Volume Title: Seasonal Analysis of Economic Time Series

Volume Author/Editor: Arnold Zellner

Volume Publisher: NBER

Volume URL: http://www.nber.org/books/zell78-1

Publication Date: 1978

Chapter Title: Seasonal Adjustment When the Seasonal Component Behaves Neither Purely Multiplicatively nor Purely Additively

Chapter Author: J. Durbin, P. B. Kenny

Chapter URL: http://www.nber.org/chapters/c4325

Chapter pages in book: (p. 173 - 198)

SEASONAL ADJUSTMENT WHEN THE SEASONAL COMPONENT BEHAVES NEITHER PURELY MULTIPLICATIVELY NOR PURELY ADDITIVELY

J. Durbin
London School of Economics
and Political Science

and

P. B. Kenny
Central Statistical Office

INTRODUCTION

Let z_t denote a time series of monthly data, and let ξ_t denote the underlying trend at time t. Most methods of seasonal adjustment are based implicitly or explicitly either on an additive model, e.g.,

$$z_t = \xi_t + \alpha_t + \epsilon_t \quad (1)$$

or on a multiplicative model, e.g.,

$$z_t = \xi_t(1+\beta_t) + \epsilon_t \quad (2)$$

where α_t is an additive seasonal factor, β_t is a multiplicative seasonal factor, and ϵ_t is an irregular component. While methods based on (1) and (2) are satisfactory for the majority of series, experience has shown that some unemployment series behave in a way that is neither purely additive nor purely multiplicative. With applications to unemployment series in mind, techniques of seasonal adjustment based on a mixed additive-multiplicative model

$$z_t = \xi_t + \alpha_t + \beta_t \xi_t + \epsilon_t \quad (3)$$

were developed. (See [2].) Procedures for testing whether a series could be regarded as behaving purely additively, purely multiplicatively, or in mixed additive-multiplicative mode were also suggested and were applied to British, American, Dutch, and Irish unemployment series for 1956–72. Particular attention was devoted to situations in which the seasonal pattern or amplitude was changing over time. The work was undertaken because of difficulties which arose in 1968 in the adjustment of British unemployment series by straightforward application of model (3). These difficulties were mainly due to the fact that seasonal behaviour was changing, and, as a result, the series were being overadjusted to a substantial extent.

In this paper, we review the problem afresh and bring up-to-date the evidence on the additive-multiplicative behaviour of the four unemployment series mentioned previously. We examine also the behaviour of the Canadian unemployment series.

A brief summary of the techniques of seasonal adjustment, based on the mixed model (3) developed in [2], will be presented in the section "Techniques for Adjusting Mixed Additive-Multiplicative Series." In the section "Testing for Additivity and Multiplicativity," we discuss the problem of diagnosing whether a series is behaving multiplicatively, additively, or as a mixture of both. The diagnostic techniques considered are applied to unemployment series from Great Britain, the United States, the Irish Republic, Holland, and Canada, 1958–76. It is found that the series show a variety of forms of seasonal behaviour; the Great Britain series has, at various times, been multiplicative, mixed, and additive; the U.S. series appears to be a mixture of components, some of which are additive and some multiplicative; the Irish and Canadian series are multiplicative in the earlier years and change to additive; and the Dutch series changes from purely multiplicative to mixed behaviour. In the section "Adjustment of Dutch Male Series by Three Different Methods," we take a particular series that shows evidence of mixed additive-multiplicative behaviour, the Dutch males unemployment series, and adjust it by three different methods: X–11 additive, X–11 multiplicative and mixed additive-multiplicative adjustment using the Durbin-Murphy [2] program. The results indicate that the X–11 multiplicative method leads to severe overadjustment and is, therefore, very unsatisfactory. The performance of the X–11 additive method is more acceptable, but there seem to be signs of underadjustment in the winter months. The mixed model, however, appears to give reasonably satisfactory results. The paper concludes with a general discussion of the problem of seasonally adjusting series that are neither purely additive nor purely multiplicative.

TECHNIQUES FOR ADJUSTING MIXED ADDITIVE-MULTIPLICATIVE SERIES

In this section we summarize the adjustment techniques suggested in [2], referring the reader to the original paper for further details.

Let x_t, a_t, and b_t denote estimates of ξ_t, α_t, and β_t, obtained from a sample of monthly data. Setting $t=12(i-1)+j$ and assuming, for the moment, that the seasonal variation is constant, we may write the fitted version of the model (3) in the form

$$z_{ij}=x_{ij}+a_j+b_j x_{ij}+r_{ij}$$

or

$$z_{ij}-x_{ij}=a_j+b_j x_{ij}+r_{ij},\ i=1,2,\ldots, j=1,2,\ldots,12 \tag{4}$$

where r_{ij} is the estimated irregular component. The form (4) clearly exhibits the fitting problem as that of estimating the constants in a regression of deviations from trend $z_{ij}-x_{ij}$ on estimated trend x_{ij}. The additive and multiplicative factors are constrained by the relations

$$\sum_{j=1}^{12} a_j=\sum_{j=1}^{12} b_j=0 \tag{5}$$

and the seasonally adjusted values $\hat{z}_{ij}$ are defined by the relation

$$\hat{z}_{ij}=\frac{z_{ij}-a_j}{1+b_j} \tag{6}$$

The first step in the Durbin-Murphy method is to obtain a preliminary estimate of trend x_{ij}; this is done by means of a specially constructed 21-point filter. The filter was designed to pass a cubic polynomial unchanged while eliminating all seasonal components locally and minimising the amount of the irregular component passed through. We wish next to fit the model (4), and we immediately recognise a difficulty in that the model contains 24 constants a_j, b_j, which seems a large number to be fitted from what may be a relatively short stretch of data. The remedy adopted was to express the constants a_j, b_j in terms of the Fourier components

$$\begin{aligned} f_{1p} &= 1/\sqrt{12} \\ f_{kp} &= (1/\sqrt{6})\cos(k\,p\,\pi/12),\ k=2,4,\ldots,10 \\ &= (1/\sqrt{6})\sin[(k-1)p\,\pi/12],\ k=3,5,\ldots,11 \\ f_{12,p} &= (1/\sqrt{12})\cos(p\pi) \end{aligned} \tag{7}$$

for $p=1, 2, \ldots, 12$ and then to fit a stepwise regression on these components in such a way that only those components whose coefficients are statistically significant are included, subject to the requirement that both additive and multiplicative annual components are always included. The regression is fitted over a time period of fixed length, usually 7 years, which is moved along the series as new observations become available. The technique was tried on a number of series, and it was found that, taking a 10 percent significance level, the number of constants included was reduced from 24 to about 9–12.

Some workers who have used this technique have complained that it introduces instability into the adjustment procedure, since some components may move in and out of significance as more data are incorporated into the analysis. However, it is an easy matter to suppress the stepwise procedure if instability is feared merely by setting the F-ratio required for significance at zero.

The next step is to detect and modify extreme values arising from such causes as strikes or exceptional weather conditions. In the program, this is done by comparing the observed residual from regression with its estimated standard deviation s. Values k_1, k_2 are chosen, and observations corresponding to residuals between $\pm k_1 s$ are left unmodified. Observations corresponding to values outside $\pm k_2 s$ are adjusted to lie on the regression plane, while observations having residuals that are between $k_1 s$ and $k_2 s$ in absolute value are adjusted by an amount that varies linearly between the two. Values of k_1 and k_2 are optional, but $k_1=2.00$ and $k_2=3.75$ have been found satisfactory in practice. Further techniques for dealing with unusually large extremes are described in the Durbin-Murphy paper [2].

Experience with the G.B. unemployment series led to the belief that there are situations where the pattern of the seasonal variation remains relatively stable over time, but where the amplitude of the seasonal component changes quite rapidly. The concept of the local amplitude scaling factor was therefore introduced by means of which the amplitude of the seasonal variation is estimated over a short period of time while the pattern of the seasonal variation is estimated over a longer period. This is achieved in the following way. Let

$$s_{ij}=a_j+b_j x_{ij}$$

be the seasonal component estimated by means of the stepwise regression. The further regression

$$z_{ij}-x_{ij}=d_{ij}s_{ij}+r^*_{ij} \tag{8}$$

is then fitted over a relatively short period of successive observations. Options of 15 and 25 such observations are available in the current version of the program, but, in principle, any number can be taken. The seasonal variation is then estimated by $d_{ij}(a_j+b_j x_{ij})$, giving, for the seasonally adjusted value, the quantity

$$\hat{z}_{ij}=\frac{z_{ij}-d_{ij}a_j}{1+d_{ij}b_j}$$

Having obtained a preliminary seasonally-adjusted series in this way, we obtain a new, and hopefully superior, estimate of trend using Burman's 13-point filter [1]. The stepwise regression, modification of extreme values and local amplitude scaling factor procedures are then applied to the new set of deviations from trend. Having obtained a second and final set of modified extreme values, the Burman filter is applied for a second time, and the model is fitted for the third and last time.

A number of minor variants of this procedure are described in the Durbin-Murphy paper [2]. In the most important of these, the seasonal pattern is permitted to evolve, following a linear or quadratic function of time. A further modification is to restrict trend values to which the estimated seasonal factors are applied to the range found within the regression base over which the factors have been estimated. Details are given in the Durbin-Murphy paper [2].

It is envisaged that the sequence of operations just described would normally be carried out once a year. The program for implementing the procedure is called the Main Program. In addition a further program, called the Current Updating Program, is required to perform the adjustment of each new observation as it becomes available.

TESTING FOR ADDITIVITY AND MULTIPLICATIVITY

An important first step, before seasonally adjusting a series by any method which allows a choice of additive, multiplicative, or mixed additive-multiplicative models, is to investigate the degree of additivity and multiplicativity in the data; this is necessary if we are to choose the appropriate class of model. What we wish to know is whether the amplitude of the seasonal variation increases proportionately as the trend increases, or less than proportionately, or not at all; we also wish to examine whether the relationship between amplitude and trend remains constant through time.

A simple, but highly informative, graphical analysis is obtained by taking, as a measure of amplitude, the mean absolute deviation from trend over each year

$$A_i=\frac{1}{12}\sum_{j=1}^{12}|z_{ij}-\hat{x}_{ij}|$$

and plotting each value of A_i against the corresponding value of mean trend, T_i

$$T_i=\frac{1}{12}\sum_{j=1}^{12}\hat{x}_{ij}$$

(The estimated trend $\hat{x}_{ij}$ is obtained using the 21-point filter of Durbin and Murphy [2]). If the points for successive years are joined in chronological sequence, it is possible to obtain a visual impression both of the relationship between A_i and T_i and of any change through time in this relationship.

A program called CSOPLOT has been written that prints out the amplitude–trend diagram and two other diagrams which have been found useful. The first shows parallel plots of successive sets of 12 monthly deviations from trend, $z_{ij}-\hat{x}_{ij}$, arranged so that corresponding months lie immediately below one another; comparison of successive plots indicates whether there are variations over time in the seasonal pattern. The second gives a separate plot for each month, showing the relationship between deviation from trend and trend level for all January values, all February values, etc.; these plots indicate whether the behaviour varies from month to month.

For the purposes of the present paper, we concentrate on the amplitude-trend plot, which is the most useful way of obtaining a general indication of the additivity and multiplicativity of a series. In figures 1–10 we present the amplitude-trend plots for various unemployment series from Great Britain, the United States, Irish Republic, Holland, and Canada.

