right) + z_t \\ &= \frac{\beta_t}{\alpha_t}\left(1 - \frac{\sigma^2(\beta_t/\alpha_t - \hat{p}_t)}{ts_t^2\alpha_t\beta_t}\right) - \frac{\Delta S_t}{\alpha_t}\left(1 - \frac{\sigma^2}{ts_t^2\alpha_t^2}\right) + z_t \\ &= \left(\frac{\beta_t}{\alpha_t} - \frac{\Delta S_t}{\alpha_t}\right) + \frac{\sigma^2}{ts_t^2\alpha_t^2}\left(\hat{p}_t - \frac{\beta_t}{\alpha_t} + \frac{\Delta S_t}{\alpha_t}\right) + z_t \end{aligned} \tag{7}$$

where z_t represents various quantities of the order $o(1/ts_t^2)$.

Note that except for the term $\Delta S_t/\alpha_t$, (7) is identical to (3.7′). Thus, except for this term, (7) gives rise to a price adjustment scheme quite analogous to that of minimizing $E(x_t^2|\mathscr{H}_t)$.

From the definition

$$\Delta S_{t+1} = \frac{T+1-t-\lambda}{T-t+\lambda}\Delta S_t - \frac{x_t}{T-t+\lambda}.$$

From the above and (A.4),

$$\frac{\Delta S_{t+1}}{\alpha_{t+1}} = \frac{\Delta S_t}{\alpha_t}\left(1 + \frac{\delta_2}{\alpha_t} + \frac{1}{T-t+\lambda}\right) - \frac{x_t}{\alpha_t}\left(1 + \frac{\delta_2}{\alpha_t}\right)\frac{1}{T-t+\lambda} + o\left(\frac{1}{ts_t^2}\right).$$

Then

$$p_{t+1}^* - p_t^* = \frac{\beta_t}{\alpha_t}\left(\frac{\delta_1}{\beta_t} + \frac{\delta_2}{\alpha_t}\right) - \left\{\frac{\Delta S_t}{\alpha_t}\left(\frac{\delta_2}{\alpha_t} + \frac{1}{T-t+\lambda}\right) - \frac{x_t}{\alpha_t}\left(1 + \frac{\delta_2}{\alpha_t}\right)\frac{1}{T-t+\lambda}\right\}$$
$$+ 0\left(\frac{1}{ts_t^2}\right)$$
$$= \frac{\delta_1}{\alpha_t} + \frac{\delta_2}{\alpha_t}\left(\frac{\beta_t}{\alpha_t} - \frac{\Delta S_t}{\alpha_t} + \frac{1}{T-t+\lambda}\frac{x_t}{\alpha_t}\right) - \frac{1}{T-t+\lambda}\frac{\Delta S_t - x_t}{\alpha_t} + z_t.$$

Substituting (A.5) of Appendix 1 into the above,

$$(8) \qquad p_{t+1}^* - p_t^* = k_t(x_t - \hat{x}_t) + \frac{1}{T-t+\lambda}\frac{\Delta S_t - x_t}{\alpha_t} + o\left(\frac{1}{ts_t^2}\right); \qquad \text{w.p.1}$$

where k_t is the same as in (3.3).

Equation (8) is the price adjustment equation for this case.

5. Conclusions and Discussions

The paper has comparatively discussed four price adjustment equations, one stochastic approximation type and three Bayesian schemes corresponding to three different criterion functions.

A perhaps surprising and significant conclusion is that all these four generate price adjustment mechanisms that are the same for large t with probability one (when the stock level adjustment is ignored).

The paper also established the convergence with probability one of the estimates of the unknown parameters α and β. In this sense it generalizes some results in [13], [14]. Related to the one-period policy of Section 3.1 is the estimate of θ generated by the Kalman filter, which reduces to the simple form given below because there is no dynamics involved.

$$\hat{\theta}_{t+1} = \hat{\theta}_t + k_{t+1}[x_t - (p_t, 1)\hat{\theta}_t]$$

where

$$k_{t+1} = \frac{\sum_t}{\sigma^2}\begin{pmatrix} p_t \\ 1 \end{pmatrix}, \qquad \sum_t = \mathrm{cov}\,(\theta|\mathscr{H}_t).$$

The observability condition requires that $p_{t+1} \neq p_t$ for all t.

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Appendix 1. Bayesian Price Adjustment

The marketeer's subjective knowledge on θ at time t is embodied in his posterior probability density function $p(\theta|\mathscr{H}_t)$.

