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Interest Rate Expectations and Optimal Forward Commitments for Institutional Investors

ABSTRACT: Forward commitments to lend at predetermined rates and times account for large fractions of the investments of major institutional lenders. When the commitments are made, both the flow of future investible funds, and the forgone returns on alternative future investments, are uncertain. This paper systematically analyzes the impacts of these uncertainties on optimal portfolios including forward commitments when investors have either constant absolute or constant proportional risk aversion. This analysis of the supply of forward commitments is supplemented by a corresponding model of the demand for forward commitments by risk-averse issuers of claims; and the competitive equilibrium rate on forward commitments is derived. ¶ It is shown that under realistic assumptions equilibrium rates on forward commitments will be higher than the expected future market rate for immediate investment at takedown time—and this is true even in the limiting case of purely competitive forward commitment markets. Even though lenders avoid uncertainty of return on the funds committed forward, commitment rates higher than either current or expected future market rates are not prima facie evidence of market power per se. Even when all assessments are made in terms of

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symmetrical probability distributions, skewness preferences are introduced by lenders' concern with new-money rates, and these are shown to reinforce the results of simpler models. Further implications of lenders' concerned with relative new-money rates (i.e., their own versus those realized by their competitors) are also briefly considered.

A forward commitment is a binding agreement by a lending institution to make available a given amount of funds upon given credit terms at certain specified dates in the future. (See O'Leary 1960, p. 325; also see other references cited in the review of the literature on forward commitments, below.) Normally, the commitment agreement specifies the interest rate, maturity, redemption privileges, and so on, as well as the schedule of disbursement or "takedown" of the funds. The forward commitment is binding on the lender, and the borrower is also obligated to take down the funds in the agreed amounts.[1] The latter feature distinguishes forward commitments from the "lines of credit" common in commercial bank lending which merely give the potential borrower a "call" upon any amount of credit up to a stated maximum at any time over a specified period. Also in recent years, such lines of credit as well as the longer-term loans of commercial banks have often specified floating rates that vary with changes in the prime or some other base rate. In contrast, forward commitments typically are made at *fixed* contractual rates of interest determined at the time the commitment is made rather than at the later time when the funds are drawn down.

Forward commitments as here defined are important features of the lending of mutual savings banks, savings and loan associations (S&L's), and life insurance companies on residential, industrial, and commercial mortgages; and they are an essential feature of lending in the so-called private placement market for corporate securities. The importance of the distinction between forward contracts and spot transactions in analyzing the investment behavior of each of these lending institutions and investment markets depends upon the length of the commitment involved, which in turn of course depends significantly on the type of investment and whether new construction is involved. Advance commitments to take down some part of a new underwritten issue of corporate bonds or to write a mortgage on existing homes will typically involve only a few days or weeks. In contrast, mortgage commitments to lend on new homes must cover the construction period and will normally run from six to twelve months; those on apartments, condominiums, shopping centers, office buildings, industrial plants, and so on will often run as long as two or three years, and sometimes longer.

The typical savings and loan association has always (except, of course, during and right after World War II) invested 90 percent or more of its assets in residential mortgage loans. The concentration of this lending on one-to-four–family houses and the large fraction of such lending on existing properties makes the weighted average of the forward commitment period relatively short (one to two months or less) for most S&Ls, although it has become somewhat longer for some of the larger ones in large cities in recent years during periods of heavy construction of new apartment buildings. While mutual savings banks typically invest a somewhat smaller fraction (60–70 percent) of their assets in residential mortgages, their lending practices are similar and their weighted average commitment periods for these loans are about as short as for S&Ls. The mean commitment period for the largest savings banks will again be somewhat longer because of their mortgage lending on commercial and industrial properties as well as their loans on multifamily structures.

The weighted average of the forward commitment period for life insurance companies will be very substantially longer than for either mutual savings banks or S&Ls. Over the last two decades, home mortgages have progressively declined to a small fraction of all the mortgage commitments made by insurance companies.[2] A very high fraction of their commitments for mortgages on income properties has been for "takeout" mortgages granted upon the completion of new construction typically involving two years or more before final takedown, and large fractions of their commitments in private placements involve equally long lead times. Bonds and mortgages dominate the investment portfolios of life insurance companies, and at least since 1961, ALIA data show that bonds and mortgages acquired through forward commitments have averaged over 95 percent of all such acquisitions.

When such large fractions of all new investments involve advance commitments for later delivery of funds at rates determined at the time of the commitment rather than delivery, investment managers have a primary responsibility to act on their best judgment that the funds so placed are prudently invested in this way rather than in alternative investment outlets available for immediate purchase at the *future* delivery date. The latter spot market yields will of course not be known until the latter time, but the decision must nevertheless be made now when the yield on the relevant alternative investment outlet is still uncertain. Managements' assessments (made at the time of the commitment) of the relevant spot rates that will be available when the funds are to be actually paid out must be an essential factor in the determination of the desirable scale of forward commitments for every institution investing in these markets. The only possible exception would be a small savings bank or savings and loan association in a small town with no new construction whose mortgage lending is confined to loans for the transfer of existing houses. Even in this limiting case, there would be some uncertain movement in the spot rates on alternative market investments over the few days or so between the times of the commitments and the payouts of the funds on the new loans. But for all savings banks and savings and loan associations that lend on new, single-family construction, assessments of spot market rates at least six months in the future will be involved—and for larger savings institutions and life insurance companies that lend on new multifamily residential and commercial or other "income" properties and in the private placement market, the lead time of assessment of the opportunity cost of any commitment at currently determined rates is several times longer.

Since as a general rule the uncertainty regarding any assessment increases with its futurity, we should expect that the variance of the assessment of the future spot rates relevant to lending on new income properties would be greater than on new small residential construction, and doubly greater than on loans for the transfer of existing homes. Moreover, with a given degree of risk aversion, modern portfolio and investment theory would correspondingly suggest that the assessments of probable future market interest rates and their uncertainties would play an essential role in the investment decisions of all institutional investors, with the possible exception of the smallest banks as a limiting case, and that the role of future market rates would be relatively greater on average for most life insurance companies than for even large savings banks or S&Ls.

Remarkably enough, a review of the literature indicates that there has been no systematic theoretical analysis of the extent to which investing institutions should vary the scale of their forward commitments at any given current commitment rates on the basis of their assessments of the expected values and variances of future market rates. There have indeed been several recent studies of the forward commitment behavior of institutional investors. In particular, Jaffee (1972) has analyzed the forward commitments for residential mortgages made by mutual savings banks, savings and loan associations, and life insurance companies; and Bisignano (1971), Pesando (1971), and Ribble (1973) have studied the forward commitments of insurance companies, broken down by type of property on loan underlying the commitment. In each of these four studies, the now relatively standard stock-adjustment model is adopted to explain forward commitments in terms of cash flows and discrepancies between actual and desired stocks of each type of investment. Desired future holdings of each asset (for instance, income property mortgages) in *dollars* is made a linear function of concurrent yield spreads against other assets. Neither the expected value of the difference between the current commitment rate and the future (takedown time) rate on the alternative use of the funds, nor the variance of this assessment, enters into the analysis in these studies. More recently, Fleuriet (1975), in the course of a study of changes in the spread between rates on public and private offerings of debt issues, finds that the dollar amounts of life insurance companies' total forward commitments are linearly (and significantly) related to the dollar amounts of their investible funds, the current rate on commitments, and the forward rate implicit in the yield curve; but this primarily empirical study did not carefully develop the underlying theory.[3]

The primary purpose of the present paper is to fill in the analysis of the impact of assessments of uncertain future market rates and the stochastic properties of flows of investible funds upon the optimal forward commitment positions of major institutional investors. A larger companion study (Lintner, Piper, Fortune 1976) fills out the institutional, empirical, and econometric analysis of forward commitment positions with particular reference to life insurance companies.

Specifically, this paper develops a theoretical analysis of the impact of interest rate expectations upon forward commitments by examining the behavior of a single lender who must determine the appropriate amounts of funds to commit for future takedown in the face of uncertainty about what relevant future interest rates and future investible funds will turn out to be at the time the committed funds are disbursed. To keep these essential elements in the clearest possible focus, I simplify the general forward commitment problem by supposing (i) that only *one type of asset* is available for future commitments and (ii) that there is a *given and fixed time period* of months after the initial commitment is made before the funds are disbursed.[4] In order to highlight the effects of uncertainties regarding future market interest rates (and the amount of funds available for investment), I also assume initially that (iii) the interest rate (r_c) at which commitments may be made is given by the market and does not depend on the volume of commitments any one company decides to undertake.[5]

In particular, I concentrate on the investment decision of a particular institutional investor which expects to have F funds available for investment τ months hence. At the present time, the company can make a forward commitment to deliver some or all of these funds at a known and fixed rate (r_c), but if it commits \$C forward, it will have to invest the remaining funds (\$$F$ − \$C) available τ months from now at whatever the current market rate ($\tilde{r}$) on the relevant alternative (future) "spot" investment turns out to be at that time.[6] The decision regarding the amount of forward commitments (C) must be made *now*, even though the alternative market rate ($\tilde{r}$)—and usually also the total amount of available funds ($\tilde{F}$) for investment—will not be known until τ months later.

It is assumed throughout this theoretical analysis that each management has a *risk-averse preference* ordering over different combinations of risks and returns, and that it chooses the particular level of its forward commitments to obtain the combination of risk-and-return characteristics it prefers to those associated with any other investment posture available. In field work for a larger study (Lintner, Piper, Fortune 1976), however, certain interesting differences were uncovered in the identification of the returns concerning which investing institutions were behaving in a risk-averse manner. Some considerable support was found among savings banks and insurance companies for identifying these returns with (i) the amount of income (or the level of the income stream) produced by the investments made. More generally, however, this field work indicated that these managements were primarily concerned with (ii) the *rate* of return—the "new-money rate"—obtained on the investment funds disbursed at any given time. There was also substantial evidence that several insurance companies with relatively large sales of group insurance were giving a heavy weight to measures of (iii) their *relative performance*—i.e., their own rate of return in comparison with that realized by other companies.[7]

In section 1, I analyze the supply of forward commitments from institutional lenders that are risk averse with respect to the level of the income stream provided by the investments made. This simple model readily lends itself to a rather transparent analysis of the effects of several important features of commitment markets, including equilibrium commitment rates in purely competitive markets of risk-averse borrowers and lenders. In section 2, all the preceding qualitative results for lenders' behavior are re-established on the assumption that lenders act in terms of rates of return on the basis of preference functions exhibiting constant proportional risk aversion, and I also establish the relevance and consequences of "skewness preference" when investors are concerned with rate of return on uncertain flows of investible funds. In section 3, I develop additional implications of lenders' concern with relative rate-of-return performance criteria.

It will be useful to summarize the principal conclusions reached before the more detailed analysis is presented. In particular, when the contract yield on commitments is known but the market rate on the relevant alternative asset at the time of takedown is uncertain, the following analysis establishes that:

i. *If* the amount of investible funds were known ahead of time, risk-averse institutional investors using either an investment income or new-money-rate performance criterion would undertake to be fully committed forward *even if* the expected future market rate were as high as the rate available on commitments.

ii. Using either criterion, (a) penalty costs incident to any shortfall of investible funds insure that the optimal level of forward commitments will be reduced by uncertainty at the time of commitment regarding the volume of funds that will be available for investment at the time of takedown. Moreover, even with no penalty costs (b) the (empirically strong) *inverse* relation between future investible funds and changes in the market rate *always reduces* the forward commitment position that would otherwise be optimal. Indeed, with both criteria, (c) a given degree of negative covariance reduces the optimal commitments more sharply the higher the level of market rates.

iii. Other things equal, optimal forward commitment positions vary directly with the rate (r_c) (including fees) on forward commitments and inversely with changes in the level of the expected market rate ($\bar{r}$). In both respects, however, the extent of the adjustment in forward commitment positions as a result of any change in r_c or $\bar{r}$ will be smaller, the more risk averse the lender and the greater the uncertainty in its assessment of the future market rate.

iv. In view of (ii) and (iii) above, institutional lenders risk averse with respect to either their levels of investment income as such or the levels of their new-money rates will choose a high ratio of forward commitments to their *expected* investible funds *only if commitment rates are higher* than expected market rates at takedown time. Moreover, to support a given commitment ratio of say 90 percent or so, this positive "commitment premium" will have to be larger: (a) the greater their uncertainty regarding market rates, (b) the greater the (negative) dependence of investible funds on movements in interest rates, (c) the greater their risk aversion; and also (assuming the appropriate covariance is stable), (d) the higher the level of interest rates.

Comment: These conclusions are particularly important because they show that under empirically relevant assumptions we should observe commitment rates *higher* than expected (or current) market rates (for direct investment) *even if* forward commitment markets were purely competitive in the strictest and most ideal sense and no lender had any "market power" whatsoever. The fact that commitment rates have been considerably higher than current market rates is not in itself prima facie evidence of market power. (In this connection, recall that ALIA data show that life insurance companies since 1961 have been acquiring more than 90 percent of their long-term assets through forward commitments.)

v. The above results follow from mean-variance portfolio theory based on the assumption of symmetric probability distributions. But when the amount of investible funds is ex ante a stochastic variable negatively correlated with the uncertain future market rate, the *new-money rate* (defined as the ratio of income produced to the actual amount invested) has a *negatively skewed* distribution even when both the numerator and denominator are symmetrically distributed. Also, this negative skewness will be increased by any penalty costs associated with shortfalls of investible funds below prior commitments. Institutional investors acting on a new-money-rate performance criterion and who are *averse to negative skewness*[8] will, consequently, have a lower forward commitment position, ceteris paribus, than those who are merely risk-averse in a mean-variance context.[9] As a corollary, it also follows that when such lenders act on the basis of preferences over new-money rates, the excess of commitment rates over expected market rates observed in purely competitive markets will be *larger* the more prevalent and intense the lenders' aversion to downside skewness.

vi. In section 3, it is established that companies that are risk-averse with respect to *relative* new-money rates[10] as the measure of their investment performance will tend to be relatively less heavily committed forward than their major competitors *unless* commitment rates are sufficiently higher than expected market rates. The amount of "commitment premium" required to bring their forward commitment position up to their competitors' will be larger (a) the greater their uncertainty over interest rate movements, (b) the stronger the negative covariance between interest rate movements and their flows of investible funds, (c) the more intense their aversion to risk and uncertainty, and (d) the greater their dislike of negative skewness in the distribution of relative new-money rates.

In the companion manuscript (Lintner, Piper, Fortune 1976), it is established that each of these effects of interest rate expectations and uncertainties, derived here in the simple context of a one-period model, continue to hold in the more general and realistic context of a multiperiod, stock-adjustment model of the forward commitment process. In that manuscript it is also established that the results derived here on the assumption that only one asset is available for forward commitment carry over to *total* forward commitments of an institution and to its commitments in each type of asset subject to forward commitment (different types of mortgages and private placements).

[1] INSTITUTIONAL INVESTORS RISK AVERSE WITH RESPECT TO INVESTMENT INCOME

It is assumed that management uses its assessments of the basic uncertainties of its investment situation to form estimates of the expected returns and risks which would be associated with any possible level of future commitments (C) it might choose to undertake, and that the particular level of forward commitments (C^*) it chooses is the one that in its judgment has the *preferred* combination of risk-and-return characteristics. In this section, it is assumed that the institution acts in terms of a risk-averse *preference ordering* (or "utility function") over the possible outcomes of its decisions which depend essentially on the stream of investment income (or on the rates of return) associated with the different possible decisions it may take. Without loss of generality for present purposes, it can also be assumed that risks are adequately indexed by the variances of the uncertain outcomes, i.e., that all stochastic variables are normally distributed.

Formally, the forward commitment decision problem is analyzed in the context of a standard mean-variance model of portfolio theory. The list of variables used is defined as follows:

- $F \equiv$ the cash flow that will be available for investment τ months in the future. (When this term is uncertain at the time the decision is made, it is written as $\tilde{F}$.)
- $r_c \equiv$ the rate of interest *currently* available on funds committed now for delivery at the time τ months from now.
- $\tilde{r} \equiv$ the uncertain rate at which funds for immediate delivery may be invested τ months from now.
- $C \equiv$ the amount of funds committed now for forward delivery τ months from now.
- $\tilde{Y} \equiv$ the size of the investment income stream produced, beginning at time τ, by the current decision regarding forward commitments.
- $\bar{Y}, \bar{r}, \bar{F} \equiv$ the expected values of the indicated variables.
- $V_Y, V_r, V_F \equiv$ the variances of the indicated variables.
- $\tilde{r} \equiv N(\bar{r}, V_r)$ and $\tilde{F} = N(\bar{F}, V_F)$.
- $\sigma_{rF} \equiv$ the covariance between $\tilde{r}$ and $\tilde{F}$.
- $U(Y) \equiv$ the utility function of the lending institution that exhibits risk aversion, i.e., $U'(Y) = \partial U/\partial Y > 0$ and $U''(Y) = \partial^2 U/\partial Y^2 < 0$.

Given management's assessments of the underlying uncertainties in interest rates ($\tilde{r}$) and funds ($\tilde{F}$), each possible choice of a level of forward commitments (C) will be associated with a different distribution of investment income ($\tilde{Y}$), and the optimal choice (C^*) can be found by choosing C to maximize $E[U(\tilde{Y})]$. But it is well known that with normally distributed random elements the effort to maximize $E[U(\tilde{Y})]$ is strictly equivalent as a decision criterion to the simpler

(1.1) *criterion:* choose C to maximize $W(\bar{Y}, V_y)$

where the welfare index $W(\bar{Y}, V_y)$ is

(1.2) $W = \bar{Y} - \gamma V_y/2$

and where

$$\text{(1.3)} \quad \gamma = -\frac{U''(Y)}{U'(Y)} > 0$$

is the measure of the investor's *degree of risk aversion*, which was first suggested in the pioneering work of Pratt (1964) and Arrow (1965). More-risk-averse investors have larger values of γ and require larger increments of expected return to justify (on the basis of their subjective preferences) any action that increases the variance ("risk") of the distribution of their income. Although the degree of risk aversion generally varies somewhat with the level of income, the effects of any such variations on the level of forward commitments are of secondary importance and small relative to the other considerations that will be emphasized. Consequently, for convenience I treat the risk-aversion parameter as a constant for any given institution, even though different institutions will be more or less risk averse.

Given the functional relation between the levels of C and the values of $\bar{Y}$ and V_Y, the optimal amount of forward commitments will be that value C^* for which the derivative of (1.2) with respect to C equals zero, i.e.,

$$\frac{\partial W}{\partial \bar{Y}} \frac{\partial \bar{Y}}{\partial C} + \frac{\partial W}{\partial V_Y} \frac{\partial V_Y}{\partial C} = 0;$$

which simplifies to

$$\text{(1.4)} \quad \frac{\partial \bar{Y}}{\partial C} - \frac{\gamma}{2} \frac{\partial V_Y}{\partial C} = 0.$$

To illustrate the solution in the simplest possible situation, I first consider the decision that would be made if the level of F were known in advance. This case will also serve as a benchmark to bring out most explicitly the impact of uncertainties about the supply of investible funds (introduced in section 1.2 below).

[1.1] The Commitment Decision When F Is Not Uncertain

When F, the total amount of investment funds available for delivery at time τ, is known in advance, the only uncertainty involves the interest rate on spot investment at time τ. The uncertain investment income which will be realized by a *total* investment of \$$F$, with \$$C$ committed in advance, will thus be

$$\text{(1.5)} \quad \tilde{Y} = Cr_c + (F - C)\tilde{r};$$

and the expected values and variance of $\tilde{Y}$ are respectively

(1.6) $\bar{Y} = Cr_c + (F - C)\bar{r},$

and

(1.7) $V_y = (F - C)^2 V_r.$

When (1.6) and (1.7) are differentiated, and the results are substituted into (1.4), it is found that the optimal level of C must satisfy

(1.8) $r_c = \bar{r} + \gamma(F - C)V_r = 0.$

The general solution for the optimal level of C (denoted C^*) is given by

(1.9) $C^* = F + \dfrac{r_c - \bar{r}}{\gamma V_r}$

when (i) the total amount of funds (F) that will be available for investment is known in advance, and (ii) the rate (r_c) available on new commitments is given and does not depend on the amount of commitments made by the given company, and (iii) the interest rate that will be available on direct investments in the future is uncertain.

Within its relevant range,[11] equation 1.9 shows, as expected, that the supply of forward commitments varies directly with the commitment rate (r_c) and inversely with the *expected* future alternative rate ($\bar{r}$). Moreover, in both respects, the absolute rate of response in the volume of commitments is inversely proportional to both risk aversion and the uncertainty of the interest rate forecast. But equation 1.9 also establishes the important conclusion [(i) in the Introduction] that even when the *expected* value of the uncertain return on future direct investments is as high as the current forward commitment rate,[12] companies acting *as if* they were sure of the amount of funds they will have available at takedown time (but uncertain of the market rates which will be available at that time) will make forward commitments equal to the (known) amount of their available funds ($C^* = F$). If a risk-averse institutional lender knew F in advance, it would be fully invested in commitments whenever $r_c \geq \bar{r}$ because the rate on forward commitments is a known value while the *certainty equivalent* of the *uncertain* future market rate is *less than its expected value* in the assessment of all risk averters. Under such favorable conditions a "fully committed investment policy" is a consequence of the concern of risk-averse lenders with the expected level and uncertainty of future rates on alternative investments.[13]

[1.2] The Commitment Decision When the Supply of Investible Funds Is Uncertain (with No Penalties for Shortfalls)

When the total amount of funds that will be available for delivery at time τ is also uncertain, the random amount of investment income which will be realized from a *total* investment of $\$\tilde{F}$, including $\$C$ committed in advance, will be

(1.5a) $\tilde{Y} = Cr_c + (\tilde{F} - C)\tilde{r}$.

In the present subsection, I assume for simplicity that the same rate $\tilde{r}$ represents both (i) the relevant future yield that will be available in the spot market for any excess of available funds ($\tilde{F}$) over prior commitments C and (ii) the yield given up on sales of previously acquired assets (or the rate paid on new borrowing) to cover any excess of C over the flow of investible funds ($\tilde{F}$) which become available at takedown time, τ periods hence. Such an assumption of equal borrowing and lending rates is common in theoretical work on capital markets and yields significant results in the present instance, even though in the context of commitment markets represents only a rather special limiting case. The effects of "penalty costs" for shortfalls ($\tilde{F} < C$) will be analyzed in subsection 1.4 below.

With no penalty costs for shortfalls, the expected value and variance of $\tilde{Y}$ become

(1.6a) $\bar{Y} = (r_c - \bar{r})C + \bar{r}\bar{F} + \sigma_{F,r}$,

and

(1.7a) $V_Y = C^2V_r - 2C\sigma_{Fr,r} + V_{Fr}$.

Now it can be shown[14] that the covariance of the product $\tilde{F}\tilde{r}$ with $\tilde{r}$ itself can be simplified to

$\sigma_{Fr,r} = \bar{r}\sigma_{F,r} + \bar{F}V_r$,

with the result that (1.7a) reduces[15] to

(1.7b) $V_Y = (C^2 - 2C\bar{F})V_r - 2\,C\bar{r}\sigma_{F,r} + V_{Fr}$.

Proceeding as before, differentiating (1.6a) and (1.7b) and substituting these results into (1.4), we find that the optimal level of C must now satisfy

(1.8a) $r_c - \bar{r} - \gamma(C - \bar{F})V_r + \gamma\bar{r}\sigma_{F,r} = 0$.

When this equation is solved, it is seen that when the amount of investible funds is also uncertain, the optimal level of forward commitments is given by

(1.9a) $C^* = \bar{F} + \dfrac{r_c - \bar{r}}{\gamma V_r} + \bar{r}\,\dfrac{\sigma_{F,r}}{V_r}$.

