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5 Policy coordination and dynamic games

MARCUS MILLER AND MARK SALMON*

Introduction

The conventional theory of economic policy has been revised to take account of forward-looking expectations formed in the private sector and John Driffill (1982) has shown how 'rational' expectations in the foreign exchange market affect optimal monetary policy in a small open economy. For large open economies with substantial spillover effects, policy may need to take into account overseas reactions. National policy makers may thus find themselves in a strategic relationship with each other as well as with private market speculators.

In a number of papers Koichi Hamada analysed the relations between national economic policy makers as a static game, contrasting non-cooperative Nash or Stackelberg outcomes with those which might be achieved by co-operation, see Hamada (1979) and references therein. Recently, Oudiz and Sachs (1984) have made a bold attempt to estimate the potential benefits of international policy co-ordination, treating policy formation as a static game.

The dynamic aspects of economic interdependence emphasised by Hamada and Sakurai (1978) have invited the application of dynamic game theory, where policy makers minimise costs over an extended period of time. The papers by Sachs (1983) and Turner (1984) effectively complement the work of Hamada and Sakurai by treating the policy variables as the instruments in a Nash dynamic game.

While it is true that these models include dynamic elements, they exclude strategic 'asymmetries': no country acts as a Stackelberg leader, and there are no forward-looking elements in private sector behaviour to take account of in designing policy. (The exchange rate is determined only by the current account, assuming zero capital mobility.) In an earlier paper, Miller and Salmon (1983), methods for solving dynamic games with both of these characteristics were described, with particular emphasis on the analogy between a Stackelberg leader in an open loop dynamic game and a government announcing policy to a market with forward looking expectations. It was noted that such 'asymmetries' lead to the *time inconsistency* of optimal policy – the temptation for a leader to depart from previously announced plans when the announcement effects of policy upon 'forward looking' followers have been achieved.

In the absence of precommitment, however, such time inconsistent optimal policy lacks credibility. In this paper, therefore, we focus on how various *time consistent* solutions may be obtained and computed for situations involving both strategic asymmetry and forward-looking expectations.

To characterise the relations between policy makers, we examine first the symmetric equilibria of Nash differential games, both open and closed loop. We then consider the open and closed loop Stackelberg equilibria arising when one of the players is elevated to the status of leader, but constrained to implement time consistent policies. As between the governments and forward looking markets (here the foreign exchange market) two alternatives are considered. Either policy makers treat the path of the exchange rate as given in determining their policy, or the exchange rate is taken to be a given linear function of the state of the system.

A general description of the various solutions obtained in this way is provided in Section 1 with technical details available in the Annex. In the following section a two-country version of the model proposed in Buiter and Miller (1982) is introduced and a number of these equilibria computed and compared given fairly standard objectives with respect to the control of inflation and output. This relatively unfavourable performance of co-ordinated policy in this application doubtless reflects the absence of any long run 'conflict of objectives', a point to which we return in conclusion.[1]

I Time consistent equilibria

After a brief account of the linear dynamic system and the quadratic costs to be minimised, we describe a number of time consistent equilibria, which follow from varying both relationships prevailing between the policy makers themselves as well as those prevailing between the policy makers and private markets. Technical details are available in the Annex.

(a) Linear dynamics and quadratic costs

Throughout this paper we assume a constant parameter linear differential equation system of the form

$$Dx(t) = Ax(t) + B_1 u(t) + B_2 v(t) \tag{1}$$

where $x(t)$ is a vector of 'state' variables

$u(t)$ is the vector of 'control' variables associated with player 1

$v(t)$ is the vector of 'control' variables associated with player 2

D is the differential operator, $Dx \equiv \frac{d}{dt}(x)$.

The state vector $x(t)$ is partitioned between those variables $x_1(t)$ which are predetermined at time t, and those $x_2(t)$ which are not. The latter represent forward-looking asset prices which discount expected future events and move flexibly as 'news' about such events arrives. (Time subscripts may be omitted from the vector notation where convenient.)

Each of two policy makers ('players') minimises a quadratic cost function over an infinite horizon, viz.

$$V_i(t_0) = \int_t^\infty [x^T(s)\, u^T(s)\, v^T(s)]\, Q_i \begin{bmatrix} x(s) \\ u(s) \\ v(s) \end{bmatrix} ds \quad i = 1, 2 \tag{2}$$

where Q_i is positive semi definite, $i = 1, 2$

Q_{1uu}, Q_{2vv} positive definite

and the integral is assumed to converge without discounting.

(b) Strategic relations between policy makers

To focus upon the relationships between policy makers, we first describe open and closed loop Nash and Stackelberg differential games on the assumption that the entire state vector is predetermined.

The Nash equilibria are familiar, see Basar and Olsder (1982), and can be briefly described. The first occurs when each player takes the entire future path of the other's controls as given when computing his or her own controls. Thus in choosing $u(t)$, $t \geqslant t_0$ to minimise $V_1(t_0)$ subject to the dynamics of the system described by equation (1) player one takes $x(t_0)$ and $v(t)$, $t \geqslant t_0$ as given; while player two in minimising $V_2(t_0)$ takes as given the path for $u(t)$, $t \geqslant t_0$. The *Open Loop Nash* equilibrium is defined by the requirement that the control path which each player takes as predetermined should match what the other player chooses as optimal.

The second form of Nash equilibrium which we compute occurs when each player responds to the current state of the system and recognises that the other player is responding likewise. Specifically player 1 minimises V_1 by choosing a closed loop or feedback rule for u on the hypothesis that player two operates a linear rule $v = R_2 x$, while player 2 minimises V_2 assuming $u = R_1 x$. The *Closed Loop Nash* equilibrium is defined by the requirement that the feedback rule either player assumes for the other should in fact be the optimal current state feedback rule for that other player. While it is straightforward to compute the open loop Nash solution for a linear quadratic game, the feedback rules required for the closed loop solution have to be found iteratively. The procedure used in this paper is to compute optimal rules $\hat{R}_1$, $\hat{R}_2$ for players one and two conditional on some arbitrary initial values for R_1 and R_2 (typically zero). These optimal rules replace the initial estimates, and this procedure is repeated until convergence is achieved.

So far both players have been treated symmetrically but we now consider the consequences of one player acting as a Stackelberg leader. One immediate consequence is that if player 1 is in the position of announcing at t_0 a path for his controls $u(t)$, $t \geqslant t_0$, conditional upon which player 2 will choose $v(t)$, $t \geqslant t_0$, the optimal path for u is *time inconsistent*; so that recalculating an optimal plan at a later date $t_1 > t_0$ will not produce a continuation of the chosen path for u.

Simaan and Cruz (1973), who used the maximum principle to compute the time inconsistent optimal solution for the linear quadratic Stackelberg differential game, argued that *time consistent* optimal plans could be obtained by using dynamic programming methods (cf. also Cruz (1975) for a general discussion of the class of Stackelberg equilibria generated in this way).

However, Cohen and Michel (1984) have recently shown that the maximum principle may also be used to obtain the time consistent solution for the *Open Loop Stackelberg* game by imposing an appropriate constraint on the leader's optimisation. Thus while choosing u to minimize $V_1(t_0)$ subject to

$$Dx = Ax + B_1 u + B_2 v \tag{3}$$

given $x(t_0)$ and the follower's first order conditions, namely

$$\partial H^2/\partial v = 0 \rightarrow v = R_{vx} x + R_{vu} u + R_{vp_2} p_2 \tag{4}$$

$$Dp_2 = \partial H^2/\partial x \tag{5}$$

where p_2 are the follower's costate variables and

$$H^2 = [x^T u^T v^T] Q_2 \begin{bmatrix} x \\ u \\ v \end{bmatrix} + p_2^T Dx$$

is the follower's Hamiltonian generates the time inconsistent solution, they argue that the optimal time consistent solution can be obtained by choosing u to minimise $V_1(t_0)$ given $x(t_0)$ subject to (3), (4) and the constraint that

$$p_2 = \theta x \tag{6}$$

where θ is chosen so that the solution also satisfies equation (5).

Table 5.1. *Varieties of non-cooperative behavior*

	Objectives[1]	Strategic constraints[2]			
		Symmetric Games		Asymmetric Games[3]	
	(Value functionals to be minimised)	Open Loop Nash	Closed Loop Nash	Open Loop Stackelberg	Feedback Stackelberg
Player 1	$\min_u V_1(t_0) = \int_{t_0}^{\infty} [x^T u^T v^T] Q_1 \begin{bmatrix} x \\ u \\ v \end{bmatrix}$	path for $v(t)$, $t \geqslant t_0$ treated as open loop	$\left.\frac{\partial v}{\partial u}\right\|_x = 0$ $v = R_2 x$	$v = [R_{vu}\ R_{vx}\ R_{vp_2}] \begin{bmatrix} u \\ x \\ p_1 \end{bmatrix}$ $p_2 = \theta x$	$v = [R_{vu}\ R_{vx}\ R_{vp_2}] \begin{bmatrix} u \\ x \\ p_2 \end{bmatrix}$ $v = R_2 x$
Player 2	$\min_v V_2(t_0) = \int_{t_0}^{\infty} [x^T u^T v^T] Q_2 \begin{bmatrix} x \\ u \\ v \end{bmatrix}$	path for $u(t)$, $t \geqslant t_0$ treated as open loop	$\left.\frac{\partial u}{\partial v}\right\|_x = 0$ $u = R_1 x$	path for $u(t)$, $t \geqslant 0$, treated as open loop	$\left.\frac{\partial u}{\partial v}\right\|_x = 0$ $u = R_1 x$

Notes: [1] The symmetric matrices Q_1, Q_2 are subject to positive definiteness conditions
[2] Other non-strategic constraints include the state equation and the initial conditions
[3] Player 1 acting as leader

It is clear that the result is time consistent, since substitution of (4) and (6) into (3) implies that the leader faces an orthodox single controller problem. The remarkable feature of Cohen and Michel's procedure is that it generates the 'dynamic programming' solution, because the constraint on p_2 reflects the restriction on the leader's choice of controls imposed by Bellman's principle of optimality. (The calculation of the appropriate value for θ may be achieved by using the iterative procedures already mentioned.)