In interpreting these plots, we look for groups of consecutive years in which the relationship between amplitude and trend is approximately linear. If the line passes through the origin, we have an indication of multiplicative seasonality; if the line is horizontal, this indicates additive seasonality; any other straight line indicates mixed additive-multiplicative seasonality. A curved relationship, or an abrupt change from one straight line to another, suggests seasonal behaviour that is changing over time. We must emphasise that all these patterns are only approximations to the actual plots; the true relationship between trend and seasonal is unlikely to conform to our idealised representations, and the observed values of the amplitude will, in any case, be influenced by the irregular as well as by the seasonal.

Figures 1 and 2 show the amplitude-trend diagrams for GB total unemployment and GB female unemployment. In the total diagram (fig. 1), we see that the points up to 1962 are approximately represented by a straight line passing above the origin—although 1957 is something of an outlier. For this period we seem to have mixed seasonality, therefore. The point for 1963 is a considerable distance above the previous line, which is attributed to an exceptionally severe winter; the points for 1964 and 1965 conform closely to the previous mixed pattern. From 1966 onward, however, we have a new pattern; although there are considerable variations from year to year, the general tendency is clearly horizontal, and the indication is that the seasonal behaviour is additive. For the female series (fig. 2), the year-to-year variations are much greater, and the linear patterns are harder to detect. There is a clear multiplicative tendency until about 1969, with a suggestion of a different, but still multiplicative, pattern until 1973; thereafter, the relationship seems additive.

Figures 3 and 4 show the amplitude-trend diagrams for U.S. male unemployment, subdivided into 16-19 years old (fig. 3) and 20 years old and over (fig. 4). Figure 3 clearly has two periods of different behaviour, although it is not

Figure 1. **GREAT BRITAIN UNEMPLOYMENT**

(Seasonal amplitude against trend)

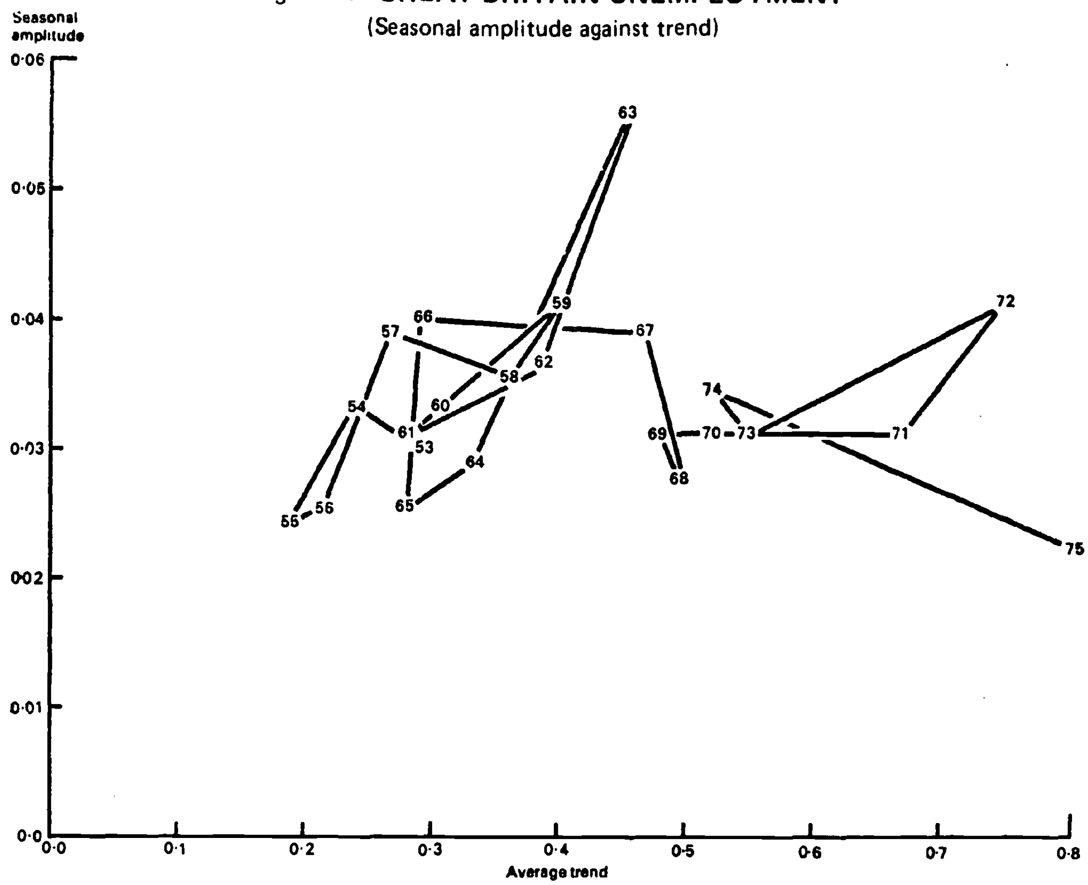


Figure 2. **U.S. UNEMPLOYMENT FOR FEMALES**

(Seasonal amplitude against trend)

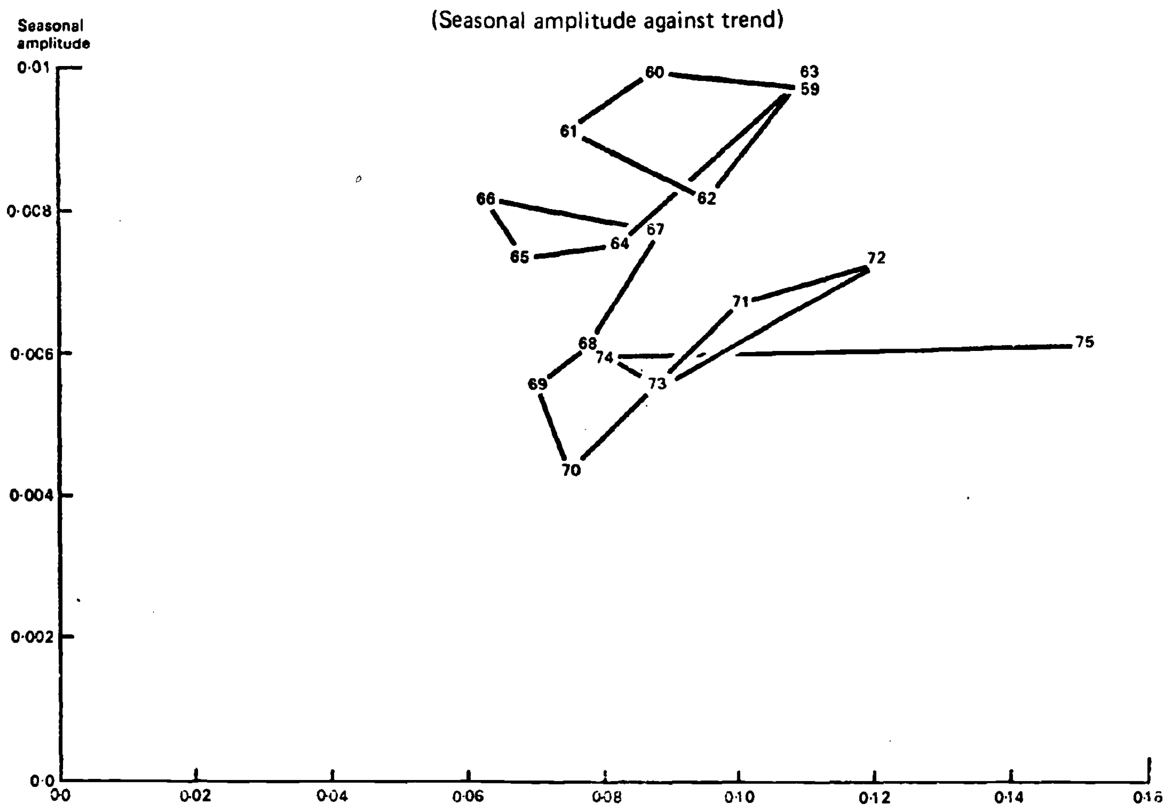


Figure 3. **U.S. UNEMPLOYMENT FOR MALES 16 TO 19 YEARS OLD**

(Seasonal amplitude against trend)

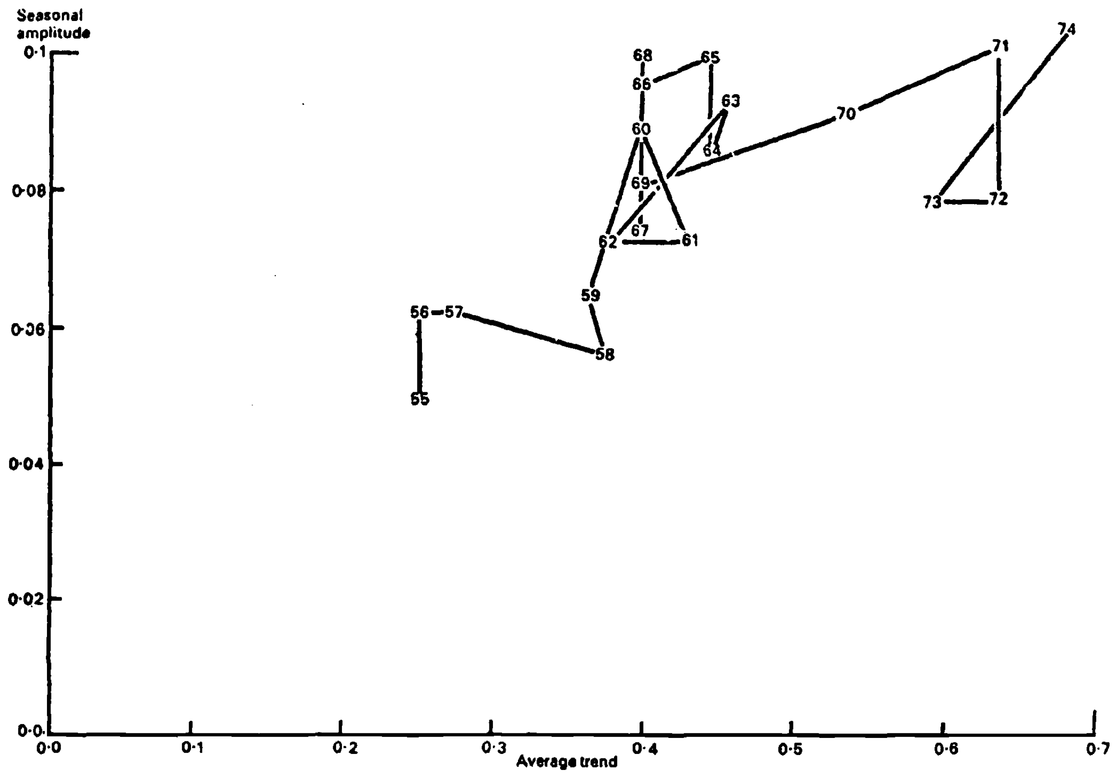
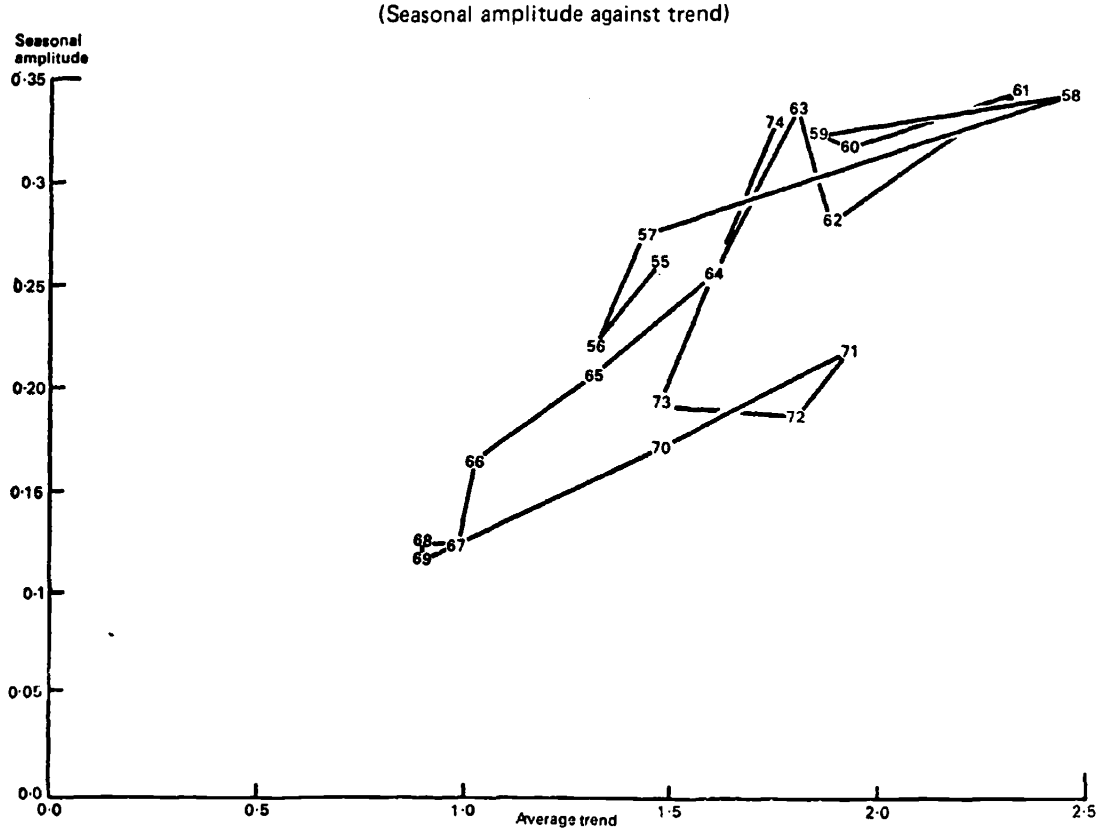


