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CHAPTER II

Graduating by Simple Moving Averages and by the Mid Ordinates of Third-Degree Parabolas Fitted by the Method of Least Squares.

If the data be such that it seems undesirable to fit any curve or graduation, the points of which will be appreciably [1] affected by the position of data points a long distance away from the graduated point, it would seem rather natural to consider using some form of "moving average." If the data are monthly and if the cycles are generally long and the erratic fluctuations are quite small, a simple 2-months moving average of a 12-months moving average [2] may give a comparatively smooth graduation which may be a fair approximation to the underlying curve desired. However, unless the cycles are long (72 months or more) such a simple moving average will not reach up into the maximum regions of the desired curve nor down into the minimum regions in the way that it should.

[1] Though each point on the Whittaker-Henderson curve (discussed below) is affected by the position of each point of the data, the influence of any distant data point is entirely negligible. This is unlike harmonic analysis where near and distant points have equal effects on the curve.

[2] Which eliminates seasonal fluctuations.

Moreover, unless the erratic fluctuations are quite small, a 2 of a 12-months moving average[1] will not contain enough terms to smooth the data at all successfully. Some longer moving average, such as an 8 of a 12, must then be used. Such a longer moving average will, however, require still longer cycles in the data if its "dampening" effect is not to be too great. For such reasons, moving averages are generally unsatisfactory.

In the preceding paragraph, we have referred to the fact that an 8-months moving average of a 12-months moving average will give a smoother curve than a 2 of a 12. The fact that the 8 of a 12 contains more terms than the 2 of a 12 is not the only reason why it gives a smoother result. Another reason is that its "weight diagram" is smoother. Before passing on to the discussion of types of smoothing other than simple moving averages, it seems desirable to explain the significance of smoothness in the weight diagram. A simple moving average may be thought of as being fitted to the data by the use of a set of weights whose total is unity. In a 13-months simple moving average, each weight would be $+1/13$. The weight diagram is a rectangle 13 units long and $1/13$ of a unit high. A 2-months moving average of a 12-months mov-

[1] That is, a 2-months moving average of a 12-months moving average. Similar abbreviations are used freely throughout the discussion which follows.

ing average also contains 13 weights, but the weight diagram is a little smoother. The end weights are each $+\frac{1}{24}$, while the other eleven weights are each $+\frac{1}{12}$. A commonly used criterion of smoothness is based on the sum of the squares of the third differences. Now the sum of the squares of the third differences of the 13-months moving average set of weights is $\frac{12}{169}$ while the sum of the squares of the third differences of the 2-months moving average of a 12-months moving average set of weights is only $\frac{1}{72}$.[1] A 4 of a 5 of a 6 contains only 13 terms but gives a still smoother weight diagram than a 2 of a 12. The sum of the squares of its third differences is only $\frac{1}{800}$.

A smooth weight diagram leads to smoothness in the resulting graduation because smoothing by means of any weighted or unweighted moving average amounts to distributing each observation over a region as long as the weight diagram and of the same shape as the weight diagram. For example, if we take a 13-months simple moving average of a series of observations, we may think of this 13-months moving average as being constructed as follows: Each individual observation is divided by 13 and these 13 equal fractional values

[1] In all discussions of weights (or weight diagrams) the reader must think of such weights as infinite in number; to the right and left of the actual values used, he must think of an infinite number of zero values. In connection with the discussion above, the reader may refer to the first few diagrams in Chart I.

are distributed along 13 adjacent points on the x axis, of which 6 are on each side of the observation.

A little thought or experimentation will quickly convince the reader that, so long as we restrict ourselves to *positive* weights, no moving average, weighted or unweighted, will exactly fit any mathematical curve except a straight line. If, to consecutive and equally spaced points on a second (or third) degree parabola, we fit a moving curve each point of which is the mid ordinate of a second-degree parabola fitted to n consecutive observations by the method of least squares, the fitted moving curve will, of course, fall on the original parabola.[1]

The mid ordinate of a second-degree parabola fitted by the method of least squares to n consecutive observations may be computed by means of a weighted moving average with a particular set of weights. For example, if n equals 13, the weights are:

$$-\frac{11}{143},\ 0,\ +\frac{9}{143},\ +\frac{16}{143},\ +\frac{21}{143},\ +\frac{24}{143},\ +\frac{25}{143},\ +\frac{24}{143},$$
$$+\frac{21}{143},\ +\frac{16}{143},\ +\frac{9}{143},\ 0,\ -\frac{11}{143}.$$ [2]

[1] If fitted to a *third*-degree parabola, it will fall exactly on that curve, as the mid ordinate of a second-degree parabola fitted to data by the method of least squares is the same as the mid ordinate of a third-degree parabola fitted to the same data.