The last time consistent equilibrium to be described is for a Stackelberg game where each player is aware of the other's feedback rule, but the leader is, in addition, able to exploit the follower's reaction function, see Basar and Haurie (1982). It is the leader's ability, by his current choice of u, to affect the follower's choice of v that makes the game asymmetric and distinguishes it from the closed loop Nash equilibrium. Thus while player 2 minimises V_2 subject to $u = R_1 x$, the leader minimises V_1 subject to both $v = R_2 x$ and (4), the follower's reaction function. The *Feedback Stackelberg* equilibrium is defined by the requirement that the feedback rules assumed for the other player are optimal for that player.

In table 5.1 we summarise the strategic constraints which these four relationships impose on the optimisation problem facing each player, in addition to the state equation (3) and the initial value $x(t_0)$ common to both players. The conditions which must be satisfied in equilibrium by the feedback rules, R_1, R_2, and the time consistency constraint, θ, appearing in the Table are spelt out in Annex 1.

(c) Policy co-operation with market anticipations

In this section we consider co-operative equilibria calculated by postulating a single 'policy-coordinator' who minimises the weighted sum of V_1 and V_2 given a state vector which contains 'forward-looking' asset prices, x_2, as well as predetermined variables, x_1.

If the policy co-ordinator, starting at time t_0 and acting as a Stackelberg leader *vis-à-vis* private markets, were to choose paths $u(t)$, $v(t)$, $t \geqslant t_0$ so as to minimise the weighted sum of $V_1(t_0)$ and $V_2(t_0)$ subject to all of the state equations

$$Dx_1 = A_{11}x_1 + A_{12}x_2 + B_{11}u + B_{12}v \tag{3a}$$

$$Dx_2 = A_{21}x_1 + A_{22}x_2 + B_{21}u + B_{22}v \tag{3b}$$

given $x_1(t_0)$, with $x_2(t_0)$ responding to the announced policy, then the resulting plan would be time inconsistent in the absence of precommitments. So we seek time consistent alternatives.

One is the 'loss of leadership' solution proposed by Buiter (1983) where the policy coordinator treats the path of asset prices as predetermined when designing policy. Specifically the weighted sum of $V_1(t_0)$ and $V_2(t_0)$ is minimised subject to (3a) (the state equations for x_1), but given $x_1(t_0)$ and $x_2(t)$, $t \geqslant t_0$. Equilibrium is defined by the requirement that the assumed path of asset prices be a correct discounting of the chosen policy, so that the remaining state equations for x_2 will also be satisfied. As the label suggests, this approach achieved time consistency by denying the leader the asymmetric position initially assumed, and postulating instead a type of Nash equilibrium.

Time consistent policy which is compatible with strategic asymmetry may, however, be determined by applying Bellman's principle of optimality. Proceeding by analogy with Cohen and Michel (1984), we assume that co-operative policy will, in this infinite horizon linear-quadratic context, be constrained to a fixed linear feedback rule, which may be found by replacing the state equations (3b) in the time inconsistent solution by an appropriate constraint of the form

$$x_2 = \theta_0 x_1 \tag{7}$$

so co-operative policy will minimise the weighted sum of $V_1(t_0)$ and $V_2(t_0)$ subject to (3a) and (7), given $x_1(t_0)$. As for the analogous open loop Stackelberg constraint, equation (7) must in equilibrium generate values for x_2 which correctly discount the optimal policy, so that the remaining state equations are satisfied. The calculation of θ_0 is discussed further below and in Miller and Salmon (1985).

(d) Non-cooperative policy with market anticipations – a summary

Finally we consider the nature of the time consistent equilibria which arise from various forms of non-cooperative behaviour in a dynamic setting with forward-looking financial markets. A summary is presented in Table 5.2, which indicates the constraints on optimisation.

The derivatives appearing in the first two rows come from the strategic constraints shown in Table 1 above. Thus the entries in the top of column (1) indicate that, for the Open Loop Nash game where player one takes the other's control $v(t)$, $t \geqslant t_0$ as predetermined, both $\partial v/\partial u$ and $\partial v/\partial x_1$ are assumed to be zero by player one. In columns (2) and (4) we show the feedback coefficients assumed by each player to characterise the behaviour of the other in closed loop equilibrium.

Where player one acts as a Stackelberg leader, he or she can exploit the reaction function of the follower, namely

$$v = R_{vu}\, u + R_{vx}\, x + R_{vp_2}\, p_2$$

so that the term $\partial v/\partial u$ is no longer zero in columns (3) and (4). In the open

Table 5.2. *Non-cooperative time-consistent solutions*

Strategic Constraints on Optimisation[a]		Relation between policy makers: Symmetric (1) Open Loop Nash	Symmetric (2) Closed Loop Nash	Asymmetric[b] (3) Open Loop Stackelberg	Asymmetric[b] (4) Feedback Stackelberg
On Player One	$\left.\frac{\partial v}{\partial u}\right\|_{x_1}$	0	0	R_{vu}	R_{vu}
	$\frac{\partial v}{\partial x_1}$	0	R_2	R	R_2
On Player Two	$\left.\frac{\partial u}{\partial v}\right\|_{x_1}$	0	0	0	0
	$\frac{\partial u}{\partial x_1}$	0	R_1	0	R_1
On Both Players[c]	$\frac{\partial x_2}{\partial x_1}$	$0/\theta_0$	$0/\theta_0$	$0/\theta_0$	$0/\theta_0$

Notes: [a] x_1 denotes predetermined state variables, x_2 'asset prices'
[b] Player one acting as leader
[c] Arising from alternative assumptions as to the relationship between policy makers and forward looking asset prices.

loop Stackelberg case, the leader uses this reaction function together with the constraint that

$$p_2 = \theta x$$

in calculating the feedback rule for the follower.

It is important to note that the relevant state vector for all the feedback rules is only x_1. As we indicate symbolically in the last line the variables x_2 are *either* treated as predetermined *or* as linear functions of x_1, given by

$$x_2 = \theta_0 x_1$$

as discussed in the preceding section.

II Anti-inflationary monetary policy in a two-country setting

The notion that a floating exchange rate would ensure that money was neutral in its effects was challenged by Dornbusch (1976) who, in a model with floating rates and perfect capital mobility, showed that temporarily rigid ('sticky') goods and/or factor prices were enough to ensure substantial non-neutrality in the short run. Hamada and Sakurai (1978), in a two-country model with the same short run stickiness of nominal factor prices but with zero capital mobility, showed how domestic monetary policies could generate spillover effects overseas despite freely floating rates.

It is naturally a matter of some interest to see what such interdependence implies for the design of policy in general and for the gains to policy co-ordination in particular. While non-cooperative commercial policy can evidently inflict considerable welfare losses, recent analysis using the Hamada/Sakurai framework (of countries linked *only* by trade in goods) suggests that this may not carry over to macroeconomic policy coordination, see Paul Turner (1984). In this paper, therefore, we investigate some implications of policy interdependence on an environment where two countries (with a floating exchange rate but sticky factor/goods prices) are linked by both trade and capital flows – indeed capital is assumed to be perfectly mobile.

The structure used is that developed for one country in Buiter and Miller (1982), to which the reader is referred for more complete discussion. Briefly, there is a 'Keynesian' determination of aggregate production, an augmented Phillips curve governing inflation, and perfect capital mobility with forward looking expectations in the foreign exchange market. (The design of time inconsistent optimal policy in a single open economy of this sort has already been comprehensively analysed by Driffill (1982)).

(a) Economic model and policy objectives

Definition of variables, and of notation used

- i rate of change of consumer price index
- π 'core' inflation
- y output (in logs), measured from 'natural rate'
- z integral of past output
- c 'competitiveness' for home country (in logs), i.e. real price of foreign goods
- r real consumption rate of interest
- p_s costate (for state variable s)
- H Hamiltonian

Table 5.3. *Economic model and policy objectives*

Home Country	Overseas Country
Model Structure	
Static Equations	
Aggregate Demand[a]	
$y = -\gamma r + \delta c + \eta y^*$	$y^* = -\gamma r - \delta c + \eta y$
Phillips Curve	
$i = \phi y + \sigma Dc + \pi$	$i^* = \phi y^* - \sigma Dc + \pi^*$
Core Inflation	
$\pi = \xi\phi z + \xi\sigma c$	$\pi^* = \xi\phi z^* - \xi\sigma c$
Dynamic Equations	
Accumulation $Dz = y$	$Dz^* = y$
Arbitrage $E[Dc] = r - r^*$	
Economic Policy	
Loss Function	
$\min_{r} V \equiv \frac{1}{2}\int_t^\infty \beta\pi^2 + y^2$	$\min_{r^*} V^* \equiv \frac{1}{2}\int_t^\infty \beta\pi^{*2} + y^{*2}$
Hamiltonian	
$H = \frac{\beta\pi^2 + y^2}{2} + p_z Dz + p_{z^*} Dz^*$	$H^* = \frac{\beta\pi^* + y^{*2}}{2} + p^*_{z^*} Dz^* + p^*_z Dz$

Note: [a] By substitution $y = -\kappa\gamma r - \kappa\eta\gamma r^* + \kappa(1-\eta)\delta c$, $y^* = -\kappa\gamma r^* - \kappa\eta\gamma r - \kappa(1-\eta)\delta c$ where $\kappa \equiv (1-\eta^2)^{-1}$

- V Loss function, integral of costs
- D differential operator, $Dx = dx/dt$
- E expectations operator

subscript a denotes average
subscript d denotes difference
superscript * denotes variable pertaining to foreign country

The equations of the model are listed in Table 5.3. The first pair of equations show local output being 'demand determined' where demand depends on the real consumption rate of interest, on the real exchange rate and on the level of real output overseas. (This is something of a 'reduced form' where the dependence of demand on local output has been solved out.) The next pair of equations show that the rate of change of the consumer price index in each country depends on demand pressure, on 'core' inflation and on the change in the real exchange rate (where σ represents the share of imports in the price index).

Core inflation is itself determined as the weighted sum of two components:

a backward looking integral of past output and the current level of the real exchange rate. The latter is in turn a 'forward looking' integral of expected international real interest rate differentials, as implied by the arbitrage condition.

We characterise the stance of domestic monetary policy simply by the level of the domestic real interest rate, and it is assumed that policy makers aim to minimise the undiscounted integral of a quadratic function of output and core inflation. Although no direct 'costs' are attached to the level of the real interest rates, the structure of the model implies that welfare in each country depends on domestic and overseas interest rates, in addition to the 'state variables' z, z^* and c.