Since the first two terms of (1.9a) are the same as those in (1.9), the set of conclusions in point iii of the Introduction is again confirmed. Moreover, even when the amount of investible funds is uncertain but no penalty costs are involved in shortfalls, institutional investors that are risk-averse optimizers of the income streams produced by their investments will still follow a "fully committed investment policy" in making their forward commitments (i.e., $C^* = \bar{F}$) *unless* they expect future "opportunity cost" rates to be higher than the rates currently available on new commitments, and *provided* they do not allow for (or expect) any association between future levels of market rates and the amount of total funds they will have for investment. Once again, under these favorable conditions, a "full investment" operating policy follows from their risk-averse objective of optimizing the returns on their investments.

The third term of (1.9a) may be written in two equivalent alternative forms which have interesting interpretations. First, it should be noted that the term $\sigma_{F,r}/V_r$ is just the slope coefficient in a least squares regression of the amount of investible funds ($\bar{F}$) on the level of the market interest rate ($\bar{r}$). Consequently, the optimizing condition may also be written as

$$(1.9b) \quad C^* = \bar{F} + \frac{r_c - \bar{r}}{\gamma V_r} + \bar{r} b_{Fr}$$

where $b_{Fr} = \sigma_{F,r}/V_r$. We also observe that $b_{Fr} = \partial \bar{F}/\partial \bar{r}$, with the result that the elasticity (η_{Fr}) of expected funds relative to expected interest rates is $-(\partial \bar{F}/\bar{F})/(\partial \bar{r}/\bar{r}) = -\bar{r} b_{Fr}/\bar{F}$. Again substituting, we have

$$(1.9c) \quad C^* = (1 - \eta_{Fr})\bar{F} + \frac{r_c - \bar{r}}{\gamma V r}.$$

These equations clearly show that, even when there are no penalties in investment income when shortfalls ($\bar{F} < C^*$) occur, uncertainties regarding the amount of funds that will be available for investment will also produce further adjustments in the level of forward commitments whenever management believes there is any negative association between higher or lower interest rates and the amount of its investible funds. Such an association will certainly be negative, i.e., $b_{Fr} < 0$ and $\eta_{Fr} > 0$, for several related reasons. Lower interest rates will induce advance refunding of securities originally floated at higher rates and stimulate accelerated repayments on outstanding mortgages, both of which serve to increase the amount of funds that must be reinvested when interest rates are low and particularly when they are falling. This negative association carries through for both savings institutions and insurance companies when rates are rising because of the absence of these incremental inflows of funds. Moreover, when interest rates are higher than the contractual rates on loans against insurance policies, an increase in market rates will then lead to substantial increases in the policy loans of insurance companies, and these prior claims serve to reduce, dollar for dollar, the pool of funds otherwise available for satisfying forward commitments or normal market investments (O'Leary 1960; Lintner, Piper, Fortune 1976). Correspondingly, high and rising market rates induce large and increasing net outflows of savings deposits and withdrawals—popularly termed "disintermediation." Results of numerous econometric studies strongly confirm that each of these effects produces a negative association between interest rate movements and the funds otherwise available for investment.

Our equations clearly show that when commitment rates just match the expected levels of future market rates, i.e., when $r_c = \bar{r}$, allowance for such a negative association between interest rates and investible funds will lead risk-averse income-maximizing companies to back off from a fully invested forward commitment policy. Moreover, if we concentrate the formulation in equation 1.9b, we see that the amount of "backing off" will vary with the product of the expected interest rate and the (negative) slope (b_{Fr}) of a regression of investible funds on the interest rate. Since b_{Fr} measures the expected number of dollars $\bar{F}$ lost per unit of change in $\bar{r}$, this product will be a large number for large institutions. Alternatively, we see in (1.9c) that the optimal forward commitment position of any lender, expressed as a fraction of its expected investible funds ($\bar{F}$), will vary *inversely* with the *elasticity* (η_{Fr}), and econometric estimates show that in many cases this elasticity ranges as high as 0.5 or 0.6.[16]

This analysis of the extent to which the volume of forward commitments supplied by any lender will fall short of the amount of his expected investible funds when $r_c = \bar{r}$ provides one useful benchmark for the interpretation of this model. Another is provided by defining the commitment premium ($r_c - \bar{r}$) as the excess of the current rate on forward commitments over the expected future rate on the appropriate alternative asset at takedown time.[17] We can determine the commitment premium required to bring forth a volume of forward commitments equal to the volume of funds expected at takedown time by setting $C = \bar{F}$ in equations 1.9b and 1.9c and solving for $r_c - \bar{r}$. This gives us[18]

$$(1.10a) \quad r_c - \bar{r} = -\overset{+}{\gamma}\overset{+}{V_r}(\overset{+}{\bar{r}}\bar{b}_{Fr}) > 0 \text{ when } C^* = \bar{F},$$

and

$$(1.10b) \quad r_c - \bar{r} = +\overset{+}{\gamma}\overset{+}{V_r}\overset{+}{\eta}_{Fr}\overset{+}{\bar{F}} > 0 \text{ when } C^* = \bar{F}.$$

In both formulations, we have a general proof that when (i) suppliers are risk-averse, (ii) are uncertain regarding the amount of their future investible funds, and (iii) allow for negative covariance between such funds and interest rates, the supply of new forward commitments made will be as large as the expected volume of investible funds *only if* the yield on forward commitments is greater than the expected interest rate on open-market purchases of long-term securities at the time of takedown.

[1.3] Equilibrium Commitment Premiums in Purely Competitive Markets

The C in equations 1.9b and 1.9c represents the total volume of forward commitments a given lender would want to have outstanding at a given time on the basis of the expected value ($\bar{F}$) of the investible funds to be available at the time of takedown and the values of the other parameters and variables in the equation. It thus represents the supply of forward commitments from a given institution, and by aggregation the same equation can be used to represent the supply of forward commitments from all institutional lenders on the basis of given assessments[19] of the other variables. The supply of forward commitments from any institutional lender will be a rising function of the commitment premium, with its slope inversely proportional to the product of the risk-aversion coefficient and the variance (uncertainty) of the future rate. Summing over all lenders, the aggregate supply of forward commitments will have a similar[20] positive slope of $1/\gamma V_r$.

The aggregate demand for forward commitments will correspondingly have a negative slope. For most developers of apartment houses, condominiums, office buildings, shopping centers, and the like, obtaining an advance guarantee of permanent "takeout" mortgage financing is usually essential,[21] not just a matter of convenience or monetary advantage, and similar considerations apply to the financing of the construction of single-family houses.[22] A declining (aggregate market) demand curve for such forward commitments follows from the inverse relation (ex ante and other things equal) between the net profitability of the operation of the completed property and its financing cost. With each increase in r_c, ceteris paribus, more potential borrowers drop out of the market.

The rationale for the negatively sloped demands of industrial and other users of forward commitments can take two somewhat different forms which can be sketched briefly. In the first form (admittedly not strictly relevant in "perfect" capital markets), a potential borrower knows that he will need a certain amount of funds at some given time in the future; arranging an advance commitment from a lender to provide the funds will substantially reduce his risks that the funds may not be available when they are needed because of credit rationing. Lane (1974) has shown that in such circumstances, the volume of forward commitment that risk-averse borrowers will want declines with increases in the cost of commitments.[23] The second (which does hold even in idealized competitive markets) derives from the insurance that forward commitments provide to risk-averse borrowers against increases in market rates by the time the funds are drawn down. Let

$B \equiv$ amount of total borrowing required
$C_d \equiv$ amount of forward commitment obtained
$\tilde{r}_b \equiv$ uncertain rate on borrowing other than through commitments
$\tilde{m} \equiv$ uncertain rate of profit from use of funds before financing costs
$\tilde{\Pi} \equiv$ uncertain rate of profit after finance charges

Then

$$(1.11) \quad \tilde{\Pi} = C_d(\tilde{m} - r_c) + (B - C_d)(\tilde{m} - \tilde{r}_b) = B(\tilde{m} - \tilde{r}_b) + C_d(\tilde{r}_b - r_c).$$

The expected value $(\bar{\Pi})$ and variance (V_{Π}) of net profits are then:[24]

$$(1.12) \quad \bar{\Pi} = B(\bar{m} - \bar{r}_b) + C_d(\bar{r}_b - r_c).$$

$$(1.13) \quad V_{\Pi} = B^2 V_{m-r_b} + C_d^2 V_{rb} + 2BC_d\sigma_{r,m-r} = B^2 V_m + (B - C_d)^2 V_r + 2B(C_d - B)\sigma_{mr}.$$

Risk-averse borrowers will choose C_d to maximize $E[U(\bar{\Pi})]$, which by familiar derivation is the value satisfying

$$(1.14) \quad \partial E[U(\bar{\Pi})]/\partial C_d = 0 = \bar{r}_b - r_c - \gamma_b[B\sigma_{mr} - (B - C_d)V_r]$$

or

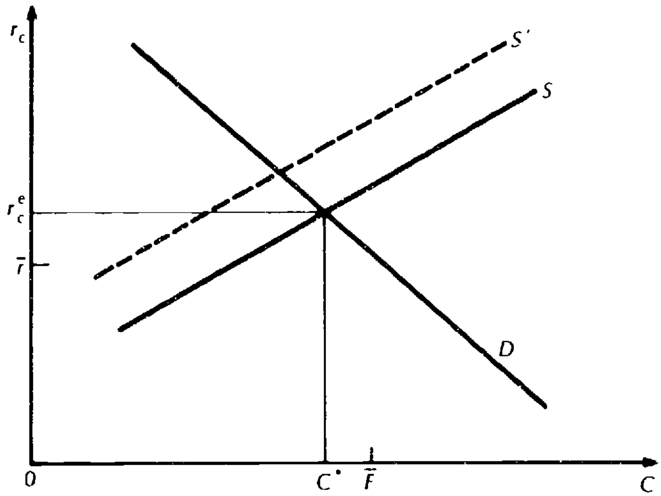
$$(1.15) \quad C_d^* = B - [(r_c - \bar{r}_b)/\gamma_b V_r] - \beta_{mr}$$

where $\beta_{mr} = \sigma_{mr}/V_r$ and γ_b is the borrower's risk-aversion parameter. Again, because of risk aversion, the demand for forward commitments (C_d^*) varies inversely with the commitment rate. Since each type of demand for forward commitments is negatively inclined, the aggregate demand will also be a declining function of the commitment rate r_c.

The preceding analysis of the supplies and demands for forward commitments may be combined as illustrated in Figure 1, and the equilibrium rate r_c^e in a purely competitive forward commitment market will then be given by the intersection of these supply and demand curves. The position of these lines as drawn reflects (i) the proof, given at the end of subsection 1.2, that the S curve must be higher than $\bar{r}$ at the point above expected investible funds $(\bar{F})$, and (ii) the empirical fact that the commitment premium has remained positive in practice, which implies that the demand curve has to be high enough to intersect the S curve at a level above $\bar{r}$.

Over time the position of the demand curve will shift to the right or left depending on business conditions and changes in potential borrower's

FIGURE 1 Supply and Demand Curves of Forward Commitments



NOTE: r_c^e is the market-clearing or equilibrium rate on forward commitments, and C^* is the dollar volume relative to $\bar{F}$, the aggregate expectation of investible funds.

needs for funds. Correspondingly, the position of the supply curve will shift equally with changes over time in the expected volume of investible funds $\bar{F}$, as well as any changes in the current assessments of the other elements in equations 1.9b or 1.9c. The empirical observation that commitment premiums have remained positive over time[25] indicates that throughout their respective cyclical shifts to the right or the left, the position of the demand and supply curves relative to each other has always involved an intersection at which the market-clearing rate on forward commitments (r_c^e) exceeded the currently assessed expectation of the relevant future market rate ($\bar{r}$).

But independent of past shifts of demands relative to supplies, it is particularly important to observe that r_c^e *must always lie on the supply curve*. We can consequently use equation 1.9b or 1.9c, or both, to conclude further that for any given level of $\bar{F}$, the commitment premium ($r_c - \bar{r}$) required to draw forth any given volume of forward commitments (C) will have to be *algebraically larger*

i. The more risk averse the companies;
ii. The more uncertain the future interest rate;
iii. The greater the negative covariance between changes in available funds and changes in interest rates; and,
iv. The higher the level of expected future interest rates.

Each of these changes in conditions or assessments, in other words, will raise the height of the supply curve of forward commitments (or, equivalently, move the S curve to the left).

Moreover, it follows as a further corollary that even under idealized conditions, in which forward commitment markets would be purely competitive in the strictest sense and no individual lender would have any market power whatsoever, we would nevertheless observe forward commitment rates higher than current or expected market rates (for direct investment) whenever (and at all times that) the aggregate demands for forward commitments at equilibrium rates (r_c^e) are even approximately as large as the supplies forthcoming at those rates from risk-averse companies facing uncertain future interest rates and negative covariances with their flows of investible funds. These conditions are sufficient analytically to explain the observed persistence of substantially positive commitment premiums over time. In particular, the existence of such premiums is not prima facie evidence of market power in the hands of any lender or group of lenders. (There is other evidence indicating the existence of such bargaining power, but the observation of positive commitment premiums per se is not a sufficient condition for the inference of market power.) This completes my proof of general conclusion iv in the Introduction and the Comment that follows it.

Addendum Up to this point the focus has been on the total forward commitment position (C_t) which the institutional lender (or industry) would want to have outstanding at any given time t relative to the expected volume of investible funds $\bar{F}_{t+\tau}$ to be available at the time of takedown, τ months later. (Subscripts have been added merely to delineate the calendar dates involved.) The analysis has also been simplified by assuming that all commitments are in a single asset with a fixed takedown lag. It can now be recognized that there will also be other loans (e.g., industrial and multifamily mortgages) for which commitments have been made some time earlier but which are also expected to be drawn down at time $t + \tau$. We thus have

$$(1.16)\quad C_t = OC_t + NC_t$$

where $OC_t \equiv$ commitments already outstanding at the beginning of time t for drawdown at $t + \tau$;[26] and $NC_t \equiv$ *new* commitments made at time t. If now we also define

$$(1.17)\quad \bar{NF}_{t+\tau} \equiv \bar{F}_{t+\tau} - OC_t$$

it is immediately apparent that we can subtract OC_t from both sides of equation 1.9b and rewrite the latter as

$$(1.9b')\quad NC_t = \bar{NF}_{t+\tau} + \frac{\bar{r}_c - \bar{r}}{\gamma V_r} + \bar{r}b_{Fr}.$$

Moreover, all our earlier analysis relating C_t to $\bar{F}_{t+\tau}$ carries over without any other change to the new commitments currently entered into relative to the net expected fund position ($\tilde{NF}_{t+\tau}$) after allowing for the already outstanding commitments due to be drawn down at this future date. In particular, all the above conclusions regarding conditions for positive commitment premiums and conditional shifts in commitment supply functions apply directly and equivalently to new commitments ($NC_{t+\tau}$) as well as to total commitment positions and funds (C_t and $\bar{F}_{t+\tau}$).

[1.4] Allowance for Added Costs When the Volume of Available Investible Funds Falls Short of the Volume of Forward Commitments

In the formal analysis so far, it has been assumed that a single market rate r (see equation 1.5a) measures both the rate of return on the volume of funds (i.e., $\bar{F} - C$) directly invested in the market when $\bar{F} > C$ and the opportunity cost of the funds obtained to satisfy shortfalls ($\bar{F} < C$). In the former circumstance, it is clear that the relevant market rate is the yield on publicly issued long-term bonds. The assumption made to this point—that the same rate measures the cost of funds required to cover shortfalls—clearly involves no serious distortions so long as the shortfalls are relatively small, infrequent, and of short duration. But in general, the costs involved in shortfalls will be greater than the direct investment rate—and usually, by amounts which increase with the amount of the shortfall. Companies are most reluctant to permit any large reduction in liquidity stocks because such a development would entail sharply rising opportunity costs. When the shortfall is large relative to the small cushion provided by liquidity, the companies will consequently be forced to resort either to emergency sales of long-term bonds from existing portfolios or to borrowing. Moreover, such large shortfalls are most likely to occur when long-term rates are high relative to previous norms and rising rapidly.[27] Heavy sales made within a limited period of time into such weak markets must almost always be made on a yield basis higher than existing rates on new issues; and the yields forgone on the existing assets sold to cover the shortfall will be considerably higher[28] than the yield which could have been realized on any positive level of $\bar{F} > C$, that might have occurred at the same time.[29] Similarly, borrowing from commercial banks to cover such shortfalls would have to be done at rates higher than the current rate on long-term market securities—again because shortfalls appear when rates are higher than had been anticipated and in periods of tight money. At such times, short rates in general and prime commercial bank lending rates in particular (even before allowance for added compensating balances) are higher than long rates.

In order to introduce these considerations into the formal analysis, it is necessary to rewrite equation 1.5a thus:

$$(1.5a') \quad \tilde{Y} = C r_c + (\tilde{F} - C)^+ \tilde{r}_m + (\tilde{F} - C)^- \tilde{r}_j$$
$$= C r_c + (\tilde{F} - C)\,\tilde{r}_m + (\tilde{F} - C)^- (\tilde{r}_j - \tilde{r}_m)$$

where

$$(\tilde{F} - C)^+ \equiv \begin{cases} (\tilde{F} - C) & \text{when } \tilde{F} - C \geq 0, \\ 0 & \text{when } \tilde{F} - C < 0. \end{cases}$$

$$(\tilde{F} - C)^- \equiv \begin{cases} 0 & \text{when } \tilde{F} - C \geq 0, \\ (\tilde{F} - C) & \text{when } \tilde{F} - C < 0. \end{cases}$$

$\tilde{r}_m \equiv$ the random yield on new issues of securities in the market at the time of takedown on commitments.

$\tilde{r}_j \equiv$ either the yield forgone on sales of portfolio securities (measured in terms of net sales proceeds after deducting all costs attributable to the sale) or the interest cost of borrowing calculated in terms of net usable funds.

It will be observed that the final form of equation 1.5a′ is the same as equation 1.5a except for the additional term. So long as forward commitments are kept low enough to preclude any chance of a shortfall, this final term can be ignored (because then $(\tilde{F} - C)^- = 0$ with certainty). But with any given distribution of $\tilde{F}$, further increases in forward commitments will involve increasingly large chances of shortfalls, and the exiected value of $(\tilde{F} - C)^-$ will become an increasingly large *negative* number. Moreover, since the principal source of the random fluctuations in $\tilde{F}$ about its expected value is the random fluctuations in interest rates about their expected values, this inverse relation and the risk aversion of the companies make it almost certain that significant negative values of $(\tilde{F} - C)^-$ will be associated with upward movements in interest rates when the latter are already at a relatively high level.[30] But at such times, $\tilde{r}_j > \tilde{r}_m$ because the sale of portfolio assets will be taking place in a weak securities market and short-term rates for borrowing will be high and will exceed the open-market rates for purchases of long-term bonds. Indeed, the short-long spread in rates will itself be an increasing function of the movements in the general level of rates.[31]

These considerations lead directly to the conclusion that so long as there is any chance of a shortfall, an allowance for the added costs which would be involved necessarily reduces the expected value of the income stream produced by any forward commitment position. Formally, we have

$$\bar{Y} = E[C\tilde{r}_c + (\tilde{F} - C) r_m] + E[(\tilde{F} - C)^- (\tilde{r}_j - \tilde{r}_m)].$$

The expectation of the first expression within brackets is given by equation

1.6a; the expectation of the second is negative for two compounding reasons. First, the product of the expectations, over all values of $\tilde{F} < C$, of the two terms in parentheses is negative because the first is inherently negative and the second is positive; and the expectation is further reduced by the negative covariance between the first and second terms. Second, the negative expected value of the second bracketed expression falls (becomes negatively larger) at an increasing rate as forward commitment positions are increased in the relevant range under any given set of circumstances. Similarly, it is shown in Appendix A that allowance for the incremental losses of investment income involved in shortfalls of investible funds necessarily increases the variance of the level of investment income involved in any forward commitment position so long as there is any chance of such a shortfall occurring. And once again, the increase in the variance of investment income will grow at an increasing rate as forward commitment positions are progressively raised beyond the point where shortfalls involving penalty costs may occur.[32]

It is obvious from the earlier analysis that even with no risk aversion, the expected value of the investment income associated with any given commitment position will be reduced by the amount of penalty costs ($\tilde{r}_j - \tilde{r}_m$) associated with any shortfall of investible funds—and these reductions in $\bar{Y}$ will, of themselves, reduce the size of the commitment position which would otherwise be optimal in any given set of circumstances. Similarly, because of risk aversion, these increases in the variance of the return would of themselves reduce the commitment position that would otherwise be taken. In fact, both effects operate simultaneously to restrain forward commitment positions in the range where there is any significant chance of a substantial shortfall. Moreover, increasing probabilities of shortfalls involve reductions in the expected value and concurrent increases in the variance of any given commitment position, and both effects increase at an increasing rate. The costs and risks of shortfalls thus induce reductions in optimal forward commitment positions, and those reductions increase at an increasing rate as the probability of a shortfall increases. But for any given expected $\bar{F}$ and probability distribution over investible funds, the chances of a shortfall become larger as C increases (or as the "margin of safety," $\bar{F} - C$, becomes smaller). Consequently, *with a given probability distribution over investible funds* ($\tilde{F}$), *the costs and risks of shortfalls of* $\tilde{F}$ *below prior commitments induce reductions in forward commitment positions (from the levels indicated by equation 1.9b, above) that are both absolutely and relatively greater as the volume of forward commitments (C) approaches the expected inflow of investible funds* ($\bar{F}$), *i.e., as the safety margin between* C *and* $\bar{F}$ *narrows.*

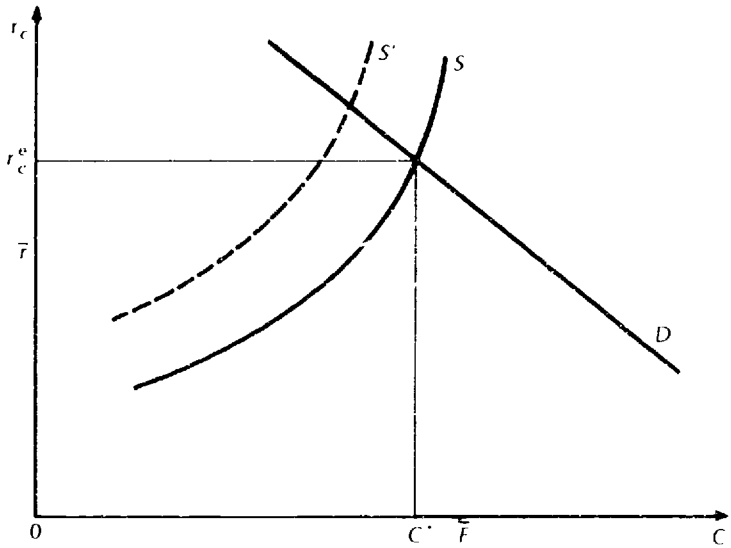
Thus, conclusion ii(a) in the Introduction is established even in the case when the distribution of $\tilde{F}$ is independent of market rates. The analysis in Appendix A shows that these nonlinearities are compounded by allowance for the empirically strong negative dependence of $\tilde{F}$ on $\tilde{r}_m$. After incorporating our earlier analysis, we have established in general that *the reduction in optimal forward commitment levels due to shortfall costs and risks will be absolutely and relatively greater, (i) the higher the expected level of market interest rates; (ii) the greater the (negative) covariance between interest rates and the inflow of investible funds; and (iii) the greater the (ex ante) variance of interest rates and (iv) the greater the conditional variance of investible funds, given interest rates.*

It is important to observe further that explicit allowance for the losses of investment income or incremental costs incurred in the event of a shortfall serves to reinforce my earlier conclusion that even in purely competitive markets, positive commitment premiums are required if there are high ratios of forward commitments in the presence of risk aversion and negative covariances between investible funds and interest rate changes. Even though allowance for shortfall costs and risks reduces the optimal commitment level (C^*) associated with any given $\tilde{F}$, other things equal, it remains true that with any given probability distribution over $\tilde{F}$, C^* will be a rising function of the relevant commitment premium $(r_c - \bar{r})$—just as in the simpler model. However, in the simpler model of subsection 1.2, C^* rose linearly with increases in r_c. We now see that with any given assessment of future market rates $(\bar{r})$ and probability distribution $\tilde{F}$, ever larger increases in r_c will be required to bring forth successive (equal-sized) increments of forward commitments whenever an allowance is made for penalty costs associated with shortfalls of investible funds available from normal sources. Moreover, since these costs and risks make the supply function of forward commitments from each lender concave upward, the aggregate supply of commitments from all lenders in the market will have the same characteristic, as illustrated in Figure 2. My earlier conclusions regarding the existence of a positive equilibrium commitment premium even in purely competitive markets are thus reinforced when an explicit allowance is made for shortfall costs and risks. I therefore revert to the simpler competitive model of section 1.2 as I explore the impact of still other elements of the forward commitment decision.