Figure 4. **U.S. UNEMPLOYMENT FOR MALES 20 YEARS OLD AND OVER**

(Seasonal amplitude against trend)



easy to date precisely the change from one to the other. In the earlier period, extending up to 1966, the amplitude is roughly proportional to trend, indicating multiplicative behaviour. From 1969 onwards, the amplitude shows very little increase as the trend rises, and the pattern is approximately additive. Since the trend level is almost constant from 1966 to 1969, it is not possible to observe the form of the relationship over this period; the change from multiplicative to additive could be dated anywhere between 1966 and 1969. Figure 4 shows that adult male unemployment has remained predominantly multiplicative throughout; although there are substantial variations from year to year, these seem to have no systematic pattern, and there is no indication of a need to consider different models for different periods.

For comparison, we show, in figures 5 and 6, the amplitude-trend plots for U.S. female unemployment, subdivided in the same way as for male unemployment. Figure 5 (females 16–19 years old) clearly has the same general form as figure 3, being multiplicative in the earlier years and additive for the most recent period; however, in this case it is possible to be more precise in the dating of the change, which seems to have taken place in 1968 or 1969. Figure 6 (females 20 years old and over) is very difficult to interpret; the erratic year-to-year variations are very large, and the only visible pattern is a general multiplicative tendency, which is particularly noticeable since 1970.

In view of the results presented here for U.S. unemployment, it is of interest that the Bureau of Labour Statistics has recently changed its procedures so that teenage unemployment (male and female) since 1967 is adjusted additively, while multiplicative adjustment continues to be used for adult unemployment.

Figures 7 and 8 show the comparison of Dutch male and female unemployment. Figure 7 (male unemployment) seems to be explicable in terms of two fairly homogenous periods, 1959–66 and 1968–74, with 1967 forming an intermediate stage; in the earlier period, the behaviour is clearly multiplicative, while in the later period, it is mixed, with additive predominating. Figure 8 (female unemployment) is clearly multiplicative since 1965. Up to 1965, the range of trend variation is so small that it is difficult to identify a relationship, although there is a suggestion of mixed behaviour with multiplicative predominating.

Figure 9 shows the amplitude-trend plot for Canadian total unemployment. This can be split into three phases. From 1957 to 1962, the results are consistent with mixed behaviour, while from 1962 to 1966, the behaviour is still mixed, with additive and multiplicative components of opposite sign. Since 1966, there is negligible multiplicativity. An alternative explanation of the period 1957–66 would be a multiplicative seasonality whose amplitude also showed a downward-time trend. (From information obtained at the conference, it seems likely that the apparently anomalous value for 1974 may be the result of a change in the Canadian Labour Force Survey which occurred at that time.)

The Irish Republic total unemployment series, shown in figure 10, clearly behaves in a predominantly multiplicative manner up to 1966–67 and additively thereafter.

These comparisons reveal a number of points of interest. First, they show that behaviour of all three types, purely-additive, purely-multiplicative, and mixed additive-multiplicative, exists in series in the one field of unemployment. Secondly, the type of the seasonal behaviour can vary over time. In this connection, it is remarkable that in about 1966–67 there appeared to be a worldwide shift from multiplicative towards additive behaviour in some components of unemployment series. The third point is that, within the one country, different components of unemployment series can behave differently.

Further tests of additivity and multiplicativity are obtained from the Model-Test Program described in [2]. The main test in this program is based on the analysis of variance. To begin with, an additive model is fitted, and the additional contribution to the regression sum of squares obtained by including multiplicative components is tested by means of an F test. The test is then repeated the other way round, i.e., first fitting a multiplicative model and testing the increase in the regression sum of squares produced by adding additive components. The tests then determine whether an additive-only model (A), a multiplicative-only model (M), a mixed model containing both additive and multiplicative components (B), or either an additive or a multiplicative model (E) suffices to explain the observed variation. The tests were carried out for successive positions of a moving 7-year regression base.

The program gives a further way of estimating the relative strengths of additive and multiplicative components by calculating the degree of multiplicativity

$$M=\left[\frac{A_H-A_L}{A_H+A_L}\right]\left[\frac{H+L}{H-L}\right] \tag{9}$$

where A_H is the seasonal amplitude at high trend-level H, and A_L is the seasonal amplitude at low trend-level L. Here, H and L are defined to be the seventh highest and seventh lowest trend values found within the regression base (usually a 7-year period). The value A_H is the maximum difference in the seasonal variation $a_j+b_j H$ over any 12-month period, and the value A_L is similarly defined at low trend-level L. The value of M is then zero for a purely additive series (with all b_j zero) and is one for a purely multiplicative series (with all a_j zero).

The results obtained from the model-test program for the G.B., U.S., Dutch, Irish, and Canadian unemployment series are given in tables 1–5. As for figures 1–10, the series were run without modifications for extreme values. As before, we note the overall reduction for most series in degree of multiplicativity over the period. For the United States, and for Britain in the earlier years, the series for females were substantially less multiplicative than the series for males, but, for the Dutch data and the recent British experience, the tendency is reversed.

Figure 5. **U.S. UNEMPLOYMENT FOR FEMALES 16 TO 19 YEARS OLD**

(Seasonal amplitude against trend)

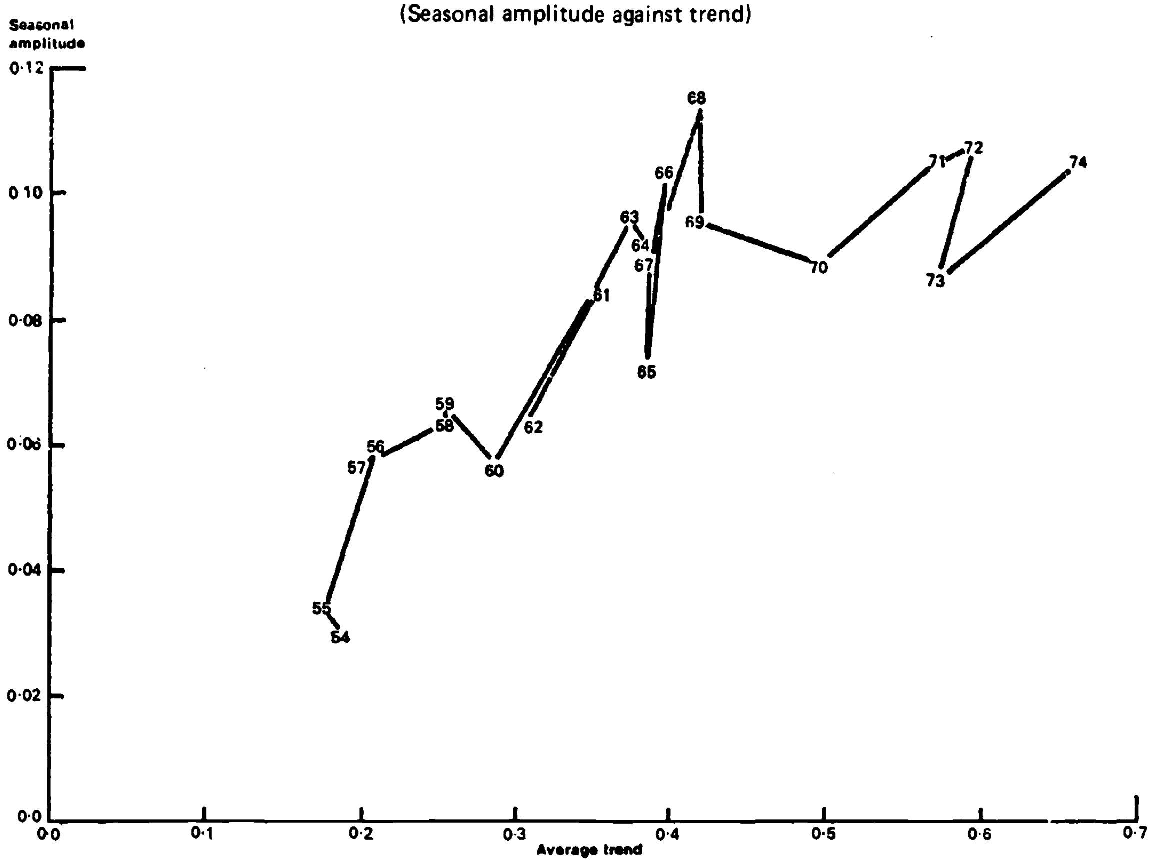


Figure 6. **U.S. UNEMPLOYMENT FOR FEMALES 20 YEARS OLD AND OVER**

(Seasonal amplitude against trend)