(b) Co-ordinated Policy

It is easiest to begin the study of policy design with cooperative behaviour for, as we have seen, this can be treated as a single-controller problem by the convenient fiction that a 'policy co-ordinator' chooses r and r^* to minimise the weighted average of national welfare costs, V and V^*.

The necessary conditions for such a coordinator minimising an equally weighted sum are shown in Table 5.4, for the time-inconsistent optimal policy on the right and for time consistent policy on the left. Note that time consistency has been achieved by simply dropping p_c, the costate variable for the real exchange rate, and replacing this by the assumption that the latter is a stable function of the state variables, i.e.

$$c = \theta_1 z + \theta_2 z^*$$

(The appropriate choice of θ_1, θ_2 to ensure that Bellman's principle is satisfied are discussed below). For the 'Nash' time consistent alternative, where the policy co-ordinator treats the real exchange rate as completely predetermined independently of his actions and of the state variables, the necessary conditions are obtained by setting θ_1 and θ_2 to zero, in Table 5.4.

The numerical results[2] produced by the choice of an arbitrary, but plausible, set of parameter values namely

$$\beta = \phi = \xi = 1,\ \gamma = \delta = \tfrac{1}{2},\ \eta = \tfrac{1}{3},\ \sigma = \tfrac{1}{10},$$

are shown in Table 5.5. While all three policies possess a stable root of unity, time inconsistent policy has two others larger in absolute size. Each of the two time consistent policies has only one other stable root, less than unity in absolute size in both cases.

The Riccati coefficients in the table describe how the real exchange rate is related to the state variables along the stable path associated with these roots, and the so-called reaction coefficients provide the same information

Table 5.4. *Coordinated policy*

Time consistent	Time inconsistent
Hamiltonian	
$H=\frac{1}{2}\left(\frac{\beta^2+y^2}{z}\right)+\frac{1}{2}\left(\frac{\beta\Pi^{*2}+y^{*2}}{z}\right)$ $+p_z y+p_{z^*}y^*$	$H=\frac{1}{2}\left(\frac{\beta^2+y^2}{z}\right)+\frac{1}{2}\left(\frac{\beta\Pi^{*2}+y^{*2}}{z}\right)$ $+p_z y+p_{z^*}y^*+p_c c$
First order conditions	
$\frac{\partial H}{\partial r}=-\kappa\gamma(\frac{1}{2}y+p_z)$ $-\kappa\eta\gamma(\frac{1}{2}y^*+p_{z^*})=0$	$\frac{\partial H}{\partial r}=-\kappa\gamma(\frac{1}{2}y+p_z)$ $-\kappa\eta\gamma(\frac{1}{2}y^*+p_{z^*})+p_c=0$
$\frac{\partial H}{\partial r^*}=-\kappa\gamma(\frac{1}{2}y+p_{z^*})$ $-\kappa\eta\gamma(\frac{1}{2}y+p_z)=0$	$\frac{\partial H}{\partial r^*}=-\kappa\gamma(\frac{1}{2}y^*+p_{z^*})$ $-\kappa\eta\gamma(\frac{1}{2}y+p_z)-p_c=0$
$-Dp_z=\frac{1}{2}\beta\xi\phi\pi+\theta_1 H_c$	$-Dp_z=\frac{\partial H}{\partial z}=\frac{1}{2}\beta\xi\theta\pi$
$-Dp_{z^*}=\frac{1}{2}\beta\xi\phi\pi^*+\theta_2 H_c$	$-Dp_{z^*}=\frac{\partial H}{\partial z^*}=\frac{1}{2}\beta\xi\phi\pi$
where	
$H_c\equiv\frac{\partial H}{\partial c}=\frac{1}{2}\beta\xi\sigma(\pi-\pi^*)$ $+\kappa\delta(1-\eta)\left(\frac{y+y^*}{2}+p_z-p_{z^*}\right)$	$-Dp_c=\frac{\partial H}{\partial c}=\frac{1}{2}\beta\xi\sigma(\pi-\pi^*)$ $+\kappa\delta(1-\eta)\left(\frac{y+y^*}{2}+p_z-p_{z^*}\right)$

Notes: State variables z, z^*, c	State variables z, z^*, c
Costate variables p_z, p_{z^*}	Costate variables p_z, p_{z^*}, p_c
In equilibrium $c=\theta_1 z+\theta_2 z^*$	

for the real interest rates. Thus for the 'Nash' equilibrium tabulated in column (a) we found

$$\begin{bmatrix} c \\ r \\ r^* \end{bmatrix} = \begin{bmatrix} -0.835 & 0.835 \\ 1.048 & 0.285 \\ 0.285 & 1.048 \end{bmatrix} \begin{bmatrix} z \\ z^* \end{bmatrix}$$

As might be expected from the symmetry of the two economies, for all three solutions we find z and z^* have Riccati coefficients which are equal but of opposite sign and possess symmetric reaction coefficients.

These coefficients are used to calculate the initial values shown in the lower half of the table, assuming an inherited rate of core inflation of 10% in the home country and zero overseas. All three policies respond with real interest rates of about 10% at home and 3% overseas, which implies an initial loss of competitiveness of about 8% for the inflationary economy.

From the last line of the table is is apparent that there is little difference

Table 5.5. *Coordinated policy*

	Time consistent				Time inconsistent optimal policy		
	(a)		(b)				
Roots							
Averages	−1.0		−1.0		−1.0		
Differences	−0.913		−0.842		−1.200, −1.667		
Ricatti Coefficients							
$\theta_1, \theta_2, (\theta_3)$	−0.835	0.835	−0.790	0.790	−0.833	0.833	3.333
Reaction Coefficients							
$\rho_1, \rho_{12}, (\rho_{13})$	1.048	0.285	0.999	0.334	0.994	0.388	−6.444
$\rho_{21}, \rho_{22}, (\rho_{23})$	0.285	1.048	0.334	0.999	0.388	0.944	6.444
Initial Value							
$z, z^*, (p_c)$	10	0	0	0	10	0	0
c	−8.36		−7.90		−8.33		
r, r^*	10.48	2.85	9.99	3.34	9.44	3.89	
y, y^*	−9.56	−0.44	−9.21	−0.79	−9.17	−0.83	
π, π^*	9.16	0.84	9.21	0.79	9.17	0.83	
Welfare Costs							
Average of V and V^*	22.96		23.03		22.92		

Notes: (a) Path of real exchange rate assumed to be exogenously given in designing policy. (b) Path of real exchange rate assumed to be a linear function of stable state variables.

Table 5.6. *Time consistent coordinated policy*

Averages	Differences
Model Properties	
$\pi_a = \xi\phi z_a$	$\pi_d = \xi\phi z_d + 2\xi\sigma c$
$Dz_a = y_a$	$Dz_d = y_d$
	$Dc = r_d = \frac{2\sigma c}{g} - \frac{(1+\eta)}{\delta} y_d$
First Order Conditions	
$y_a = -2p_c$	$y_d = -2p_d$
$-Dp_a = \frac{1}{2}\beta\xi\phi\pi_a$	$-Dp_d = \frac{1}{2}\beta\xi\phi\pi_d + 2\theta_1 H_c$
	$H_c = \frac{1}{2}\beta\xi\sigma\pi_d + \frac{\delta}{(1+\eta)}(\frac{1}{2}y_d + p_d)$
Adjoint Equations	
$\begin{bmatrix} Dz_a \\ Dp_a \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -\frac{1}{2}\beta\xi^2\phi^2 & 0 \end{bmatrix} \begin{bmatrix} z_a \\ p_a \end{bmatrix}$	$\begin{bmatrix} Dz_d \\ D_c \\ -Dp_d \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 2\delta/\gamma & 3(1+\eta) & 0 \\ \frac{1}{2}\beta\xi^2\phi^2 & \beta\xi^2\phi\sigma & 0 & 2\theta_1 \\ \frac{1}{2}\xi^2\sigma\phi & \beta\xi^2\sigma^2 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_d \\ c \\ p_d \\ H_c \end{bmatrix}$
so $\lambda_a = -\sqrt{\beta\xi^2\phi^2}$	
Stable roots (see text for parameter values)	
$\lambda_a = -1$	$\begin{cases} \lambda_d = -0.913 \text{ where } \theta_1 = 0 \\ \lambda_d = -0.842 \text{ where } \theta_1 = C_{21} C_{11}^{-1} = -0.79 \end{cases}$

between the welfare costs arising under these policies. That the time inconsistent policy should generate the least cost is only to be expected. What is of more interest is that the time consistent 'Stackelberg' equilibrium is dominated by the 'Nash' equilibrium in this case.

The distinction between these last two policies can best be seen by a transformation of variables.[3] By forming 'averages', so $y_a = \frac{1}{2}(y+y^*)$, and 'differences', $y_d = y-y^*$, and assuming $\theta_1 = -\theta_2$, the system under control is decomposed into two separate blocs each involving but a single stable root, as shown in Table 5.6.

The system of averages with its root of unity is common to both time consistent solutions. Since the stable eigenvector is $\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$ this implies that $y_a = -z_a$; so the world average recession is equal in percentage terms to world average inflation, both declining with a unit root. Where the policies are distinct, of course, is in connection with the exchange rate which enters the other sub-system. To obtain the 'Nash' equilibrium, the parameter θ_1 in Table 5.6 is set equal to zero, which generates a root of -0.913, as we have seen, and a Riccati coefficient of -0.835, as $c = -0.835$ $(z-z^*) = -0.835\ z_d$ along the stable path. If one constrains the value of θ_1 $(= -\theta_2)$ assumed by the policy co-ordinator to match the Riccati coefficient of the system, one obtains the root of -0.842 and the Riccati coefficient of -0.79 which characterise the time consistent Stackelberg equilibrium.

(c) Non-cooperative policy

The solutions which emerge when the national policy makers set real rates in a non-cooperative fashion are examined in this section and provide some surprises.

Where the two monetary authorities are on an equal strategic footing the necessary conditions for optimisation for the resulting Nash games are given in Table 5.7. For the *Closed Loop Nash* equilibrium the overseas authority acts as if $r = \rho_{11} z + \rho_{12} z^*$ and the home government acts as if $r^* = \rho_{21} z + \rho_{22} z^*$, where these feedback coefficients satisfy constraints discussed in the Annex. For the *Open Loop Nash* game, of course, these feedback coefficients are omitted.