It is computed by the Bayes rule recursively from $p_0(\theta)$ by

$$p(\theta|\mathscr{H}_{t+1}) = \frac{p(\theta|\mathscr{H}_t)p(x_t|\mathscr{H}_t, \theta, p_t)}{p(x_t|\mathscr{H}_t, p_t)} = \frac{p(\theta|\mathscr{H}_t)p(x_t|\theta, p_t)}{p(x_t|\mathscr{H}_t, p_t)}$$

where

$$p(\theta|\mathscr{H}_0) = \frac{p(x_0|\theta; p_0)p_0(\theta)}{\int_\theta p(x_0|\theta, p_0)p_0(\theta)\,d\theta}$$

where we compute $p(x_t|\theta, p_t)$ from our knowledge of the probability density function for the noise ξ_t. For example, when ξ_t is Gaussian, with mean 0 and standard deviation σ, then we have

$$p(x_t|\theta, p_t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}[x_t - f(\theta, p_t)]^2\right).$$

It was shown in [3] that

$$\hat{\theta}_{t+1} = (I - K_{t+1})\left[\hat{\theta}_t + \frac{\Lambda_t}{\sigma^2}\begin{pmatrix} p_t \\ 1 \end{pmatrix} x_t\right] \tag{A.1}$$

where

$$\Lambda_t = \mathrm{cov}\,(\theta|\mathscr{H}_t)$$

$$K_{t+1} = \Lambda_t \tilde{p}_t \tilde{p}_t'/(\sigma^2 + \tilde{p}'\Lambda_t\tilde{p}).$$

Denote the elements of Λ_t by

$$\Lambda_t/\sigma^2 = \begin{pmatrix} \lambda_{1t} & \lambda_{2t} \\ \lambda_{2t} & \lambda_{3t} \end{pmatrix}.$$

Let $\hat{\theta}_t = \begin{pmatrix} -\alpha_t \\ \beta_t \end{pmatrix}$ and write (A.1) in terms of components as

$$\alpha_{t+1} = \alpha_t - (\lambda_{1t}p_t + \lambda_{2t})\frac{x_t - \hat{x}_t}{1 + \tilde{p}_t'\Lambda_t\tilde{p}_t/\sigma^2} \tag{A.2}$$

$$\beta_{t+1} = \beta_t + (\lambda_{2t}p_t + \lambda_{3t})\frac{x_t - \hat{x}_t}{1 + \tilde{p}_t'\Lambda_t\tilde{p}_t/\sigma^2}$$

where

$$\hat{x}_t = \tilde{p}_t'\hat{\theta}_t = -\alpha_t p_t + \beta_t.$$

This term is zero for the pricing policy $p_t = \beta_t/\alpha_t$, but non-zero for other policies.

We have computed in [3] that

$$\sigma^2\Lambda_t^{-1} = \begin{pmatrix} \lambda_1 + \sum_{s=0}^{t-1} p_s^2 & \lambda_2 + \sum_{s=0}^{t-1} p_s \\ \lambda_2 + \sum_{s=0}^{t-1} p_s & \lambda_3 + t \end{pmatrix}$$

where

$$\sigma^2\Lambda_1^{-1} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 \end{pmatrix}.$$

There is a very close and interesting relation with the so-called input-signal synthesis problem of control theory. The problem is to design input signals to excite the dynamic systems so as to minimize some measure of estimation error. While this problem makes sense in a control context, it is not too appropriate in an economic context, since there is a real cost and information (search) cost associated with changing price in an economic context. See for example [2] on the search cost associated with changing price. We do not explore this aspect in this paper, since it will be outside the immediate concern of this paper. See for example [4].

Inverting the above matrix we obtain

$$\frac{\Lambda_t}{\sigma^2} = \frac{1}{\Delta}\begin{pmatrix} \lambda_3 + t & -(\lambda_2 + \sum p_s) \\ -(\lambda_2 + \sum p_s) & \lambda_1 + \sum p_s^2 \end{pmatrix}$$

with

$$\begin{aligned}\Delta &= (\lambda_1 + \textstyle\sum p_s^2)(\lambda_3 + t) - (\lambda_2 + \sum p)^2 \\ &= (\lambda_1\lambda_3 - \lambda_2^2) + \lambda_1 t + \lambda_3 \textstyle\sum p^2 - 2\lambda_2 \sum p + t \sum p^2 - (\sum p)^2.\end{aligned}$$

For simplicity, assume $\lambda_2 = 0$.[3] Then

$$\lambda_{1t} = \frac{1}{\Delta}(\lambda_3 + t), \qquad \lambda_{2t} = -\frac{1}{\Delta}\sum_{s=0}^{t-1} p_s$$

and

$$\lambda_{3t} = \frac{1}{\Delta}\left(\lambda_1 + \sum_{s=0}^{t-1} p_s^2\right)$$

with

$$\Delta = \left(\lambda_1 + \sum_s p_s^2\right)(\lambda_3 + t) - \left(\sum_s p_s\right)^2.$$