[2] Compare E. T. Whittaker and G. Robinson, *The Calculus of Observations*, Blackie & Son, 1924, p. 295. Also see Chart I.

This weighted average must, of course, be centered on the middle, or 7th, month. The points on such a graduation will naturally lie exactly on the original data points, as these data points are themselves points on a second (or third) degree parabola. We notice that there are two minus weights in the set of 13 weights. The graduation could not fall on the original parabola if it did not have these minus weights. A graduation of this type, each of whose points is the mid ordinate of a second-degree parabola fitted by the method of least squares, does not however necessarily give ideal results. There are at least four reasons for not using it to smooth such a time series as monthly Call Money Rates.

A first reason is that such a graduation will entirely eliminate seasonal fluctuations only by the most improbable accident. If, neglecting for the moment erratic fluctuations, the original monthly data be thought of as made up of two parts, (1) a smooth curve and (2) a seasonal fluctuation superposed on the smooth curve, the results of fitting a parabola to the original data are the same as if we fitted a parabola to the smooth curve and another parabola to the seasonal fluctuations and added together, each month, the pairs of resulting ordinates. Now, if the seasonal fluctuations were constant from year to year, the smooth curve fitted to them should, by the definition of seasonal fluctuations, be simply $y = 0$.[1] In general, a curve fitted to seasonals will give continuous zero values only if its weight diagram is such that equal weights are given to each *nominal* month. A simple 12-months moving average gives such equal weights to each nominal month. Many other graduations may be constructed which will do the same. Any moving average of a 12-months moving average will give equal weights to each nominal month. The 43-term graduation emphasized in this book gives such equal weights to each nominal month, but the weights for the mid ordinate of a second-degree parabola *fitted by the method of least squares* do not do so.

A second reason for not graduating such data as Call Money Rates by formulas for the mid ordinates of second-degree parabolas fitted by the method of least squares is that this method of graduation does not give the smoothest possible results. The weight diagrams are not even moderately smooth. For example, in the weight diagram

[1] The assumption is here made that in a seasonal movement the sum of the negative values in any 12-months period equals the sum of the positive values. If seasonal movement be defined in any other way, for example, as that the *product* of any 12 consecutive monthly seasonal values equals 1, the further assumption would have to be made that such a function of the data had been taken before smoothing as to permit the sum of the positive and negative values of the seasonal of this function to equal 0. For example, in the interest rate and security price series, this very product assumption is made and the 43-term curves are fitted not to the original data but to the logarithms of the original data. See Appendix VIII.

for the mid ordinate of a second-degree parabola fitted to 13 observations by the method of least squares, there are two distinct cusps at the $-{}^{11}/_{143}$ values. Unless a weight diagram be itself a smooth curve, it will not give the smoothest possible results in fitting.

A third reason for not using a graduation based upon the mid ordinates of second-degree parabolic curves is that such a graduation is poorly adapted to describing periodic functions. Second (or third) degree parabolic fitting may in exceptional cases be useful to describe long time trends but it is not adapted to the adequate description of cyclical or wave-like movements. If a curve based on moving parabolas be fitted to a sine curve by the method of least squares, the fitted curve will always be too low at maximum points and too high at minimum points. It is true that the smaller the number of terms to which such a parabola is fitted, the less will be the deviations of the fitted curve from the original sine curve. However, if we are not fitting to a sine curve but to actual irregular data, the taking of a small enough number of terms to allow the fitted parabola to reach up adequately into maximum sections of the data and down into minimum sections, will introduce the same difficulty encountered in the case of a simple moving average—though of course to a much smaller degree. Unless the erratic fluctuations of the data are very small as compared with the amplitude of the cyclical movements, a large number of terms will have to be used in the parabolic set of weights or the data will not be adequately "smoothed." However, unless the cycles of the original data have very long periods, it will not be possible to use a large number of terms without departing too far from the underlying fundamental curve.[1]

A fourth and very cogent reason for not smoothing by means of any graduation based on the mid ordinates of least squares parabolas is that the weights do not lend themselves to easy computation. For easy computation, the weights should always be such that they can be broken up into a short series of simple moving averages. If this cannot be done, we are faced with the multiplication of each datum point by its proper weight in order to get each separate point on the curve. If the number of weights in the weight diagram be large, the computation of the smooth curve then becomes extremely laborious.

[1] The negligible seasonal fluctuation in Railroad Bond Yields and the small size of the minor fluctuations as compared with the cyclical movements would have permitted the use of least squares second-degree parabolic formulas for smoothing that particular series.