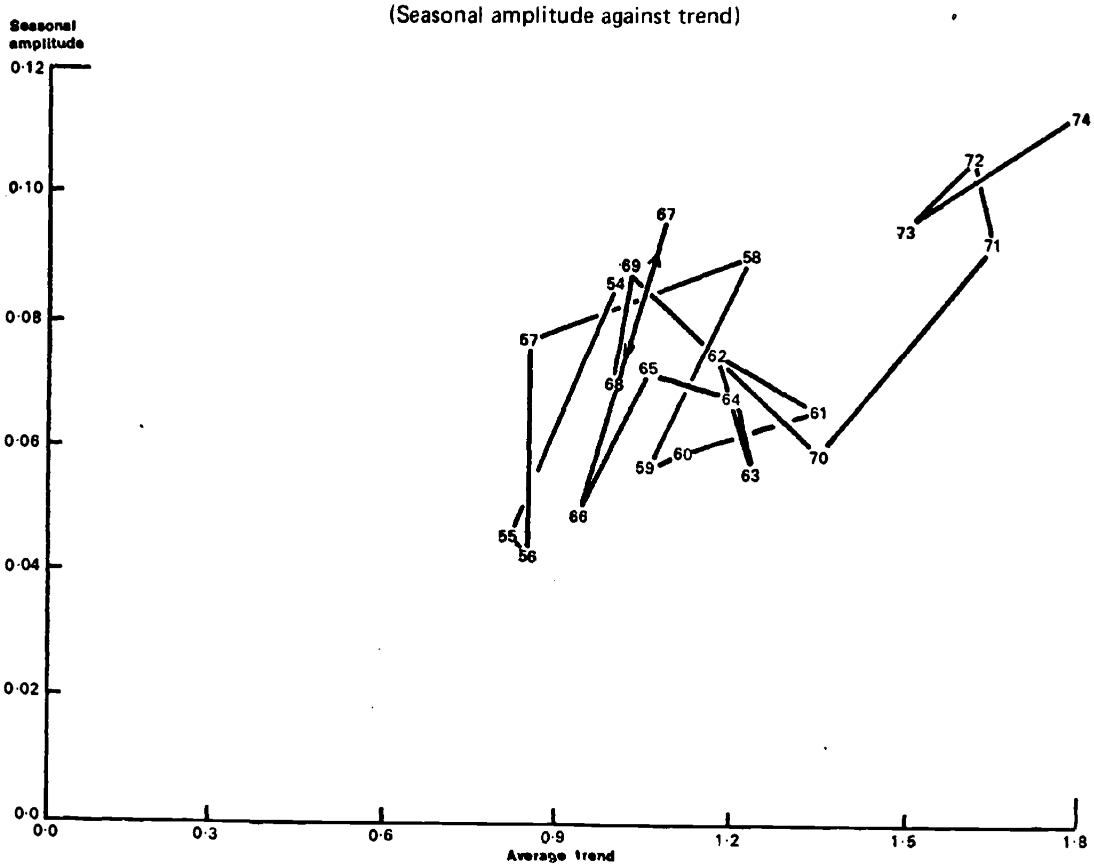


Figure 7. **DUTCH UNEMPLOYMENT FOR MALES**

(Seasonal amplitude against trend)

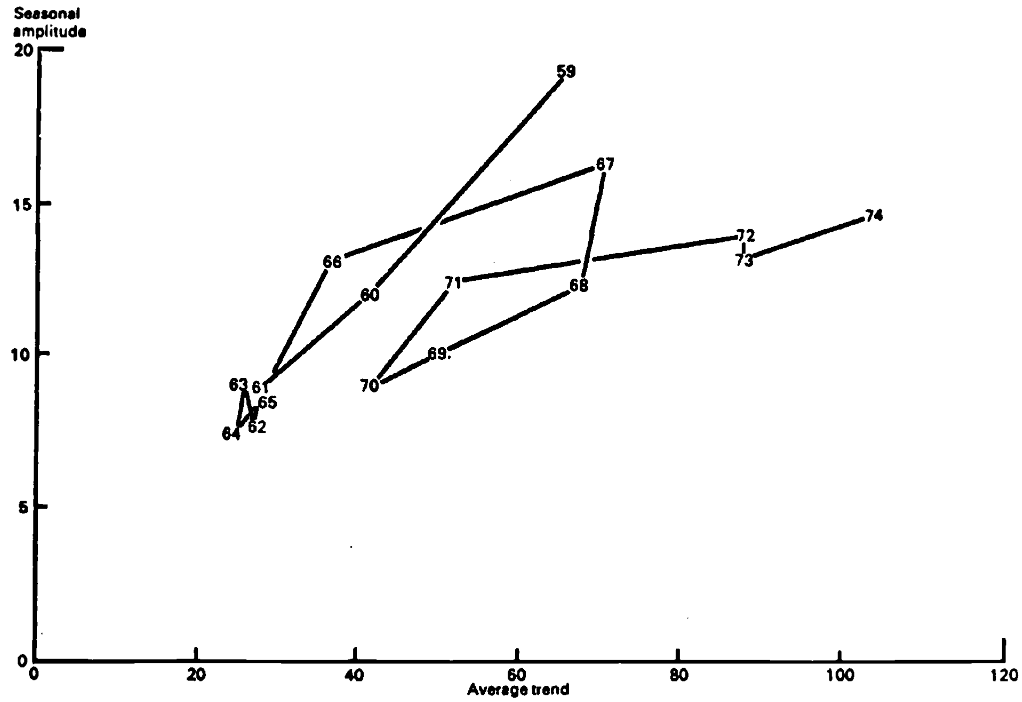


Figure 8. **DUTCH UNEMPLOYMENT FOR FEMALES**

(Seasonal amplitude against trend)

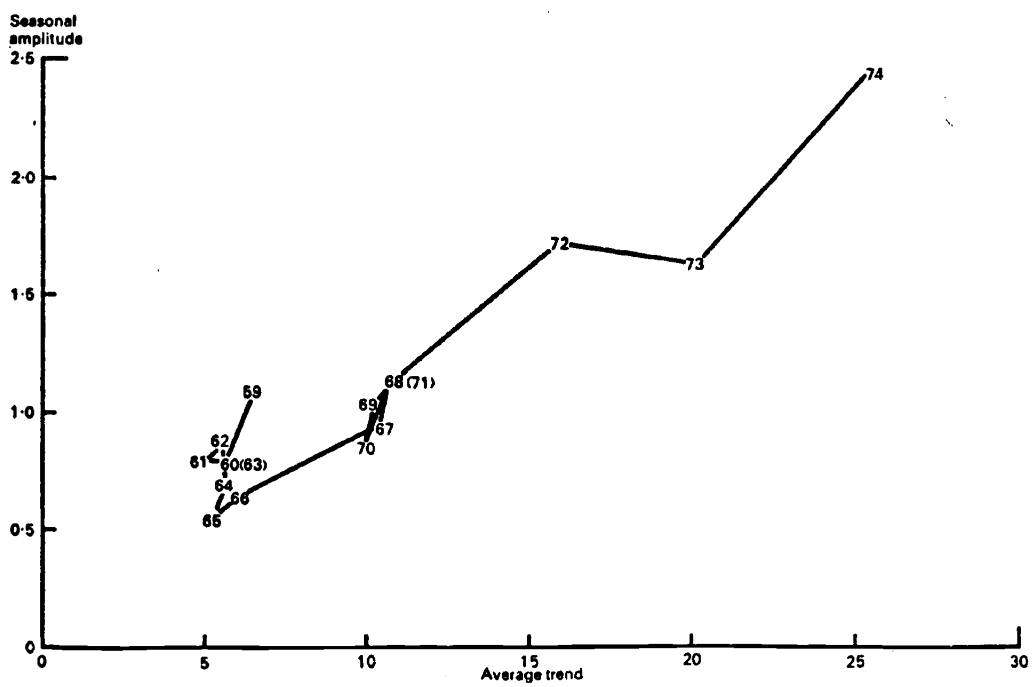


Figure 9. **CANADIAN UNEMPLOYMENT**

(Seasonal amplitude against trend)

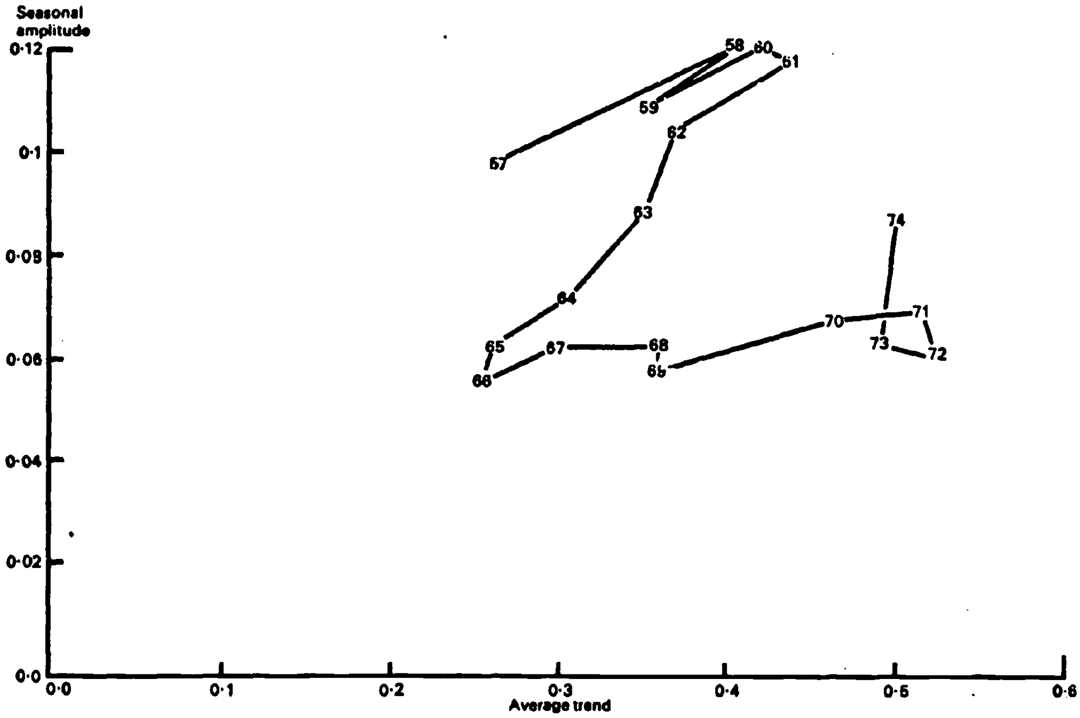


Figure 10. **IRISH REPUBLIC UNEMPLOYMENT**

(Seasonal amplitude against trend)