[1.5] Optimal Policies When the Commitment Rate Must Be Reduced to Increase Volume

To this point, I treated the forward commitment rate available to any one insurance company as a given market datum. But if r_c is treated as a constant that is independent of the commitment level to which it applies, then we are implicitly assuming not only that all insurance companies are price takers in the market but that each company could increase the

FIGURE 2 Supply and Demand Curves of Forward Commitments (With Costs and Risks of Fund Shortfalls Recognized)



volume of its own commitments as much as it might wish without having to cut its rate (or relax the "term and conditions"—which amounts, in effect, to reducing its interest charge). In other words, as in all standard models of freely competitive markets, I assumed that the "demand" curve facing the individual firm is horizontal, even though the aggregate demand curve for commitments from the whole industry (as drawn in figures 1 and 2, above) is a declining function of whatever rate is determined in the market as a whole.

In this subsection, the earlier analysis is modified to allow for less than perfectly elastic demands for forward commitments from individual companies. While the demands for commitments from any given lender will of course be considerably more elastic than the demands facing all lenders as a group, the former almost surely have less than infinite elasticity whenever the borrowers are facing a substantial but still rather limited number of alternative suppliers of funds.[33]

In equation 1.5a above, the rate (r_c) an institution could obtain on the forward commitments it made during any relatively short period of time was fully determined by the supply-demand interactions in the entire commitment market (as described in Section 1.3, above) and, at any given time, was exogenously determined so far as an individual firm was concerned. The firm could make all the commitments it wanted at this rate (so it did not need to lower its quotation), and it could not make any at a higher rate. This simplification can be readily removed by introducing a declining demand curve[34] for each institution's commitments, such as

$$(1.18)\quad r_c = r_0 - (cC/2),\ c > 0,$$

in place of the (hitherto exogenous) r_c in equation 1.5a and then tracing the consequences of the substitution. In particular, expected income now becomes

$$(1.6b)\quad \bar{Y} = [r_0 - (cC/2) - \bar{r}]C + \bar{r}\bar{F} + \sigma_{F,r}$$

but equations 1.7a and 1.7b, for the variances of the return, will be unaffected. Proceeding as before, it is found that the optimal level of C must satisfy

$$(1.8b)\quad r_0 - cC - \bar{r} - \gamma(C - F)V_r + \gamma\bar{r}\sigma_{F,r} = 0.$$

After relation $\sigma_{F,r} = b_{Fr}V_r$ is used (see equation 1.9b, above), the optimal level of forward commitments is given by

$$(1.19)\quad C^* = (F + \bar{r}b_{Fr})\frac{\gamma V_r}{c + \gamma V_r} + \frac{r_0 - \bar{r}}{c + \gamma V_r}.$$

In the limiting case, when c = 0, no price-cutting is required, and the optimal level of commitments here is the same as in (1.9b). The optimal level of forward commitments available from any company will always be reduced by any need to reduce rates in order to get desired business, and its C* will fall as c increases, i.e., as the volume of available commitments becomes more dependent on the rate charged.

Even in the absence of any uncertainties about future fund flows, this dependence of commitment demand upon rate would lead institutional lenders to be less than fully committed, but as long as any company held to the same estimates of the size of c and the other parameters in equation 1.19, the ratio of its commitments to its expected funds (i.e., $C^*/\bar{F}$) would be stable over time. Moreover, for any given expected volume ($\bar{F}$) of investible funds, the volume of its commitments will be reduced by any increase in its expectation of future market rates ($\bar{r}$), or any increase in its uncertainty regarding future rates, or any increase in its assessment of the negative dependence of its available funds on the level of future rates—all as in the simpler cases considered earlier in subsection 1.2.

Because the primary focus of the analysis here is upon the effects on forward commitments of these latter considerations, it will be assumed hereafter that the commitment rate (r_c) is given by the market, independent of the decisions of any single lender. The optimal commitment levels found on the basis of a fixed r_c would be reduced by any appropriate allowance for downward sloping demand curves in exactly the same way as in the case just considered, but this part of the analysis will not be repeated at each step.

[2] INSTITUTIONAL LENDERS RISK AVERSE WITH RESPECT TO THE RATE OF RETURN OR NEW-MONEY RATE

In the previous section, the forward commitment policies of savings banks and life insurance companies were examined under the assumption that their objective was to optimize in terms of the level of the *stream of income* that would be provided by their investments, taking account as risk averters of the uncertainties in their assessments of those income streams. I now analyze the forward commitment decision in parallel fashion under the alternative assumption that the objective of lenders as risk averters is to optimize in terms of the average rate of return—or new-money rate—realized on their investments. Although this new form of the objective function leads to somewhat more difficult mathematics when the amount of investible funds is not well known in advance, *all major qualitative conclusions* of the simpler model already considered are found *to hold*. In particular, changes in uncertainties and the assessment of covariances will all affect the investing institution's forward commitment position in the same direction (up or down) as before. It is also especially noteworthy that with new-money-rate criteria it is again found that, with negative covariances between investible funds and interest rates, companies will be fully committed only if the yield on forward commitments is *greater* than expected future market interest rates. Indeed, I show that with new-money-rate criteria, the excess of commitment rates over expected market rates must be even larger than that required with the earlier investment income criterion to justify any given level of forward commitments relative to expected flows of investible income.

All the definitions of variables and specifications of the forward commitment problem used in section 1 are maintained in this section except that the company's risk-averse utility function is now denominated in terms of the average rate of return on its investments:

- $\tilde{y} \equiv \tilde{Y}/\tilde{F}$, the average rate of return earned on funds disbursed at time t, i.e., the new-money rate.
- $\bar{y}$, $V_y \equiv$ the expected value and variance of $\tilde{y}$.
- $h \equiv C/\bar{F}$, the fraction of total funds expected to be available for investment and which are committed in advance.
- $U(\tilde{y}) \equiv$ the company's (risk-averse) utility function over the average rate of return on all funds disbursed at time t.

Just as it was appropriate in section 1 to analyze the behavior of lenders concerned with the level of their investment income in terms of some given level of absolute risk aversion, I now analyze the behavior of lenders concerned with *rates* of return in terms of some given level of *proportionate* risk aversion. Although the intensity of any given lender's proportional risk aversion (and his assessments of the distribution of $\tilde{y}$) may vary over time or as a function of wealth levels, changes in the degree of relative risk aversion will in practice be of second order in terms of the purposes of the present analysis, and I simply assume that each lender acts at each point in time in terms of a given proportionate risk aversion, λ (which is likely, however, to have a different value for different companies or banks).

In sections 2.2, and 2.3, I analyze the optimal forward commitment decisions of lenders with constant proportional risk aversion in terms of the means and variances of their overall rate of return, $\tilde{y}$. In this context, decisions to maximize expected utility, $E[U(\tilde{y})]$, can readily be shown to be those which are optimal in terms of the simpler[35]

(2.2) *criterion*: choose h^* to obtain max $W(\bar{Y}, V_y)$

where the welfare index, $W(\bar{Y}, V_y)$, is

(2.2) $W = \bar{Y} - \lambda V_y/2,$

and $\lambda + 1 > 0$ is the relevant measure of constant proportional risk aversion, with larger values indicating greater risk aversion: a higher expected return will be required to justify any action involving any given increase in risk. Given the functional relation of $\bar{Y}$ and V_y, respectively, with h, optimal h^* will be that value for which

$$\frac{\partial W \partial \bar{y}}{\partial \bar{y} \partial h} + \frac{\partial W \partial V_y}{\partial V_y \partial h} = 0,$$

and using (2.2), this simplifies to

$$(2.3) \quad \frac{\partial \bar{y}}{\partial h} - \frac{\lambda}{2}\frac{\partial V_y}{\partial h} = 0.$$

[2.1] The Commitment Decision When There Is No Uncertainty Regarding $\tilde{F}$

If the amount of investible funds is known in advance, and the only uncertainty involves the interest rate ($\tilde{r}$) that will be realized on the funds remaining available for spot investment after taking care of commitments, the uncertain new-money rate will be given by

$$(2.4) \quad \tilde{y} = \frac{\tilde{Y}}{F} = \frac{Cr_c + (F - C)\tilde{r}}{F} = hr_c + (1 - h)\tilde{r},$$

and the expected values ($\bar{y}$) and variance (V_y) of $\tilde{y}$ are,

$$(2.5) \quad \bar{y} = \bar{r} + h(r_c - \bar{r})$$

and

$$(2.6) \quad V_y = (1 - h)^2 V_r.$$

Differentiating these expressions and substituting into (2.3), we find that optimal h^* must satisfy

$$r_c - \bar{r} + \lambda(1 - h)V_r = 0,$$

which simplifies to

$$(2.7)\quad h^* = 1 - \frac{\bar{r} - r_c}{\lambda V_r} = 1 + \frac{r_c - \bar{r}}{\lambda V_r}.$$

Equation 2.7 may be compared with (1.9). It is immediately apparent that when expected interest rates are equal to current commitment rates and there is *no uncertainty over future investment funds,* companies with a new-money-rate objective will be fully invested just as will those with an investment income objective. Moreover, the forward commitment position of any lender will respond in the same way as before to changes in current commitment rates and/or expected future market rates. Finally, the extent of the adjustment in commitments from any given change in expectations of market rates also continues to be smaller, the more uncertain the company feels about its projections and the greater its risk aversion.

I now introduce uncertainty regarding the amounts of investible funds and show that not merely do the earlier conclusions continue to be valid, but that emphasis on new-money-rate performance serves to strengthen those earlier results.

[2.2] The Commitment Decision When the Amount of Investible Funds Is Also Uncertain (Preliminary Analysis)

When companies have a new-money-rate objective and both the amount of funds ($\bar{F}$) available for investment and the future market rate ($\bar{r}$) are uncertain, the uncertain new-money-rate is given by

$$(2.8)\quad \bar{y} = \frac{\bar{Y}}{\bar{F}} = \frac{Cr_c + (\bar{F} - C)\bar{r}}{\bar{F}}$$
$$= \bar{r} + (C/\bar{F})(r_c - \bar{r})$$

or

$$(2.9)\quad \bar{y} = \bar{r} + h\bar{w},$$

where, as before,

$$(2.10)\quad h = C/\bar{F},$$

i.e., the ratio of the amount of commitments to the amount of funds expected to be available at takedown; and now

$$(2.11)\quad \tilde{w} = \frac{r_c - \bar{r}}{\bar{F}/\bar{F}},$$

i.e., the ratio of (i) the uncertain difference between the known rate of return on commitments and the uncertain market rate available at the time of takedown, to (ii) the ratio of the uncertain amount of investible funds to their expected value.

In order to get a more explicit indication of the interactions involved, let ϵ represent the random deviations of realized market rates from their expected values

(2.12) $\tilde{\epsilon} = \tilde{r} - \bar{r}$,

and let $\beta > 0$ represent the negative slope coefficient or (standardized) covariance between $\tilde{\epsilon}$ and the percent deviations of $\tilde{F}$ around $\bar{F}$:

(2.13) $\tilde{F}/\bar{F} = 1 - \beta\tilde{\epsilon}$.

Also, to simplify notation, let x_0 represent the excess of commitment rates over the expected value of market interest rates:

(2.14) $x_0 = r_c - \bar{r}$.

Then

(2.11a) $$\tilde{w} = \frac{r_c - \bar{r} - \tilde{\epsilon}}{1 - \beta\tilde{\epsilon}} = \frac{x_0 - \tilde{\epsilon}}{1 - \beta\tilde{\epsilon}}.$$

When F and $\tilde{r}$ are negatively correlated and companies use a new-money-rate criterion, the analysis of optimal forward commitment positions becomes more complex. In section 1.2, the covariances between $\tilde{F}$ and $\tilde{r}$ entered the expressions for Y and V_y directly and affected the optimal level of forward commitments (C^*), given $\bar{F}$, in a simple linear fashion (see equations 1.6a, 1.7a, and 1.9a). When the new-money-rate objective is used, however, equations 2.8 and 2.11 show that the negative covariance ($\sigma_{F,r}$) involves an interaction between the numerator and denominator of $\bar{w}$. In consequence, its effect upon $\bar{y}$ and thereby upon the optimal commitment ratio ($h^* = C^*/\bar{F}$) becomes nonlinear and variable.[36] Indeed, with this new-money-rate criterion, the optimal commitment ratio will also respond to most other changes in assessments and conditions in a variable, nonlinear fashion. Moreover, these nonlinearities are introduced into the reactions even when the basic uncertainties are assessed to be perfectly symmetrical—i.e., when equally large deviations above and below the expected values of market interest rates or the expected amounts of investible funds are thought to be equally likely. Finally, it may be observed from equations 2.8 and 2.11 that even when $\tilde{r}$ and $\tilde{F}$ have perfectly symmetrical distributions about their own means, the distributions of the "payoff" variables, y and w, will not be symmetrically distributed about their expected values, $\bar{y}$ and $\bar{w}$. Indeed, as I show below, the distributions of $\tilde{y}$ and $\tilde{w}$ will be negatively skewed, i.e., the probabilities will be considerably greater that the new-money rate will fall short of its expected value than that it will be greater than expected; and correspondingly, the odds will be better than even that $\tilde{w}$ will fall short of $\bar{w}$.

In view of these considerations, when the amount of investible funds is uncertain and a new-money-rate criterion is used, it is desirable to supplement our analysis of optimal forward commitment policies in the usual mean-variance context with an analysis that explicitly allows for skewness preference as well as risk aversion in the usual sense. Because of the complexity of the interactions involved, in the rest of this section I work through a relatively simple illustration in some detail in order to lay bare the economic rationale of the results of the following general analysis. In section 2.3 I then develop the formal analysis of optimal forward commitment policies, when F is uncertain and new-money-rate criteria are used, in a mean-variance context. Specifically, I find h^*, the forward commitment position that will maximize $E[U(\tilde{y})]$ in equation 2.2 and thereby satisfy equation 2.3. The results obtained in this mean-variance context are of interest because they can be readily compared with those obtained in sections 1 and 2.1, which were all also in a mean-variance context. Finally, since even symmetrical assessments of $\tilde{r}$ and $\tilde{F}$ now introduce asymmetries in the distribution of rates of return ($\tilde{y}$), in section 2.4, I also allow for investor preferences with respect to the skewness (i.e., third moment) of $\tilde{y}$, along with their preferences regarding its mean and variance. In this broader context, decisions to maximize expected utility $E[U(\tilde{y})]$ are identical to those which are optimal under the following:

(2.15) *criterion:* choose h^* to maximize $W(\bar{y}, V_y, S_y)$

where V_y and S_y are the respective variance and skewness (third central moment) of $\tilde{y}$, and the welfare index $W(\bar{y}, V_y, S_y)$ is

(2.16) $W = \bar{y} - \lambda V_y/2 + \phi S_y/6,$

where[37] $\lambda > 0$ measures the degree of risk aversion, and $\phi > 0$ measures the degree of preference for positive skewness (distribution of outcomes "stretched" toward higher values) and/or the degree of dispreference for smaller positive—or larger negative—outcomes).[38]

As already suggested, the *size* of the reaction of forward commitments to any change in circumstances will generally be somewhat different in these two formulations of the new-money-rate criterion when investible funds are uncertain, and they will also differ somewhat in size from the effects found earlier (section 2) when an investment income objective was assumed.[39] *Nevertheless*, more detailed analysis of both these new-money-rate models shows that the direction of the effect of any change in conditions on the level of forward commitments (i.e., the sign of the derivative of h^* with respect to each variable) is still the same whether or not skewness preference is allowed for—and also whether the objective is taken to be the new money rate or the level of the stream of investment income produced (as in section 1).

Consequently, conclusions ii, iii, and iv of the Introduction are re-established in both new-money-rate models, and the "reinforcing" conclusion (v) is also proved. In the rest of this subsection, I illustrate these conclusions in ordinary language with the aid of a simple numerical example. Rigorous proofs of the generality of the results are given in subsections 2.3 and 2.4.

The *economic rationale* underlying the foregoing analytical results builds upon the combination of two basic factual considerations: Changes in market rates affect not only (i) the yields obtainable in the future on the direct investment of funds not committed in advance, but also (ii) the total amount of funds available for investment at the time the funds on forward commitments are to be disbursed. If market rates ease over the commitment period, the supply of funds for investment will almost surely be larger than if rates had held steady. Correspondingly, if market rates should increase sharply over the commitment period, the induced increases in policy loans and reductions in prepayments on outstanding mortgages will reduce the supply of funds available for investment.

The important interactions between these two effects can be brought out by considering the very simple situation in which current commitment rates are equal to expected future market rates (and all forward commitments have the same takedown period). Suppose that a company in this situation makes forward commitments equal to X percent of the total amount of funds it expects to have available for investment at the time of takedown. It could of course have committed less than X percent forward now and had a correspondingly larger amount of funds remaining at the end of the takedown period for direct investment at the market rate prevailing at that time. If market yields *have* declined meanwhile, the larger commitment would look good with hindsight because some funds will have been invested at the higher rates available when the commitment was made, but the fall in rates will have made the amount of funds available for investment larger than anticipated. Because of this negative correlation between available funds and market rates, the funds committed forward turn out to be less than X percent of the actual supply of investment funds and the fraction of high-yielding investments is smaller than expected. Correspondingly, if interest rates increase between the time commitments are made and taken down, forward commitments will have incurred the opportunity loss of the subsequent rise in yields. In addition, the rise in market yields will have made the amount of funds available for investment smaller than expected, with the result that the (relatively) low-yielding investments will turn out to represent a larger fraction of the total funds used to compute the new-money rate. Combining the two possibilities, it is apparent that if commitment rates are equal to expected future market rates, the relative gains in new-money rates obtained by committing forward before interest rates decline are always (and may be substantially) smaller than the losses incurred by having committed forward before interest rates increase. However, any excess of currently available commitment rates (including fees) over expected future market rates will of course improve the new-money-rate performance of any forward commitment position. We thus have the general proposition: *Unless current commitment rates (and fees) are sufficiently higher than the market rates expected at time of takedown, the expected new-money rate will fall as forward commitment positions are increased.*[40]

A simple numerical example will illustrate the countervailing forces just described and confirm their negative net effect on expected new-money rates as forward commitment positions are extended in the absence of sufficient premiums in rates on commitments. Suppose that commitments with a nine-month takedown can be made now at an 8.5 percent yield, and that current judgments are that it is equally likely that market yields on direct investments will either go up to 9.5 percent or stay at 8.5 percent or fall to 7.5 percent by the end of the nine-month period. At that point, the company expects to have $100 million to invest, as it will if rates turn out to be 8.5 percent; but if interest rates fall to 7.5 percent it will wind up having to invest $120 million, and if rates go up to 9.5 percent it will have only $80 million available. The results of alternative decisions to commit forward 0, $20 million, $40 million, $60 million, or $80 million of the expected $100 million of investible funds are given in Table 1. The dollar amounts committed forward at an 8.5 percent yield are shown in column 1; and columns 2, 4, and 6 give the amounts of funds remaining for investment at the respective later market rates of 7.5 percent, 8.5 percent, or 9.5 percent. The new-money rates for the resulting investment positions conditional on the level of the later market rate are given in columns 3, 5, and 7. Finally, in column 8, the average or expected value of the new-money rate is shown, with equal weight given to the three outcomes thought to be equally likely. (The remaining columns will be considered later.) It is apparent from this example that when (or if) the new commitment rate is the same as the expected value of the future market rate, larger forward commitment positions produce progressively lower expected (average) new-money rates (but with the benefit to risk-averse investors of lower risk, as explained below).

Why then do lenders relying on a new-money-rate criterion heavily use forward commitments in their investment operations? One reason is clearly that the yields available on forward commitments have been higher than the rates currently available in the open market—and also higher than the open-market rates expected at the time of takedown.[41] And Table 2 illustrates the general principle that when *forward commitment yields are*

TABLE 1 Expected Values, Standard Deviations, and Skewness[a] of the New-Money Rate for Different Levels of Forward Commitments When $r_c = \bar{r} = 8.5$ Percent (dollars in millions)

	If $\tilde{r}$ = 7.5 %		If $\tilde{r}$ = 8.5%		If $\tilde{r}$ = 9.5%		Descriptive Measures of New-Money Rate (y)			
Funds Committed at 8.5% (1)	Funds Invested at 7.5% (2)	New-Money Rate (3)	Funds Invested at 8.5% (4)	New-Money Rate (5)	Funds Invested at 9.5% (6)	New-Money Rate (7)	Expected (Average Value ($\bar{y}$) (8)	Standard Deviation (σ_y) (9)	Skewness s_y (10)	Skewness s_y'' (11)
$ 0	$120	7.500%	$100	8.5%	$80	9.50%	8.500%	.817%	0	0
20	100	7.667	80	8.5	60	9.25	8.472	.647	−.257%	−.063
40	80	7.833	60	8.5	40	9.00	8.444	.478	−.265	−.170
60	60	8.000	40	8.5	20	8.75	8.417	.312	−.227	−.384
80	40	8.167	20	8.5	0	8.50	8.389	.157	−.140	−.707

NOTE: In the notation of equations 2.9–2.14 and 2.17, this table is based on the following assumptions: $x_0 = 0$; $\beta = 0.2$; and $e = 1$ percent (which with $p = 1/3$, implies $V_e =$ 0.66 percent). The values of x_0 and e are purely illustrative, but the value of β approximately equals the (negative) slope coefficient (0.21) found on the interest rate term (in percent units) in a regression on LIAA industry aggregate data (in millions of dollars) fitted quarterly after allowing for a quadratic time trend for the years 1965–1970. In Table 2, I illustrate the effect of raising x_0 to 0.5; a general analysis of the effects of changing each of the underlying parameters is given in the text.

[a] In column 9, $\sigma_y = (V_y)^{1/2}$; and in column 10, in percent units, $s_y = (S_y)^{1/3}$, where S_y is the third central moment of y. This serves to make the measures of expected values in column 8 and of dispersion and skewness all comparably of the first order. Support for the use of these measures in a portfolio–market equilibrium context is found in Kraus and Litzenberger (1972). In column 11, $s_y'' = S_y/\sigma_y^3$, which is the usual nondimensional measure of skewness, reflecting shifts in the shape of the distribution.

TABLE 2 Expected Values, Standard Deviations, and Skewness[a] of the New-Money Rate for Different Levels of Forward Commitments, When r_c = 9.0 Percent and $\bar{r}$ = 8.5 Percent (dollars in millions)

	If $\bar{r}$ = 7.5%		If $\bar{r}$ = 8.5%		If $\bar{r}$ = 9.5%		Descriptive Measures of New-Money Rate (y)			
Funds Committed at 9.0% (1)	Funds Invested at 7.5% (2)	New-Money Rate (3)	Funds Invested at 8.5% (4)	New-Money Rate (5)	Funds Invested at 9.5% (6)	New-Money Rate (7)	Expected (Average) Value ($\bar{y}$) (8)	Standard Deviation (σ_y) (9)	Skewness s_y (10)	Skewness s''_y (11)
$ 0	$120	7.50%	$100	8.5%	$80	9.50 %	8.50%	.816%	0	0
20	100	7.75	80	8.6	60	9.375	8.575	.664	−.255%	−.056
40	80	8.00	60	8.7	40	9.250	8.650	.512	−.269	−.144
60	60	8.25	40	8.8	20	9.125	8.725	.361	−.242	−.302
80	40	8.50	30	8.9	0	9.000	8.800	.216	−.182	−.595

[a]See notes to Table 1.