As before, two variants of the perceived relationship between policy makers and the foreign exchange market may be considered. If the monetary authorities take the exchange rate as predetermined in designing interest rate policy (so θ_1 and θ_2 are zero in Table 5.7), the result of decentralising policy is easy to describe. It has no effect! Both non-cooperative Nash equilibria will be the same as that shown for co-ordinated policy in the first column of Table 5.5. The reason for this is

Table 5.7. *Time consistent symmetric (Nash) game*[a]

Home Country	Overseas Country
First Order Conditions (10 equations)	
$\dfrac{\partial H}{\partial r} = -\kappa\gamma(y+p_z) - \kappa\eta\gamma p_{z^*} = 0$	$\dfrac{\partial H^*}{\partial r^*} = -k\gamma(y^*+p^*_{z*}) - \kappa\eta\gamma p^*_z = 0$
$-Dp_z = \beta\xi\phi\pi + \theta_1 H_c + \rho_{21} H_{r^*}$	$-Dp^*_{z^*} = \beta\xi\phi\pi^* + \theta_2 H^*_c + \rho_{12} H^*_r$
$-Dp_{z^*} = \theta_2 H_c + \rho_{22} H_{r^*}$	$-Dp^*_z = \theta_1 H^*_c + \rho_{11} H^*_r$
where	where
$H_{r^*} \equiv \kappa\eta\gamma(y+p_z) - \kappa\gamma p_{z^*}$	$H^*_r \equiv \kappa\eta\gamma(y^*+p^*_{z^*}) - \kappa\gamma p^*_z$
$H_c \equiv \beta\xi\sigma\pi + \delta(1+\eta)^{-1}(y+p_z-p_{z^*})$	$H^*_c = -\beta\xi\sigma\pi^* - \delta(1+\eta)^{-1}(y^*+p^*_{z^*}-p^*_z)$
Model Structure (7 equations)	
(IS)[b] $y = -\gamma r + \delta c + \eta y^*$	$y^* = -\gamma r^* - \delta c + \eta y$
(CI) $\pi = \xi\phi z + \xi\sigma c$	$\pi^* = \xi\phi z^* - \xi\sigma c$
(Accumulation $Dz = y$	$Dz^* = y^*$
(Arbitrage) $Dc = r - r^*$	

Notes:
[a] *VARIABLES*
'State' variables (3) z, z^*, c
'Costate' variables (4) $p_z, p_{z^*}, p^*_z, p^*_{z^*}$
'Output' variables (10) $r, r^*, y, y^*, \pi, \pi^*, H_{r^*}, H_c, H^*_r, H^*_c$
[b] By substitution: $y = -\kappa\gamma r - \kappa\eta\gamma r^* + \kappa(1-\eta)\delta c,$

$$y^* = -\kappa\eta\gamma r - \kappa\gamma r - \kappa(1-\eta)\delta c, \; \kappa \equiv \frac{1}{(1-\eta)^2}$$

that ignoring the impact of interest rates on the exchange rate leaves the authorities free to neutralise the impact of foreign rates on domestic output. The latter is then steered along the optimal path, whatever the behaviour of foreign rates.

Where the policy makers do take account of the effect of interest rates on the exchange rate, then the outcomes are as shown in the first two columns of Table 5.8, where the time consistent constraint on optimisation $c = \theta_1 z + \theta_2 z^*$, is restricted to match the Riccati coefficients (and the parameter values are as before). In comparison with the coordinated policy outcome shown in column 2 of Table 5.5, the roots have moved closer together and the matrix of the reaction coefficients has a more dominant diagonal. The general nature of the policy response to $z(t_0) = 10\%$, $z^*(t_0) = 0$ is, however, much as before, with real rates being set at 10%

Table 5.8. *Time consistent non-cooperative solutions*[a]

	Symmetric equilibria				Asymmetric equilibria			
	Open-Loop Nash		Closed-Loop Nash		Open-Loop Stacikelberg		Feedback Stackelberg	
Roots								
Averages	−0.981		−0.972		−0.9719		−0.9669	
Differences	−0.896		−0.882		−0.8873		−0.8839	
Ricatti Coefficients								
θ_1, θ_2	−0.825	0.825	−0.820	0.820	−0.812	0.826	−0.814	0.820
Reaction Coefficients								
ρ_{11}, ρ_{12}	1.024	0.284	1.012	0.284	0.994	0.282	1.000	0.283
ρ_{21}, ρ_{22}	0.284	1.024	0.284	1.012	0.285	1.024	0.284	1.012
Initial Values								
z, z^*	10	0	10	0	10	0	10	0
c	−8.25		−8.20		−8.14		−8.14	
r, r^*	10.24	2.84	10.12	2.84	9.94	2.85	10.00	2.84
y, y^*	−9.38	−0.42	−9.30	−0.42	−9.21	−0.42	−9.21	−0.42
π, π^*	9.17	0.83	9.18	0.82	9.19	0.81	9.19	0.81
Welfare Costs								
V, V^*	45.79	0.14	45.81	0.14	45.85	0.16	46.06	0.14
Average of V and V^*	22.97		22.98		23.00		23.09	

Notes: [a] Real exchange rate assumed to be a linear function of the stable state variables ($c = \theta_1 z + \theta_2 z^*$).

at home and about 3% overseas, leaving the domestic economy 'uncompetitive' by about 8%.

The welfare costs are correspondingly not very different, but they are in fact marginally *lower* for decentralised policy that was the case for coordinated policy. The reason why the best efforts of the policy coordinator may fail to improve welfare is that the values of θ_1, θ_2 taken as predetermined do in fact vary with the structure of decision making.

In the remainder of Table 5.8 we report the time consistent solutions for asymmetric games with the home country acting as Stackelberg leader. In these cases $\theta_1 \neq \theta_2$, and the reaction coefficients are no longer symmetric.

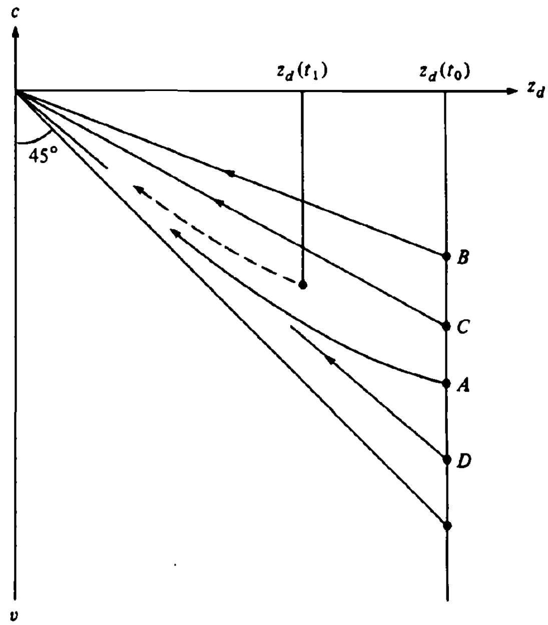
Given that the overseas reaction function is of the form $r^* = -\eta r + \left(\frac{1-\eta^2}{\gamma}\right)(p^*_{z^*} + p^*_z) + \frac{(1-\eta)}{\gamma}\delta c$ the necessary conditions for the *Feedback Stackelberg* solution are obtained from those for Closed Loop Nash by adding an extra term $-\eta H_{r^*}$ to the expression for $\partial H/\partial r$ in the top left of the Table 5.7. For *Open Loop Stackelberg* the necessary conditions for Open Loop Nash are modified in the same way and in addition terms equivalent to ρ_{21} and ρ_{22} in Table 5.7 are calculated from the reaction function above and the Riccati coefficients for $p^*_{z^*}$, p^*_z and c. The numerical results for the asymmetric equilibria hardly differ from those for their symmetric counterparts also appearing in Table 5.8. Perhaps the most interesting feature to note is that the costs to the Stackelberg leader *increase* in each case (compare values for V in columns one and three, and in columns two and four). As in the policy co-ordination problem treated earlier, it seems preferable *not* to adopt a leadership role.

(d) Summary and Interpretation of Results

Key features of four of the outcomes can be seen by charting the loss of competitiveness in the home country against z_d the 'difference' between cumulated excess demand at home and overseas. (Only those solutions which share a common unit root for the 'averages' are shown in Figure 5.1, which is not to scale.)

In all four cases there is a significant initial loss of competitiveness. The path starting from A representing the time inconsistent policy is, unlike the others, governed by two stable roots and consequently shows a changing ratio of competitiveness to z_d. This ratio moves towards (minus) 1 as the trajectory approaches asymptotically the 45° line (which happens to be the eigenvector associated with the smaller of the two stable roots).

The time inconsistency of this policy is immediately apparent: reoptimisation at t_1, when z_d has reached $z_d(t_1)$, would lead to the selection of a new path, shown as a dotted line, with a starting point on a ray joining



loss of competitiveness

Key:
A: Time Inconsistent Optimal Co-ordinated Policy
B: Time Consistent Co-ordinated Policy ($z_d = \theta_1 c$)
C: Time Consistent Co-ordinated Policy (c given)
D: Time Consistent Open Loop Nash Game ($z_d = \theta_1 c$)

5.1 Policy coordination and the real exchange rate

A to the origin. As this involves some restoration of competitiveness it is evident that, in reoptimising, the policy co-ordinator is tempted to reduce interest differentials below what were promised at time t_0. Consequently optimal policy appears to involve a cut in the initial level of competitiveness and core inflation in the home economy achieved by a path of interest differentials skewed into the future so as to limit the impact on current domestic output.

Imposing the constraint of time consistency on the policy co-ordinator prevents this skewed pattern of interest differentials. As a result, the initial spread of rates is increased to $6\frac{1}{2}\%$ though the initial loss of competitiveness falls to 7.90%, see Table 5.5 and the path starting at B. The use of interest differentials to affect c directly is ruled out by the assumption that $c = \theta_1 z_d$, and the indirect effect of high interest differentials is to reduce the loss of competitiveness by reducing z_d. If the coordinator treats the exchange rate as given, this inhibition is removed and both the initial interest differential and the initial loss of competitiveness increase (see path from point C in the figure).

Since the welfare costs fall when the exchange rate is taken as given in this way, it appears that the perceived ability to exert an influence on the exchange rate is, in the absence of precommitment, counter-productive. This may help to explain why welfare costs also fall in the non-cooperative Open Loop Nash case as lack of co-ordination seems to 'weaken' this time consistency constraint and allows for the choice of policies with higher initial interest differentials, and a greater loss of competitiveness (see path from point D).