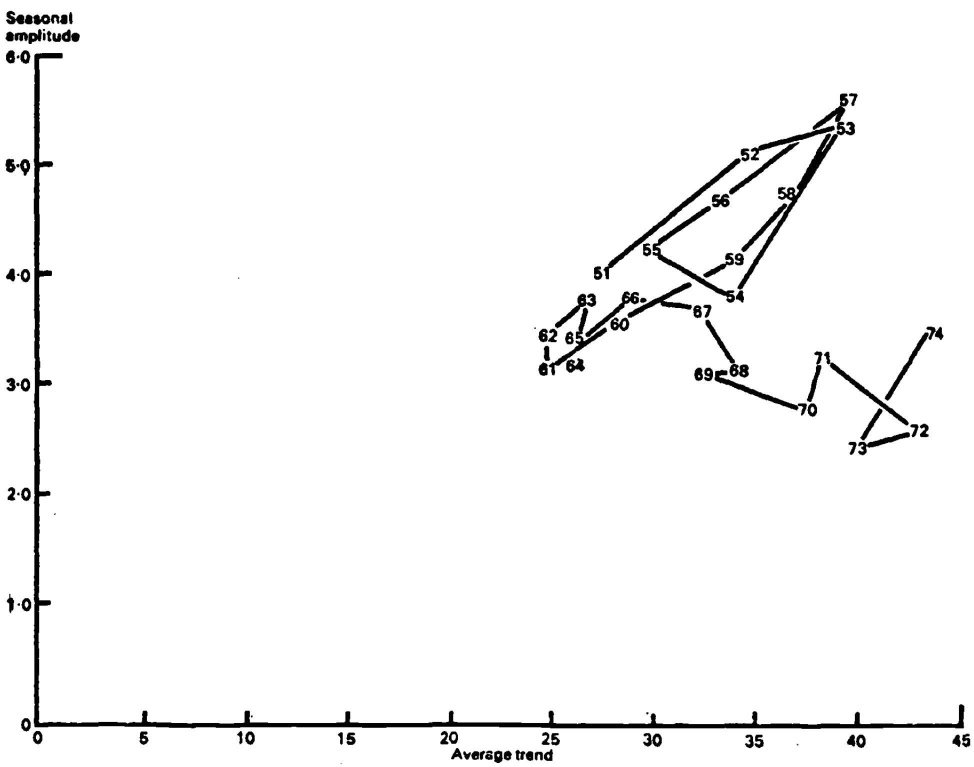


Table 1. ADDITIVITY/MULTIPLICATIVITY OF THE GREAT BRITAIN UNEMPLOYMENT SERIES

(Excluding school leavers and adult students)

Base position	Males				Females				Total			
	F-values[1]		Test result (1-percent level)[2]	Degree of Multiplicativity	F-values[1]		Test result (1-percent level)[2]	Degree of Multiplicativity	F-values[1]		Test result (1-percent level)[2]	Degree of Multiplicativity
	+Add	+Mult			+Add	+Mult			+Add	+Mult		
1959 to 1965	5.4	15.9	B	1.45	5.3	4.9	B	0.46	4.1	13.5	B	1.36
1960 to 1966	3.9	11.9	B	1.36	5.9	4.9	B	.51	3.2	9.8	B	1.21
1961 to 1967	1.2	5.8	M	1.04	6.3	4.0	B	.43	1.7	5.9	M	.90
1962 to 1968	.8	1.5	E	.58	5.6	4.9	B	.52	1.1	1.9	E	.60
1963 to 1969	1.2	1.0	E	.46	3.7	4.0	B	.62	1.3	1.3	E	.43
1964 to 1970	5.8	2.3	A	.11	2.3	1.2	E	.22	6.3	1.8	A	.04
1965 to 1971	7.2	1.8	A	.04	2.8	.7	A	.22	7.7	1.5	A	−.01
1966 to 1972	6.0	1.9	A	.15	3.2	1.7	A	.51	6.3	2.0	A	.16
1967 to 1973	1.8	1.1	E	.45	.6	1.9	E	.93	1.7	1.3	E	.50
1968 to 1974	1.4	1.6	E	.72	.6	1.4	E	.79	1.3	1.6	E	.72
1969 to 1975	1.5	.6	E	.33	1.8	1.8	E	.68	1.6	.7	E	.20

See footnotes below.

Table 2. ADDITIVITY/MULTIPLICATIVITY OF THE U.S. UNEMPLOYMENT SERIES

Base position	Males				Females				Total			
	F-values[1]		Test result (1-percent level)[2]	Degree of Multiplicativity	F-values[1]		Test result (1-percent level)[2]	Degree of Multiplicativity	F-values[1]		Test result (1-percent level)[2]	Degree of Multiplicativity
	+Add	+Mult			+Add	+Mult			+Add	+Mult		
1959 to 1965	0.7	3.2	M	1.36	0.8	1.9	E	1.71	1.2	2.8	M	1.65
1960 to 1966	2.3	9.6	M	1.30	1.2	1.3	E	.32	4.0	7.3	B	.96
1961 to 1967	5.2	18.4	B	1.33	.9	.9	E	.68	5.4	9.0	B	1.16
1962 to 1968	7.1	16.8	B	1.30	.8	.5	E	.07	7.6	8.9	B	.76
1963 to 1969	8.0	16.3	B	1.35	.9	.6	E	.23	7.3	8.2	B	.68
1964 to 1970	2.4	5.2	M	.96	1.3	.8	E	−.64	2.3	1.9	E	−.25
1965 to 1971	2.4	4.9	M	.80	2.5	1.4	E	−.23	2.8	1.8	A	.28
1966 to 1972	3.5	6.6	B	.71	2.2	1.5	E	−.10	3.8	2.3	A	.12
1967 to 1973	3.4	6.5	B	.76	2.6	.9	A	−.22	4.9	2.7	B	−.08
1968 to 1974	1.6	6.0	M	1.03	3.1	1.3	A	−.12	3.1	4.0	B	.11

A Additive.
B Both additive and multiplicative.
E Either additive or multiplicative.
M Multiplicative.

[1] F-test has 11 and 62 degrees of freedom.
[2] 1-percent significance level is 2.55.

In view of the earlier discussion of changes in the seasonal behaviour of U.S. unemployment, it may be of interest to consider in more detail table 3, showing model test results for components of U.S. unemployment, in conjunction with figures 3–6, which show amplitude-trend plots for the same components. The components used here are, we understand, those which Bureau of Labour Statistics adjust separately in constructing a seasonally adjusted series for total unemployment.

The succession of model test results for moving 7-year bases, reading down the column of table 3, shows several instances, particularly for the more erratic series, where the result changes to the indefinite conclusion E and then changes back to its previous state after 1 or 2 years. This phenomenon serves as a warning that the test results will always be subject to some degree of random variation and, hence, that we will need to have several successive concordant results before being confident that the appropriate model has been established.

The series for males 16–19 years old has 9 indefinite results out of 16 tests, which indicates the extent to which erratic variations have affected this component. The results that are conclusive are multiplicative for bases up to 1965, additive for bases after 1966, and one mixed result for 1963–69. On this evidence, we would suggest a model that changes from multiplicative to additive in 1965 or 1966, although the conclusion must be rather tentative.

The series for adult males is very consistent; apart from one indefinite result and two mixed results, the indications are unambiguously multiplicative throughout.

The series for females 16–19 years old also gives a clear picture. For bases ending in 1966 or earlier, the results are multiplicative (except for one indefinite), while, for bases beginning in 1966 or later, the results are additive; all bases spanning 1966 give indefinite results. On these figures, we could confidently state that the model changes from multiplicative to additive in 1966.

The series for adult females gives indefinite results for all base positions except one, and nothing can be said about the appropriate choice of model.

If these conclusions are compared with the earlier discussion of figures 3–6, it will be seen that the agreement is close. The only minor divergence is over the dating of the change from multiplicative to additive for teenage unemployment, where the model test results indicate dates 1 or 2 years earlier; taking all the evidence into account, a reasonable compromise is to take 1966 as the date of the change for male and female teenagers.

ADJUSTMENT OF DUTCH MALES SERIES BY THREE DIFFERENT METHODS

We chose the Dutch series for males as an example of data that behave neither purely multiplicatively nor purely additively in order to compare the results obtained by adjusting it by the X–11 additive procedure, the X–11 multiplicative procedure, and the Durbin-Murphy [2] mixed method. Figure 11 displays graphs of the seasonally adjusted series produced by each of the three methods. In each case, the adjustment was performed under operational conditions, i.e., each value was adjusted using only observations available up to the time at which the value was observed. Standard options were used for both X–11 adjustments. The mixed-model regressions were fitted without time-varying terms, time variation being allowed for by the use of a local amplitude scaling factor.

The results show that the X–11 multiplicative adjustment is completely unsatisfactory, particularly towards the end of the time period when the trend is rising fairly rapidly. The presence of the additive component in the seasonal variation then leads to overadjustment. The performance of the X–11 additive procedure is more acceptable, but there is a slight suggestion of some residual seasonality being left in the series due to underadjustment, particularly in the winter months. On the other hand, the adjustment based on the mixed model appears satisfactory, since it does not seem to give rise to residual seasonality.

GENERAL DISCUSSION

The evidence presented earlier demonstrates that some unemployment series behave, at times, in a way that is neither purely additive nor purely multiplicative. In consequence, at times of rising or falling trend, standard methods of adjustment based on purely additive or purely multiplicative techniques, can give rise to serious over- or under-adjustment. For situations where there is doubt if a series is behaving additively, multiplicatively, or a mixture of both, we have suggested some diagnostic techniques for investigating the matter. Methods for adjusting series displaying mixed behaviour were suggested by Durbin and Murphy, but we would not suggest that their approach to the problem is the only possible one. It may be that relatively simple modifications to standard procedures, such as those in the X–11 package, can be devised.

As a contribution to the discussion of possibilities in this direction, we will suggest two such modifications. The first is based on the idea of estimating the underlying pattern of the seasonal variation by the X–11 program used additively and then adjusting the amplitude of the seasonal variation by means of a local amplitude scaling factor estimated as described in [2, sec. 4.2], i.e., if z_{ij} is the observation, x_{ij} is the estimated trend, and a_{ij} is the estimated additive seasonal factor in the j^{th} month of the i^{th} year, the regression

$$z_{ij}-x_{ij}=d\, a_{ij}+\text{residual}$$

of deviation from trend $z_{ij}-x_{ij}$ on a_{ij} is fitted over a short stretch of data terminating (for adjustment of the current observation) at the value to be adjusted. The revised seasonal factor $d\, a_{ij}$ is then used instead of a_{ij} itself to make the seasonal adjustment of the current value.

An objection to this proposal is that the dependence of amplitude on trend is not explicit. The following modified