Define

$$\hat{p}_t = \frac{1}{t}\sum_{s=0}^{t-1} p_s: \quad \text{average price over } [0, t-1],$$

and

$$s_t^2 = \frac{1}{t}\sum_{s=0}^{t-1} (p_s - \hat{p}_t)^2: \quad \text{sample variance over } [0, t-1].$$

Using these quantities, we can express the elements of Λ_t/σ^2 as

$$\lambda_{1t} = (\lambda_3 + t)/\Delta, \qquad \lambda_{2t} = -t\hat{p}_{t-1}/\Delta$$

and

(A.3a) $$\lambda_{3t} = \left[\lambda_1 + \sum_{s=0}^{t-1} p_s^2\right]\Big/\Delta$$

with

$$\Delta = t^2 s_t^2 + t[\lambda_1 + \lambda_3(\hat{p}_t^2 + s_t^2)] + \lambda_1\lambda_3.$$

[3] It is reasonable to assume that the marketeer's *a priori* knowledge of the slope of the excess demand curve and the point of intercept are uncorrelated.

Hence for large t if $ts_t^2 \to \infty$ (with Probability 1), then

(A.3b)
$$\lambda_{1t} = \frac{1}{ts_t^2} + r_{1t}$$

where the remainder term is $r_{1t} \sim o(1/ts_t^2)$ (with Probability 1).

(A.3c)
$$\lambda_{2t} = -\frac{\hat{p}_t}{ts_t^2} + r_{2t}$$

(A.3d)
$$\lambda_{3t} = \frac{\hat{p}_t^2 + s_t^2}{ts_t^2} + r_{3t},$$

where r_{2t} and r_{3t} are both $o(1/ts_t^2)$ with probability 1.

From the consideration of information search cost, it is reasonable to assume that p's will not be violently changing for large t [2]. Then $\hat{p}_t$ will be nearly a constant and ts_t^2 will be only slowly growing for large t. See Appendix 2 for precise statements on this point.

Expressions similar to those below can be easily obtained for p_s = const. From (A.3),

$$\begin{aligned}\lambda_{1t}p_t + \lambda_{2t} &= \{(\lambda_3 + t)p_t - t\hat{p}_t\}/\Delta \\ &= [\lambda_3 + t(p_t - \hat{p}_t)]/\Delta\end{aligned}$$

and

$$\lambda_{2t}p_t + \lambda_{3t} = [\lambda_1 + ts_t^2 - t\hat{p}_t(p_t - \hat{p}_t)]/\Delta.$$

We have also

$$\tilde{p}_t'\Lambda_t\tilde{p}_t/\sigma^2 = [\lambda_1 + \lambda_3 p_t^2 + t(p_t - \hat{p}_t)^2 + ts_t^2]/\Delta.$$

Therefore, from the above and (A.2),

(A.4)
$$\begin{aligned}\alpha_{t+1} &= \alpha_t - \delta_2 \\ \beta_{t+1} &= \beta_t + \delta_1\end{aligned}$$

where

$$\begin{aligned}\delta_2 &= \frac{[\lambda_3 + t(p_t - \hat{p}_t)](x_t - \hat{x}_t)}{t(t+1)s_t^2 + t[(p_t - \hat{p}_t)^2 + \lambda_1 + \lambda_3(\hat{p}_t^2 + s_t^2)] + \lambda_1(1 + \lambda_3) + \lambda_3 p_t^2} \\ &= \frac{[(p_t - \hat{p}_t) + \lambda_3/t](x_t - \hat{x}_t)}{(t+1)s_t^2 + (p_t - \hat{p}_t)^2 + \lambda_1 + \lambda_3(\hat{p}_t^2 + s_t^2) + \lambda_1(1 + \lambda_3)/t + \lambda_3 p_t^2/t} \\ &= \frac{(p_t - \hat{p}_t)}{(t+1)s_t^2}(x_t - \hat{x}_t) + u\end{aligned}$$

and

(A.5)
$$\delta_1 = \frac{s_t^2 - \hat{p}_t(p_t - \hat{p}_t)}{(t+1)s_t^2}(x_t - \hat{x}_t) + v$$

where $u \sim o(1/ts_t^2)$ and $v \sim o(1/ts_t^2)$ (with probability 1). Therefore,

$$(p_{t+1} - p_t)/p_t = \delta_1/\beta_t + \delta_2/\alpha_t$$

or

$$p_{t+1} - p_t = k_t(x_t - \hat{x}_t) \tag{A.6}$$

where from $p_t = \beta_t/\alpha_t$, we have

$$k_t = \frac{1}{(t+1)\alpha_t} + \frac{1}{(t+1)\alpha_t}\frac{(p_t - \hat{p}_t)^2}{s_t^2} + w_t, \tag{A.7}$$

where $w_t \sim o(1/ts_t^2)$ (with probability one).

Appendix 2. Almost Sure Convergence

The Nonlinear Recursion Equation

The equation numbers refer to equations in this Appendix unless specified otherwise.