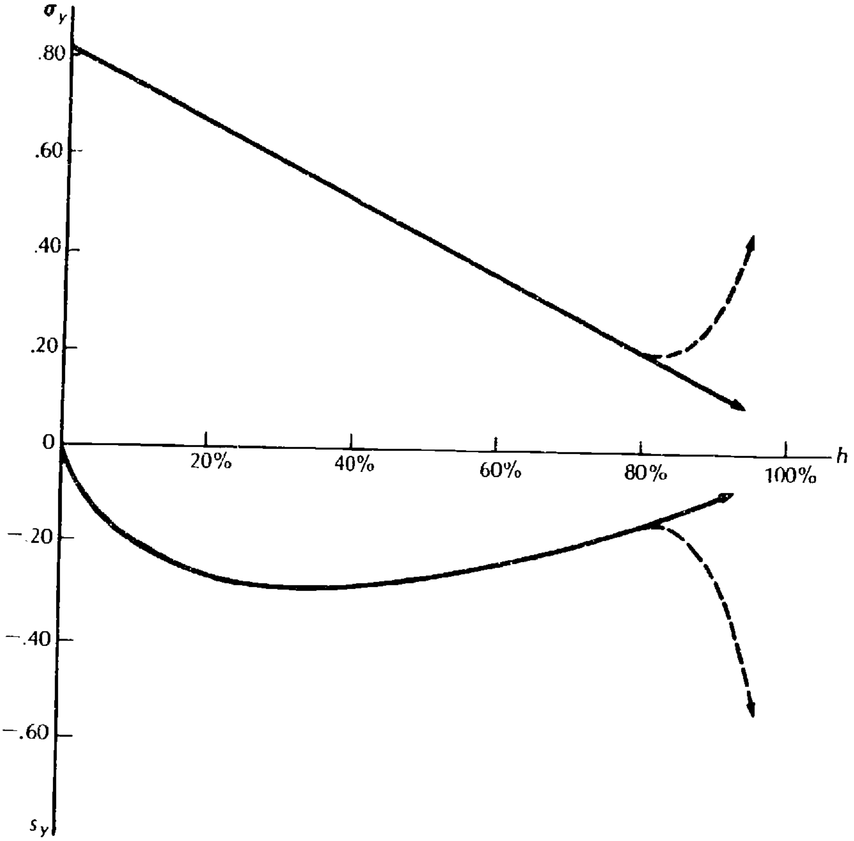
sufficiently higher than expected future open-market rates, the *expected new-money rate will increase,* rather than decrease, as the *forward commitment* position increases. Just how much premium makes commitment yields enough higher to prevent expected new-money rates from falling as forward commitment positions are extended of course depends on other conditions—as analyzed algebraically below.

Another reason why institutional lenders commit forward rather heavily even at times when expected new-money yields are decreased or are unaffected by such action[42] is that (as discussed later) the *dispersion of new-money rates is progressively reduced by increases in forward commitment positions up to the point where risks of absolute shortfalls of investible funds become important.* This reduction in the dispersion of new-money rates is produced by the negative covariance between interest rates and investible funds. Since the risks of shortfalls are generally quite small until commitments reach relatively high levels, the relation between new-money-rate dispersion and commitment levels over all commitment levels up to perhaps 80 percent can be determined to a good approximation without considering the added effect of possible shortfalls; column 9 in tables 1 and 2 has been calculated on this basis. As illustrated there and clarified in the notes to Table 1, this reduction in σ_y as forward commitments are increased up to high levels is found whether expected new-money rates are rising or falling with commitments. Indeed, over this range, increased commitments reduce the uncertainties regarding new-money rates somewhat more rapidly at times when expected new-money rates will be reduced rather than increased by larger commitment positions. Since all the evidence indicates that institutional lenders are risk-averse investors, the reduction in the risks and uncertainties of realized rates of return resulting from heavier commitment positions serves as a pervasive and strong stimulus for those lenders to increase their forward commitment positions, at least up to the level where shortfall considerations become important.

The stimulus to forward commitments as a means of risk reduction has of course been strong in all the models developed in this paper. As explained in sections 1.1 and 2.1, above, the essential reason why lenders would be fully committed (even with no commitment premium) if they knew in advance the amount of investible funds they would have available at takedown time is that they could then reduce their uncertainties regarding investment returns when $C \leq F$ by adding to their commitment positions. We now see that the same risk-reducing effect on new-money rates is importantly reinforced when there is a strong negative effect of increased interest rates on amounts of investible funds up to the commitment level, where the probabilities of shortfalls of investible funds become significant. A review of section 1.4 above, however, shows that the dispersion of the new-money rate will begin to turn up at increasing rates beyond the point where these "fund shortfall costs and risks" become important. The dispersion risks in terms of which institutional lenders make their decisions has the latterly rising pattern of the dashed line in Figure 3 (rather than the solid line which depicts the data in column 9 of Table 1; in the latter, shortfall sources of dispersion are ignored).

But when institutional lenders judge performance by new-money rates and allow for the negative dependence of investible funds on uncertain future interest rates, there is more to the story. (For simplicity, shortfall effects are for the moment again ignored.) As illustrated in the last columns

FIGURE 3 Dispersion and Skewness of New-Money Rates As a Function of Forward Commitment Position Relative to Expected Investible Funds



σ_y = standard deviation of new-money rates.
s_y = skewness of new-money rate.
h = forward commitments (C) as a percent of the expected supply of investible funds ($\bar{F}$).
SOURCE: Solid line in upper quadrant: Table 2.1, column 9. Solid line in lower quadrant: Table 2.1, column 10. Dashed lines show results when allowance is made for added costs and risks of shortfalls of investible funds.

of tables 1 and 2, the usual statistical measure of skewness (see notes to Table 1) applied to the new-money rates in our examples is negative and becomes progressively more so at an increasing rate as forward commitment positions are increased. This statistic is a normalized measure which, when negative, indexes the extent to which the probabilities of shortfalls from expected values exceed the chances of better-than-expected outcomes. Institutional lenders are not only risk-averse investors but they also have a marked *dispreference for negative skewness*. Other things equal, they would clearly prefer investments offering greater chances of higher returns to those on which the betting odds favored returns below expected values. Forward commitments shift the odds toward shortfalls in realized returns ($\tilde{y}$), and this fact must also be recognized as an important consideration in decisions on appropriate forward commitment levels.

The *relative* importance of this consideration, however, will depend on the strength of the dispreference for negative skewness (measured by ϕ, derived from the utility function); on how much the odds are tilted or stretched toward shortfalls; and, also, on how large the negative shortfalls are likely to be. We have already seen that since forward commitments are made at a known yield, increasing commitments serves to reduce the share of investible funds subject to uncertainties in rates. Even though an increase in forward commitments tilts the relative odds toward shortfalls in new-money rates, it also reduces the relative amount of funds subject to either higher or lower returns than expected.

These two countervailing skewness effects are combined in column 10 of each table. In the context of a portfolio or investment position, the figures represent the downward pressure on forward commitments arising from the negative asymmetries in the distribution of new-money rates, given the dispreference of lenders for negative skewness. The data indicate that the downward pressure builds up until forward commitments have reached a moderate level and that commitments above that level are associated with reduced levels of "skewness disutility." We found earlier that institutional lenders, being risk averters, would be stimulated to take higher commitment positions by the opportunity of reducing the dispersion of new-money rates. Similarly, the reduction in the average net shortfalls involved in negative skewness of returns, also disliked, would stimulate lenders to take higher forward commitment positions so long as s_y took on smaller negative values as commitment levels were increased. But the impact on returns of any possible shortfall of investible funds ($\tilde{F}$) below their expected level ($\bar{F}$) has, for simplicity, been left out of the declining pattern in Table 1. Once again, a review of section 1.4 will show that allowance for the added costs involved in such shortfall risks will increase the negative skewness of the distribution of realized new-money rates (or rates of return, $\tilde{y}$). Moreover, since the chance of a shortfall becomes significant as commitments increase beyond some reasonably high level (relative to $\bar{r}$)—and indeed increase at an increasing rate—the pattern of s_v, which actually affects commitment decisions, will decline when commitment levels range between about 33 percent to 80 or 90 percent of expected funds, and will thereafter rise rapidly[43] as indicated by the sashed line in Figure 3.

The particular level of forward commitments that will be best for any given institutional lender in any given set of circumstances of course will depend not only on prevailing conditions and on the particular set of assessments used, but also and very importantly, on the degree of risk aversion of the lender and on how much it is willing to give up in expected new-money yields in order to reduce the undesired downtilt in the distribution of its uncertain new-money rates. The numerical illustration can do no more than lay bare some of the more important currents and counterforces at work, and provide a concrete setting for my verbal explanation and discussion of the significant influences that bear on the final choices made.

In the rest of this section I consequently provide a more detailed and more general mathematical analysis of optimal forward commitment positions when the new-money-rate criterion is applied and there is a significant negative covariance between interest rates and the amount of investible funds.[44] Using equations 2.8–2.14, I focus on a *symmetric* three-point distribution of possible interest rates,[45] in which there is a

$$(2.17)\quad \begin{cases} 100\,p\% \text{ chance of } \tilde{\epsilon} = +e \\ 100\,p\% \text{ chance of } \tilde{\epsilon} = -e, \text{ and a} \\ 100\,(1-2p)\,\% \text{ chance of } \tilde{\epsilon} = 0, \end{cases}$$

where $p < 0.5$ and $\tilde{\epsilon} = r - \bar{r}$, as shown in equation 2.12, and e is the absolute number of units of displacement of $\tilde{\epsilon}$ from zero.[46] With these distributions, we now have the variance of the interest rate given by

$$(2.18)\quad V_r = 2pe^2.$$

Also the expected value of $\bar{w}$ in (2.11a) is given by

$$(2.19)\quad \bar{w} = p\left(\frac{x_0 - e}{1 - \beta e}\right) + (1 - 2p)x_0 + p\left(\frac{x_0 - e}{1 + \beta e}\right)$$

$$= x_0 - 2p\beta e^2 K = x_0 - \beta K V_r$$

where

$$(2.20)\quad K \equiv \frac{1 - \beta x_0}{1 - \beta^2 e^2} > 0;$$

x_0 is the commitment premium ($r_c - \bar{r}$) as in equation 2.14: and $\beta > 0$ represents the *negative* slope coefficient (standardized covariance) between the percent changes in investible funds and market interest rates, as in equation 2.13 above. Finally, the covariance between w and r is found to be

(2.21) $\sigma_{wr} = E(\tilde{w} - \bar{w})(\tilde{r} - \bar{r}) = E[\tilde{e}(\tilde{w} - \bar{r})] = -KV_r,$

and the variance of w,

(2.22) $V_w = E(\tilde{w} - \bar{w})^2 = K^2 V_r M$

where

(2.23) $M \equiv 1 + (1 - 2p)\beta^2 e^2 > 1; p \leq 0.5.$

From equation 2.9, we also have

(2.24) $\bar{y} = \bar{r} + h\bar{w},$

and it may be noted that the expected value of new-money rates will decline (or rise) with increasing forward commitment positions as long as $\bar{w} <$ (or $>$) 0. From (2.19) and (2.24) we thus have

(2.25) $\frac{\partial \bar{y}}{\partial h} > 0 \text{ as } \bar{w} \gtreqless 0$

and

(2.25a) $\bar{w} \gtreqless 0 \text{ as } x_0 \gtreqless \frac{\beta V_r}{1 - \beta^2(e^2 - V_r)} = \frac{2pe^2\beta}{1 - (1 - 2p)\beta^2 e^2}$

or

(2.25b) $\bar{w} \gtreqless 0 \text{ as } x_0 \gtreqless \frac{\beta V_r}{2 - M}$

Observe that the critical value of the commitment premium, $x_0 = r_c - \bar{r}$, increases (or decreases) with the (negative) covariance indexed by β.[47]

For simplicity, I ignore the effects of shortfall risks[48] on the variance of new-money rates (V_y) until later. From (2.9), we have directly

(2.26) $V_y = V_r + h^2 V_w + 2h\sigma_{wr}.$

[2.3] Optimal Forward Commitments with Uncertain $\bar{F}$ under a Mean-Variance Criterion for New-Money Rates

I now derive the optimal commitment position, using the mean-variance criterion in equation 2.2, by finding the value of forward commitments (h) that will satisfy

(2.27) $\frac{\partial W}{\partial h} = \frac{\partial \bar{y}}{\partial h} - \frac{\lambda}{2} \frac{\partial V_y}{\partial h} = 0.$

When the values of $\bar{y}$ and V_y from (2.24) and (2.26) are substituted, and derivatives are taken, it is found that

(2.28) $\partial W/\partial h = \bar{w} - \lambda(hV_w + \sigma_{wr}) = 0,$

with the result that[49] the optimal forward commitment position is given by[50]

$$(2.29)\quad h_q^* = \frac{\bar{w} - \lambda\sigma_{wr}}{\lambda V_w}.$$

This expression may be written in more transparent form by substituting (2.18)–(2.23), which show that

$$(2.29a)\quad h_q^* = \frac{x_0 - K\beta V_r + \lambda K V_r}{\lambda K^2 V_r M} = \frac{\bar{w}}{\lambda V_w} - \frac{1}{KM}$$

$$= \frac{1}{KM}\left[\frac{x_0}{\lambda K V_r} + (1 - \beta/\lambda)\right]$$

$$= \frac{x_0}{\lambda V_w} + \frac{1}{KM}[1 - \beta/\lambda].$$

We first observe that in the limit, as the covariance β between available funds and interest rates approaches zero, these expressions for h^* reduce to equation 2.7, so that firms using this rate-of-return criterion will be fully invested (in the absence of penalty costs on shortfalls) when commitment rates are non-negative and $\beta = 0$.[51] But when the commitment premium (x_0) is zero and $\beta > 0$ [so that by (2.13), there is a negative covariance between available funds and interest rates], the companies will be *less* than fully invested, as before, since the first term in the last line is then zero by assumption, and the β value will reduce the rest of the expression below unity.[52] Moreover, other things equal, forward commitment positions vary inversely with the intensity of the risk aversion of the lenders (see Note 3 in Appendix B), and as β rises (reflecting more negative covariance of $\bar{F}$ with $\bar{r}$) optimal commitment positions will be further reduced. Optimal forward commitment positions clearly vary directly with the expected commitment premium ($x_0 = r_c - \bar{r}$); and the amount of added commitment premium required to raise h^* by any given amount will be greater: (i) the greater the uncertainty of interest rates (V_r), (ii) the greater the negative covariance (β) between investible funds and interest rates, (iii) the greater the lenders' proportionate risk aversion (λ), and (iv) the higher the level of the expected market rate (see Note 4 in Appendix B).

These conclusions for institutional lenders pursuing new-money-rate objectives are based on the mean-variance equation (2.29a); they are, of course, qualitatively the same as those obtained more simply from the investment income model in subsection 1.2. In addition, since the supply of forward commitments of institutional lenders pursuing new-money-rate objectives is simply $C^* = h_q^*\bar{F}$, the general conclusions of the analysis in section 1.3 continue to hold. In particular, whenever the market clearing demand for commitments approaches $\bar{F}$, the equilibrium values of r_c and $\bar{r}$ in purely competitive markets for forward commitments must satisfy the inequality $r_c > \bar{r}$; and this is true even if no allowance is made for the penalty costs that are generally involved ex ante in possible shortfalls.

Finally, in this mean-variance new-money-rate model, any allowance for increased variances in new-money rates due to the risks and costs of shortfalls at high commitment levels, i.e., use of a variance function that behaved like the dashed line (rather than the solid line) in Figure 3—would simply reduce optimal commitment positions below the levels they otherwise would have had, so long as commitment levels were high enough anyway for lenders to begin to incur shortfall risks.[53] The expected value of the new-money rate would also be lowered by the added costs associated with such shortfalls. But although those costs and added variances would lower the levels of forward commitment positions, ceteris paribus, the conclusions regarding the effects on optimal h_q^* resulting from changes in the levels of β, x_0, and λ are all unchanged (see Note 5 in Appendix B). Moreover, the ceteris are not paribus, and allowance for the risks and costs associated with shortfalls requires that the purely competitive market equilibrium rate on forward commitments (r_c) be still higher (relative to the expected future rate, $\bar{r}$, on new direct investments).

[2.4] Optimal Forward Commitments When the Amount of Investible Funds Is Uncertain and the Skewness Preference on New-Money Rates Is Taken Account of

I first establish that even with symmetrical probability distributions of interest rates ($\tilde{r}$), and of investible funds ($\tilde{F}$) conditional on interest rates, the new-money rate ($\tilde{y}$) will be subject to negative skewness throughout the relevant region when the covariance $\sigma_{Fr} < 0$. I then prove the impact of such skewness on the optimal forward commitment positions of investors who are not only risk averse in the usual sense but also have a dispreference for negative skewness.

From equation 2.9, the third moment (which determines the skewness) of $\tilde{y}$ is given by

$$S_y \equiv E[\tilde{r} + h\tilde{w} - (\bar{r} - h\bar{w})]^3$$

$$\equiv E[\tilde{e} + h(\tilde{w} - \bar{w})]^3$$

$$\equiv 3hA + 3h^2B + h^3S_w \qquad \text{using (2.12)} \tag{2.30}$$

where E represents mathematical expectation (note that for any symmetrical distribution of $\tilde{r}$, $E(\tilde{e}) = 0$ and also $E(\tilde{r} - \bar{r})^3 = E(\tilde{e})^3 = 0$, and is omitted from the final statement above), and

$$A = E(\tilde{e}^2\tilde{w}) - \bar{w}E(\tilde{e}^2) = E(\tilde{e}^2w) - \bar{w}V_r \tag{2.31}$$

(2.32) $B = E(\tilde{e}\tilde{w}^2) - 2\bar{w}E(\tilde{e}\tilde{w}) + \bar{w}^2E(\tilde{e})$

$= E(\tilde{e}\tilde{w}^2) - 2\bar{w}\sigma_{wr} + 0$

and

(2.33) $S_w \equiv E(\tilde{w} - \bar{w})^3.$

For the representative symmetrical probability distribution in (2.17), it may be shown that the values of A, B, and S_w in equations 2.30–2.33 are as follows:

(2.31a) $A = -\dfrac{2p(1-2p)\beta e^4(1-\beta x_0)}{(1-\beta^2e^2)}$

$= -KV_rG < 0;$ using (2.18) and (2.20)

where

(2.34) $G \equiv (1-2p)\beta e^2 > 0;$

(2.32a) $B = \dfrac{4p(1-2p)\beta e^4(1-\beta x_0)^2}{(1-\beta^2e^2)^2}$

$= -2AK = +2K^2V_rG > 0.$ using (2.31a)

(2.33a) $S_w = \dfrac{2p(1-2p)\beta e^4(1-\beta x_0)^3}{(1-\beta^2e^2)^3}[3+(1-4p)\beta^2e^2]$

$= AK^2[3+(1-4p)\beta^2e^2] < 0$

$= -V_rGK^3H < 0.$ using (2.31a)

where

(2.35) $H = 3 + (1-4p)\beta^2e^2 > 0.$

Consequently, inserting (2.31a)–(2.33a) into (2.30), we have

(2.30a) $S_y = hA(3 - 6hK + h^2K^2H)$

$= -hKV_rG[3(1-hk)^2 + h^2K^2(1-4p)\beta^2e^2].$

As shown just below, the optimizing conditions depend critically on the sign of S_w. It is important to observe that S_w is inherently negative for all symmetrical distributions of interest rates involving three (or more) possible outcomes,[54] and S_y will also be negative for all relevant values of all other variables (see Note 6 in Appendix B). But since we have not yet introduced the added negative skewness that is induced at high levels of h by the costs and chances of shortfalls,[55] S_y in equation 2.30a will become less negative as h increases beyond relatively low levels (see Note 7 in Appendix B). (I return to this matter later.)

With this preparation, values for $\bar{y}$, V_y, and S_y from (2.24), (2.25), and (2.30a) can be substituted into the objective function 2.16 and derivatives taken with respect to h to determine the optimal forward commitment position (h^*) when the decision reflects dispreference for negative skewness as well as risk aversion in the usual sense:

$$(2.36)\quad \frac{\partial W}{\partial h} = \frac{\partial \bar{y}}{\partial h} - \frac{\lambda}{2}\frac{\partial V_v}{\partial h} + \frac{\phi}{6}\frac{\partial S_v}{\partial h} = 0$$

$$= \bar{w} - \lambda(hV_w + \sigma_{wr}) + \phi[A + 2hB + h^2 S_w]/2 = 0.$$

Equation 2.36 is a quadratic equation in h which may be written

$$(2.37)\quad ah^2 - bh + c = 0$$

where

$$(2.37a)\quad a = \phi S_w/2 = -\phi K^3 V_r GH/2 < 0 \qquad \text{using (2.33a)}$$

$$(2.37b)\quad b = \lambda V_w - \phi B = \lambda V_w - 2\phi K^2 V_r G \qquad \text{using (2.32a)}$$

and[56]

$$(2.37c)\quad c = \bar{w} - \lambda\sigma_{wr} + \phi A/2$$

$$= \lambda V_w h_q^* + \phi A/2 \qquad \text{using (2.29)}$$

$$= \lambda V_w h_q^* - \phi K V_r G/2. \qquad \text{using (2.31a)}$$

Solutions for h in equations 2.36 (or 2.37) will be real if the discriminant (D) is positive, i.e., when

$$(2.38)\quad D = b^2 - 4ac > 0$$

$$= (\lambda V_w - 2\phi K^2 V_r G)^2 + 2\phi K^3 V_r GH[\lambda V_w h_q^* - \phi K V_r G/2] > 0.$$

It is safe to assume that this condition is always satisfied, since in fact lenders do take observable (and positive) forward commitment positions (see Note 8 in Appendix B). Moreover, with $D > 0$, the relevant solution (see Note 9 in Appendix B) of equation 2.36 for the firm's optimal forward commitment position when it allows for skewness considerations is

$$(2.39)\quad h_s^* = \frac{\lambda V_w - \phi B - \sqrt{D}}{\phi S_w} = \frac{\sqrt{D} - \lambda V_w}{\phi |S_w|} + \frac{B}{|S_w|}$$

$$= \frac{\sqrt{D} - \lambda V_w}{\phi |S_w|} + \frac{2}{KH} \qquad \text{using [(2.32a), (2.33a), and (2.35)]}$$

In spite of the apparent complexity of this model, it is routine to show that all the earlier conclusions obtained without introducing skewness considerations continue to hold after skewness and skewness preferences are taken account of along with risk aversion in the more usual sense. Specifically, all the conclusions of the mean-variance new-money-rate model in section 2.3 continue to be valid for any given level of skewness preference (or dispreference for negative skewness), thereby re-establishing conclusions ii, iii, and iv of the Introduction.