Figure 11. ALTERNATIVE SEASONAL ADJUSTMENTS OF DUTCH UNEMPLOYMENT FOR MALES

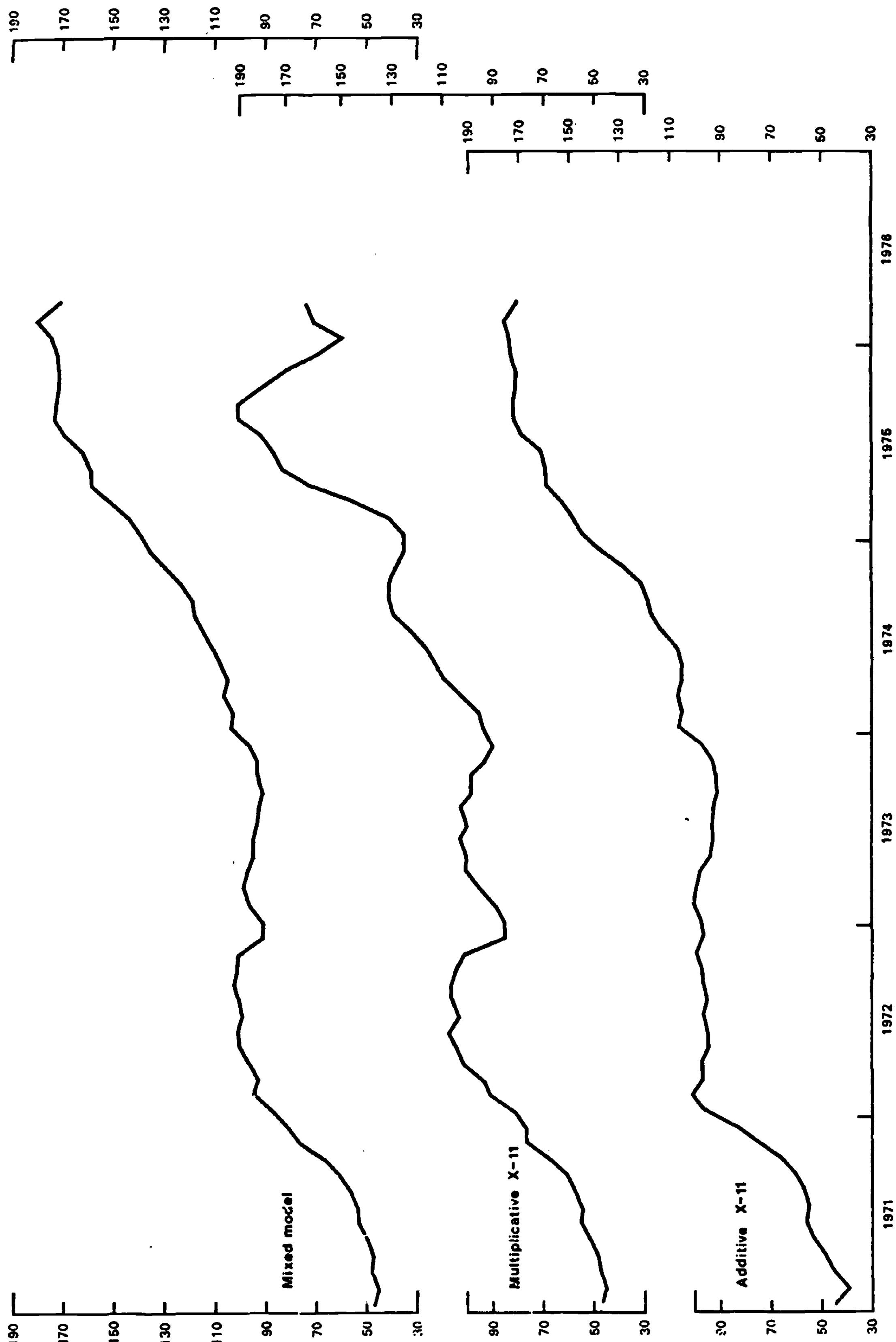


Table 3. ADDITIVITY/MULTIPLICATIVITY OF COMPONENTS OF U.S. UNEMPLOYMENT

Base position	Males 16-19 years old				Males 20 years old and over				Females 16-19 years old				Females 20 years old and over			
	F-values		Test result (1-percent level)	M	F-values		Test result (1-percent level)	M	F-values		Test result (1-percent level)	M	F-values		Test result (1-percent level)	M
	+Add	+Mult			+Add	+Mult			+Add	+Mult			+Add	+Mult		
1954 to 1960 . .	1.0	1.6	E	0.47	0.8	2.0	E	0.86	0.6	3.7	M	1.07	1.4	1.8	E	0.98
1955 to 1961 . .	1.4	2.9	M	.42	.7	2.7	M	.86	.8	5.5	M	.92	1.6	2.3	E	.84
1956 to 1962 . .	1.4	1.8	E	.36	.9	2.8	M	.92	1.3	2.3	E	.64	1.0	1.3	E	1.26
1957 to 1963 . .	.8	2.2	E	.62	1.5	2.9	M	.99	.9	4.3	M	.91	.5	.6	E	1.01
1958 to 1964 . .	1.4	2.9	M	1.49	3.0	4.4	B	1.25	.7	3.5	M	.92	.7	.9	E	1.30
1959 to 1965 . .	1.5	3.0	M	1.50	.6	3.2	M	1.14	.5	2.6	M	.60	.8	.9	E	1.23
1960 to 1966 . .	.7	1.0	E	.66	1.0	9.4	M	.95	.5	3.2	M	1.12	1.3	1.6	E	1.18
1961 to 1967 . .	.6	1.0	E	1.26	1.0	15.6	M	1.10	.6	1.5	E	.97	1.5	1.4	E	1.26
1962 to 1968 . . .	1.3	1.5	E	1.01	1.4	15.8	M	1.17	.5	1.7	E	1.22	.7	.8	E	.94
1963 to 1969 . .	3.3	3.2	B	1.37	4.0	26.6	B	1.29	.7	1.0	E	1.39	.9	1.0	E	1.05
1964 to 1970 . .	.7	.2	E	.38	.9	10.2	M	1.15	1.7	1.2	E	.07	1.0	1.0	E	.58
1965 to 1971 . .	2.2	.6	E	.07	.5	6.8	M	.93	2.2	1.1	E	.31	.7	.8	E	.49
1966 to 1972 . .	4.8	1.6	A	.08	1.1	5.7	M	.73	3.8	1.5	A	.35	.8	.8	E	.74
1967 to 1973 . .	5.9	1.2	A	.07	.9	6.0	M	.90	5.3	.9	A	.16	1.2	.8	E	.50
1968 to 1974 . .	4.9	1.2	A	-0.7	.4	5.3	M	1.08	4.4	.9	A	.09	1.6	1.3	E	.34
1969 to 1975 . .	2.0	1.8	E	.47	.5	8.3	M	.84	6.2	1.3	A	.03	3.1	2.9	B	.32

See footnotes below.

Table 4. ADDITIVITY/MULTIPLICATIVITY OF THE DUTCH UNEMPLOYMENT SERIES

(For all ages under 65 years old)

Base position	Males				Females				Total			
	F-values[1]		Test result (1-percent level)[2]	Degree of Multiplicativity	F-values[1]		Test result (1-percent level)[2]	Degree of Multiplicativity	F-values[1]		Test result (1-percent level)[2]	Degree of Multiplicativity
	+Add	+Mult			+Add	+Mult			+Add	+Mult		
1959 to 1965	1.5	62.2	M	0.85	0.7	3.3	M	1.08	1.0	61.9	M	0.92
1960 to 1966	1.4	11.4	M	.82	3.2	1.9	A	−.07	1.5	11.4	M	.85
1961 to 1967	3.5	15.4	B	.72	5.0	2.1	A	.49	2.6	14.1	B	.73
1962 to 1968	4.5	10.4	B	.64	2.8	2.9	B	.58	3.6	9.5	B	.65
1963 to 1969	4.4	7.0	B	.58	1.9	4.3	M	.60	3.6	6.1	B	.58
1964 to 1970	2.5	7.0	M	.64	1.4	3.5	M	.64	2.0	5.8	M	.65
1965 to 1971	2.1	4.8	M	.58	1.1	4.5	M	.72	1.8	4.0	M	.67
1966 to 1972	2.8	3.0	B	.50	3.3	11.6	B	.88	2.8	2.7	B	.55
1967 to 1973	2.5	2.5	E	.45	4.8	12.6	B	.97	2.9	1.8	A	.38
1968 to 1974	5.2	5.3	B	.54	6.0	31.4	B	1.03	5.7	4.7	B	.50

A Additive.
B Both additive and multiplicative.
E Either additive or multiplicative.
M Multiplicative.

[1] F-test has 11 and 62 degrees of freedom.
[2] 1-percent significance level is 2.56.

Table 5. ADDITIVITY/MULTIPLICATIVITY OF THE CANADIAN AND IRISH TOTAL UNEMPLOYMENT SERIES

Base position	Canada				Irish Republic			
	F-values[1]		Test result (1-percent level)[2]	Degree of Multiplicativity	F-values[1]		Test result (1-percent level)[2]	Degree of Multiplicativity
	+Add	+Mult			+Add	+Mult		
1959 to 1965	1.8	15.0	M	1.21	0.7	3.4	M	0.98
1960 to 1966	4.8	31.3	B	1.15	.6	1.6	E	1.13
1961 to 1967	5.2	31.8	B	1.19	.6	1.0	E	.58
1962 to 1968	.8	4.9	M	1.24	1.4	.3	E	.16
1963 to 1969	.4	1.0	E	.53	2.1	.6	E	–.08
1964 to 1970	8.0	4.9	B	.18	3.7	1.7	A	–.40
1965 to 1971	5.5	2.4	A	.32	2.7	.8	A	–.46
1966 to 1972	4.4	1.3	A	.32	3.0	.9	A	–.73
1967 to 1973	3.1	.7	A	.17	2.9	1.2	A	–.58
1968 to 1974	.8	1.0	E	.57	1.4	.9	E	–.04

A Additive.
B Both additive and multiplicative.
E Either additive or multiplicative.
M Multiplicative.

[1] F-test has 11 and 62 degrees of freedom.
[2] 1-percent significance level is 2.56.

form may, therefore, be worth considering. Suppose that a_{ij} is an estimated additive seasonal factor, as before, and assume that the amplitude of the seasonal variation varies linearly with trend. One then fits the regression

$$z_{ij}-x_{ij}=(c+d\,x_{ij})a_{ij}+\text{error}$$

estimating the two constants c and d from a suitable period of data ending at the current observation. The revised additive seasonal factor is then $(c+dx_{ij})a_{ij}$ instead of a_{ij}.

Variations of these suggestions for modifying multiplicative seasonal factors instead of additive factors are easily worked out.

It would be interesting to know whether any series, other than unemployment series, show evidence of mixed additive-multiplicative behaviour. We ourselves have not encountered any.

Enquiries about the availability of the computer programs in the paper should be addressed to the Computer and Modeling Branch of the Central Statistical Office, Great George Street, London S.W.1.

REFERENCES

1. Burman, J. P. "Moving Seasonal Adjustment of Economic Time Series." *Journal of the Royal Statistical Society,* ser. A, 128 (1965): 534–558.
2. Durbin, J., and Murphy, M. J. "Seasonal Adjustment Based on a Mixed Additive-Multiplicative Model. *Journal of the Royal Statistical Society,* ser. A, 138 (1975): 385–410.

COMMENTS ON "SEASONAL ADJUSTMENT WHEN THE SEASONAL COMPONENT BEHAVES NEITHER PURELY MULTIPLICATIVELY NOR PURELY ADDITIVELY" BY J. DURBIN AND P. B. KENNY

Neva VanPeski
U.S. Federal Reserve System

The value of the paper by Durbin and Kenny is that it deals, in a practical way, with some problems that anyone who has tried to seasonally adjust a variety of series will recognize. It has been particularly helpful to me, because the method that it describes for seasonally adjusting monthly data is very similar to, although more highly developed than, a method for seasonally adjusting daily data that was developed by myself and other members of the Federal Reserve Board staff. The development of this daily seasonal method was suggested by Milton Friedman as an alternative to X–11 that could be used to adjust the money supply series, M1, for which daily data are available. Thus far, the method has been used only on the two components of the M1 series, demand deposits and currency. I would like to present a short description of the daily seasonal method and to discuss some of the problems still remaining in the method developed thus far. I will then compare the daily method to that of Durbin and Kenny and point out the differences between the two methods, which in large part stem from the greater number of observations dealt with in the daily seasonal method.

The first step in the daily adjustment method is to remove the intraweekly movement in the series. This is done by computing seven day-of-week factors and dividing them into the original series to get a day-of-week adjusted series. The day-of-week factors are computed as follows: First, the ratio of each day's observations to a 7-day centered moving average is computed. Next, the ratios for each day are averaged by quarters and analysis of variance tests are made for changes in the ratios between years and between the quarters within a year. If the tests show no significant change, seven day-of-week factors are computed by averaging ratios for all Mondays, all Tuesdays, etc.