We have from (5) of Section 3

$$\alpha_t = \alpha(1 - D_t), \qquad D_t = \frac{\sum_{s=1}^{t-1}(p_s - \hat{p}_t)\zeta_s}{\sum_{s=1}^{t-1}(p_s - \hat{p}_t)^2}$$

where $\zeta_s = \xi_s/\alpha, s = 1, \ldots.$

We have

$$\alpha a_t = \frac{1}{(t+1)} + \frac{t}{t+1}f_t \tag{1}$$

where

$$f_t = \frac{t}{(t+1)}\frac{q_t^2/\sum_{u=1}^{t}(p_u - \hat{p}_t)^2 + D_t/t}{1 - D_t}. \tag{2}$$

Substituting into the recursion equation (6) of Section 3, we obtain

$$r_{t+1} = \frac{t}{t+1}r_t + \frac{1}{t+1}\zeta_t + \frac{t}{t+1}f_t(\zeta_t - r_t). \tag{3}$$

Note that f_t is a function of $\zeta_s, s < t$.

Since f_t depends on $r_s, s < t$ and $\hat{r}_t$, (2) and (3) are rather complex nonlinear recursion equations. We use some order of magnitude estimate of f_t to circumvent the complexity. We carry out first order analysis to see that q_t^2 is less than one for t sufficiently large and $\sum q_t^2 = \infty$ a.s., where $q_t = r_t - \hat{r}_t$. We also obtain from the first order analysis that $f_t = o(t^{-1})$.

Let $y_t = tr_t$. The recursion formula (3) may be rewritten as

$$y_{t+1} = (1 - f_t)y_t + (1 + tf_t)\zeta_t. \tag{4}$$

Define $\hat{r}_t$ by $\hat{p}_t - \beta/\alpha$. Letting $z_t = t\hat{r}_t$, its recursion equation is

(5) $$z_{t+1} = z_t + \frac{(1-f_t)}{t+1}y_t + \frac{(1+tf_t)}{t+1}\zeta_t.$$

Recall q_t is $r_t - \hat{r}_t$. Let $\sigma_t = t^2 q_t$. Then its recursion equation is given by

(6) $$\sigma_{t+1} = (1-f_t)\sigma_t + (1+tf_t)(t\zeta_t - z_t).$$

Solving these recursion equations, we obtain

(7) $$y_t = c_{t,1}y_1 + \sum_{s=1}^{t-1} c_{t,s+1}(1+sf_s)\zeta_s$$

where

(8) $$c_{t+1,s} = (1-f_t)c_{t,s}, \qquad c_{t,s} = (1-f_{t-1})\dots(1-f_s).$$

Also

$$z_t = z_1 + \sum_{s=1}^{t-1}\frac{1-f_s}{1+s}y_s + \sum_{s=1}^{t-1}\frac{1+sf_s}{1+s}\zeta_s.$$

Substituting the expression for y_s into that for z_t, we have

(9) $$z_t = z_1 + \sum_{s=1}^{t-1}\frac{1+sf_s}{1+s}\zeta_s + \sum_{u=1}^{t-2}\zeta_u(1+uf_u)\sum_{\tau=u+1}^{t-1}\left(\frac{1-f_\tau}{1+\tau}c_{\tau,u+1}\right) + \sum_{s=1}^{t-1}\frac{1-f_s}{1+s}c_{t,1}y_1$$

and substituting the expression for z_t into that of σ_t, we get

(10) $$\begin{aligned}\sigma_t &= c_{t,1}\sigma_1 + \sum_{s=1}^{t-1} c_{t,s+1}(1+sf_s)(s\zeta_s - z_s)\\ &= \sum_{s=1}^{t-1} c_{t,s+1}(1+sf_s)s\zeta_s + \sum_{u=1}^{t-2}\frac{1-f_u}{1+u}c_{u,1}y_1\sum_{s=u+1}^{t-1} c_{t,s+1}(1+sf_s)\\ &\quad - \sum_{s=1}^{t-1} c_{t,s+1}(1+sf_s)z_1 - \sum_{u=1}^{t-2}\zeta_u\frac{1+uf_u}{1+u}\left(\sum_{s=u+1}^{t-1} c_{t,s+1}(1+sf_s)\right)\\ &\quad - \sum_{u=1}^{t-2}\zeta_u(1+uf_u)\left(\sum_{s=u+1}^{t-1} c_{t,s+1}(1+sf_s)\right)\left(\sum_{\tau=u+1}^{t-1}\frac{1-f_\tau}{1+\tau}c_{\tau-1,u+1}\right).\end{aligned}$$

The term $c_{t,1}\sigma_1$ vanishes since $\sigma_1 = 0$. As will be shown later f_s is small for large s, and sf_s will be shown to be $o(1)$, hence these equations show the relative magnitudes of approximation conveniently.