Incremental Effects of the Dispreference for Negative Skewness Since it has been shown that the skewness of the new-money rate (S_y) is inherently negative, it might appear that institutional investors having a dispreference for negative skewness ($\phi > 0$ in criterion function 2.16) will necessarily adopt smaller forward commitment positions than they would have chosen, other things equal, if they had simply used a mean-variance criterion and ignored skewness effects. The optimal forward commitment position (h_s^*), however, depends upon the marginal effects of a small increase in h on the level of the criterion, and will consequently be the value of h that satisfies (2.36), not (2.16). Examination of the total differentials of the optimizing condition (2.36) readily establishes that *the effects of allowance for skewness preferences upon the optimal forward* commitment *position almost certainly have the same sign as* $\partial S_y/\partial h$ *which measures the change in new-money-rate skewness as* h *is increased,*[57] i.e.,

$$(2.40) \quad \text{sign}\,(dh_s^*/d\phi) = \text{sign}\,(\partial S_y/\partial h).$$

The analysis in Note 7 of Appendix B shows that if there are no penalty costs associated with shortfalls, the value of the negative algebraic skewness of new-money rates increases with increasing h (i.e., $\partial S_y/\partial h > 0$) so long as the lender is more than about one-third committed (see the solid portion of the S_y curve in Figure 3). On the basis of this condition,

$$(2.41) \quad h_s^* > h_q^*$$

when $h_q^* > 0.35$ (approximately), i.e., in this case, lenders wary of skewness will be more heavily committed than they would have been if mean-variance criteria had been used and skewness disregarded, other things being equal. However, if and whenever significant penalty costs are associated with shortfall risks, it can be shown that increasing commitments increases the absolute value of the negative skewness, i.e., $\partial S_y/\partial h < 0$. Indeed, both the probabilities and special costs of a shortfall increase rapidly as commitment positions increase, resulting in $\partial^2 S_y/\partial h^2 < 0$ as well as $\partial S_y/\partial h < 0$ whenever h would otherwise be relatively large (see the more analytic development of essentially the same point in Note 4 of Appendix C). Consequently, when there are significant penalty costs and commitment positions would otherwise be high (apart from skewness considerations), $h_s^* < h_q^*$; and the extent of the reduction in the commitment positions which otherwise would have been taken will be larger at higher commitment levels. Moreover, it follows that the commitment premium otherwise required to induce any given and relatively high forward commitment position will be increased by skewness considerations whenever significant penalty costs are attached to potential shortfalls.

[3] FORWARD COMMITMENT POSITIONS OF LENDERS RISK AVERSE TO RELATIVE NEW-MONEY-RATE PERFORMANCE

In the previous section we examined the optimal forward commitment positions of risk-averse institutional lenders who were judging their performance by their rates of return (new-money rate) on the funds invested. While field work indicated that this was the predominant objective of most saving banks and insurance companies during most of the last decade, increasing concern with the *relative* new-money rate (their own rate in comparison with their competitors') in the latter part of the period was also observed. This concern was especially strong among life insurance companies which were heavily involved in group insurance and actively seeking to increase their market share in this type of business.[58]

I now analyze the optimal forward commitment positions of institutions that focus on a relative new-money-rate objective. In keeping with the field work and as in the previous section, I assume that institutions are risk-averse optimizers of this objective and that their decisions also reflect a dispreference for negative skewness: They seek to increase their relative new-money rates, but are averse to uncertainty regarding their *relative* performance and to actions that raise the odds that their *relative* returns will be less favorable than expected. I move immediately to the case in which the amount of every firm's investible funds is uncertain.[59]

All the definitions of variables and specification of the forward commitment problem used previously are maintained. In addition, subscript i denotes the institution making the decisions, and subscript 0 the other institution (or relevant average of other institutions) which i regards as its relevant competition. More important, the focus of the analysis shifts to differences in new-money rates:

$h_i \equiv C_i/\bar{F}_i \equiv$ the fraction of the expected total investible funds of the ith company which it chooses to commit in advance;

$h_0 \equiv C_0/F_0 \equiv$ the corresponding fractional forward commitment position of the ith company's "competition";

$\tilde{y}^* \equiv \tilde{y}_i - \tilde{y}_0 \equiv$ the difference between the uncertain new-money earnings rate of the ith company and its competition;

$\bar{y}^*, V_{y^*}, S_{y^*} \equiv$ the expected value, variance, and third moment of the random difference in return;

$U(\tilde{y}^*) \equiv U(\tilde{y}_i - \tilde{y}_0) \equiv$ the ith institution's utility function over the differences between its new-money rate and that of its competition, where $U'(\tilde{y}^*) > 0$, $U''(\tilde{y}^*) < 0$, and $U'''(\tilde{y}^*) > 0$; i.e., it seeks to increase $\tilde{y}^*$, but is averse to uncertainty regarding its relative new-earnings rate, and likes positive skewness but correspondingly dislikes negative skewness in its relative new-money rates.

For later reference, I observe that decisions to maximize the expected utility of *relative* new-money-rate performance $E[u(\bar{y}^*)]$ are identical to those which are optimal under the analytic criterion:

(3.1) *criterion:* for any given value of h_0, choose h_i^* to max $W(\bar{y}^*, V_{y^*}, S_{y^*})$

where

$$(3.1a)\quad Q^* = W(\bar{y}^*, V_{y^*}, S_{y^*}) = \bar{y}^* - \lambda V_y^*/2 + \phi S_y^*/6,$$

with $\lambda > 0$ again measuring the ith company's aversion to risk or uncertainty per se and $\phi > 0$ measuring the strength of its preference for positive skewness and, also, its *dispreference for negative skewness* in its relative performance.

At any given time, the ith institution and its competitors can obtain approximately the same rates on forward commitments and, at the end of the period, will face the same current market rate. Also, the negative covariance between interest rate movements and the percent deviation of investible funds will often be similar in magnitude. For convenience, I simply assume that the compound variable $\tilde{w}$ given in equation 2.11, above, has the same distribution for the ith institution and its competitors.[60] Then from equation 2.9, we have its new-money rate:

$$(3.2a)\quad \tilde{y}_i = \bar{r} + h_i\tilde{w},$$

and its competitor's

$$(3.2b)\quad \tilde{y}_0 = \bar{r} + h_0\tilde{w};$$

hence, its *relative* new-money rate is

$$(3.2c)\quad \tilde{y}^* = \tilde{y}_i - \tilde{y}_0 = (h_i - h_0)\tilde{w} = z_i\tilde{w},$$

where for convenience we write

$$(3.3)\quad z = h_i - h_0.$$

For any given commitment position of its competition, the ith institution will choose its h_i^* (or z_i^*) to maximize Q^* in equation 3.1a. But from (3.2c) we know that

$$(3.4)\quad \bar{y}^* = z\bar{w};\; V_y^* = z^2V_w;\; \text{and } S_y^* = z^3S_w.$$

When we set the derivatives of (3.1a) to zero, we consequently find the optimal z^* as the solution to

$$(3.5)\quad \frac{\partial Q^*}{\partial z} = \frac{\partial \bar{y}^*}{\partial z} - \frac{\lambda}{2}\frac{\partial V_y^*}{\partial z} + \frac{\phi}{6}\frac{\partial S_y^*}{\partial z} = 0,$$

or[61]

$$(3.5a)\quad \frac{\partial Q^*}{\partial z} = \bar{w} - \lambda V_w z + \frac{\phi S_w z^2}{2} = 0.$$

This equation has real roots only if its discriminant is non-negative, i.e., if

$$(3.6)\quad D = \lambda^2 V_w^2 - 2\bar{w}\phi S_w \geq 0$$

and this condition will always be satisfied whenever $\bar{w} \geq 0$ since, as shown in equation 2.33a, the skewness term (S_w) is inherently negative.[62] While effective restrictions are placed on the variables when $\bar{w} < 0$, it is safe to assume that condition 3.6 will be always satisfied, since, in fact, institutions do take forward commitment positions.[63] Moreover, with $D > 0$, the relevant solution (see Note 1 of Appendix C) of equation 3.5 for the firm's optimal relative forward commitment position is

$$(3.7)\quad z^* = \frac{\lambda V_w - \sqrt{\lambda^2 V_w^2 - 2\bar{w}\phi S_w}}{\phi S_w} = \frac{\sqrt{\lambda^2 V_w^2 - 2\bar{w}\phi S_w} - \lambda V_w}{\phi |S_w|}.$$

We can readily establish the following patterns of behavior of risk-averse institutions that also dislike negative skewness and that choose their forward commitment levels to optimize their position in terms of their new-money rate relative to their competition:

i. If commitment premiums were negative (or sufficiently *small*), with the result that $\bar{w} < 0$, institutions focusing on relative new-money rates would be less heavily committed ahead than their competitors.[64]

ii. In particular, if the rate available on forward commitments is no higher than the expected value of the future market rate for immediate purchases, i.e., if $r_c < \bar{r}$, then every institution acting on a relative new-money-rate criterion will undertake to be less heavily committed ahead than its competition. This is true because $\bar{w}$ will always be negative whenever commitment rates (r_c) fail to exceed $\bar{r}$ (see last six columns in upper panel of Table 3 and equations 2.25a and 2.25b).

iii. The drying up of forward commitments under these circumstances will of course serve to raise the available commitment rate relative to the expected future market rate. Consequently, the following important further conclusion is established: As in the cases examined in sections 1.2, 1.4, and 2.2,[65] in which firms seek to optimize absolute new-money rates or investment income, the behavior of firms with a relative new-money-rate objective will insure that commitment rates will be higher than expected future market rates, except in unusual and generally short-lived periods that are essentially transitory in character, i.e., whenever the aggregate demand

TABLE 3 Formulas[a] and Illustrative Values for $\bar{w}$ and Values of Commitment Premiums Required for $\bar{w} = 0$ for Four Probability Distributions (percent)

Name	Probabilities on $+\epsilon\%$	Probabilities on 0	Probabilities on $-\epsilon\%$	Formula	Values of $\bar{w}$ when $x_0 = r_c - \bar{r} = 0$, and $\beta = .2$, $\epsilon = 1\%$	$\beta = .2$, $\epsilon = 75\%$	$\beta = .1$, $\epsilon = 1\%$	$\beta = .1$, $\epsilon = .75\%$	$\beta = .3$, $\epsilon = 1\%$	$\beta = .3$, $\epsilon = .75\%$
Two-point	$\frac{1}{2}$	0	$\frac{1}{2}$	$\bar{w} = \frac{x_0 - \beta\epsilon^2}{1 - \beta^2\epsilon^2}$	−.208	−.113	−.101	−.057	−.330	−.178
Flat three-point	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\bar{w} = \frac{x_0(3 - \beta^2\epsilon^2) - 2\beta\epsilon^2}{3(1 - \beta^2\epsilon^2)}$	−.139	−.075	−.067	−.038	−.220	−.119
Peaked three-point	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\bar{w} = \frac{x_0(2 - \beta^2\epsilon^2) - \beta\epsilon^2}{2(1 - \beta^2\epsilon^2)}$	−.104	−.057	−.053	−.029	−.165	−.089
Uniform or rectangular	Range: from $-\epsilon\%$ to $+\epsilon\%$			$\bar{w} = \frac{(1 - \beta x_0) B + 1}{\beta}$	−.067	−.054	−.015	−.02	−.108	−.059

TABLE 3 (concluded)

					Values of $r_c - \bar{r}$ Required for $\bar{w} = 0$ When					
					$\beta = .2$		$\beta = .1$		$\beta = .3$	
					$\epsilon = 1\%$	$\epsilon = .75\%$	$\epsilon = 1\%$	$\epsilon = .75\%$	$\epsilon = 1\%$	$\epsilon = .75\%$
Two-point	$\frac{1}{2}$	0	$\frac{1}{2}$	$\bar{w} = \frac{x_0 - \beta\epsilon^2}{1 - \beta^2\epsilon^2}$	+.200	+.113	+.100	+.056	+.300	+.169
Flat three-point	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\bar{w} = \frac{x_0(3 - \beta^2\epsilon^2) - 2\beta\epsilon^2}{3(1 - \beta^2\epsilon^2)}$	+.135	+.076	+.067	+.037	+.206	+.115
Peaked three-point	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\bar{w} = \frac{x_0(2 - \beta^2\epsilon^2) - \beta\epsilon^2}{2(1 - \beta^2\epsilon^2)}$	+.103	+.057	+.051	+.028	+.157	+.087
Uniform or rectangular	Range: from $-\epsilon\%$ to $+\epsilon\%$			$\bar{w} = \frac{(1 - \beta x_0) B + 1}{\beta}$	+.067	+.054	+.05	+.02	+.108	+.059

[a]In all the formulas, $x_0 = r_c - \bar{r}$. In the formula for the uniform distribution,

$B = [\ln(1 - \beta\epsilon) - \ln(1 + \beta\epsilon)]/2\beta\epsilon$.

A more general formula for a peaked three-point distribution is available. Assume that the probability of a deviation of either $+\epsilon$ percent or $-\epsilon$ percent is p; then the probability of a zero deviation is $(1 - 2p)$. Then,

$$\bar{w} = \frac{x_0(1 - \beta^2\epsilon^2) - 2p\beta\epsilon^2(1 - \beta x_0)}{1 - \beta^2\epsilon^2} = x_0 - \beta KV_r$$

as in equation 2.29. Each of the first three formulas in the table is a particular case of this general expression when p = 1/2, 1/3, and 1/4, respectively for commitments from all lenders is a large fraction of their aggregate expected investible funds.

iv. The size of the forward commitment premium [$(r_c - \bar{r}) = x_0$] required for institutions focusing on relative new-money rates to be willing to be as heavily committed ahead as their competitors will depend on the probability distribution implicitly used, but in every case this commitment premium will have to be larger the greater its uncertainty regarding interest rates and, also, the greater the (negative) effect of interest rate changes on available funds. Both effects are illustrated in the bottom half of Table 3, and are clear from equations 2.25a and 2.25b, since in general, $z^* \geq 0$ if and only if $\bar{w} > 0$.[56]

v. Moreover, other things equal, the optimal forward commitment position of such institutions relative to that of their competitors will vary directly with the size of the commitment premium available in the market.[67] In the context of the whole forward commitment market, the added supplies of commitments from institutions responding in this way to large commitment premiums [the demand (schedule) for commitments being given] will tend to lower the commitment rate available and, hence, the commitment premium (although it would still remain positive). These marketwide reactions in the present context parallel those analyzed above in section 1.3. Moreover, the extent to which institutions will allow their commitment positions to get ahead of those of their competitors is moderated by the factors described in conclusions vi and vii, below.

vi. Other things equal, the more risk averse an institution focusing on relative new-money-rate performance may be, the more closely will it align its forward commitment position with that of its principal competitors. For instance, if commitment premiums are too small and $\bar{w} < 0$, h_i^* will be less than its competitor's position h_0; but the larger its λ (measuring risk aversion), the more closely its h_i^* will approach h_0. Similarly, if $\bar{w} > 0$, $h_i^* > h_0$, but $h_i^* - h_0$ will again be smaller with larger λ.[68]

vii. The above conclusions regarding the properties of the model for relative new-money-rate performance hold whether or not an allowance is made for the added costs and risks induced by considerations of potential fund shortfalls, when an institution's absolute commitment levels begin to approach its expected volume of investible funds. Allowance for the latter considerations (which can become important when the institution's commitment position is large relative to its own $\bar{F}$) reduces the relative forward commitment position, z_i (and thereby its own absolute commitment position, h_i) which would otherwise have been optimal (see Note 2 of Appendix C); and insures that its optimal z_i and h_i will each vary inversely with the intensity of its dispreference for negative skewness, γ (see Note 3 of Appendix C).

APPENDIX A: EFFECTS OF PENALTY COSTS OF SHORTFALLS OF INVESTIBLE FUNDS BELOW PRIOR COMMITMENTS

The purpose of this appendix is to provide formal proofs of the conclusions stated in section 1.4 of the text. Specifically, I now establish that when the supply of funds available for future investment is uncertain and penalty costs ($\tilde{r}_j - \tilde{r}_m > 0$) will be incurred in the event of fund shortfalls ($\tilde{F} < C$): (i) the expected returns associated with any forward commitment position (C) are a declining function of the penalty costs and (ii) the variance of the corresponding returns is an increasing function of the penalty costs; consequently (iii), the optimal forward commitment position (C^*) is a fortiori monotonically reduced by these penalty costs.

I first establish these propositions under the assumption that the distributions of available funds and market interest rates are independent (section A.1), and then under the more realistic assumption (section A.2) that fund flows are negatively correlated with market interest rates. In order to reserve the usual notation $F(\cdot)$ for the left tail of the cumulative probability of any distribution, the uncertain amount of investible funds is denoted by $\tilde{X}$, instead of, as in the text, by $\tilde{F}$. For simplicity, however, I assume that penalty costs are some non-negative constant independent of the size of the conditional shortfall:[69]

$$\text{(A.1)} \quad \tilde{r}_j - \tilde{r}_m = \alpha \geqq 0$$

and $\tilde{r}$, rather than $\tilde{r}_m$, denotes the uncertain market rate; r_c is retained for the fixed rate on funds under forward commitment. Otherwise, the notation is unchanged from that in the previous text.

Let us assume that both $\tilde{X}$ and $\tilde{r}$ are normally distributed:

$$\text{(A.2)} \quad \tilde{r} = N(\bar{r}, V_r) \text{ and } \tilde{X} = N(\bar{X}, V_X).$$

The following normalized variates are introduced:

$$\text{(A.3)} \quad \tilde{m} = (\tilde{r} - \bar{r})/\sigma_r$$

$$\text{(A.4)} \quad \tilde{v} = (\tilde{X} - \bar{X})/\sigma_X$$

$$\text{(A.5)} \quad u = (C - \bar{X})/\sigma_X$$

$$\text{(A.6)} \quad w = (r_c - \bar{r})/\sigma_r$$

Let $f(\cdot)$ be the normal density function; then

$$\text{(A.7)} \quad f(\tilde{X}) = \sigma_X^{-1}(2\pi)^{-0.5} \exp\left[-(\tilde{X} - \bar{X})^2/2\sigma_X^2\right]$$
$$= f(v) = (2\pi)^{-0.5} \exp(-v^2/2)$$

(and similarly for $\tilde{r}$ and $\tilde{m}$).

The cumulative density function of $\tilde{v}$, when $\tilde{v}$ takes on the value u, is written

(A.8) $F(u) = \int_{-\infty}^{u} f(v)dv = \int_{-\infty}^{C} f(\tilde{X})dX.$

It will also be convenient to use the following supplementary or derived functions:

(A.9) $G(u) = 1 - F(u) \equiv \int_{u}^{\infty} f(v)dv$

and the "linear loss function"

(A.10) $L(u) = \int_{u}^{\infty} (v - u)dF(v) = f(u) - uG(u)$

which follows from the relation

(A.11a) $\int_{u}^{\infty} vdF(v) = f(u).$

The following integrals are stated for later reference:[70]

(A.11b) $\int_{-\infty}^{u} vdF(v) = -f(u)$

(A.12a) $\int_{u}^{\infty} v^2dF(v) = uf(u) + G(u)$

(A.12b) $\int_{-\infty}^{u} v^2dF(v) = -uf(u) + F(u)$

(A.13a) $\int_{u}^{\infty} v^3dF(v) = f(u)(u^2 + 2)$

(A.13b) $\int_{-\infty}^{u} v^3dF(v) = -f(u)(u^2 + 2)$

(A.14a) $\int_{u}^{\infty} v^4dF(v) = uf(u)(u^2 + 3) + 3G(u)$

(A.14b) $\int_{-\infty}^{u} v^4dF(v) = -uf(u)(u^2 + 3) + 3F(u).$

As a final preliminary, we should note that (see Pratt, Raiffa, Schlaifer 1965, p. 17, eq. 9-67)

(A.15) $L(-u) = u + L(u)$

and that

(A.16) $f(u), F(u), G(u), L(u), L(-u) > 0$ for all $-\infty < u < \infty$.

[A.1] Penalty Costs of Shortfalls When the Distributions of $\tilde{X}$ and $\tilde{r}$ Are Independent

From equation 1.5a′, with $\sigma_{Xr} = 0$, we have[71]

$$\bar{Y} = Cr_c + (X - C)\bar{r} + \int_{-\infty}^{\infty}\int_{-\infty}^{C} \alpha(\bar{X} - C)dF(X)dF(r)$$

$$= Cr_c + (\bar{X} - C)\bar{r} + \alpha\sigma_X \int_{-\infty}^{u} (v - u)dF(v)$$

$$\text{(A.17)} \quad = Cr_c + (\bar{X} - C)\bar{r} - \alpha\sigma_X L(-u).$$

Note that, when $\sigma_{Xr} = 0$, as we are now assuming, (A.17) is identical to equation 1.5a in the text so long as $\alpha = 0$. But whenever the investor will incur penalty costs on shortfalls, i.e., $\alpha > 0$, the expected return $(\bar{Y})$ conditional on any level of forward commitments varies inversely with α.[72] Moreover, the reduction in $\bar{Y}$ due to any positive level of conditional penalty costs progressively increases at an increasing rate with the level of forward commitments (C).[73] To determine the variance, note that $V_Y = V_Z$, where

$$\text{(A.18)} \quad \tilde{Z} = \tilde{Y} - Cr_c.$$

Consequently, with $\sigma_{Xr} = 0$,

$$V_Y = V_X = \int_{-\infty}^{\infty}\int_{C}^{\infty} (X - C)^2 r^2 dF(X)dF(r) + \int_{-\infty}^{\infty}\int_{-\infty}^{C} (X - C)^2 (r + \alpha)^2 dF(X)dF(r) - \bar{Z}^2$$

$$= (\bar{r}^2 + V_r)\int_{-\infty}^{\infty} (X - C)^2 dF(X) + (2\alpha\bar{r} + \alpha^2)\int_{-\infty}^{C} (X - C)^2 dF(X) - \bar{Z}^2$$

$$= (\bar{r}^2 + V_r)V_X\int_{-\infty}^{\infty} (v - u)^2 dF(v) + (2\alpha\bar{r} + \alpha^2)V_X\int_{-\infty}^{u} (v - u)^2 dF(v) - \bar{Z}^2.$$

Hence, using (A.12c), (A.12b), (A.11b), and (A.8)

$$\text{(A.19)} \quad V_Y = V_Z = (\bar{r}^2 + V_r)V_X(1 + u^2) + (2\alpha\bar{r} + \alpha^2)V_X[(1 + u^2)F(u) + uf(u)] - \bar{Z}^2.$$

But from (A.17) and (A.18)

$$\text{(A.20)} \quad \bar{Z}^2 = (\bar{X} - C)^2\bar{r}^2 + 2\alpha\bar{r}V_X uL(-u) + \alpha^2 V_X L^2(-u):$$

and from (A.5),

$$\text{(A.21)} \quad u^2 V_X = (\bar{X} - C)^2.$$

Consequently, we have

$$\text{(A.22)} \quad V_Y = V_Z = \bar{r}^2 V_X + V_r[V_X + (\bar{X} - C)^2] + \alpha V_X H:$$

where after simplification[74]

$$\text{(A.23)} \quad H = 2\bar{r}F(u) + \alpha[F(u) - L(u)L(-u)].$$

We observe that when $\sigma_{Xr} = 0$, as we are still assuming, (A.22) is identical to (1.7b) in the text so long as $\alpha = 0$.[75] But it can be shown that $H > 0$ for all $C \geqslant 0$.[76] Consequently, whenever any shortfall of investible funds entails positive penalty costs ($\alpha > 0$), the variance of returns (V_Y) conditional on any level of forward commitments increases directly with the penalty cost rate (α).[77] Moreover, the increase in the variance due to any positive level of penalty costs progressively increases with the level of forward commitments (C), and does so at an increasing rate until commitments are well in excess of the expected level of investible funds ($\bar{X}$).[78]

Finally, when we differentiate (A.17) and (A.22) with respect to C and substitute the results into our criterion equation 1.4 in the text, we find that with $\alpha > 0$ but $\sigma_{Xr} = 0$, the optimal level of commitments (C^*_α) must now satisfy

$$r_c - \bar{r} - \alpha F(u) - \gamma[(C - \bar{X})V_r + \alpha \bar{r} f(u)\sigma_X + \alpha^2 G(u)L(-u)\sigma_X] = 0.$$

Consequently, when $\sigma_{Xr} = 0$,

$$\text{(A.24)} \quad C^*_\alpha = \bar{X} + \frac{r_c - \bar{r} - \alpha F(u)}{\gamma V_r} - \frac{\alpha \sigma_X[\bar{r} f(u) + \alpha G(u)L(-u)]}{V_r}.$$

Once again, with $\sigma_{Xr} = 0$, equation A.24 reduces to equation 1.9a in the text when there are no penalty costs, but whenever $\alpha > 0$ the optimal level of forward commitments will be lower than would otherwise be the case. Indeed, since $G(u) > 0$ and $L(-u) > 0$ for all finite u, the optimal commitment position (C^*_α) declines at an increasing rate with larger α.