It is not, of course, always true that the power to affect the exchange rate proves counter-productive in this way. In an example of fiscal policy co-ordination, Miller and Salmon (1985), we find that it pays the policy co-ordinator to act as a (time-consistent) leader in the foreign exchange markets; and in this case coordinated policy dominates the non-cooperative Nash alternative.

Conclusion

The focus of this paper is on the computation of time consistent equilibria in continuous time 'rational expectations' models where there is more than one decision maker. As an illustration, we have examined the choice of interest rate policies in two countries linked by trade and capital movements with a perfectly flexible exchange rate.

The loss function which policy makers seek to minimise includes only the squared deviation of domestic output from its stable inflation level and the square of core inflation, defined as a moving average of inflation. Such objectives do not, of course, involve any long-run international 'conflict of interest': in each country the equilibrium level of output is determined by the long-run Phillips curve and the floating rate allows for the desired inflation rate to be achieved without coordination.

What gains there may be from coordination must come therefore, in choosing the path towards equilibrium and not in the final equilibrium itself. For the model specified here, however, when the path of the exchange rate is taken to be predetermined in designing policy, there are no such dynamic gains to coordination. And where effects of interest rates on the exchange rate are recognised, coordination so alters the time consistency constraint that average welfare declines marginally, cf. Rogoff (1983) where coordination of monetary policy delivers worse outcomes than decentralised policy.

The failure to reap dynamic benefits from coordination in this case doubtless depends on the particular features of the model, especially the feature that time consistent Stackelberg leadership in setting interest rates appears to be uniformly unattractive in this floating rate environment.

While it is surely useful to focus on these purely dynamic aspects of the problem, it has to be recognised that a more realistic appraisal of the merits of coordination should also make due allowance for international 'conflicts of interest' which persist even in equilibrium, as discussed, for example, by Canzoneri and Gray (1983) and Turner (1983).

Annex. Deriving time consistent equilibria in symmetric and asymmetric dynamic games

In this Annex we derive the four equilibria for full information linear quadratic differential games discussed in the text. In order to focus on the strategic relations between the players, the state vector is taken to be entirely predetermined. The first of the symmetric solutions is for the Open Loop Nash game, where each player takes as given the other's sequence of policy *actions*. This is followed by Closed Loop Nash game where each player takes the other's policy *rule* as given (such rules being restricted to current state feedback only). Where one player is dominant, there are two analogous time consistent solutions, the Open Loop and Feedback Stackelberg equilibria.

The integrals to be minimised are assumed to converge without discounting. By using the 'current value' Hamiltonian and 'current value' shadow prices, these solutions can without difficulty be extended to incorporate discounting if required. Likewise the restriction to only two players is not essential.

1.1 Open loop Nash

Each player is assumed to choose the time path of his or her own control variable(s) so as to minimise the integral of a quadratic cost function defined on the variables of a linear differential equation system, conditional on the assumption that the entire path of the other player's control variable is given.

Since the behaviour of the two players is symmetric, the analysis need only be carried out in detail for the one of them, denoted 1, with a quadratic cost function defined on the set of 'state' variables x, his own controls u, and those of the second player, v. Note that the weights in this function, denoted by the matrix Q_1 below, incorporate any costs attached to other ('output') variables which have been eliminated by substitution in reducing the cost function to one involving only the state variables and the controls.

Thus, for the infinite horizon case, player 1 has to solve the following standard dynamic optimisation problem:

$$\min_u \frac{1}{2}\int_{t_0}^{\infty} w^T(s)\, Q_1\, w(s)\, ds, \text{ where } w(t) \equiv \begin{bmatrix} x(t) \\ u(t) \\ v(t) \end{bmatrix}, \tag{1}$$

subject to the linear differential 'state' equation

$$\dot{x}(t) = Ax(t) + B_1 u(t) + B_2 v(t), \text{ where } \dot{x} \equiv \frac{dx}{dt}, \tag{2}$$

given $x(t_0)$ and $v(t)$, $t \geqslant t_0$. The coefficients of the symmetric matrix Q_1 are assumed to satisfy the usual positive definiteness conditions to ensure that costs are positive and that the values of the control are bounded. (The explicit time-indexing of the elements of w is, for convenience, dropped henceforth).

Using Pontryagin's maximum principle to solve this problem, the appropriate Hamiltonian for player 1 is

$$H^1 = \tfrac{1}{2}(w^T Q_1 w) + p_1^T(Ax + B_1 u + B_2 v) \tag{3}$$

where p_1 is a vector of 'costates', and hence the first order conditions (FOC's) for a minimum may be written

$$\frac{\partial H^1}{\partial x} = Q_{1xx}\, x + Q_{1xu}\, u + Q_{1xv}\, v + A^T p_1 = -\dot{p}_1 \tag{4}$$

$$\frac{\partial H^1}{\partial p_1} = Ax + B_1 u + B_2 v = \dot{x} \tag{5}$$

$$\frac{\partial H^1}{\partial u} = Q_{1xu}\, x + Q_{1uv}\, v + Q_{1uu}\, u + B_1^T\, p_1 = 0 \tag{6}$$

Since the second player is, in fact, choosing his path for v conditional on u being given, the Nash equilibrium may be obtained as the simultaneous solution of these two optimisation problems. To this end, the state equation and the FOC's of both players (bearing in mind that those of the second player will be symmetric to those of player 1 already described) are first collected together as follows:

$$\begin{bmatrix} \dot{x} \\ -\dot{p}_1 \\ -\dot{p}_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A & 0 & 0 & B_1 & B_2 \\ Q_{1xx} & A^T & 0 & Q_{1xu} & Q_{1xv} \\ Q_{2xx} & 0 & A^T & Q_{2xu} & Q_{2xv} \\ Q_{1ux} & B_1^T & 0 & Q_{1uu} & Q_{1uv} \\ Q_{2vx} & 0 & B_2^T & Q_{2vu} & Q_{2vv} \end{bmatrix} \begin{bmatrix} x \\ p_1 \\ p_2 \\ u \\ v \end{bmatrix} \tag{7}$$

The paths for the states and costates can be more simply described, however, if the control variables are eliminated by substitution. Thus from (7) we obtain

$$\begin{bmatrix} u \\ v \end{bmatrix} = -\begin{bmatrix} Q_{1uu} & Q_{1uv} \\ Q_{2vu} & Q_{2vv} \end{bmatrix}^{-1} \begin{bmatrix} Q_{1ux} & B_1^T & 0 \\ Q_{2vx} & 0 & B_2^T \end{bmatrix} \cdot \begin{bmatrix} x \\ p_1 \\ p_2 \end{bmatrix} \tag{8}$$

and so

$$\begin{bmatrix} u \\ v \end{bmatrix} \quad \begin{bmatrix} J_{ux} & J_{up_1} & J_{up_2} \\ J_{vx} & J_{vp_1} & J_{vp_2} \end{bmatrix} \begin{bmatrix} x \\ p_1 \\ p_2 \end{bmatrix} \tag{9}$$

Hence we can express (7) alternatively as an adjoint system involving only states and costates, which we denote

$$\begin{bmatrix} \dot{x} \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} M_{xx} & M_{xp_1} & M_{xp_2} \\ M_{p_1x} & M_{p_1p_1} & M_{p_1p_2} \\ M_{p_2x} & M_{p_2p_1} & M_{p_2p_2} \end{bmatrix} \begin{bmatrix} x \\ p_1 \\ p_2 \end{bmatrix} \equiv M \begin{bmatrix} x \\ p_1 \\ p_2 \end{bmatrix} \tag{10}$$

where, for example,

$$M_{xx} = A + B_1 J_{ux} + B_2 J_{vx}$$

Since the convergent solution to this infinite time problem involves only the stable roots of this adjoint system, it is straightforward to express the paths for the states and costates in terms of the initial conditions for x and the stable eigenvalues and vectors of the adjoint matrix M in (10).

It is convenient for this purpose first to define a vector of canonical variables, z, each associated with a single root, thus

$$\begin{bmatrix} x \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix} \begin{bmatrix} z_s \\ z_n \end{bmatrix} = Cz \tag{11}$$

where

$$\begin{bmatrix} \dot{z}_s \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \Lambda_s & \\ & \Lambda_n \end{bmatrix} \begin{bmatrix} z_s \\ z_n \end{bmatrix} = \Lambda z$$

letting s, and n denote stable and unstable roots. The matrix C appearing in (11) is simply the matrix of column eigenvectors of M, so $MC = C\Lambda$, where Λ is the diagonal matrix of eigenvalues of M. Partitioning C as shown and setting $z_n = 0$ provides the solution for the states and costates as follows:

$$\left.\begin{aligned} x(t) &= C_{11} z = C_{11} e^{A_s(t-t_0)} C_{11}^{-1} x(t_0), \\ p_1(t) &= C_{21} C_{11}^{-1} x. \\ p_2(t) &= C_{31} C_{11}^{-1} x \end{aligned}\right\} \tag{12}$$

The time paths for u and v in this open loop Nash equilibrium may now be obtained by substitution of (12) into (9).

1.2 Closed loop or feedback Nash

In contrast to the open loop case, it is now assumed that in minimising the same integral of costs each player observes and reacts to the current state, x. In the Closed Loop Nash game each player takes as given the *feedback rule* of the other, and these closed loop rules, $u = R_1 x$, $v = R_2 x$, alter the players' first-order conditions for cost minimisation. Specifically, when calculating the effect of a change in the state on each player's Hamiltonian, account must be taken of the response of the other player to the change in the state.

Thus for player 1, given (3), and $v = R_2 x$, one calculates

$$\frac{\partial H^1}{\partial x} = Q_{1xx} x + Q_{1xu} u + Q_{1xv} v + A^T p_1 + R_2^T H_v^1 = -\dot{p}_1 \tag{13}$$

where

$$H_v^1 \equiv Q_{1vx} x + Q_{1vu} u + Q_{1vv} v + B_2^T p_1,$$

which, in comparison with equation (4) above, contains an extra expression representing $\dfrac{\partial H^1}{\partial v}\dfrac{\partial v}{\partial x}$. Player 2 similarly takes into account this closed loop behaviour of player 1, represented by $u = R_1 x$.