First-order Approximation

For example, collecting terms not involving f's, we obtain the expressions for r_t and $\hat{r}_t$ when the gain $1/[(t+1)\alpha]$ is used for a_t, i.e.,

$$r_t^{(1)} = \frac{1}{t}\sum_{s=1}^{t-1} \zeta_s + \frac{r_1}{t}$$

$$\hat{r}_t^{(1)} = \frac{1}{t}\sum_{s=1}^{t-1} \frac{1}{1+s}\zeta_s + \frac{1}{t}\sum_{u=1}^{t-2} \ln\left(\frac{t}{u+1}\right)\zeta_u + \frac{r_1}{t} + \frac{1}{t}\sum_{s=1}^{t-1}\frac{1}{1+s}r_1$$

since $\hat{r}_1 = r_1$.

We have

$$\begin{aligned} q_t^{(1)} &= r_t^{(1)} - \hat{r}_t^{(1)} \\ &= \lambda_t - \mu_t \end{aligned}$$

where

$$\lambda_t = \frac{1}{t}\sum_{s=1}^{t-1}\left(1 - \frac{1}{s+1}\right)\zeta_s$$

$$\mu_t = \frac{1}{t}\sum_{u=1}^{t-2}\ln\left(\frac{t}{u+1}\right)\zeta_u + r_1\frac{\ln t}{t}.$$

We evaluate how fast they approach zero. We prove two lemmas for that purpose.

Lemma 1

$t^{1/2-\delta}\lambda_t \to 0$ a.s. for arbitrarily small $\delta > 0$.

Proof. The proof is by the method of subsequences. We first show that $t^{\alpha}\lambda_t \to 0$ a.s. for appropriate chosen α. Since the variance of λ_t, denoted by V_t is given by

$$V_t = \frac{1}{t^2}\sum_{s=1}^{t-1}\left(1 - \frac{1}{s+1}\right)^2 \leq \frac{1}{t},$$

we have by the Chebychev inequality

$$p[|t^{\alpha}\lambda_t| > \varepsilon] \leq \frac{t^{2\alpha}}{\varepsilon^2 t} = \frac{1}{\varepsilon^2 t^{1-2\alpha}}.$$

Choose a subsequence $t = n^2$. Then for α satisfying $0 < \alpha < \frac{1}{4}$, $\sum_n 1/n^{2-4\alpha} < \infty$. Thus applying the Borel–Cantelli lemma, we see that $t^{\alpha}\lambda_t \to 0$ a.s. for $0 < \alpha < \frac{1}{4}$ along the subsequence $t = n^2$, $n = 1, 2 \ldots$. Let

$$D_n = \max_k |(\lambda_k - \lambda_{n^2})|$$

where k ranges over $n_2 < k < (n+1)^2$. We have

$$\operatorname{var}(\lambda_k - \lambda_{n^2}) \leq \frac{k - n^2}{n^4} + \frac{(k^2 - n^4)}{k^2 n^2} = 0\left(\frac{1}{n^3}\right).$$

Thus

$$p[|D_n| > \varepsilon/n^{2\alpha}] \leq \frac{\text{const.}}{\varepsilon^2 n^{3-4\alpha}}.$$

With $\alpha < \frac{1}{4}$, $\sum 1/n^{3-4\alpha} < \infty$, hence

$$n^{4\alpha} D_n \to 0 \text{ a.s. as } n \text{ increases.}$$

Since

$$\begin{aligned} t^\alpha \lambda_t &= n^{2\alpha}\lambda_{n^2} + (t^\alpha \lambda_t - n^{2\alpha}\lambda_{n^2}) \\ &= n^{2\alpha}\lambda_{n^2} + n^{2\alpha}(\lambda_t - \lambda_{n^2}) + (t^\alpha - n^{2\alpha})\lambda_t \end{aligned}$$

where

$$\operatorname{var}(t^\alpha - n^{2\alpha})\lambda_t \le (t^\alpha - n^{2\alpha})^2 \frac{1}{t} \le \frac{\text{const.}}{n^{3-4\alpha}}, \qquad \text{for } n^2 < t < (n+1)^2.$$

Therefore, each of the three terms on the right converges to zero, a.s. for $0 < \alpha < \frac{1}{4}$. The a.s. convergence can be established by using a subsequence, $t = n^3$ or $t = n^m$ for $m = 3, 4, \ldots$, in similar manners. With $t = n^m$,

$$p(|n^{\alpha m}\lambda_{n^m}| > \varepsilon) \le \frac{\text{const } n^{2\alpha m}}{\varepsilon^2 n^m} = \frac{\text{const}}{n^{m(1-2\alpha)}}.$$

Thus $t^\alpha \lambda_t \to 0$ a.s. along the subsequence $t = n^m$, $n = 1, 2, \ldots$, for

$$0 < \alpha < \frac{m-1}{2m} = \frac{1}{2} - \frac{1}{2m}.$$

This establishes the assertion.