[A.2] Expectation and Variance of Returns and Optimal Commitment Positions When $\sigma_{Xr} < 0$

To establish that the empirically observed negative correlation of $\tilde{X}$ with $\tilde{r}$ reinforces each of the three conclusions obtained above under the assumption that $\tilde{X}$ and $\tilde{r}$ were independently distributed, let it be assumed that

$$\text{(A.25)} \quad \tilde{X} = K - \beta \tilde{r},$$

and

$$\text{(A.26)} \quad C = K - \beta r_c,$$

where $\beta > 0$. For simplicity I continue to assume, as in equation A.1, that the penalty costs are some non-negative constant independent of the size of any shortfall of investible funds below commitments.

Substituting these values into equation 1.5a′ in the text, we have

$$\bar{Y} = Cr_c + \int_{-\infty}^{r_c} (K - \beta r - C) r dF(r) + \int_{r_c}^{\infty} (K - \beta r - C)(r + \alpha) dF(r)$$

where (from A.25 and A.26)

(A.27) $r_c = (K - C)/\beta.$

Consequently,

$$\bar{Y} = Cr_c + \beta \int_{-\infty}^{r_c} (r_c - r) r\,dF(r) + \beta \int_{r_c}^{\infty} (r_c - r)(r + \alpha)\,dF(r)$$

$$= Cr_c + \beta\sigma_r \int_{-\infty}^{\infty} (w - m)(\bar{r} + m\sigma_r)\,dF(m) + \alpha\beta\sigma_r \int_{w}^{\infty} (w - m)\,dF(m)$$

(A.28) $$= Cr_c + \beta\bar{r}w\sigma_r - \beta V_r - \alpha\beta\sigma_r L(w)$$

after using (A.3), (A.6), (A.12a), and (A.10). But from (A.25) and (A.26) we have

(A.29) $\sigma_{Xr} = E[K - \beta r - (K - \beta\bar{r})](r - \bar{r}) = -\beta V_r$

and

(A.30) $\beta w\sigma_r = \beta(r_c - \bar{r}) = (\bar{X} - C),$

Hence,

(A.31) $\bar{Y} = C(r_c - \bar{r}) + \bar{r}\bar{X} + \sigma_{Xr} - \alpha\beta\sigma_r L(w).$

When there are no penalty costs ($\alpha = 0$), (A.31) is thus identical to (1.6a) in the text, but whenever the investor will incur positive penalty costs on any shortfall of investible funds, the expected return ($\bar{Y}$) associated with any forward commitment position varies inversely with the penalty cost (α).[79] Moreover, once again, the reduction in $\bar{Y}$ due to any positive level of conditional penalty costs progressively increases at an increasing rate with the level of forward commitments.[80]

To determine the variance when $\sigma_{Xr} < 0$, (A.18) is again used to obtain

$$V_Y = V_Z = \int_{-\infty}^{r_c} [(\bar{X} - C)^+ r]^2\,dF(r) + \int_{r_c}^{\infty} [(\bar{X} - C)^- \bar{r}]^2\,dF(r) - \bar{Z}^2$$

$$= \beta^2 \int_{-\infty}^{r_c} (r_c - r)^2 r^2\,dF(r) + \beta^2 \int_{r_c}^{\infty} (r_c - r)^2 (r + \alpha)^2\,dF(r) - \bar{Z}^2$$

(A.32) $$= \beta^2 \int_{-\infty}^{\infty} (r_c - r)^2 r^2\,dF(r) + 2\alpha\beta^2 \int_{r_c}^{\infty} (r_c - r)^2 r\,dF(r)$$

$$+ \alpha^2\beta^2 \int_{r_c}^{-\infty} (r_c - r)^2\,dF(r) - \bar{Z}^2.$$

This equation can be written

(A.32′) $$= A + B + C - \bar{Z}^2,$$

where it can be shown that[81]

(A.33) $A = \beta^2 \int_{-\infty}^{\infty} (r_c - r)^2 r^2\,dF(r) = \beta^2\bar{r}^2 V_r(w^2 + 1) - 4\beta^2\bar{r}w\sigma_r^3 + \beta^2 V_r^2(w^2 + 3)$

and[82]

(A.34) $B = 2\alpha\beta \int_{r_c}^{\infty} (r_c - r)^2 r dF(r) = 2\alpha\beta^2\bar{r}V_r[G(w) - wL(w)] + 4\alpha\beta^2\sigma_r^2 L(w)$

and[83]

(A.35) $C = \alpha^2\beta^2V_r \int_{r_c}^{\infty} (r_c - r)^2 dF(r) = \alpha^2\beta^2V_r[G(w) - wL(w)]$

and from (A.28),

(A.36) $\bar{Z}^2 = (\bar{Y} - Cr_c)^2 = \beta^2V_r[\bar{r}w - \sigma_r - \alpha L(w)]^2.$

Because these expressions are complex, I first establish that when $\alpha = 0$, the equations are equivalent to equation 1.7b in the text, in which no penalty costs were assumed for shortfalls but $\sigma_{Xr} < 0$ was allowed for. I then establish that V_Y and V_Z in (A.32) are monotone increasing functions of α and C. For this purpose, I write V_{Y0}, V_{Z0} and $\bar{Z}_0$ to denote the value of those variables when $\alpha = 0$. From (A.32) and (A.36) we have $V_{Y0} = V_{Z0} = A - \bar{Z}_0^2$, which reduces to

(A.37) $V_{Y0} = V_{Z0} = \beta^2V_r[\bar{r}^2 - 2\bar{r}w\sigma_r + V_r(w^2 + 2)]$

and this is strictly equivalent to equation 1.7b in the text.[84] Next, from (A.32) we have

(A.38) $V_Y = V_Z = V_{Z0} + B + C - (\bar{Z}^2 - \bar{Z}_0^2)$

where from (A.36)

(A.39) $\bar{Z}^2 - \bar{Z}_0^2 = \beta^2V_r[-2\alpha L(w)(\bar{r}w - \sigma_r) + \alpha^2L^2(w)]$

hence

(A.40) $V_Y = V_Z = V_{Y0} + \alpha\beta^2V_r[2\bar{r}G(w) + 2\sigma_rL(w) + \alpha\phi(w)]$

where

(A.41) $\phi(w) = G(w) - wL(w) - L^2(w).$

But since $G(w)$ and $L(w) > 0$ for all finite w, and since[85] $\phi(w) > 0$ for all $w > -\infty$, it is apparent that the variance (V_Y) is an increasing function of the penalty costs (α), as was to be shown.[86]

Moreover, it is again found that the increase in the variance due to any positive level of penalty costs progressively increases with the level of forward commitments (C) and does so at an increasing rate until commitments are well in excess of the expected level of investible funds ($\bar{X}$). From (A.40) the increase in variance is

(A.42) $\Delta = \alpha\beta^2V_r[2\bar{r}G(w) + 2\sigma_rL(w) + \alpha\phi(w)].$

Using (A.30),

$$\frac{\partial \Delta}{\partial C} = \frac{\partial \Delta}{\partial w}\frac{\partial w}{\partial C} = -\frac{\partial \Delta}{\partial w}\frac{1}{\beta\sigma_r}$$

$$= -\alpha\beta\sigma_r[-2\bar{r}f(w) - 2\sigma_r G(w) - 2\alpha L(w)F(w)],$$

Since, from footnote 85, $\phi'(w) = -2L(w)F(w)$. Consequently,

$$\text{(A.43)}\quad \frac{\partial \Delta}{\partial C} = 2\alpha\beta\sigma_r[\bar{r}f(w) + \sigma_r G(w) + \alpha L(w)F(w)] > 0$$

for all finite w. The proof that $\partial^2\Delta/\partial C^2 > 0$ as well for all $C < \bar{X}$ (and somewhat beyond) follows similarly.[87]

Finally, to prove that when $\sigma_{Xr} < 0$, positive penalty costs on fund shortfalls will reduce the optimal level of forward commitments, (A.31) and (A.40) are differentiated, using (A.37), and the results are substituted into equation 1.4 in the main text. It is thereby found that the optimal forward commitment position with $\alpha > 0$ must satisfy

$$\text{(A.44)}\quad r_c - \bar{r} - \alpha G(w) - \gamma[(C - \bar{X})V_r - \bar{r}\sigma_{Xr} + (\partial\Delta/\partial C)] = 0$$

so that

$$\text{(A.45)}\quad C^*_\alpha = \bar{X} + \frac{r_c - \bar{r} - \alpha G(w)}{\gamma V_r} + \bar{r}\frac{\sigma_{Xr}}{V_r} - \frac{\partial\Delta/\partial C}{V_r}.$$

When $\alpha = 0$, this expression directly reduces to that for the optimum in the absence of penalty costs (equation 1.9a), but in view of (A.43) it is apparent that any positive penalty costs will reduce the optimal commitment position (C^*_α) and that the reduction becomes progressively larger as the penalty cost differential becomes larger.

APPENDIX B:[88] FURTHER NOTES FOR SECTIONS 2.3 AND 2.4

Note 1

Equation 2.26 (which ignores the added variance of V_y induced at high levels of h by risks of fund shortfalls) has the properties illustrated by the data in column 9 of tables 1 and 2 and by the solid (variance) line in Figure 3. From (2.26), we find that this variance of the new-money rate will decline with increasing forward commitment positions as long as

$$\text{(i)}\quad \frac{\partial V_y/\partial h}{\partial h} = 2(hV_x + \sigma_{xr}) < 0 = 2KV_r(hKM - 1) < 0$$

using (2.21) and (2.22), i.e.,

(ii) $\frac{\partial V_y/\partial h}{\partial h} < 0$

for all $h < 1/KM$, which surely covers the relevant range of forward commitment positions. With $\beta = 0.2$, $x_0 = 0$, and $e = 1$ (see note to Table 1), we have $K = 1.04166$, $M = 1.0133$, and $1/KM = 0.947$. With these values, the variance declines with h for all forward commitment positions up to about 95 percent of expected funds. Increasing x_0 to 0.5 reduces K and increases this figure to 0.957. From equations 2.20 and 2.23, it is also clear that smaller values of e (and V_r) would always reduce both K and M and raise the range still further.

We may also observe that the decline in the variance of the new-money rate will be more rapid when the commitment premium $x_0 = (r_c - \bar{r})$ is larger. Note that the derivative of the right side of (2.27) with respect to x_0 is $= V_r(4hKM - 1)(\partial K/\partial x_0)$. Since K and M are both roughly equal to 1.0, the entire expression will have the sign of the final derivative for all h greater than about 0.25; but from (2.20), the latter derivative is clearly negative, thereby increasing the negative value of the right side of (2.27). Q.E.D.

Note 2

From the second line of (2.29a),

$$\text{(A)}\quad \frac{\partial h_q^*}{\partial B} = -\frac{1}{\lambda KM} - \frac{x_0}{\lambda K^3 M V_r}\frac{\partial K}{\partial \beta} - \frac{1}{(KM)^2}\left[\frac{x_0}{\lambda K V_r} + (1 - \beta/\lambda)\right]\frac{\partial KM}{\partial \beta}$$

All three terms act to reduce h_q^* as long as $\partial K/\partial\beta > 0$ and $\partial KM/\partial\beta > 0$. Since $\partial M/\partial\beta = 2(1 - 2p)\beta e^2 = 2(M - 1)/\beta > 0$, and $\partial K/\partial\beta$ reduces to $(-x_0 + 2\beta K e^2)/(1 - \beta^2 e^2)$, after reduction

$$\text{(B)}\quad (1 - \beta^2 e^2)^2 \partial K/\partial\beta = 2\beta e^2 - x_0(1 + \beta^2 e^2)$$

and

$$\text{(C)}\quad (1 - \beta^2 e^2)^2 \partial KM/\partial\beta = 4p\beta e^2 - x_0[3M - M\beta^2 e^2 - 2(1 - \beta^2 e^2)]$$
$$\sim 4p\beta e^2 - x_0(1 + 4\beta^2 e^2).$$

Both derivatives are positive until x_0 reaches significant positive values; but with even larger commitment premiums, these derivatives will remain relatively small in absolute size. The full effect of the second and third terms in equation A will consequently no more than partially offset the dominant effect of the first term, which (with sign) is inherently negative.

Note 3

In the "investment income" model (equation 1.9a), increasing λ reduces forward commitment positions so long as $x_0 = r_c - \bar{r} > 0$. It was concluded that this inverse relation has held in practice because it could be established that $x_0 > 0$ in practice throughout the period covered by the study (see Fleuriet 1975, and Lintner, Piper, Fortune 1976). In the present new-money-rate model, the corresponding condition is the somewhat more stringent requirement that $\bar{w} = x_0 - \lambda K V_r > 0$. (Note that in the right-hand expression on the first line of equation 2.29a, increasing λ reduces the denominator of $\bar{w}/\lambda V_w$ and so it reduces h_q^* *if and only if* $\bar{w} > 0$.) This difference between the models in the condition for $\partial h^*/\partial\lambda < 0$ arises simply because when new-money rates (or rates of return) are the criterion of performance, the variable $\bar{w}$ in equation 2.11 plays the same role as the variable $x_0 = r_c - \bar{r}$ in the investment income model. (Specifically, commitments affect expected income ($\bar{y}$) linearly with x_0 in equation 1.6a, while the impact of the commitment ratio on expected rates of return is linear in $\bar{w}$ in equation 2.19.)

We can assert that $\partial h_q^*/\partial\lambda < 0$ in the new-money-rate model in practice because the evidence clearly indicates that $\bar{w} > 0$ throughout the period covered by this study. Since, as noted above, x_0 has been positive by significant margins throughout the period, and I now show that $\bar{w} - x_0 = \beta K V_r$ was very small in the early 1960s and continued to be relatively small thereafter. In the early and middle 1960s, $\beta < 0.15$ (as shown by the long-term regression for equation 2.13 in Note 4, below). With K approximately 1, $\beta K V_r < 3$ basis points while x_0 at all times was substantially larger. In the late 1960s, ex ante assessments of β may have risen to as much as 0.4 or 0.5 and V_r may have risen toward 1; but since high β and x_0 imply $K < 1$, we would have $\beta K V_r < 0.5$ (and probably considerably lower for most lenders most of the time). In any event, $\bar{w} > 0$, since the commitment premiums (x_0) were running above 70 basis points in these years, while $|\beta K V_r| < 0.5$.

Note 4

All the other conclusions are unambiguous, but this final inference depends on just how the interactions between stochastic changes in investible funds and interest rates are assessed. If managements assess the covariance between investible funds and interest rates by assuming that there is a stable linear relation between $\tilde{F}/\bar{F}$ and the absolute changes in interest rates (as implied for simplicity by the form of equation 2.13), then there would be no additional effect of the level of expected interest rates as such. But, if they assume a stable linear relationship between percent changes in funds ($\tilde{F}$) and *percent* deviations of rates, then the coefficient b would be stable in

$$\text{(2.13a)} \quad \tilde{F}/\bar{F} = 1 - b(\tilde{\epsilon}/\bar{r}).$$

The rest of the analysis continues as before where this "percent covariance" b is substituted for β in all the previous derivations and discussion. But upon comparing (2.13a) with the original (2.13), it is found that $b = \beta\bar{r}$; hence, there is an additional multiplicative depressive effect of the expected level of the interest rate ($\bar{r}$) as asserted in the text on the basis of this assessment.

The field interviews confirmed that the ex ante variances of the interest rate had grown larger during the later 1960s roughly in line with the higher levels of interest rates, and that managements' assessment associated these deviations in rates with larger percent changes in fund flows, as implied by the latter formulation; and the results of econometric tests of the two forms generally confirmed this preference. Each form was fitted to aggregate industry data, using quarterly observations for the entire period, 1957–1971, and for the later years, 1965–1971, separately. A quadratic time trend was allowed for in each case, and the following results were obtained:

	1957–1971			1965–1971		
Equation	R^2	**Slope Value**	***t* Ratio**	R^2	**Slope Value**	***t* Ratio**
2.13	.92	$-\beta = .14$	4.28	.88	$-\beta = 0.21$	4.16
2.13a	.92	$-b = .91$	4.75	.89	$-b = 1.84$	4.63

Although both forms explain about 90 percent of the total variance of fund flows, the t values are somewhat higher in the latter form. Moreover, it will be observed that the slope is more negative in both forms for the later years alone than for the longer period. This also, and even more strongly, suggests using the product form βr; and the conclusions in the text follow immediately.

Note 5

For greater generality, instead of (2.26) in the text, let the variance after recognizing fund shortage risks be

$$(2.26\text{b}) \quad V_v^* = V_r + h^2 V_w + 2h\sigma_{wr} + 2f(h)$$

where $f(h) = 0$ for all $h < h_0$; but $f(h) > 0$, $f''(h) > 0$ and $f'(h) > 0$ for all $h > h_0$. If we also let $\bar{w}^* = \bar{w}_0 - g(h)$, $g'(h) > 0$, to reflect the expected value of the costs associated with the fund shortages, the optimizing equation becomes

$$(2.28\text{b}) \quad \partial W/\partial h = 0 = \bar{w}^* - \lambda(\partial V_v^*/\partial h)/z$$
$$= \bar{w} - g'(h) - \lambda[hV_w + \sigma_{wr} + f'(h)]$$

instead of (2.28) in the text. When total differentials are taken, the previous results for x_0 and λ are readily established. In the same way, it can be shown that previous conclusions with respect to the sign of $dh/d\beta$ continue to hold apart from (undoubtedly exceptional) situations in which $\partial f'(h)/\partial\beta$ is both large and positive.

Note 6

Since the outside sign is negative for S_v, its negative sign can be established by showing that the bracketed expression is positive for all relevant values of the variables. This condition is clearly satisfied for all $p \leq 0.25$, but the situation is less obvious for all $0.25 < p \leq 0.5$. Note first that K is inherently positive and that we are only concerned with positive levels of forward commitments (h); since h and K only appear as a product we write $X = hK$. Next note that since [ ] > 0 for all $p < 0.25$ it will now suffice to show that the expression in brackets does not change sign for all $0 < p < 0.5$. Since [ ] is a quadratic in X, it can only change sign if it has a "solution value" of X. Such solution values turn out to be

$$X^* = \frac{3 \pm \sqrt{3(4p - 1)\beta^2 e^2]}}{3 + (1 - 4p)\beta^2 e^2}.$$

With the illustrative values of $\beta = 0.2$, e = 1, and $p = 1/3$ in Table 1, S_v is negative for all X except $0.95 < X < 1.08$; and smaller values of e (or β) would further and rapidly reduce the exceptional range. (On the other hand, a $p = 0.4$ would increase the range to $0.92 < X < 1.10$.) Not only is the exceptional range narrow and limited in our model as it stands, it is obvious without going into mathematical details that any allowance for the "costs and risks of shortfalls" in this model (as in section 1.4 for that simpler model) would add substantial amounts of negative skewness as X (and h) rose toward unity, thereby insuring that $S_v < 0$ for all relevant values of all other variables.

Returning to the illustration in Table 1, I note further that

$$(s_v^n)^3 = \frac{S_v}{(V_v)^{3/2}} = \frac{-V_r G[3hK(1 - hK)^2 + h^3K^3(1 - 4p)\beta^2 e^2]}{[V_r + h^2K^2V_rM - hKV_r]^{3/2}}$$

using (2.21), (2.22), and (2.25)

$$= \frac{-G[3hK(1 - hk)^2 + h^3K^3(1 - 4p)\beta^2 e^2]}{\sqrt{V_r}\,[(1 - hK)^3 + h^3K^3\beta^3 e^3(1 - 2p)^{3/2}]}.$$

The first term in both the numerator and denominator strongly dominates the second terms, so we have as a good approximation: $-3GhK/\sqrt{V_r}(1 - hK)$, which becomes increasingly negative as h increases, as illustrated in the final column of the table. With respect to the contrasting pattern of S_y itself, see Note 7.

Note 7

From (2.30a) as it stands, it can be seen that the maximum negative value of S_y will occur for the value of h that maximizes

$$3h(1 - hK)^2 + h^3K^2(1 - 4p)\beta^2e^2$$

and this occurs when h is about ⅓; and thereafter S_y becomes less negative as h increases. The small second term may be ignored; the derivative of the first term with h reduces to $(1 - 3Kh)(1 - Kh)$, and since $K \sim 1$, this will be negative for all $h > ⅓$ roughly. But since outside the bracket we have $A < 0$, therefore $\partial S_y/\partial h > 0$ for all $1 > h > ⅓$ approximately.

Note 8

Mathematical analysis is nevertheless informative. After multiplying out (2.38), we find that

$$D = \lambda^2V_w^2 - 2[K^2V_rG\lambda V_w(2 - HKh_q^*)]\phi + K^4V_r^2G^2g\phi^2$$

where

$$g = 4 - H = 1 + (1 - 4p)\beta^2e^2 > 0.$$

When $\phi = 0$ we have $D = \lambda^2V_w^2 > 0$, and this quadratic in ϕ will not change its sign so long as $b_0^2 < 4a_0c_0$ [where $a_0 = K^4V_r^2G^2g$; $b_0 = 2K^2V_rG\lambda V_w(2 - KHh_q^*)$; and $c_0 = \lambda^2V_w^2$]. But $b_0^2 < 4\, a_0c_0$ reduces by massive cancellation to $(2 - HKh_q^*)^2 < g$, which in turn (using the approximate values $g \sim 1$, $K \sim 1$ and $H \sim 3$) is satisfied for all $h_q^* > 0.33$. Consequently, D remains positive for all values of ϕ so long as the companies would have had forward commitment positions greater than roughly one-third of their expected (future) investible fund flows in the absence of any concern with skewness per se. Not only have forward commitment positions uniformly exceeded this level in practice, but as I show in note 9, the observed h's are lower than they would have been in the absence of dispreference for negative skewness when risks and costs of fund shortfalls are significant—as they are in practice.

Note 9

Rewrite (2.36) using (2.37) as $q = (\partial W/\partial h) = ah^2 - bh + c$; and note that q is a vertical parabola that is open downward since $a < 0$ (because $\phi > 0$

and $S_w < 0$). Max $q = c - (b^2/4a) > 0$ because $D > 0$. Max q occurs when $h_s^* = b/2a = (\lambda V_w - \gamma B)/\gamma S_w$. On either side of max q, there are two roots for which $q = 0$. The left-hand root (with smaller h) is given by $h_1 = (b + \sqrt{D})/2a$—this is the smaller root because $a < 0$—and the larger root is $h_2 = (b - \sqrt{D})/2a$ which is (2.39) in the text. h_1 is not a relevant root, because at h, the necessary second-order conditions for a solution of (2.36) to represent a true maximum of (2.16) are not satisfied, since at this point $\partial q/\partial h = \partial^2 W/\partial h^2 = \sqrt{D} > 0$. If the ith company's forward commitment position equals h_1, it can increase its utility level by increasing h; and this will continue to be true until its h reaches the higher level given by the right-hand root, h_2, shown in (2.39). Moreover, with its forward commitment position at this level, the second-order conditions for a true relative maximum are satisfied, since we then have $\partial q/\partial h = \partial^2 W/\partial h^2 = -\sqrt{D} < 0$, and any further increase in h beyond this point would reduce its utility level.