As there are no other changes to the FOCs, then collected results for the closed loop Nash equilibrium can be shown by augmenting those for the open loop case as follows

$$\begin{bmatrix} \dot{x}_1 \\ -\dot{p}_1 \\ -\dot{p}_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \left[\begin{array}{ccccc|cc} & & & & & 0 & 0 \\ & & S^{OLN} & & & R_2^T & 0 \\ & & & & & 0 & R_1^T \\ & & & & & 0 & 0 \\ \hline & & & & & 0 & 0 \\ Q_{1vx} & 0 & 0 & Q_{1vu} & Q_{1vv} & -1 & 0 \\ Q_{2ux} & 0 & 0 & Q_{2uu} & Q_{2uv} & 0 & -1 \end{array}\right] \begin{bmatrix} x \\ p_1 \\ p_2 \\ u \\ v \\ H_v^1 \\ H_u^2 \end{bmatrix} \tag{14}$$

where S^{OLN} denotes the matrix appearing in equation (7) and H_v^1, H_u^2 are the partial derivatives of the Hamiltonians with respect to the *other* player's controls.

For any given values of R_1, R_2, the system can be reduced by substitution to the adjoint equations in states and costates

$$\begin{bmatrix} \dot{x} \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = M^{CLN} \begin{bmatrix} x \\ p_1 \\ p_2 \end{bmatrix} \tag{15}$$

and the solution paths may be obtained from the initial condition $x(t_0)$ and the stable eigenvalues and eigenvectors of this adjoint system as described in the last section. It is clear that the eigenvectors so determined will depend on the assumed reaction functions R_1, R_2. Given equation (9) above, it is necessary for the closed loop Nash equilibrium that these reaction functions should simultaneously be related to the eigenvectors as follows:

$$\left.\begin{aligned} R_1 &= J_{ux} + J_{up_1} \Pi_1 + J_{up_2} \Pi_2 \\ R_2 &= J_{vx} + J_{up_2} \Pi_2 + J_{up_2} \Pi_2 \end{aligned}\right\} \tag{16}$$

where Π_1, Π_2 are the Ricatti matrices relating the costates to the states, so

$$\left.\begin{aligned} p_1 &= \Pi_1 x = C_{21} C_{11}^{-1} x \\ p_2 &= \Pi_2 x = C_{31} C_{11}^{-1} x \end{aligned}\right\} \tag{17}$$

The values for R_1, R_2 which characterise the closed loop Nash equilibrium may be determined by applying Jacobi's iterative method to the system after 'modal decomposition', as follows. Given starting values for R_1 and R_2, one may calculate the eigenvectors of M^{CIN}. These can then be used to compute the Ricatti matrices, Π_1 and Π_2, and so a new set of values for R_1 and R_2 (using equations (16) and (17) above). These new values may be substituted back into (14) and the procedure repeated until convergence is obtained. This iterative method was used to compute the closed loop solutions for the example in this paper. Typically R_1 and R_2 were set to zero in the first iteration (which then generates the eigenvectors of the open loop Nash solution in the first round).

An alternative procedure would be to apply numerical methods to the coupled Riccati equations associated with such a closed loop Nash game, namely

$$\left.\begin{aligned} Q_{1xx} + Q_{1xu} R_1 + Q_{1xv} R_2 + A^T \Pi_1 + \Pi_1 A + \Pi_1 B_1 R_1 + \Pi_1 B_3 R_2 &= 0 \\ Q_{2xx} + Q_{2xu} R_1 + Q_{2xv} R_2 + A^T \Pi_2 + \Pi_2 A + \Pi_2 B_1 R_1 + \Pi_2 B_2 R_2 &= 0 \end{aligned}\right\} \tag{18}$$

where R_1 and R_2 are defined as in (16) above.

Time consistent Stackelberg equilibria

The dynamic games described so far have been symmetric. In many situations, however, one player may be dominant and such a 'leader' can design his policy in part with a view to inducing others ('followers') to act in a manner beneficial to him.

The optimal strategy for the Stackelberg leader in an open loop dynamic game turns out to be time inconsistent, see Simaan and Cruz (1973), and we discussed such time inconsistent strategies in an earlier paper, Miller and Salmon (1983). Here, however, we focus on those time consistent policies which may be obtained using the recursive techniques of dynamic programming.

In what follows, we first describe the time consistent Stackelberg equilibrium derived for an asymmetric *open loop* game along the lines developed by Cohen and Michel (1984); then the *closed loop* equivalent reported earlier by Basar and Haurie (1982).

1.3 Open loop Stackelberg equilibrium

For the linear quadratic problem described in equations (1) and (2) above, Hamiltonians are formed as shown in (3), for the leader, player 1, and likewise for the follower. The first order conditions for optimisation will, of course, be affected by strategic asymmetry.

We begin with the conditions describing how the controls, u and v, are described as functions of the state and costate variables chosen so as to minimise the respective Hamiltonians. While the first order conditions for the follower are unchanged from the symmetric case, the leader, in setting u, will typically take advantage of the follower's reaction to his choice of u.

Thus, for the follower, v will be determined as in equation (7) above, namely

$$\frac{\partial H^2}{\partial v} = Q_{2vx}\, x + Q_{2vu}\, u + Q_{2vv}\, v + B_2^T p_2 = 0 \tag{19}$$

This implies that

$$\left.\begin{aligned} v &= -Q_{2vv}^{-1}(Q_{2vx}\, x + Q_{2vu}\, u + B_2^T p_2). \\ &\equiv R_{vx}\, x + R_{vu}\, u + R_{vp_2} p_2 \end{aligned}\right\} \tag{20}$$

which will be taken into account by the leader, whose first order condition for setting u therefore becomes

$$\frac{\partial H^1}{\partial u} = Q_{1ux}\, x + Q_{1uu}\, u + Q_{1uv}\, v + B_1^T p_1 + R_{vu}^T H_v^1 = 0 \tag{21}$$

where

$$R_{vu} \equiv \frac{\partial v}{\partial u} = -Q_{2vv}^{-1} Q_{2vu}$$

as shown above, and

$$H_v^1 \equiv \frac{\partial H^1}{\partial v} = Q_{1vx} x + Q_{1vu} u + Q_{1vv} v + B_2^T p_1$$

As this asymmetry exists at a point in time, it does not lead to time inconsistency associated with intertemporal asymmetries.

Turning now to the conditions governing the evolution of the costate variables p_1 and p_2, we note that, as the follower treats the leader's actions as an open loop, the condition for $\dot{p}_2$ will be as in equation (9) above, namely

$$-\dot{p}_2 = \frac{\partial H^2}{\partial x} = Q_{2xx} x + Q_{2xu} u + Q_{2xv} v + A^T p_2 \tag{22}$$

Cohen and Michel have argued that Bellman's Principle of Optimality in the open loop Stackelberg game implies that the leader takes account of the follower's behaviour as summarised here in equation (20), subject *not* to equation (22) but to the 'time consistency' restriction that

$$p_2 = \theta x \tag{23}$$

where θ is taken as given by player 1 but is endogenous to the system. Formally, player 1 chooses u to minimise V_1 subject to (2), (20) and (23).

As a consequence the first order condition describing the evolution of p_1 is

$$\frac{\partial H^1}{\partial K} = Q_{1xx} x + Q_{1xu} u + Q_{1xv} v + A^T p_1 + R^T H_v^1 = -\dot{p}_1 \tag{24}$$

where

$$R \equiv R_{vx} + R_{vp_2} \theta \tag{25}$$

so R is similar to (but not identical with) a closed loop reaction function.

We thus have

$$\begin{bmatrix} \dot{x}_1 \\ -\dot{p}_1 \\ -\dot{p}_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \left[\begin{array}{ccccc|c} & & & & & 0 \\ & & & & & R^T \\ & & S^{OLN} & & & 0 \\ & & & & & R_{vu}^T \\ & & & & & 0 \\ \hline Q_{1vx} & 0 & 0 & Q_{1vu} & Q_{1vv} & -1 \end{array}\right] \begin{bmatrix} x \\ p_1 \\ p_2 \\ u \\ v \\ H_v^1 \end{bmatrix} \tag{26}$$

where S^{OLN} denotes the matrix appearing in (7) above, and R is defined in (25).

Equation (26) can be reduced to an adjoint equation in the state and costate variables of the form

$$\begin{bmatrix} \dot{x} \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = M^{OLS} \begin{bmatrix} x \\ p_1 \\ p_2 \end{bmatrix} \tag{27}$$

using the definition of H_v^1 and the reduced form equations for u and v obtained from (19) and (21) which we denote

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{32} \end{bmatrix} \begin{bmatrix} x \\ p_1 \\ p_2 \end{bmatrix} \tag{28}$$

using a slightly different notation to distinguish this form equation (9).

The equilibrium is defined by the requirement that the matrix θ incorporated in M^{OLS} satisfies the condition that

$$\theta = C_{31} C_{11}^{-1} \tag{28}$$

where $\begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix}$ denote the stable eigenvectors of (27). This solution may be obtained either by the Jacobi method used here, or alternatively by solving the non-linear Riccati equations associated with this case.

1.4 Feedback Stackelberg equilibrium

In much the same way, the closed loop asymmetric equilibrium derived using dynamic programming methods by Basar and Haurie can be described as an augmented version of the closed loop Nash game.