The variance of μ_t is

$$\operatorname{var}(\mu_t) = \frac{1}{t^2}\sum_{u=1}^{t-2} (\ln t/u + 1)^2 \sim \frac{t}{t^2}(\chi(\ln \chi)^2 = 2\chi \ln \chi + 2\chi + \text{const})$$

where $\chi = (t-1)/t$. Therefore $\operatorname{var}(\mu_t) \sim 1/t$, and the virtually same proof of Lemma 1 establishes $t^{1/2-\delta}\mu_t \to 0$ a.s. for arbitrarily small δ. From this fact and Lemma 1, we establish Lemma 3. We use Lemma 2 to prove Lemma 4.

Lemma 2

$$\sum_{u=2}^{t} (p_u - \hat{p}_t)^2 \ge \sum_{2}^{t} (p_u - \hat{p}_u)^2, \qquad \text{for all } t \ge 2 \text{ a.s.}$$

Proof. Let the sum on the left hand side be named C_t. Then

$$C_{t+1} = (p_{t+1} - \hat{p}_{t+1})^2 + \sum_{u=2}^{t} (p_u - \hat{p}_{t+1})^2.$$

Substitute

$$\hat{p}_{t+1} = (t\hat{p}_t + p_{t+1})/t + 1$$

in the above to obtain

$$C_{t+1} = C_t + (p_{t+1} - \hat{p}_{t+1})^2 + \left(\frac{1}{t+1}\right)^2 (\hat{p}_t - p_{t+1})^2 \ge C_t + (p_{t+1} - \hat{p}_{t+1})^2.$$

By iterating the above, we obtain the lemma.

Lemma 3

$t^{1/2-\delta}q_t^{(1)} \to 0$ for arbitrarily small $\delta > 0$, a.s.

From Lemma 3, we see that

$$q_t^{(1)} = o(t^{-1/2+\delta}), \text{ a.s.}$$

Cor. f_t is $o(t^{-1})$, a.s.

Proof

Claim

$$q_t^2 \Big/ \sum_{u=1}^{t} q_u^2 \simeq \frac{d}{dt} \ln \sum_{1}^{t} q_u^2 = o(2\delta/t), \text{ a.s.}$$

Proof of Claim. From Lemma 3, $q_t^2 = o(t^{-1+2\delta})$, a.s. Thus, $\sum q_t^2$ is divergent but $q_t^2/\sum^t q_u^2 \to 0$, a.s.

Define a positive monotonically decreasing function $h(t)$ and set $h(n) = q_n^2$, $n = 1, 2, \ldots$.

From Theorem 3 in Section 3.3 of Knopp [15], we have

$$\sum^{t} q_u^2 \simeq \int_1^t h(\tau)\, dt = o(t^{2\delta}).$$

Therefore

$$\frac{d}{dt} \ln \int_1^t h(\tau)\, d\tau = \frac{h(t)}{\int_1^t h(\tau)\, d\tau} \simeq \frac{q_t^2}{\sum^t q_u^2}.$$

Let

$$\hat{f}_t = \left[q_t^2 \Big/ \sum_{u=1}^{t} (p_u - \hat{p}_t)^2 \right] t/t + 1.$$

From Lemma 2, we see that

$$\hat{f}_t \le q_t^2 \Big/ \sum^{t} q_u^2 \simeq \frac{d}{dt} \ln \int^t q_u^2\, du.$$

By Claim we see that

$$t\hat{f}_t = o(2\delta), \text{ a.s.}$$

The assertion follows since f_t and $\hat{f}_t$ are equivalent sequences as proved at the end of Appendix 2.

Higher Order Terms

With this first-order approximation, we are able to show that the higher order effects on y_t and z_t are at most the same order of magnitudes as first-order effects.

For example, with $tf_t \le k$, a.s., the higher order term in r_t,

$$\sum_{s=1}^{t-1} c_{t,s+1} s f_s \zeta_s / t,$$

shows the same convergence behavior as the first order term,

$$\sum_{s=1}^{t-1} \zeta_s/t,$$

since $c_{t,s+1} s f_s \leq$ const., a.s. Actually we need only $t^{1-\delta} f_t$ bounded for all t a.s. for some small $\delta > 0$ to obtain the results.

Convergence of Prices

Proposition 1

Assume $t^{1-\delta} f_t$ is bounded for all t, a.s. for very small $\delta > 0$. The z_t/t and y_t/t converge to zero a.s., i.e., r_t and $\hat{r}_t$ both converge to zero a.s.