Next, the trend is removed from the day-of-week adjusted series. Trend is computed as a 365-day centered moving average; the weekday-adjusted observations are divided by the trend to obtain ratios that combine the daily seasonal factor and the irregular. Because the next stage involves a Fourier transform of the data, the February 28 ratio is computed as the average of February 28 and 29, and February 29 is omitted. At this point, the program provides for the reduction of values that lie outside a selected multiple of the standard deviation of the seasonal irregular ratios. In adjusting the M1 series, this option was not used, because it had little effect on the final results.

The next step, which is quite parallel to Durbin and Kenny's procedure, is to make a Fourier transform of the seasonal irregular ratios. The coefficients A_k and B_k of the sine and cosine terms in the Fourier sequence

$$y(t)=1/2A_0+\sum_{k=1}^{182} A_k \cos\frac{(2\pi kt)}{365}+\sum_{k=1}^{182} B_k \sin\frac{(2\pi kt)}{365} \qquad t=1,\ 2 \ldots .\ 365 \tag{1}$$

are estimated as

$$A_k=\frac{2}{N}\sum_{t=1}^{N} y(t)\cos\frac{(2\pi kt)}{365},\quad k=0,\ 1,\ \ldots .\ 182$$

$$B_k=\frac{2}{N}\sum_{t=1}^{N} y(t)\sin\frac{(2\pi kt)}{365},\quad k=1,\ 2,\ \ldots .\ 182 \tag{2}$$

where N is 365 times the number of years in the series.

It would be possible to use these trigonometric coefficients, to construct daily seasonal factors, were it not for the fact that it is necessary to make allowance for abrupt spikes in the seasonal irregular series. These spikes, which can be easily spotted by charting the seasonal irregular series, occur in the M1 series on holidays and tax dates. It is necessary to make special allowance for them, both because they are not well approximated by a Fourier series and because they occur on different dates each year.[1] In order to take account of their effects, a regression is fitted that has the seasonal irregular ratios as the dependent variable and those trigonometric terms having the largest coefficients in (2), plus dummy variables for holidays and other special days as independent variables.

[1] Even holidays which occur on the same date each year, such as Christmas, will affect the money supply components on different dates if they fall on a weekend. One of the virtues of the daily method is the flexibility it allows in handling holidays, such as Easter, which may fall in a different month each year.

Neva VanPeski is an economist, Board of Governors, Federal Reserve System, Washington, D.C.

The number of sine and cosine terms used in the regression varies with the series being adjusted, inasmuch as only those terms are included that would have significant coefficients if all the trigonometric terms were included in the regression (leaving aside, for the moment, the influence of the dummy variables). In general, the greater the irregular component in a series, the smaller the number of trigonometric terms that will be used.

The results of this regression are used to construct daily seasonal factors which are the same for all years in the series.[2] Daily factors are multiplied times day-of-week factors to obtain final seasonal factors,[3] which are divided into the original observations to arrive at daily seasonally adjusted figures. Weekly and monthly seasonally adjusted values are computed as averages of the daily figures, and, thus, the weekly and monthly seasonally adjusted series are consistent with each other. Implied weekly and monthly seasonal factors are computed by dividing the unadjusted by the adjusted series. Factors for future periods may be projected by averaging projected daily factors; these will vary somewhat from the factors computed after the actual data become available, but the differences, at least for the M1 components, are too small to affect published figures.

The daily adjustment method was tested on the components of M1 for 1969–74, in which there was known to be little if any change in seasonal patterns. When comparisons were made between monthly averages of the adjusted daily series and an X–11 multiplicative adjustment with factors constrained to be stable, the results were quite similar. There were somewhat larger differences when the daily seasonal method was compared with a standard X–11 multiplicative adjustment.[4]

Further improvements are needed in the method, especially if it is to be used on series with changing seasonal patterns. The most important improvements needed are a better trend estimator (a 365-day moving average is obviously an inflexible trend estimator) and a method of allowing for changing seasonal patterns. The present method can handle a sudden shift in seasonal factors with dummy variables, but the only way to allow for gradual changes is to adjust successive spans of, e.g., 6 years, and use the new factors for the most recent years. These problems have been solved in the additive-multiplicative method described in Durbin and Kenny's paper; however, the techniques used cannot be easily transferred to the daily seasonal method.

Although the broad outlines of the daily seasonal method and the additive-multiplicative method are similar, a step-by-step comparison shows that the larger number of observations in a 1-year period when using daily data necessitates a difference in the techniques used for trend removal and for seasonal factor computation. In addition, there are differences in the two methods that are dictated by the assumptions made about the data. The additive-multiplicative method described by Durbin and Kenny is based on the assumption that error is unrelated to trend, i.e., the series is of the form

$$y=s_1+(1+s_2)t+e \qquad (3)$$

where y is the observed value, s_1 and s_2 are respectively the additive seasonal component and the multiplicative seasonal factor, t is trend and e is irregular; thus trend is first removed by subtraction, and then used as an independent variable in the regression which estimates s_1 and s_2. In contrast, the daily seasonal method is most appropriate for a series of the form

$$y=t(s+e) \qquad (4)$$

in which trend is assumed multiplicatively related to the irregular, and is removed by division.

In the additive-multiplicative method, trend removal is done in two stages. The filter used to estimate preliminary seasonal factors eliminates seasonal frequencies; it, therefore, performs the same job that a 365-day moving average does for a daily seasonal, but, in addition, it estimates a cubic, while a 365-day average only estimates a straight line. The second filter is applied to the seasonally adjusted series, and, while it does not eliminate seasonal frequencies, it is more effective in estimating low frequencies and eliminating high ones. Both filters are designed to come as close as possible to a specified frequency response function; this determines the criteria used for selecting the length and weights of the moving average. The same technique could (at least, theoretically) be used for the daily seasonal adjustment, but the length of the moving averages involved and the number of weights to be estimated makes it a much more formidable project.

What is left after trend removal is, in the daily seasonal method, a seasonal factor plus an irregular factor, while in the additive-multiplicative method, it is an additive seasonal component, a multiplicative factor times trend and an irregular component. Both methods use a Fourier transform of the data to separate the seasonal from the irregular; both methods do this by selecting from a Fourier sequence trigonometric terms having significant coefficients to represent the seasonal. However, the methods used to select the terms are different. In the daily seasonal method, selecting the terms is quite simple; only those are used that would have significent coefficients in a regression fitted with the seasonal irregular ratios as the

---

[2] It has been pointed out by Pierce [3] that, if the underlying model is assumed to be of the form $y=t(s+e)$, the seasonal factor should be $(s+e)/(1+e)$ in order to avoid overadjustment; however, in adjusting the M1 components, the seasonal factor was simply s, the sum of all trigonometric terms and holidays fitted by the regression.

[3] At this point, the seasonal factors should be forced to sum to 365 (or 366) to ensure that the overall level of the series remains the same after seasonal adjustment. This was not done in our work with the M1 series.

[4] Six years is too short a period of time to make a meaningful comparison with an X–11 moving adjustment, because a large portion of the series is adjusted with the special routines for terminal years at the beginning and end of the series.

dependent variable and all 365 trigonometric terms as independent variables. Because the residual in such a regression is simply the difference between the total variance of the observations and half the sum of the squared Fourier coefficients calculated in (2) and because the trigonometric terms are orthogonal, it is possible to test for the significance of the coefficients without actually fitting the regression. In the additive-multiplicative method, however, the selection is complicated by the fact that there are both additive and multiplicative seasonal components to be estimaed. Thus, each term in the Fourier sequence must be considered twice (once for its significance in estimating the additive component and again for the multiplicative), and the consequent loss of orthogonality means that another way must be found to select those terms that are significant. A stepwise regression is used for this purpose, and, for most series considered, the number of terms selected was 9 to 12 out of the 24.

One advantage of a seasonal adjustment method that estimates both additive and multiplicative seasonal components is that stable factors can take account of a gradually changing seasonal, which cannot be done using only stable multiplicative factors, as is done in the daily seasonal method. Since, for some series, this could be a major problem, one might want to consider the possibility of applying the method used by Durbin and Kenny to the daily series to select daily additive and multiplicative seasonal factors. However, it must be kept in mind that instead of selecting 9 to 12 coefficients out of a possible 24, it would be necessary to select possibly 50 or 60 out of 730, using stepwise regression.

REFERENCES

1. Burman, J. P. "Moving Seasonal Adjustment of Economic Time Series." *Journal of the Royal Statistical Society,* ser. A., pt. 4, 128 (1965): 534–558.

2. Durbin, J., and Murphy, M. J. "Seasonal Adjustment Based on a Mixed Additive-Multiplicative Model." *Journal of the Royal Statistical Society,* ser. A, pt. 3, 138 (1975): 385–410.

3. Pierce, David A. "Seasonal Adjustment of Daily Aggregates," January 24, 1975. (Unpublished paper.)

COMMENTS ON "SEASONAL ADJUSTMENT WHEN THE SEASONAL COMPONENT BEHAVES NEITHER PURELY MULTIPLICATIVELY NOR PURELY ADDITIVELY" BY J. DURBIN AND P. B. KENNY

Craig F. Ansley, W. Allen Spivey, and William J. Wrobleski
University of Michigan

The central feature of the paper under discussion and its predecessor, the paper by Durbin and Murphy [2], is the development of a time series components model that permits joint consideration of two sources of seasonal variation, one attributed to purely additive seasonal influence and one to a purely multiplicative seasonal influence. The model is written as

$$z_t=\xi_t+\alpha_t+\beta_t\,\xi_t+\epsilon_t \tag{1}$$

where z_t is an observation on the time series, ξ_t is the trend (a deterministic or random function of time), α_t is an additive seasonal factor, β_t is a multiplicative seasonal factor, and ϵ_t is the irregular component. Note that each component enters into (1) additively. Special cases of this model are the purely additive model

$$z_t=\xi_t+\alpha_t+\epsilon_t \tag{2}$$

in which $\beta_t=0$, and the purely multiplicative model

$$z_t=\xi_t+\beta_t\,\xi_t+\epsilon_t \tag{3}$$

in which $\alpha_t=0$. Note also that the purely multiplicative model (3) is not the same as the conventional multiplicative model used in component time series analysis and in Census II, X–11 type procedures.

Four modifications of the model (1) are available. A two-stage procedure, which utilizes a stepwise regression, can be used for estimating the trend and seasonal components. A local amplitude scaling factor, based on the work of Wald [3], can be employed when seasonal amplitudes change, but the seasonal pattern remains constant, and a further modification is available to accommodate a changing seasonal pattern in addition to changing amplitudes. Finally, procedures or options for dealing with extreme values or outliers can be employed.

Properties possessed by the error term ϵ_t are not explicitly stated in the papers; we presume that they are: $E(\epsilon_t)=0$; $Var[\epsilon_t]=\sigma^2_\epsilon$; $Cov[\epsilon_{t_i},\epsilon_{t_j}]=0$, $t_i\neq t_j$; and the ϵ_t must be uncorrelated with the trend component ξ_t and the seasonal component $\alpha_t+\beta_t\,\xi_t$. In addition, normality of the ϵ_t may be required for F-tests that the authors employ in choosing between models (1), (2), and (3).