The asymmetry will as before mean that v and u are set as in (19) and (21) above. But the closed loop assumption now implies that both players are aware of the other's response to the state, so V_1 is minimised subject to (1) and $v = R_2 x$ and V_2 is minimised subject to (2) and $u = R_1 x$, with each player taking as given the other's reaction function. The resulting first order conditions are almost identical to those for the closed loop Nash game in (4). Specifically

$$\begin{bmatrix} \dot{x}_1 \\ -\dot{p}_1 \\ -\dot{p}_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A & 0 & 0 & B_1 & B_2 & 0 & 0 \\ Q_{1xx} & A^T & 0 & Q_{1xu} & Q_{1xv} & R_2^T & 0 \\ Q_{2xx} & 0 & A^T & Q_{2xu} & Q_{2xv} & 0 & R_1^T \\ Q_{1ux} & B_1^T & 0 & Q_{1uu} & Q_{1uv} & R_{vu}^T & 0 \\ Q_{2vx} & 0 & B_2^T & Q_{2vu} & Q_{2vv} & 0 & 0 \\ Q_{1vx} & 0 & 0 & Q_{1vu} & Q_{1vv} & -1 & 0 \\ Q_{2ux} & 0 & 0 & Q_{2uu} & Q_{2uv} & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ p_1 \\ p_2 \\ u \\ v \\ H_v^1 \\ H_u^2 \end{bmatrix} \tag{29}$$

Of course if the term $R_{vu} = -Q_{2vv}^{-1} Q_{2vu}$ is zero then this solution will coincide with the closed loop Nash. Once again the system may be reduced to the form

$$\begin{bmatrix} \dot{x} \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = M^{CLS} \begin{bmatrix} x \\ p_1 \\ p_2 \end{bmatrix} \tag{30}$$

while the reaction functions both R_1 and R_2 appearing in M^{CLS} must, for the Feedback Stackelberg equilibrium, satisfy the conditions

$$\left.\begin{aligned} R_1 &= J_{11} + J_{12} C_{21} C_{11}^{-1} + J_{13} C_{31} C_{11}^{-1} \\ R_2 &= J_{21} + J_{22} C_{21} C_{11}^{-1} + J_{23} C_{31} C_{11}^{-1} \end{aligned}\right\} \tag{31}$$

defined with respect to the stable eigenvectors of M^{CLS} in (30).

NOTES

* In revising this paper for publication we have benefitted substantially from comments received at the Conference.

1 The reader is invited to compare the results obtained here for differential games (and the methods used to obtain them) with those presented in the excellent paper by G. Oudiz and J. Sachs included in this volume, which is cast in discrete time and uses explicit dynamic programming methods to obtain time consistent equilibria.

2 Obtained using 'Saddlepoint'; see Austin and Buiter (1982).

3 Cf. Aoki (1981).

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COMMENT RALPH C. BRYANT

I have mixed reactions about the paper by Miller and Salmon. On the favorable side, the authors make an interesting contribution to a growing technical literature. Researchers working intensively on this topic will benefit from the paper (and the authors' earlier research on which it draws). On the negative side, the authors are not sufficiently clear about *why* the particular aspects of coordination studied here are the issues that most merit analytical attention. Many readers will have difficulty extracting the basic ideas from the technical presentation.

In this written version of my comments I omit specific comments about the authors' model and their empirical results. Instead, I include only some general points that were sparked by my reading of the Miller–Salmon paper and some of the other papers presented at the conference. These points fall under three headings: the characterization of 'solutions' to problems of strategic interactions among national governments; the information and uncertainty aspects of strategic interactions; and some questions about the recent preoccupation with 'time-consistent' policy strategies.

Cooperative and non-cooperative games

The conventional characterization of solutions to problems of strategic interactions distinguishes between 'cooperative' and 'noncooperative' games. I formerly believed that this distinction was clear and that it turned on whether the players in a strategic situation enter into binding agreements with each other. More recently, following conversations with others much more conversant with game theory than I am (for example, Edward Green), I have come to doubt the clarity of this distinction. In particular, the concept of cooperation as customarily used in game theory tends not to highlight the 'enforceability' and 'credibility' aspects of an agreement. And it also fails to pay enough attention to the 'information structure' of the strategic situation.

Every strategic situation can be interpreted as having both 'efficiency' and enforceability aspects. And every game has a particular information structure.[1] The relative importance of the enforceability and the efficiency aspects varies from one game to another, depending on the information structure and the sequencing of decisions. For example, as brought out in the Miller-Salmon paper, a 'Stackelberg' solution can be interpreted as just another Nash solution with a different information structure.

It is interesting to ask whether a steep tradeoff exists between efficiency and enforceability. Does it become more difficult to enforce a solution as the players cooperate to reach an efficient outcome? Alternatively, are the solutions that are most easily enforced and credible likely to be less efficient? For example, we know that a 'noncooperative' Nash solution may be enforceable, yet quite inefficient.

A pessimist about cooperation would probably argue that there is, inevitably, such a tradeoff. I am not clear about this question myself, and my main purpose here is to identify the issue as one deserving careful study. At this rudimentary stage of our knowledge, I tentatively hold an optimistic view. That is, I nourish the hope that negotiators and analysts can be innovative, finding solutions for many types of games that are fairly efficient *and* reasonably enforceable.

International cooperation about national macroeconomic policies is a prime example where I hope an optimistic attitude can be justified. As pointed out in the earlier literature on this subject, the basic inhibitions to and potential gains from collective action apply to this area as well.[2] There is no 'market' in the policy actions of national governments. Governments cannot therefore feasibly 'trade' policy actions with each other. As a result, significant external economies or diseconomies can arise. There thus exists, in principle, a collective ('nonmarket') approach that could bring about more efficient outcomes.

With strong political leadership, a particular efficient solution might be found through bargaining that could make each nation better off. The optimistic view asserts that such a solution not only could be found but could also be enforced. Enforceability requires innovative ideas for containing the 'free-rider' incentives to stay out of the bargaining or to renege on the other players after a putative efficient solution has been bargained and agreed. For any foreseeable future, no international authority will be given strong enough powers to act as a policeman to enforce international agreements. Hence agreements must be 'self-enforcing' – consistent with continued decentralized decisionmaking by national governments.

Information and uncertainty aspects of strategic behavior

Research on international coordination of economic policies could develop in (at least) two ways. Analysts could explore the theoretical aspects of strategic behavior using highly simplified models. This approach – call it type 1 – permits a sharp focus on particular analytical issues, a few at a time. Alternatively, analysts could try to develop the empirical aspects of the issues. As a necessary counterpart to this other approach – type 2 – analysts will have to use empirical structural models of how economies interact; these models must adequately capture the size and nature of transmission of economic forces from one economy to another.

This research by Miller and Salmon, most of the work being discussed at this conference, and indeed the bulk of the interesting recent work by others, has largely been type-1 in nature. This theoretical emphasis has been useful. We certainly have to be clearer about the conceptual and theoretical aspects before we can make significant progress in applying the ideas empirically. At the same time we should not lose sight of the vast need for type-2 work. Theoretical clarity is a necessary, but very far from sufficient, condition for successful empirical application of the ideas.

In real-life discussions among national governments, the most fundamental obstacle to more cooperation is not a lack of awareness of the potential gains from coordination. Nor is it merely a lack of political will. To be sure, both those lacunae are important – especially in the last five years. Yet a still more important obstacle is the tremendous uncertainty about the magnitudes, and even the signs, of cross-border transmissions of economic forces.

In the research discussed at this conference, the players *know* how economic forces are transmitted from one nation to another. So to speak, everyone knows all the details of the relevant matrices in the correct structural model. And each player has the same model in mind. The practical situation confronting policymakers in national governments is of course entirely different.

Even if we cannot yet expect substantial progress in type-2 research, it would be a big advance if we could learn how to put some explicit recognition of uncertainty into our type-1 research. For example, could we incorporate some variances and covariances in our simplified models? Following the lead of the literature started by Brainard (1967), could we try to see how our inferences about strategic behavior and the potential gains from cooperation may be altered if we allow explicitly for uncertainty about what the true model is? A related approach would start by recognizing that two countries (players) disagree about the relevant model characterizing their interdependence. Could we construct and then employ a 'common model' that puts low or zero variance around the parameters not in dispute, but high variances around the parameters where disagreement is strongest?

In a similar vein, I offer a final comment about information and uncertainty. Perhaps we are underplaying the importance of the 'mere' exchange of information among countries? It is true that an exchange of information and forecasts is now virtually all that happens in international discussions about macroeconomic policies. But it does not follow, as is often assumed, that this activity has negligible consequences. When basic uncertainty is so high, 'mere' exchanges of information about recent developments and about national forecasts may be no small thing.

Credibility and time consistency

The credibility and 'time-consistency' aspects of strategic interactions have recently drawn the lion's share of attention in type-1 research. I am doubtful, however, that those aspects most warrant our analytical attention – or at least in the narrow way typical of the recent work.

The questions about time consistency that bother me can be identified with the use of an analogy, the interactions between a parent and young child. One may suppose that the interests of these two players are partly in conflict: what the parent wants for the child at any point in time may differ from what the child wants. Yet there is also a latent commonality of objectives.

In particular, the child dislikes doing homework, likes to watch television, and is often tempted to watch television before the homework is done. The parent lays down behavior rules, including the requirement that all homework has to be done before the TV can be turned on. It is a feature of the situation that the game is 'repeated'; each day is potentially a new test of the homework-before-TV rule.

If the parent were to enforce the rule rigidly, the strategies of both parent and child would be 'time-consistent' (in the sense now popular in the technical literature). Now ask, however, whether it could be sensible for the parent to alter the rule, and if so in what circumstances.

If the child has a tendency to procrastinate, behavior by the parent that 're-optimizes' in a 'time-inconsistent' way can lead to trouble. For example, suppose the parent approves of the child watching certain educational TV programs. If the child procrastinates with his homework and yet at 9:00 p.m. the parent relaxes the rule to permit the child to watch the educational program, the parent's re-optimization can create a credibility problem. On future days the child may again procrastinate, hoping to get the parent to relent again, even for non-educational programs. This result would be an example where re-optimization can induce poor outcomes averaged over longer runs. It is situations of this type into which the recent literature has given us insights.

But now consider a different set of circumstances. Suppose at 3:00 p.m. there is a power failure, which lasts until 8:45 p.m.; suppose the child's homework is to practice typing, and the only typewriter in the house is an electric typewriter. Suppose again at 9:00 p.m. there is an educational TV program that the parent would like the child to watch. Given the unexpected event of the power failure, over which neither player had any control, would a credibility problem be created if the parent permits the child to watch TV? This particular re-optimization could be a constructive breach of the homework rule.

For completeness, imagine a variant of the preceding case. Suppose there is a past history of the child frequently procrastinating. Imagine again the power failure, and again the desirable program on evening TV. How should the parent react to the surprise power failure and its consequent effects on the homework? The parent's decision is more difficult because of the past history. A possible relaxation of policy tonight may induce more procrastination in the future – even though the occasion for contemplating the breach of the rule has nothing to do with the child's behavior today.

My analogy highlights credibility issues. Arguably, child-rearing may be dominantly influenced by such issues.

I now want to ask whether macroeconomic policy is very much, or only partly, like child rearing. One important difference, I would assert, is the greater relative significance in macroeconomic policy of uncertainty about how the rules affect behavior (how policy actions affect performance of the economy). And a second important difference is the much greater prevalence in macroeconomic policy of 'surprises' (like the power failure) that are exogenous from the perspective of individual decisionmaking agents.