Proof. From the expression for z_t, one needs to verify the almost sure convergence of h_t defined below,

$$h_t/t = \frac{1}{t} \sum_{u=1}^{t-2} \zeta_u (1 + u f_u) D_{t-1,u+1},$$

where

$$\begin{aligned} D_{t-1,u+1} &= \sum_{s=u+1}^{t-1} \frac{1 - f_s}{1 + s} c_{s,u+1} \\ &\leq \sum_{s=u+1}^{t-1} \frac{1}{1 + s} \\ &\sim \ln \frac{t}{u + 1}, \end{aligned}$$

since $f_s > 0$ and $c_{s,u+1} \leq 1$ for all s and u, a.s. Let c be a positive constant such that $1 + u f_u \leq c u^\delta$ for all u, a.s. by assumption. We have

$$\sum_{u=1}^{t-2} u^{2\delta} D_{t-1,u+1}^2 \sim \int_1^{t-2} u^{2\delta} \left(\ln \frac{t}{u + 1} \right)^2 du \leq t^{1+2\delta}(1 + o(t)).$$

Then for any $\varepsilon > 0$,

$$p\left(\frac{1}{t}\left|\sum_{u=1}^{t-2} \zeta_u (1 + u f_u) D_{t-1,u+1}\right| > \varepsilon\right] \leq \frac{v_t}{\varepsilon^2 t^2}$$

where

$$\frac{\alpha^2}{\sigma^2} v_t = \sum_{u=1}^{t-2} (1 + u f_u)^2 D_{t-1,u+1}^2 \leq \text{const } t^{1+\delta}(1 + o(t)).$$

Take the subsequence $t = n^2$, $n = 1, 2, \ldots$. Then

$$\sum_n \frac{v_n^2}{\varepsilon^2 n^4} \leq \frac{2c^2}{\varepsilon^2} \sum \frac{1}{n^{2-2\delta}} < \infty, \text{ where } c \text{ is some const.}$$

Thus, $h_{n^2}/n^2 \to 0$ a.s., by Borel–Cantelli lemma. Let

$$d_n = \max_{n^2 \le k < (n+1)^2} |h_k - h_{n^2}|$$

$$= \max_k \left| \sum_{n^2+1}^{k} \zeta_u(1 + uf_u)D_{t-1,u+1} \right|.$$

Then

$$Ed_n^2 \le c^2 \sum_{n^2+1}^{(n+1)^2-1} D^2_{(n+1)^2-1,u+1}$$

$$\le c^2((n + 1)^2 - n^2 - 1)$$

$$= 2c^2 n.$$

Therefore,

$$p[d_n > n^2\varepsilon] \le \frac{2c^2 n}{n^4 \varepsilon^2}$$

hence

$$d_n \to 0 \text{ a.s.}$$

For

$$n_2 \le k < (n + 1)^2,$$

$$|h_k|/k \le \frac{|h_n^2| + d_n}{n^2}.$$

The identical technique proves the almost sure convergence of y_t/t to zero also. This proves the proposition.

Convergence of Estimates

To establish the convergence of $\hat{\theta}_t$ to θ with the adjustment equation (2) of Section 3, we note that

$$\alpha_t = \alpha(1 - D_t)$$

where

$$D_t = \frac{\sum_{s=1}^{t-1} (p_s - \hat{p}_t)\zeta_s}{\sum_{u=1}^{t} (p_u - \hat{p}_t)^2}$$

$$= D_t^1 + D_t^2,$$

where

$$D_t^1 = \frac{\sum_{s=1}^{t-1} (p_s - \hat{p}_s)\zeta_s}{\sum_{u=1}^{t} (p_u - \hat{p}_t)^2},$$

and

$$D_t^2 = \frac{\sum_{s=1}^{t-1} (\hat{p}_s - \hat{p}_t)\zeta_s}{\sum_{u=1}^{t} (p_u - \hat{p}_t)^2}.$$

Claim. D_t^1 converges to zero a.s.

Proof. Let $X_t = \sum_{s=1}^{t-1} (p_s - \hat{p}_s)\zeta_s/\sum_{u=1}^{s} (p_u - \hat{p}_s)^2$. Then it is a supermartingale and converges to a finite random variable from Kushner's lemma [11]. The conclusion follows from lemma 2 and the Kronecker's lemma since $\sum^t q_u^2 \uparrow \infty$ a.s.

Lemma

D_t^2 converges to zero a.s.