A problem arises when we rewrite equation (1) as follows:

$$\frac{z_t-\alpha_t}{1+\beta_t}=\xi_t+\frac{\epsilon_t}{1+\beta_t} \tag{4}$$

The right-hand side consists of trend plus an irregular component, and Durbin and Murphy take the left-hand side to define the seasonally adjusted series. However, under the assumption of homoscedasticity of ϵ_t, the variance of the seasonally adjusted series about the trend is

$$Var\left[\frac{\epsilon_t}{1+\beta_t}\right]=\frac{\sigma^2_\epsilon}{(1+\beta_t)^2} \tag{5}$$

In other words, the seasonally adjusted series has a seasonally varying variance about the trend, a most ingenious paradox, in the words of Sir W.S. Gilbert.

This leads to a fundamental question: Should one define a seasonally adjusted series to be one for which only the first moment of the irregular component need be free of seasonality or to be one for which all moments of this component are free of seasonality? We favor the latter definition and believe that the former leads to unacceptable practical difficulties.

One could avoid this complication by assuming that $Var[\epsilon_t]=(1+\beta_t)^2\sigma^2_\epsilon$. However, the statistical methods used by the authors in fitting the model parameters and constructing F-tests for multiplicative and additive components require homoscedasticity of errors in (1). Such an assumption would, therefore, necessitate modifying the statistical methods employed.

A useful check on the mixed model and its fitting procedures would be to simulate time series with known additive and multiplicative structure and to investigate whether this structure is estimated satisfactorily by the procedure. Time limitations and problems posed by computer program adaptation prevented us from following this approach; we, instead, confine our comments to methodological issues arising in the implementation of the authors' procedures.

Given a time series an important issue is, which of the models (1), (2), and (3) should be used for seasonal adjustment? The authors provide three complementary forms of assistance in dealing with this issue: F-tests [2, pp. 397–398], a multiplicativity statistic [2, pp. 398–399], and graphical methods [1]. Unfortunately, the use of these aids one at a time can lead the user into difficulties, and their joint use can prove to be inconclusive.

We consider first the F-tests, which pose problems. To see why this is so, a brief discussion of the two-stage fitting procedure recommended by Durbin and Murphy is necessary. The first of these stages consists of the application of a 21-point trend filter, followed by a stepwise regression procedure for the seasonal factors, and the second consists of a 13-point Burman filter applied to the seasonally adjusted series obtained from the first stage, followed by another stepwise regression. One justification offered for this procedure is that residuals obtained from the two-stage procedure appear to be more consistent with the model requirements of independent and homoscedastic errors than when the second stage is omitted. We recall the difficulty mentioned earlier that homoscedasticity of the errors in (1) leads to seasonally varying variance of the irregular component in (4).

Durbin and Murphy calculate F statistics in a variety of ways, none of which uses the residuals obtained from the above two-stage procedure. Tests are made using residuals calculated from a modified two-stage procedure in which stepwise regression is replaced by ordinary regression and are also made using residuals from the first stage only of the modified procedure. Moreover, Durbin and Murphy indicate that the test results vary considerably, depending upon which of the alternate procedures is applied beforehand. For example, in table 5 of the Durbin and Murphy paper, using only the first stage of the modified two-stage procedure for 1950–56, we would choose either the additive or the multiplicative models, but, if we use the modified two-stage procedure and enter the test sequence for the same period, the test indicates that a mixed model should be chosen. Indeed, for the 16 base periods of 7 years each, covering the years 1950 through 1971, in only five cases did consistent implications result from the F-tests applied to these two types of residuals. Furthermore, after noting the sensitivity of these tests to trend considerations, the authors state that tests based on the modified one-stage procedure (using only the 21-point filter) are "likely to prove the more reliable" [2, p. 401].

Apparently, another ingenious paradox confronts the user of the F-test, even if he can resolve the problem of seasonally varying variance in (4). On the one hand, a two-stage procedure is recommended for fitting the model parameters in (1), because the residuals have desirable statistical properties (these are also desirable for the F-test), but we are told to use, in the F-test, a one-stage procedure the nature of whose residuals is not discussed and which would appear to be less appropriate for the test.

Furthermore, if we examine figure 5 of the Durbin and Kenny paper, in which the ratios of mean absolute deviations from a 21-point trend over each year to the mean trend are plotted as part of the graphical output used to complement the F-test, one would conclude that additivity has set in for Dutch male unemployment for 1971–74. However, inspection of figure 9 shows that during 1971 and 1972 the mixed model resembles the multiplicative X–11 model more than the additive X–11 model, and only beyond 1972 does the mixed model resemble more closely the additive X–11 model. It would be interesting to know what the F-tests indicated for 1971–72. If they failed to detect it subsequently, we would be using a mixed model when an additive model appears to be appropriate. Although the difference between the mixed and additive models is minor beyond January 1973, during 1971–72 the difference between them is considerable.

An important feature of the first stage of the two-stage procedure is the reparameterization of the model by means of a Fourier transformation [2, p. 392]. This transformation assumes a constant seasonal pattern and amplitude. Stepwise regression is used to estimate the Fourier coefficients. Durbin and Murphy indicate that, for selected time series, this procedure reduces, by approximately one-half, the number of Fourier coefficients which must be estimated. Thus, this clever reparameterization and stepwise regression application appears to produce about the same number of fitted components for the mixed model as would be necessary for either a purely additive or purely multiplicative model fitted by ordinary regression. In essence, a more flexible model structure can be extracted from a given set of time-series data without losing degrees of freedom. The degrees of freedom, thus reserved, reduce the possibility of oversmoothing in the two-stage procedure.

It would be interesting to know whether this favorable experience with the number of degrees of freedom occurs when the mixed model is applied to other time series. The user of the model must be cautious in applying the recommended two-stage fitting procedures, because, if there is little or no reduction in the number of degrees of freedom, the procedure is likely to lead to oversmoothing and serious distortion in the representation of all components of the time series.

We now turn to a discussion of the local amplitude scaling factor. The original two-stage procedure, suggested by Durbin and Murphy, requires a constant seasonal pattern and unchanging seasonal amplitudes. To accomodate the latter, a local amplitude scaling factor is introduced, which is based on the work of Wald [3]. The mixed model with local amplitude scaling factor can be written as

$$z_{ij}=\xi_{ij}+n_{ij}(\alpha_j+\beta_j\xi_{ij})+\epsilon_{ij} \tag{6}$$

where n_{ij} denotes the local amplitude scaling factor for the j^{th} month during the i^{th} annual period.

The authors suggest fitting the more general model (6) in two steps, the first of which is to take $n_{ij}\equiv 1$ and then to fit the constant seasonal mixed model by the one- or two-stage procedure, referred to previously, obtaining

$$z_{ij}=x_{ij}+a_j+b_jx_{ij}+e_{ij} \tag{7}$$

or

$$z_{ij}=x_{ij}+s_{ij}+e_{ij} \tag{8}$$

where

$$s_{ij}=a_j+b_jx_{ij} \tag{9}$$

are estimated seasonal components. The second step involves regressing $z_{ij}-x_{ij}$ on the s_{ij}, obtaining

$$z_{ij}-x_{ij}=d_{ij}s_{ij}+e^*_{ij} \tag{10}$$

yielding the estimated local amplitude scaling factors d_{ij} for the interval of years considered. The seasonally adjusted series is then obtained from (10) as

$$\hat{z}_{ij}=\frac{z_{ij}-d_{ij}a_j}{1+d_{ij}b_j} \tag{11}$$

The calculations are made for moving intervals of years.

We believe that this is an innovative and useful adaptation of Wald's method. It should be noted that the same scaling factor is applied to both the additive and multiplicative seasonal factors in (11). The method might be further refined by developing different scaling factors for each of the two seasonal factors. Alternatively, the first stage of the two-stage procedure (specifically, the Fourier reparameterization and stepwise regression) might be modified in order to recognize a changing seasonal pattern and amplitude at the outset, rather than developing a model under the assumption of a constant seasonal pattern and amplitude and then adjusting for departures from these conditions.

Finally, we present some general evaluative comments on the Durbin-Murphy-Kenny approach. First, as we have noted, the model is formulated in such a way that a seasonally varying variance of the irregular component is implied. Resolution of this feature can be made only at the cost of introducing serious methodological difficulties into the subsequent analysis. Second, the F-tests for discerning additive and multiplicative components utilize residuals with properties that are not made clear. Furthermore, it is found that the use of these F-tests, jointly with the multiplicativity statistic, produces contradictory signals. Third, the method has been found only marginally useful for selected time series of unemployment.

Durbin and Kenny suggest that the method of local amplitude scaling might be employed with the X–11 additive or multiplicative models as a complement to the changing seasonal options available in the X–11 program. Without using this modification, they found that there was some slight residual seasonality in the series in their table 9 for the X–11 model. Use of their mixed model, with this modification, apparently resolved this difficulty for the time series they examined. We believe that, if the local amplitude scaling factors were to be employed in the X–11 additive model, it would be found to be satisfactory, and there would be no need to use the mixed model.

The basic distinction between purely additive and purely multiplicative models lies in the way in which the amplitude of the seasonal variation is related to trend. In the additive model, the seasonal pattern is assumed to have constant amplitude, regardless of the level of the trend component, while in the multiplicative model, the amplitude is assumed to vary directly with the level of the trend component. The Durbin-Murphy-Kenny method combines additive and multiplicative features through the representation (1). All three models imply linear relationships between the seasonal amplitude and the trend level that do not change with time. In practice, Durbin, Murphy, and Kenny found such a constant relationship to be inadequate and introduced local amplitude-scaling factors to force the seasonal amplitudes of their model to conform more closely to those of the actual time series. Such an adjustment could be used in a purely additive or purely multiplicative model and should produce similar results. Thus, we conjecture that the X–11 additive or X–11 multiplicative models with local amplitude-scaling factors will behave as well as the mixed model for any time series and would be much simpler to apply.

REFERENCES

1. Durbin, J., and Kenny, P. B. "Seasonal Adjustment When the Seasonal Component Behaves Neither Purely Multiplicatively Nor Purely Additively."
   Included in this report.
2. ——— and Murphy, M. J. "Seasonal Adjustment Based on a Mixed Additive-Multiplicative Model." *Journal of the Royal Statistical Society,* ser. A, pt. 3, 138 (1975): 385–410.
3. Wald, A. *Berechnung und Asschaltung von Saisonschwankungen, Beitrage zur Konjunkturforschung.* No. 9. Wien: Verlag von Julius Springer, 1936.

RESPONSE TO DISCUSSANTS

J. Durbin
London School of Economics
and Political Science
and
P. B. Kenny
Central Statistical Office

The paper by Ansley, Spivey, and Wrobleski has pointed out that, whereas the Durbin-Murphy paper recommends the use of a two-stage procedure for seasonal adjustment because of the desirable properties of the resulting residuals, it advocates the use of a one-stage procedure in the model-test program. The reason is that experience with actual series showed that the result of a two-stage F-test is biased in favour of the model assumed at the first stage. The use of a one-stage procedure avoids this bias, and, for the purpose of testing, we, therefore, regard it as the lesser of two evils.

On the Dutch series for males, we must emphasize that any operational adjustment procedure can only make use of information available at the time. It would not, in fact, have been possible to observe the shift toward additive behaviour until the end of 1972. We regard the results from the mixed model in 1971 and 1972 (fig. 11) as being intermediate between those from the multiplicative and those from the additive model, rather than showing an affinity for one rather than the other. In fact, we regard the behaviour of the graphs in figure 11 as entirely consistent with the evidence in figure 7.

We find VanPeski's description of her work on the money supply extremely interesting.

Section V.

New Methods for Analyzing Seasonal Problems