APPENDIX C: FURTHER NOTES FOR SECTION 3

Note 1

Rewrite (3.5) as

$$q = \frac{\partial Q^*}{\partial z} = \bar{w} - \lambda V_w z + \phi S_w z^2/2$$

and note that q is a vertical parabola which is open downward, since $\lambda > 0$ and $S_w < 0$. Max $q = \bar{w} - (\lambda^2 V_w^2/2\lambda S_w) > 0$ (since $D > 0$), and occurs when $z = \lambda V_w/\phi S_w < 0$ (since $S_w < 0$). On either side of max q, there are two roots at $q = 0$. The left-hand root (with smaller z) is given by $z_1 = (\lambda V_w + \sqrt{D})/\phi S_w$, since $S_w < 0$, and the right-hand root is given by (3.7). At z_1, the necessary second-order conditions for a solution of (3.5) to represent a true maximum of (3.1a) are not satisfied, since at this point, $\partial q/\partial z = \partial^2 Q^*/\partial z^2 = \sqrt{D} > 0$. When the ith company's relative forward commitment level is equal to z_1, it can increase its utility level by increasing z; and this will continue to be true until its z reaches the level given by the right-hand root, shown in (3.7). Moreover, with its relative forward commitments at this level, the second-order conditions for a true relative maximum are satisfied, since we then have $\partial q/\partial z = \partial^2 Q^*/\partial z^2 = -\sqrt{D} < 0$.

Note 2

To this point, it has been assumed that $\bar{y}^*$, V_y^*, and S_y^* in equation 3.1a are related to z_i (or h_i) only by way of equations 3.4a, 3.4b, and 3.4c and that

$\bar{w}$, V_w, and S_w are independent of z_i and h_i. But as I establish in Note 4 below, (i) all three of these variables are functions of h_i, when h is sufficiently large to involve significant costs and risks of fund shortfalls, and (ii) the properties of these functions are

$$\bar{w}'(h) < 0 \qquad V_w'(h) > 0 \qquad S_w'(h) < 0$$
$$\bar{w}''(h) < 0 \qquad V_w''(h) > 0 \qquad S_w''(h) \leq 0$$

Also note that, with competitors' commitment positions taken as given, $\partial h_i/\partial z_i = 1$. Using these relations, we rewrite (3.4)

$$\text{(3.5a')}\quad q(z_i) = \frac{\partial Q}{\partial z_i} = \bar{w} + z_i \frac{\partial \bar{w}}{\partial h_i}\frac{\partial h_i}{\partial z_i} - \frac{\lambda}{2} V_w \frac{\partial z^2}{\partial z_i} + z_i^2 \frac{\partial V_w}{\partial h_i}\frac{\partial h_i}{\partial z_i}$$
$$+ \frac{\phi}{6} S_w \frac{\partial z_i^3}{\partial_i} + z^3 \frac{\partial S_w}{\partial h_i}\frac{\partial h_i}{\partial z_i} = 0,$$

or

$$\text{(3.5b)}\quad q(z_i) = \frac{\partial Q^*}{\partial z_i} = \bar{w} + z_i\bar{w}'(h) - \lambda V_w z_i - (\lambda z_i^2 V_w'(h)/2)$$
$$+ (\phi S_w z_i^2/2) + (\phi z_i^3 S_w'(h)/6) = 0.$$

Explicit solution of (3.5b) is messy, but since $\bar{w} > 0$, there are meaningful solutions with $z_i > 0$. We have

$$\text{(3.5c)}\quad \partial q/\partial z_i = 2\bar{w}'(h) + z_i\bar{w}''(h) - \lambda V_w - 2\lambda z_i V_w'(h) + \gamma S_w z_i$$
$$- [\lambda z_i^2 V_w''(h)/2] + [\phi z_i^2 S_w'(h)/2] + [z_i S_w'(h)\phi/3] + [\phi z_i'' S_w''(h)/6],$$

and all terms (with sign) are negative when $z_i \geq 0$. [If we were to have $\bar{w} < 0$, then $q(0) < 0$, and meaningful solutions would also exist with $z_i < 0$; but the commitment position could not be too much smaller than that of the principal competitors (i.e., z_i cannot be too large, if negative) since stable solutions require that $\partial q/\partial z_i < 0$.] Since $\partial q/\partial z_i < 0$, we can determine the properties of the solutions. In particular the effects of changing the value of any variable or parameter for commitment positions will have the same sign as the derivative of q with respect to that variable. On this basis, we know that $dz_i/d\bar{w} > 0$, $dz_i/dV_w < 0$, and $dz_i/dS_w < 0$. But in (3.5a'), it is clear that allowance for the added costs and risks of fund shortages has simultaneously had the effect of algebraically reducing the equivalent (but fixed) values of $\bar{w}$ and S_w while raising the equivalent V_w. Consequently, allowance for costs and risks of fund shortages concurrently reduces the otherwise optimal relative forward commitment position (z_i^*), as asserted.

A further comment: we have sign $dz_i^*/d\lambda$ = sign $[z_i V_w + z_i^2 V_w'(h)/2]$. In Note 1, I dealt with the case where $V_w'(h) = 0$, and the same "covergent" conclusion follows since $V_w'(h) \geqslant 0$ and $z_i > 0$ will be increased only so long as $V_w + [z_i V'_w(h)/2] > 0$; otherwise the situation would be unstable downward.

Note 3

From the preceding note, we have sign $dz_i^*/d\phi$ = sign $\partial q/\partial\phi$ = sign $[3S_w + z_i S'_w(h)]$. Since $S_w < 0$ and $S'_w(h) < 0$, we have $dz_i^*/d\phi < 0$ whenever $z_i \geq 0$ (i.e., whenever $\bar{w} > 0$, as it always has been in practice). It should be noted, however, that any $z_i < 0$ would be reduced only as long as the expression in brackets remains positive. Any z_i less than $-3S_w/S'_w(h)$ would be increased by an increased dispreference for negative skewness, with a "stabilizing" movement of z_i toward zero.

Note 4 (to support Note 2 above)

To introduce the risks and costs of fund shortfalls into the analysis, I redefine the basic random variable $\tilde{w}$ to allow for these costs and risks. In particular, using (2.11a), I define

$$\text{(2.11b)} \quad \tilde{w} = \begin{cases} \dfrac{x_0 - \tilde{\epsilon}}{1 - \beta\tilde{\epsilon}} & \text{for } \tilde{\epsilon} \leq 0 \\[2ex] \dfrac{x_0 - \alpha\tilde{\epsilon}}{1 - \beta\tilde{\epsilon}} & \text{for } \alpha > 0, \tilde{\epsilon} > 0 \end{cases}$$

where $\alpha > 0$ incorporate the added costs of a positive interest rate deviation involving a fund shortfall. Define

$$\text{(a)} \quad \eta = \frac{(\alpha - 1)\epsilon}{1 - \beta\epsilon}.$$

Then, more simply

$$\text{(2.11b}'\text{)} \quad \tilde{w} = \begin{cases} = \tilde{w}_0 & \text{for } \tilde{\epsilon} \leq 0 \\ = \tilde{w}_0 - \eta & \text{for } \tilde{\epsilon} > 0 \end{cases}$$

where $\tilde{w}_0$ is the value of $\tilde{w}$ in equation 2.11a with no allowance for shortfalls. With the probability distribution of equation 2.17, it immediately follows that

$$\text{(2.19b)} \quad \bar{w} = \bar{w}_0 - p\eta$$

where $\bar{w}_0$ is the value of $\bar{w}$ given by equation 2.19 with no allowance for shortfall costs and risks.

Also define $(\sigma_{wr})_0$ and V^0_{w0} as the values of σ_{wr} and V_{w0} given respectively by equations 2.21 and 2.22 when $\alpha = 0$ and no allowance is made for the costs and risks of fund shortages. Introducing such shortage costs makes the covariance of $\tilde{w}$ and $\tilde{r}$ become

$$(2.21a)\quad \sigma_{wr} = E(\tilde{\epsilon} - \epsilon)(\tilde{w} - \bar{w}) = E(\tilde{\epsilon}\tilde{w})$$

$$\text{since } E(\tilde{\epsilon}) = 0$$

$$= p\left[\frac{\epsilon(x_0 - \alpha\epsilon)}{1 - \beta\epsilon}\right] + (1 - 2p)x_0 \cdot 0 + p\left[\frac{-\epsilon(x_0 + \epsilon)}{1 + \beta\epsilon}\right]$$

$$(2.21b)\quad \sigma_{wr} = (\sigma_{wr})_0 - p\epsilon\eta .$$

Similarly, it can be shown that allowance for the costs of such shortages makes the variance

$$(2.22b)\quad V_w = V_w^0 + 2p\eta K\epsilon[1 + \beta\epsilon(1 - 2p)] + p\eta^2(1 - p).$$

Finally, upon algebraic calculation, it is found that

$$(2.22c)\quad S_w = E(\tilde{w} - \bar{w})^3$$
$$= S_w^0 - p\eta[3(1 - 2p)\epsilon K(C + Z) - \eta^2(1 - 2p)(1 - p) + p^3(1 - \eta)]$$

where

$$C = \epsilon K[(1 + \beta\epsilon)^2 - 4p\beta^2\epsilon^2]$$
$$Z = \eta(1 + \beta\epsilon - 2p\beta\epsilon).$$

The signs asserted in Note 2 for the first and second derivatives of $\bar{w}$, V_w, and S_w with respect to h follow from the view of α as a rising function of h whenever forward commitments (C) are large enough relative to expected fund inflows ($\bar{F}$), or alternatively, whenever $h = C/\bar{F}$ becomes large enough to involve a significant chance of fund shortfalls. Indeed, for any given probability distribution over interest rates ($\tilde{r}$) and value of the (negative) covariance β between interest rates and funds, larger h positions will involve greater costs of readjustment and hence greater values of α, and these costs will increase at an increasing rate with larger values of h. But $\alpha'(h) > 0$ and $\alpha''(h) > 0$ imply $\eta'(h) > 0$ and $\eta''(h) > 0$, which in turn by the negative signs before the additional terms in equations 3.5a′, 3.5b, and 3.5c (Note 2 in the appendix) imply the indicated signs of the first two derivatives on $\bar{w}$, V_w, and S_w.

NOTES

1. The lender's obligation to provide the funds is legally binding. Although the borrower's obligation to draw down the funds is not legally enforceable in many cases, there is a strong presumption and moral obligation on the latter. American Life Insurance Association (ALIA) data show that cancellations have been only a small fraction of outstanding commitments throughout the period since 1961.
2. This was because of a progressive shift in relative yields. Before the mid-1950s, home

mortgage yields were unusually 40 basis points higher than those on income properties, but by the early 1970s, this margin had fallen to a *negative* 150 basis points. For further discussion, see Lintner, Piper, and Fortune (1976).

3. Jones (1968), in an earlier study, also undertook to test the extent to which insurance companies had acted on their expectation of future interest rates. He ran regressions in which both current and a moving average of past rates (as a proxy for the expected future rate) entered along with cash flows, but the results were inconclusive and there was no systematic development of the relevant theory.
4. The length of the period between commitment and takedown will, of course, vary with the type of underlying loan or asset being considered, as previously indicated. Use of this single-asset, single-period model facilitates the derivation of interest rate effects and can be readily generalized to multiple assets over multiple time periods.
5. The effects of a "downward-sloping demand curve" for commitments are introduced in subsection 1.4, below.
6. The relevant alternative "spot" rate will be a government bond rate for savings and loan associations, a new-issue corporate rate for insurance companies, and either a government or corporate rate for mutual savings banks (depending on their portfolio position and relative market yields at the time). Again, the futurity of the relevant spot rate will depend upon the type of loan involved in the commitment.
7. Indeed, the field work suggests that, in several important companies at least, this kind of competitive criterion may well have come to dominate the earlier concern with absolute income levels or rates of return themselves.
8. Or, equivalently, have a preference *for* (positive) skewness, other things equal.
9. In our field interviews, lending officers were quite reluctant to enter into situations involving better than even chances of shortfalls from expected values unless the expected gain were enough richer to justify the action.
10. In recent years, among insurance companies, there has been increasing emphasis upon comparisons of a company's new-money rate with that of its competitors, especially in companies heavily involved in group insurance.
11. Equation 1.9 implies $C^* > F$ for all values of $r_c > \bar{r}$; but in this limiting case with the total supply of investible funds treated as a known constant, it can be argued that such an excess of commitments over available funds would never occur, i.e., that (1.9) is relevant only in the region $C^* \leq F$ and $r_c \leq \bar{r}$. Later sections of the paper analyze more realistic cases free of these restrictions.
12. In practice, we observe $r_c > \bar{r}$, but as shown later this is a consequence of the uncertainty regarding the volume of funds available for investment and its negative covariance with the future interest rate.
13. The objective of maximizing income streams produced by lenders' investments over time, combined with risk aversion in the face of uncertainty over future alternative interest rates, can readily lead to the choice of a fully invested posture as a month-to-month or quarter-to-quarter *operating policy*. Jones's effort (1968) to distinguish between "maximizing returns over time" and "full investment policies" as characterizations of companies' investment objectives was consequently unjustified. For further discussion, see Lintner, Piper, Fortune (1976).
14. Guy Stevens has shown that when three variables, x, y, and z, are multivariate normal, the covariance of the product $\bar{x}\,\bar{y}$ with $\bar{z}$ is given by $\sigma_{xy.z} = \bar{y}\sigma_{x.z} + \bar{x}\sigma_{y.z}$. Letting $\bar{F} = \bar{x}$ and $\bar{r} = \bar{y} = \bar{z}$, we have the equation in the text. See Stevens (1971, pp. 1235–1250, especially page 1240).
15. Variances must, of course, always be positive, and it can be readily established that (1.7b) satisfies this requirement in spite of the two important negative terms. Stevens (1971) has shown that the variance of the product $\bar{F}\bar{r}$ is given by

$$V_{Fr} = V_r V_F + \sigma^2_{Fr} + \bar{r}^2 V_F + \bar{F}^2 V_r + 2\bar{r}\bar{F}\sigma_{Fr}.$$

When this expression is substituted into (1.7b), it is seen that

$$V_Y = Z + V_r V_F(1 + \rho^2_{Fr}).$$

where ρ^2_{Fr} is the squared correlation coefficient between $\bar{F}$ and $\bar{r}$, and

$$Z = (\bar{F} - C)^2 V_r + 2(F - C)\bar{r}\sigma_{Fr} + \bar{r}^2 V_F = (\bar{F} - C)^2 V_r + 2(F - C)\bar{r}\rho_{Fr}\sigma_r\sigma_F + \bar{r}^2 V_F,$$

where σ_r and σ_F are the standard deviations of $\bar{r}$ and $\bar{F}$. Now, since $\rho^2_{Fr} > -1$, $V_Y > Z$ and $Z > [(\bar{F} - C)\sigma_r - \bar{r}\sigma_F)]^2 > 0$.

Q.E.D.

16. See Lintner, Piper, Fortune (1976), where these theoretical results are shown to explain many of the more significant changes in the forward commitment positions of insurance companies over the last ten or fifteen years.
17. It is clear from the structure of the forward commitment decision that the yield on direct investments *currently* available in the market affects forward commitment positions only insofar as it influences management's assessment of the directly relevant *expected* (but uncertain) market rate which will be available at time of takedown. Moreover, once again, even this relevant expected future market rate influences forward commitment positions only by way of its yield spread from the rate currently available on forward commitments.
18. Each of the variables on the right side of equation 1.10a is inherently positive except for the negative covariance (or regression coefficient) term b_{Fr}, but this negative element is reversed by the negative sign before the whole expression.
19. For simplicity here, I assume some common set of values for all firms in the industry for all variables except C and $\bar{F}$, which are taken as straightforward aggregates of each lender's expected funds and commitments.
20. The γ in the aggregate supply equation will be the harmonic mean of the γ_r's of the individual lending institutions even when all assessments are homogeneous. When the lender's assessments of the expected future market rate and its variance differ, the corresponding terms in the aggregate supply equation are weighted averages formed as indicated in Lintner (1969).
21. Development and construction loans are usually advanced by other short-term lenders such as commercial banks, but generally only on the condition that some life insurance company or other institutional lender has already made a forward commitment (before construction has started) to provide the permanent mortgage on the property after its completion. (This was almost universal practice before the emergence, about 1970, of the real estate investment trusts (REITs), which began, as a competitive device, to extend rapidly increasing volumes of new-construction loans without prior takeout commitments; but the consequences of such practices are now clear, and they have been abandoned.)
22. It should also be observed that mortgage bankers who specialize in originating loans on existing properties as well as new construction will usually not assume the risks of long-term lending. Consequently, they also require forward commitments of permanent lenders before proceeding.
23. Although Lane set up his models for short-term commercial bank commitments, his formal analysis, in which prior commitment of rates, fees, amounts, and takedown time is assumed, applies equally well to the longer financings that are of concern here.
24. The relationships $V_{(m-r)} = V_m + V_r - 2\sigma_{mr}$ and $\sigma_{(r,m-r)} = \sigma_{mr} - V_r$ are used in reducing V_n to the form shown as equation 1.13.
25. Fleuriet (1975) compares the average rate on the forward commitments made in each quarter 1960:2 through 1970:2, as shown in ALIA data, with the expected future rates at mean takedown time, as estimated by the forward long-term rate implicit in the market yield curve. (The former series is shown in column 4 of his Table 2, pp. 45–46, and the latter is given in column 1 of the same table after deducting 1.02 as explained in the

footnote to the table and his earlier text.) Over this period, the minimum value of $r_c - \bar{i}$ was 0.52 percent in 1967:4, and the maximum commitment premium was 1.29 in 1970:3. Further evidence by type of loan and property is given in Lintner, Piper, Fortune (1976).

26. To simplify the notation the additional subscript for takedown time has been omitted at this point.
27. Empirically, the principal source of such shortfalls has been the impact of unexpected increases in market rates, causing investible funds to be lower than forecast (see Lintner, Piper, Fortune 1976).
28. This is clearly the economic reality, well-known and acted upon by investment officers. The bookkeeping conventions used in public reports are tied to amortized historical costs and distinguish capital gains from "income." The "book" income forgone in the sale of low coupon bonds purchased earlier may thus be less than the yield on new investments even when the yield at current market values (and a fortiori at the still lower value realized from any forced massive sale) is substantially higher.
29. A particularly striking instance of massive sales by life insurance companies from existing portfolios occurred during the 1966 credit crunch. The companies had not foreseen the massive decline in investible funds, especially from policy loans, and had to cover the gap by selling securities into declining markets, notably in the second quarter of the year.
30. If the level of interest rates is not already relatively high, the response of prepayments and policy loans to changes in interest rates is likely to be relatively small, as explained previously.
31. The simple additional adjustments in the formal model to accommodate the general aversion to borrowing that shows on year-end balance sheets (or the more relaxed attitudes beginning to be found in some companies and banks), are not given in the text. For the former (and generally still much the larger) group of lenders, the disutility of added borrowing that may not be repaid within the calendar year is considerably greater than the loss of income (and added variance) involved in the interest cost of the loan. Our model can easily be adapted for such lenders by letting $\bar{r}_j$ have a value sufficiently higher than the debt cost to reflect the aversion to debt per se. For the minority of institutional lenders (such as savings and loan associations) that are more willing to borrow rather freely to even out their fund flows over time, we merely need to add a coefficient, α (where $0 < \alpha < 1$) before the penalty term $(\bar{r}_j - \bar{r}_w)$ in equation 1.5a′ to reflect the fact that the expected life of the borrowing to cover commitments will only be a fraction of the expected life of the loans being bought through the commitment process.

   It should be emphasized that the qualitative conclusions drawn in the next paragraph continue to hold for both groups of lenders; and the conclusions in the text following are considerably strengthened for those who continue to be averse to debt per se.
32. It can also be shown that the additional term in (1.5a′) introduces "downside skewness" into the distribution of random investment income ($\tilde{Y}$) and that this negative skewness also increases at an increasing rate as forward commitments are raised relative to expected fund flows. Since the companies have a strong dislike of negative skewness per se (even when the means and variances are held constant), this skewness effect compounds and strengthens the qualitative conclusions drawn in the immediately following text.
33. It is not necessary to rest the case for finite elasticity of demands on deductive logic: it was brought out in field interviews that there were substantial numbers of situations in which lenders had *wanted* to make more commitments but were unable to do so without shading their charges.

34. If the demand curve is declining, then the volume of commitments any company will be able to place (the demand for its funds via commitments) will increase as it lowers its rate. Consequently, the greater the volume of its commitments, the lower will be the average rate it can get on its commitments. With $c > 0$, equation 1.18 incorporates these general relationships in a simple, linear form.

35. This is most readily seen if rates of return for the continuous compounding equivalent of the given discrete rates of return, $\tilde{y}$, are regarded as being normally distributed. Since $\tilde{z} = \ln(1 + \tilde{y})$, we have ending wealth $\tilde{X} = X_0 e^{\tilde{z}}$ where X_0 is initial wealth.

The power utility function is $U(\tilde{X}) = -X^{-\lambda}$ (where $\lambda > 0$), which may readily be shown to have constant proportional risk aversion $\rho^* = -XU''(\cdot)/U'(\cdot) = 1 - \lambda > 0$. Moreover, $E[U(\tilde{X})] = -X_0^{-\lambda}E(e^{-\lambda \tilde{z}})$ and when $\tilde{z}$ is normally distributed with mean $\bar{z}$ and variance V_z, the expectations on the right may be evaluated as the $-\lambda$th moment in the usual moment-generating function; hence

$$E[U(\tilde{X})] = -(X_0^{-\lambda}) \exp\{-\lambda[\bar{z} - \lambda V_z/2]\}$$

which is monotone increasing in $[\bar{z} - \lambda V_z/2]$, corresponding to (2.2).

When $\tilde{y}$ is the normally distributed rate of return over discrete intervals of time, the same equation 2.2 is still good to the second order. In that case,

$$E[U(\tilde{X})] = -(X_0^{-\lambda}/\sqrt{2\pi}) \int_{-\infty}^{\infty} (1 + \tilde{y})^{-\lambda} \exp[-(\tilde{y} - \bar{y})^2/2V_y]\, dy$$

$$= -(X_0^{-\lambda}/\sqrt{2\pi}) \int_{-\infty}^{\infty} \exp[-\lambda \ln(1 + \tilde{y}) - (y - \bar{y})^2/2V_y]\, dy.$$

But $\ln(1 + \tilde{y}) = \tilde{y} - \tilde{y}^2/2 + \ldots$, and all terms beyond the first will introduce third and higher moments into the solution. Consequently, ignoring such higher orders of smallness, we have

$$E[U(\tilde{X})] = -(X_0^{-\lambda}/\sqrt{2\pi}) \int_{-\infty}^{\infty} \exp[-\lambda \tilde{y} - (\tilde{y} - y)^2/2V_y]\, dy$$

$$= -(X_0^{-\lambda}) \exp\{-\lambda[\bar{y} - \lambda V_y/2]\}$$

which is monotone increasing in W as defined in (2.2). Q.E.D.

36. There is, of course, one special case that is an exception to this general statement. Specifically, if $\tilde{\epsilon}$ in (2.12) has a two-point distribution symmetrical about $\dot{\epsilon} = 0$, then $\dot{w}$ in (2.11a) will also have a two-point distribution symmetrical about $\bar{w}$. However, all more general symmetrical distributions of $\tilde{\epsilon}$ [including a three- (or more) point distribution centered on $\bar{\epsilon} = 0$] will yield distributions of $\tilde{w}$ which are *asymmetrical* about $\bar{w}$, with negative skewness (see footnote 54, below).