Proof. Let

$$x_t = \sum_{s=1}^{t-1} \frac{(\hat{p}_s - \hat{p}_t)\zeta_s}{\sum_{u=1}^{s} (p_u - \hat{p}_s)^2}.$$

Then from

$$x_{t+1} = \sum_{s=1}^{t} \frac{(\hat{p}_s - \hat{p}_{t+1})\zeta_s}{\sum_{u=1}^{s} (p_u - \hat{p}_s)^2}$$

$$= \frac{(\hat{p}_t - \hat{p}_{t+1})\zeta_t}{\sum_{u=1}^{t} (p_u - \hat{p}_t)^2} + \sum_{s=1}^{t-1} \frac{(\hat{p}_s - \hat{p}_t + \hat{p}_t - \hat{p}_{t+1})\zeta_s}{\sum_{u=1}^{s} (p_u - \hat{p}_s)^2},$$

we obtain $E(x_{t+1}|\mathscr{B}_t) = x_t + \rho_t$, where

$$-\rho_t = E\left[\frac{(\hat{p}_{t+1} - \hat{p}_t)\zeta_t}{\sum_{u=1}^{t} (p_u - \hat{p}_t)^2} + \sum_{s=1}^{t-1} \frac{(\hat{p}_{t+1} - \hat{p}_t)\zeta_s}{\sum_{u=1}^{s} (p_u - \hat{p}_s)^2}\Big|\mathscr{B}_t\right],$$

where

$$\hat{p}_{t+1} - \hat{p}_t = \hat{r}_{t+1} - \hat{r}_t$$

$$= \frac{1 + tf_t}{(t + 1)^2}\alpha\zeta_t + (\text{quantities depending on } \xi_s, s < t).$$

Thus

$$-\rho_t = \frac{1 + tf_t}{(t + 1)^2} \frac{\alpha\sigma^2}{\sum_{u=1}^{t} (p_u - \hat{p}_t)^2} + \sum_{s=1}^{t-1} \frac{(\hat{p}_{t+1} - \hat{p}_t)\zeta_s}{\sum_u (p_u - \hat{p}_s)^2}$$

$$\leq \frac{(1 + tf_t)\alpha\sigma^2}{(t + 1)^2 \sum_{u=1}^{t} q_u^2} + \sum_{s=1}^{t-1} \frac{(\hat{r}_{t+1} - \hat{r}_t)\zeta_s}{\sum_u (p_u - \hat{p}_s)^2}.$$

Now, from

$$E\left|\sum_{s=1}^{t-1} \frac{(\hat{r}_{t+1} - \hat{r}_t)\zeta_s}{\sum_u^s (p_u - \hat{p}_s)^2}\right| \leq \left\{E(\hat{r}_{t+1} - \hat{r}_t)^2 \cdot E\left(\sum_{s=1}^{t-1} \frac{\zeta_s}{\sum_{u=1}^{s} (p_u - \hat{p}_s)^2}\right)^2\right\}^{1/2},$$

where

$$E\left(\sum_{s=1}^{t-1} \frac{\zeta_s}{\sum_{u=1}^{s} (p_u - \hat{p}_s)^2}\right)^2 \leq \left[E \sum_{s=1}^{t-1} \left(\frac{1}{\sum_{u=1}^{s} (p_u - \hat{p}_s)^2}\right)^2\right]$$

$$\leq \left[E \sum_{s=1}^{t-1} \left(\frac{1}{\sum_{u=1}^{s} q_s^2}\right)^2\right],$$

where the last inequality is by Lemma 2. Note that for all $t \geq 1$,

$$\sum_{s=1}^{t-1} \left(\frac{1}{\sum_{u=1}^{s} q_u^2}\right) = o(1).$$

From (5),

$$\hat{r}_{t+1} - \hat{r}_t = -\frac{z_t}{t(t+1)} + \frac{(1-f_t)}{(t+1)^2} y_t + \frac{(1+tf_t)}{(t+1)^2} \zeta_t.$$

Substituting (7) and (9) into the above equation, after straightforward but tedious calculation we see that

$$E(\hat{r}_{t+1} - \hat{r}_t)^2 = o\left(\frac{\ln t}{t^{3-2\delta}}\right).$$

Thus, we establish that

$$E \sum_t |\rho_t| < \infty.$$

Thus x_t converges to a finite random variable a.s. The assertion follows from the Kronecker's lemma. Combining Claim with Lemma 4, we establish the next two propositions.

Proposition

$$\alpha_t \to \alpha \qquad \text{a.s. } t \to \infty.$$

Proposition

$$f_t - \hat{f}_t \to 0 \text{ a.s.} \qquad \text{where } \hat{f}_t = \frac{q_t^2 t}{\sum^t (p_u - \hat{p}_t)^2 (t+1)}.$$

In the above discussions, the distinction between f_t as defined by (2) and $\hat{f}_t$ which puts $D_t = 0$ has been ignored. This is justified because the two sequences $\{f_t\}$ and $\{\hat{f}_t\}$ can be shown to be equivalent sequences since

$$p[|f_t - \hat{f}_t| > \varepsilon] \leq \frac{\operatorname{var}(f_t - \hat{f}_t)}{\varepsilon^2} \leq \frac{1}{t^2} \operatorname{var}\left(\frac{D_t}{1 - D_t}\right) = o(t^{-2}).$$