37. λ is the same measure of (proportional) risk aversion found in equation 2.2 earlier. Arditti (1967, especially pp. 19–21) showed that ϕ will be proportional to the ratio of the third to the first derivative of $U(\tilde{y})$, and will necessarily be positive for all investors whose absolute risk aversion declines with the level of their wealth; and this condition is satisfied a fortiori by all institutional investors whose proportional risk aversion is approximately constant. See also Stone (1970, pp. 20–21); Alderfer and Bierman (1970), Jean (1971), Tsiang (1972), and Kraus and Litzenberger (1972).

38. The skewness preference parameter ϕ must of course be interpreted ceteris paribus. To illustrate: Consider three distributions of outcomes $(\tilde{Y})$, denoted A, B, and C. Let the respective means $(\bar{y})$ and the variances (V_y) be the same in all three distributions, but suppose that $S_y^A > S_y^B > S_y^C$. $\phi > 0$ then indicates that $U(A) > U(B) > U(C)$, i.e., distribution A is preferred to B, which, in turn, is preferred to C.

39. These differences in the size of the effect of a change in any stimulus variable in sections 1 and 2.2–2.4 reflect the essential linearity of the former models and the essential nonlinearity of the latter due to the presence of $\bar{F}$ in the denominator. As specific

illustrations, note that investing institutions risk averse to investment incomes will adjust their forward commitment positions in proportion to any changes in their assessments of the difference between commitment rates and the expected future market rate ($r_c - \bar{r}$) or to any changes in their assessments of the covariance ($\sigma_{F,r}$) between the amount of funds that will be available for investment and the uncertain market rate. A doubling of that covariance will lead to twice the reduction in forward commitments; and an expectation of market rates 50 basis points higher than current commitment rates will reduce forward commitments twice as much as an expected spread of only 25 basis points. In contrast, the inherent nonlinearities of the present formulation when F is uncertain mean that the reaction will in each case be more (or less) than proportional to the size of the stimulus change, and this is true whether or not skewness preferences are allowed for.

40. So far as I know, the first recognition of this phenomenon was in a paper by Light (1968), prepared for Eli Shapiro, which included a table similar to my Table 1 for comparing the relative new-money rates of a firm that was more or less heavily committed than its competitors. Since his analysis was confined to the case where $r_c = \bar{r}$ in my notation, he did not explore the variation of $\bar{y}$ with changes in the size of the commitment premium, nor did he develop the optimal commitment positions of companies concerned with risk-averse maximization of their own new-money rate.

41. Evidence for this empirical observation is given in Fleuriet (1975) and in Lintner, Piper, Fortune (1976). Theoretical proofs that $r_c > \bar{r}$ is a property of equilibrium in purely competitive commitment markets (general conclusion iv in the Introduction) were given above in section 1.4 (for risk-averse lenders who use investment income criteria) and in sections 2.3 and 2.4, below (for risk-averse lenders who use new-money-rate criteria).

42. Still another reason for institutional lenders to maintain commitment positions is that forward commitments improve new-money-rate performance on average over time (as the rest of my analysis demonstrates). Therefore, lenders want to maintain supply networks during fallow periods in order to take better advantage of commitment gains when times improve (see Lintner, Piper, Fortune 1976).

43. See the discussion accompanying equations 2.40–2.42, below. In section 2.4 below, I also show that, other things equal, forward commitment positions will be lower than they otherwise would have been, the greater the institutional lenders' dispreference for negative skewness, as long as demands for commitments are strong enough (relative to expected funds, $\bar{F}$) to involve at least some significant risk of shortfalls.

44. Subsection 2.2 provided a full analysis when the amount of investible funds was not subject to significant uncertainty.

45. Gaussian distribution theory is not available because of the random term in the denominator of (2.11), and this three-point distribution provides good closed form expressions that are quite representative of the results obtained with other symmetric distributions (such as the rectangular) that have been analyzed.

46. e is measured in the same units as r and $\tilde{\epsilon}$. For instance, in the note to Table 1, $e = 1$ when $r = 8.5$ (percent). But if the same r had been denoted 0.085, then e would have been denoted 0.01. It should also be observed that the product $\beta\tilde{\epsilon}$ (or βe) in equation 2.13 is invariant to this choice of decimal notion: β is the estimate of the slope coefficient in that (regression) equation, and any shift in the decimal place in the data for $\tilde{\epsilon}$ will produce an equal and opposite shift in the measured value of β.

47. The right side of (2.25) is greater than zero because β and V_r are each greater than zero and $1 < M < 2$. With our illustrative values of $\beta = 0.2$, $p = 1/3$, and $e = 1$, the critical value of x_0 that leaves $\bar{y}$ unchanged as h varies is 0.135. $\bar{y}$ falls with h in Table 1, in which $x_0 = 0$, and it rises with h in Table 2, in which $x_0 = 0.5$; the latter is greater than the critical value (given β, p, and e) of 0.135.

48. This is discussed in section 1.4 above. Since fund shortfalls are not introduced into equation 2.26, the latter generally declines with h (see Appendix B, Note 1, below).
49. The second-order conditions for a unique maximum are satisfied, since $\partial^2 W/\partial h^2 = -\lambda V_w < 0$.
50. This optimum forward commitment position based on a mean-variance criterion is identified as h_q^* so that it may be contrasted, later, with the corresponding best position after skewness also is allowed for.
51. When $\beta = 0, K = 1$ (by equation 2.20), and hence $\sigma_{wr} = -V_r$, while by (2.22) we then also have $V_w = V_r$, and by (2.19), $\bar{w} = x_0 = r_c - \bar{r}$. Inserting these values based on $\beta = 0$ into (2.29) or (2.29a) gives equation 2.7 exactly. Since the latter equation assumed no uncertainty regarding investible funds $\bar{F}$, this result parallels the conclusions reached in section 1 (compare equations 1.9 and 1.9b with $b_{Fr} = 0$) where firms were assumed to be risk averse with respect to levels of investment income (rather than to the rate of return, as assumed in this section).
52. It was established in the preceding footnote that $[1 - (\beta/\lambda)]/KM = 1$ when $\beta = 0$. Note 2 in Appendix B proves that $\partial h_q^*/\partial\beta < 0$. Q.E.D.
53. To give a simple illustration of the variance effect per se, suppose that the added risks from fund shortfalls add $2\alpha h V_w/(1 - h)$ to the variance as given by (2.26), where $\alpha > 0$ if $h > h_0$ and $\alpha = 0$ for $h < h_0$. Using a second-order approximation, we then have (2.26a):

$$V_y^* = h^2 V_w + 2h\sigma_{wr} + V_r + 2\alpha h(1 + h)V_w.$$

Use of derivatives of (2.24) and (2.26a) in (2.27) yields an optimum forward commitment level:

$$(2.29a)\quad h_q = \frac{\bar{w} - \lambda(\sigma_{wr} + \alpha V_w)}{\lambda V_w(1 + 2\alpha)}$$

Observe that $\partial h_q/\partial\alpha = -(1 + 2\lambda h_q)/(1 + 2\alpha) < 0$. Since $h_q = h_q^*$ (from 2.29) when $\alpha = 0$, and $\partial h_q/\partial\alpha < 0$ for all $h_q >$ some h_0, it follows, as stated, that $h_q \leq h_q^*$. In addition, of course, the lost returns and explicit costs due to the fund shortage would further reduce h_q by reducing $\bar{w}$.
54. Both S_w and S_y asymptotically approach zero in the limiting case of a symmetrical two-point distribution, since $C = 0$ if $p = 0.5$; but with all three-point distributions [i.e., for all $p < 0.5$ in (2.17)] $S_w < 0$ as a strict inequality, since the bracketed expression on the right is positive for all $p < (3 + \beta^2e^2)/4\beta^2e^2$. When we substitute the limiting value of $p = 0.5$, the bracketed expression is positive for all $\beta^2e^2 < 3$. Fitted values of β empirically are all less than 0.3—and even if β were taken to be as high as 0.5, the expression would be satisfied as long as the random deviation (e) in interest rates was less than about 3.5 percent either way, which is surely beyond the range of relevance.
55. Recall the three paragraphs in the text before equation 2.17.
56. c is written in terms of h_q^*—the optimal forward position under our earlier mean-variance criterion (where it was implicitly assumed that $\phi = 0$)—in order to facilitate later comparisons with forward commitment positions when $\phi > 0$, as is now assumed.
57. The total differential of (2.36) is

$$[-\lambda V_w + \phi(B + hS_w)]\,dh + (\partial S_y/\partial h)d\phi = 0.$$

The stated conclusion follows, because the bracketed term on the left is almost certainly negative, since after substituting (2.22), (2.32a), and (2.33a), we have

$$-\lambda V_w + \phi(B + hS_w) = K^2V_r[-\lambda M + 2\phi C - \phi hCKH].$$

But $M > 1$ from (2.23), $K < 1$ from (2.20), $H > 3$ from (2.35), and $C > 0$ but usually small from (2.34). The bracketed term consequently reduces to approximately $-[\lambda M + (3h - 2)]$, which is necessarily negative for all $h > 0.67$.

58. Group insurance is, of course, sold in large blocks with blanket policies covering large numbers of people, and the underlying contracts are usually rewritten at intervals of one, two, or three years. The buyers have a strong incentive to move the business from one company to another on a net-cost basis, and between any two companies, the one earning the higher rate of return will generally be able to provide the insurance at a lower net cost to the buyer. The competitive pressures to show *relatively higher* new-money *rates* have thereby become quite strong in this part of the insurance business.
59. A rather different treatment of the consequences of relative performance criteria than that given here will be found in section III of a working paper developed independently by Light (1973). He focuses on the supply side of a game-theoretic market equilibrium with a mean-variance criterion. The present work accounts for important skewness preferences but otherwise takes a more standard approach, allowing for demand considerations in the market at appropriate points as well as the impact of risks of shortfalls.
60. Strictly speaking, my assumption is that the *i*th institution will choose its h_i^* on the basis of an assumption that the distribution of its competition's $\bar{w}_0$ is the same as its own $\bar{w}_i$. This involves assuming that $\bar{\epsilon}_i = \bar{\epsilon}_0$ in equation 2.12 and that $\beta_i = \beta_0$ in equation 2.13.
61. To develop most of the structure and properties of this model most simply, I assume that $\bar{w}$, V_{wc}, and S_{wc} are given by the institution's underlying assessments and market conditions independent of its own commitment level. The added complexity introduced by risks and costs of potential fund shortfalls when commitment levels would otherwise be high are introduced later.
62. Indeed, equation 2.33a did not allow for the effects of shortfalls. I show below that allowance for the costs conditionally associated with such shortfalls necessarily and rapidly increases *the negative value* of S_{wc} with larger values of forward commitments relative to expected available funds.
63. In addition, as is observed in Note 3 of Appendix B, $\bar{w}$ has in fact been positive during the entire period covered by this study. In order to bring out more clearly and explicitly the interplay of the forces at work in institutional decision making under this relative performance criterion (and the marketwide responses at relevant points), in what follows I work through the policy inferences of assumed circumstances where $\bar{w} = 0$ as well as the more realistically relevant situations in which $\bar{w} > 0$.
64. From the right side of (3.7) we have $z^* < 0$; and therefore, $h_i^* < h_0$ if and only if $\sqrt{D} < \lambda V_{wc}$, which is true if and only if $-2\bar{w}\phi S_{wc} < 0$ (as seen by squaring both terms and canceling). But since $\phi < 0$ and $S_{wc} < 0$, the latter condition is satisfied if and only if $\bar{w} < 0$.
65. Allowance is of course made here for the observed negative covariance between interest rate changes and amounts of investible funds.
66. The proof is the converse of that given in footnote 64.
67. From equation 2.19, we have $\partial\bar{w}/\partial x_0 = 1$; and from Note 1 of Appendix C, $(\partial q/\partial z)dz + d\bar{w} = 0$; but $\partial q/\partial z = -\sqrt{D} < 0$; hence, that $\partial z_i/\partial x_0 = (\partial\bar{w}/\partial x_0)/\sqrt{D} > 0$. Q.E.D.
68. To prove, use the right-hand side of (3.7) and write $\sqrt{D}$ for the first term. Then the sign of $\partial z^*/\partial\lambda$ = the sign of $[(\lambda V_{wc}/\sqrt{D}) - 1]$. From (3.7), the condition $z^* < 0$ requires $\sqrt{D} < \lambda V_{wc}$; hence, the sign of $\partial z^*/\partial\lambda$ is positive. Correspondingly, $z^* > 0$ requires $\sqrt{D} > \lambda V_{wc}$; hence, the sign of $\partial z^*/\partial\lambda$ is negative. (See also the final paragraph in Note 3 of Appendix C.)
69. It is obvious that an allowance for penalty costs that increases with the size of the shortfall would merely reinforce these conclusions.
70. (A.11a) and (A.11b) follow immediately from the relation $v\,dF(v) = (2\pi)^{-0.5}e^{-z}\,dz$ after substituting $z = v^2/2$. Similarly, (A.12)–(A.14) can be readily established using integration by parts. Note that each pair of the above truncated moments satisfies the well-known values for the untruncated moments of the unit-normal distribution:

(A.11c) $\int_{-\infty}^{\infty} v dF(v) = 0$

(A.12c) $\int_{-\infty}^{\infty} v^2 dF(v) = 1$

(A.13c) $\int_{-\infty}^{\infty} v^3 dF(v) = 0$

(A.14c) $\int_{-\infty}^{\infty} v^4 dF(v) = 3.$

71. For the final form of (A.17), note that $\int_{-\infty}^{u} (v - u) dF(v) = -f(u) - uF(u)$, using (A.11b) and (A.8); and this reduces to $L(-u)$ after substituting (A.9), (A.10), and (A.15).

72. This follows since $\sigma_X > 0$ while $L(-u) > 0$ for all $u < \infty$ and in particular it follows from equation A.5 for all finite $C > 0$.

73. Let $Y_\alpha = \bar{Y}(\alpha > 0)$ and $Y_0 = \bar{Y}(\alpha = 0)$. Then $Y_\alpha - Y_0 = -\alpha\sigma_X L(-u)$, and using (A.5) and (A.15), the absolute reduction in $\bar{Y}$ increases with C, since

$$\partial\alpha\sigma_X L(-u)/\partial C = \alpha\sigma_X[1 - G(u)]du/dC = \alpha F(u) > 0$$

for all $u > -\infty$. The absolute reduction increases at an increasing rate for all finite C, since $F'(u) = f(u) > 0$ for all $u < -\infty$.

74. From (A.19) and (A.20), we have

(i) $H = 2\bar{r}[(1 + u^2)F(u) + uf(u) - uL(-u)] + \alpha[(1 + u^2)F(u) + uf(u) - L^2(-u)]$.

To simplify this, note that if (A.10) and (A.15) are used:

$$\begin{aligned} uf(u) - uL(-u) &= u\{f(u) - u - [f(u) - uG(u)]\} \\ \text{(ii)} \qquad &= -u^2[1 - G(u)] = -u^2F(u). \end{aligned}$$

Also,

$$\begin{aligned} uf(u) - L^2(-u) &= uf(u) - [u + L(u)]L(-u) \\ &= uf(u) - uL(-u) - L(u)L(-u) \\ \text{(iii)} \qquad &= -u^2F(u) - L(u)L(-u). \end{aligned}$$

The substitution of (ii) and (iii) directly into (i) reduces H to equation A.22 in the text.

75. From footnote 15, and writing X in place of F, we have that $V_{(Xr)}$ reduces to $V_X V_r + \bar{r}^2 V_r$ when $\sigma_{Xr} = 0$. Note that (1.7b) is equivalent to (A.22) with $\alpha = 0$.

76. Write $H = 2\bar{r}F(u) + \alpha h(u)$, where $h(u) = F(u) - L(u)L(-u)$. We have $H > 0$ for all $C \geq 0$ because $F(u) > 0$ and $h(u) > 0$ for all $u > -\infty$ and a fortiori (by equation A.5) for all $C \geq 0$. $F(u) > 0$ for finite u by definition. Correspondingly, $h(u) > 0$ for all $u > -\infty$ because $u > -\infty$, and $h(u) \to 0$ as $u \to -\infty$.

i. $h(u) = F(u) - uL(u) - L^2(u)$. Hence, $h'(u) = f(u) - L(u) + uG(u) + 2L(u)G(u)$, and so (after using A.10), $h'(u) = 2[uG(u) + L(u)G(u)] = 2G(u)L(-u) > 0$ for all $u > -\infty$.

ii. Since $\lim_{u\to\infty} F(u) = 0$, we have $\lim_{u\to-\infty} h(u) = 0$ if $\lim_{u\to-\infty} L(u)L(-u) = \lim_{u\to-\infty} L(u)L(-u) = 0$. But $L(u)L(-u) = uL(u) + L^2(u)$ and by series expansions it is readily shown that as $u \to \infty$, lim $L(u) = 0$, lim $uL(u) = 0$ and lim $L^2(u) = 0$. Q.E.D.

77. Indeed, since it was established in the preceding footnote that $H(u) > 0$ for all finite u, the variance increases at an increasing rate with the penalty costs α.

78. By (A.22) the increase in variance is $\alpha V_X H$, and $\partial H/\partial C = (\partial H/\partial u)(du/dC) = (\partial H/\partial C)(1/\sigma_X)$. Consequently, $\partial\alpha V_X H/\partial C = \alpha\sigma_X[2\bar{r}f(u) + \alpha h'(u)]$, and both terms in the bracket are greater than zero for all $u > -\infty$ by the previous footnote. Similarly, $\partial^2\alpha V_X H/\partial C^2 = \alpha[2\bar{r}f'(u) + \alpha h''(u)] > 0$ for all finite $u \leq 0$ and all $C \leq \bar{X}$, since $f'(u) = -uf(u) > 0$ for all finite $u < 0$ and $h''(u) = 2[G(u)F(u) - f(u)L(-u)] > 0$ for all $-\infty < u < +0.3$. (Interestingly, this same expression arose in Lintner 1976, where values are shown for various values in the range $-3.0 < u < 1.5$ in Table B1, Appendix B.)

79. The inverse relation of $\bar{Y}$ and α follows from the relationships $\beta > 0$, $\sigma_r > 0$, and $L(w) > 0$ for all $w < \infty$, and hence for all finite commitment positions (C) after using (A.6), (A.25), and (A.26).

80. The reduction in $\bar{Y}$ is $\beta\sigma_r L(w)$ and $\partial(\cdot)/\partial C = \beta\sigma_r L'(w)(dw/dr_c)(dr_c/dC)$. But $dw/dr_c = \sigma_r^{-1}$ by (A.6), and $dr_c/dC = -\beta^{-1}$ since (A.25) and (A.26) imply that $r_c = (K - C)/\beta$. Since $L'(w) = -G(w)$, we consequently have $\partial(\cdot)/\partial C = G(w) > 0$ for all $w > -\infty$ and hence for all finite C.

81. $A = \beta^2\int_{-\infty}^{\infty} (r_r - \bar{r})^2 r^2 dF(r) = \beta^2 V_r \int_{-\infty}^{\infty} (w - m)^2[\bar{r} + m\sigma_r]^2 dF(m)$ using (A.6)

$= A_1 + A_2 + A_3$

where

$$A_1 = \beta^2 V_r \bar{r}^2 \int_{-\infty}^{\infty} (w^2 - 2mw + m^2)dF(m) = \beta^2 V_r \bar{r}^2(w^2 + 1)$$

$$A_2 = 2\beta^2 V_r \bar{r}\int_{-\infty}^{\infty} (w^2 - 2mw + m^2)m\,dF(m) = -4\beta^2 \bar{r} w \sigma_r^3$$

and

$$A_3 = \beta^2 V_r^2 \int_{-\infty}^{\infty} (w^2 - 2mw + m^2)m^2 dF(m) = \beta^2 V_r^2(w^2 + 3)$$

after using (A.11c)–(A.14c), as appropriate.

82. $B = 2\alpha\beta^2\int_{r_r}^{\infty} (r_r - r)^2 r\,dF(r) = 2\alpha\beta^2 V_r \int_{w}^{\infty} (w - m)^2[\bar{r} + m\sigma_r]dF(m)$

$= B_1 + B_2$

where

$B_1 = 2\alpha\beta^2 \bar{r} V_r \int_{w}^{\infty} (w^2 - 2mw + m^2)dF(m)$

$= 2\alpha\beta^2 \bar{r} V_r[w^2 G(w) - 2wf(w) + wf(w) + G(w)]$ using (A.9), (A.11a), (A.12a)

$= 2\alpha\beta^2 \bar{r} V_r[(w^2 + 1)G(w) - wf(w)]$

$= 2\alpha\beta^2 \bar{r} V_r[G(w) - wL(w)]$

using (A.10) and

$B_2 = 2\alpha\beta^2\sigma_r^2 \int_{w}^{\infty} (w^2 m - 2m^2 w + m^3)dF(m)$

$= 2\alpha\beta^2\sigma_r^3 \{w^2 f(w) - 2w[wf(w) + G(w)] + (w^2 + 2)f(w)\}$ using (A.11a)–(A.13a)

$= 4\alpha\beta^2\sigma_r^3 L(w)$

using (A.10).

83. $C = \alpha^2\beta^2 V_r \int_{r_r}^{\infty} (r_r - r)^2 V_r \int_{w}^{\infty} (w - m)^2 dF(m)$

$= \alpha^2\beta^2 V_r[w^2 G(w) - 2wf(w) + wf(w) + G(w)]$ using (A.9), (A.11a), (1.12a)

$= \alpha^2\beta^2 V_r[G(w) - wL(w)]$ using (A.10)

84. After substituting from footnote 24, equation 1.7b was

$$V_Y = (C^2 - 2C\bar{X})V_r - 2C\bar{r}\sigma_{xr} + V_r V_x + \sigma_{xr}^2 + \bar{r}^2 V_x + \bar{X}^2 V_r + 2\bar{r}\bar{X}\sigma_{xr}$$
$$= (C - \bar{X})^2 V_r + 2(\bar{X} - C)\bar{r}\sigma_{xr} + V_r V_x + \sigma_{xr}^2 + \bar{r}^2 V_x.$$

But from (A.25) we have $V_X = \beta^2 V_r$, and using (A.29) and (A.30) we find

$$V_Y = \beta^2 w^2 V_r^2 - 2\beta^2 w\bar{r}\sigma_r^3 + 2\beta^2 V_r^2$$

which is the same as (A.37).

85. For proof, from (A.41) let

$$\phi(w) = G(w) - wL(w) - L^2(w)$$
$$\phi'(w) = -f(w) - L(w) + wG(w) - 2L(w)G(w)$$
$$= -2L(w) + 2L(w)G(w)$$
$$= -2L(w)F(w) < 0$$

for all $w < \infty$. But as $w \to \infty$

$$\lim \phi(w) = \lim G(w) = \lim wL(w) = \lim L^2(w) = 0.$$

[As $w \to \infty$, $\lim wL(w) = \lim wf(w) = \lim w^2G(w) = 0$ as may be confirmed by series expansion.] We thus have $\phi(w) > 0$ for all finite w since $\phi'(w) < 0$ throughout while as $w \to \infty$ $\lim \phi(w) = 0$.

86. Indeed, since $\phi(w) > 0$ for all finite w, V_Y increases at an increasing rate with larger penalty costs (α).

87. We have

$$\begin{aligned} \partial^2\Delta/\partial C^2 &= 2\alpha[-\bar{r}wf(w) - \sigma_r f(w) + \alpha L(w)f(w) - \alpha G(w)F(w)] \\ &= 2\alpha[(\bar{r}w + \sigma_r)f(w) + \alpha\omega(w)] \qquad \text{(A.43')} \end{aligned}$$

where $\omega(w) = G(w)F(w) - f(w)L(w) > 0$ for all $-0.37 < w < \infty$ as shown in Lintner (1976, Table B1, col. 4). The conclusion in the text follows because $C < \bar{X}$ as $w > 0$ from (A.30), and the first term of (A.43') is > 0 for all $w > -\sigma_r/\bar{r}$.

88. I wish to express my appreciation to Stephen S. Smith for checking the mathematical derivations in appendixes B and C. Any errors are of course my own.

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