Think, for example, of the dilemmas facing policymakers in a significantly open economy. Many types of disturbance originating elsewhere in the world – wars, debt crises, crop failures – will influence the home economy. Many types of nonpolicy shocks may also originate at home.

When uncertainty and unanticipated events are very important, the virtues of unwavering adherence to 'time-consistent' strategies may be much less clear than when the dominant elements of the situation are the interactions between the players themselves. In particular, I conjecture that there are many types of unexpected events for which it is desirable – from the perspective of all players – to re-optimize after the surprise has occurred. More generally, I believe that the recent literature has paid insufficient analytical attention to strategic interactions in the light of uncertainty and surprises.

I do not want to push the argument about exogenous surprises too far, because of the possibility that the subsequent endogenous interactions among the players could be affected adversely by the relaxation of time-consistent policies. Nevertheless, a great deal of macroeconomic policy has to do with responding to contingencies that cannot be anticipated. Policymakers in national governments have opportunities to get to know each others' behavior in international negotiations, and hence to form reasonable judgments about credibility and reputations. With the current state of knowledge of how the world economy functions, on the other hand, they cannot anticipate many contingencies and cannot be confident about how the consequences will be transmitted across national borders. Situations may often arise, therefore, where it could be mutually advantageous for all parties to be time-inconsistent, departing from presumptive rules agreed at an earlier time.

These considerations led me to be a skeptic about whether the recent fad in the profession – preoccupation with credibility problems, and with time-consistent strategies – is leading our research in the most fruitful direction.[3] Perhaps this trend is a bit like other aspects of the so-called rational-expectations revolution? The new emphasis corrects a significant oversight in the previous literature. It forces us to ask important questions. But we need to be careful not to get so swept up in the technically interesting aspects that we forget the old familiar problems that still need attention.

NOTES

1 An outcome or 'solution' is efficient if it is Pareto-optimal. A solution is enforceable if, once reached or agreed, the players have incentives to sustain the behavior generating the solution.
2 See, for example, Niehans (1968); Hamada (1976); Bryant (1980) chapter 25; and Oudiz and Sachs (1984).
3 A possible confusion exists about the use of the term 'time consistency.' In my comments I assert that re-optimization in response to exogenous surprises can – appropriately – lead to 'time-inconsistent' policies. At the conference, a few participants described such policies as 'innovation-contingent feedback rules' and preferred to label the policies as 'time consistent.' Terminology on these matters is still unsettled, and I have no semantic brief for my usage. My essential point is that the credibility aspects of interactions among the agents in a game situation may be receiving excessive attention relative to the aspects generated by uncertainty and exogenous surprises.

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COMMENT STEPHEN J. TURNOVSKY

At the NBER Conference on Policy Coordination held in Cambridge, Massachusetts, in August 1983, Marcus Miller and Mark Salmon presented some results from dynamic (differential) game theory.[1] This paper is an application of some of the results contained in that paper to the problem of monetary policy coordination within the context of a specific two country macroeconomic model. My comments will be in two parts. First, I will make some general remarks on dynamic game theory; secondly, I will make some specific comments on their analysis:

I. Some remarks on dynamic game theory

The analysis of dynamic games has several critical interrelated aspects. These include: (i) the type of strategic behavior being considered; (ii) the information structure available to the policy maker; (iii) the choice of policy instruments. I shall touch briefly on each in turn.

(i) *Types of strategic behavior*

Traditionally, two types of strategic behavior have been analyzed using game theory, namely, *noncooperative* and *cooperative*. In the former, each agent acts independently, under alternative assumptions regarding the interaction of his behavior with that of his competitors. The two most common assumptions are: (i) Nash and (ii) Stackelberg behavior. In the former, each agent takes the actions of his competitors as given and reacts to them. A given set of policies for the agents is in Nash equilibrium when there is no incentive for any one agent to deviate unilaterally from this equilibrium, with the other agents' policies being held fixed as given. In the latter mode of policy making, however, there is a dominance and in a two-player game one of the agents plays the role of a 'leader' and the other a 'follower.' This case applies quite naturally in a situation where the policy coordination involves a large country (leader) and a small country (follower). In determining his equilibrium policy in the Stackelberg sense, the leader anticipates possible rational reactions of the follower to his announced policy and optimizes his objective function accordingly. These two solution concepts are familiar from static game theory. However, they need refinement within a dynamic context, where a proliferation of the equilibrium concepts with dynamic information patterns is possible.

The most important consideration in an intertemporal context is that over time games are played at different stages. Thus for example, the rules of the game can, and probably will, evolve over time. Solution concepts need to be clarified. For example, within the framework of multiact dynamic games, the Stackelberg equilibrium concept is appropriate for the class of decision problems in which the leader has the ability to announce his decisions at all of his possible information sets ahead of time, a solution which forces the leader to commit himself to the actions dictated by these strategies. Thus the Stackelberg solution involves a prior commitment on the part of the leader.[2]

On the other hand, the leader may not have the ability to announce and enforce his strategy at all levels prior to the start of the game. Such a hierarchical equilibrium, which has a Stackelberg property at each level of play, is called a feedback Stackelberg solution. The leader may change over time, depending upon the policies chosen by the agents and the outcome of the dynamic process.

It is also possible that the policy makers in the two countries find it mutually beneficial to coordinate their behavior and to achieve some kind of cooperative equilibrium. This involves the optimization of some joint utility function and is one of the solutions considered by Miller and Salmon.

(ii) *Information structure*

A most important aspect of the solution involves the information structure; see Basar and Olsder (1982). In a dynamic game, a precise delineation of the information pattern, such as which economic agent knows what, how the information pattern available to each agent evolves over time, how much of this is common information shared by all policy makers, and what part of it constitutes private information for each agent, is of paramount importance. An information set is said to be open-loop if only the a priori raw data set is available at all points in time; in this case the policy variables depend only upon time and are called *open-loop* policies. On the other hand if there is some dynamic evolution of the available information and the policy variables are allowed to depend upon this dynamic information, the information pattern is said to be *closed-loop* or *feedback*, with the precise terminology depending upon what is actually available and how it is utilized in the policy making process. Each one of these information structures give rise to a different game and to a different equilibrium structure, some of which are considered in the Miller–Salmon paper.

The definition of the information set is particularly important in the application of strategic behavior to international policy problems. In the first place, certain variables such as exchange rates and interest rates are available with much greater frequency than variables such as output or employment. Secondly, policy makers are likely to have superior information on domestic economic variables than they are likely to have on analogous foreign variables.

(iii) *Choice of policy instruments*

Dynamic games are concerned with determining how policy makers within each economy acting over time choose optimally among some given set of policy instruments. A crucial point, unfamiliar to most economists, is that even within a deterministic context, the choice of policy instruments and the nature of the underlying information pattern is critical to the equilibrium outcome of the differential game. This is in contrast to a single country (agent) dynamic optimization context where, under the assumption of uncertainty, such a choice is unimportant.

Since this distinction between traditional control theory and differential games is so important, it requires further elaboration. Let us abstract from stochastic disturbances and consider for the moment a single agent optimization problem, such as a single government, the objective of which is to tradeoff optimally between the rate of inflation and the rate of unemployment. The optimal solution will be the same whether the policy maker chooses to use say (i) only a fiscal instrument, (ii) only a monetary instrument. (Of course the solution will be different if the policy maker chooses to use both instruments simultaneously.) Furthermore, the end result will be the same regardless of what observables these variables are chosen to depend upon. That is, open-loop and closed-loop policies lead to the same outcome, provided that the same instrument variables are adopted.

By contrast, in the case of a dynamic game version of this problem, involving say two policy makers, even if each policy maker uses only one instrument, the equilibrium outcome will in general depend upon which instrument is being used. Thus, for example, under given behavioral assumptions, the equilibrium outcome will be different, depending upon whether each policy maker is using a fiscal or a monetary (or for that matter some other) policy instrument, and the precise nature of this policy instrument. Consequently, the assumption of the specific policy instrument takes on added importance in a dynamic context. The reason is simply that the policy maker's reaction curve which conditions the optimization of each of the agents depends upon the choice of policy instrument.[3]

Moreover, even if the choice of policy instrument is fixed, the information pattern (whether open-loop or closed-loop feedback information is available to the agents) plays an important role in the characterization and existence of equilibria as Miller and Salmon note, again in contrast to the single-agent case.

II. Specifics of the Miller–Salmon Analysis

I now comment on certain aspects of the Miller–Salmon paper. First, the consistency of the model is critically dependent upon the 'sluggishness' introduced by the core rate of inflation. Using their notation, the critical relationships for say the domestic economy are

$$i = \phi y + \sigma Dc + \pi$$

$$D\pi = \zeta(i - \pi)$$

The first equation is very much like an expectations-augmented Phillips curve, where inflationary expectations (which we may identify with π) are formed adaptively. If one now takes the limiting case where $\zeta \to \infty$, so that the core rate of inflation converges to the current, the model tends to degenerate. With $\zeta \to \infty$, $i = \pi$ and the Phillips curve reduces to

$$\phi y + \sigma Dc = 0 \tag{1}$$

With symmetry, the analogous relationship in the rest of the world is

$$\phi y^* - \sigma Dc = 0 \tag{1'}$$

and combining these two equations yields

$$y + y^* = 0 \tag{2}$$

Further, summing the IS curves for the two countries and noting (2) one finds

$$r + r^* = 0 \tag{3}$$

In the Miller–Salmon model, r and r^* are policy instruments. However, in the limiting case we have considered, the two policy instruments cannot be chosen independently by their respective policy makers. Precisely the same difficulty obtains under the assumption of symmetric economies if one adopts an expectations-augmented Phillips curve, together with the assumption of rational expectations; indeed the two models are observationally equivalent.[4] Thus in either case, the viability of the model depends upon some sluggishness being present. Equation (1) can be viewed as being a manifestation of the ineffectiveness of policy under rational expectations in the aggregate world economy. While due to relative price effects, the output in one country can deviate from its natural level, this is offset by a deviation abroad, so that on balance total output remains unchanged.

In discussing the policy instruments, the authors comment that 'For the design of monetary policy in such a structure it is not necessary to look at the details of the monetary sector, as Driffill (1982) has pointed out.'[5]

This comment is perfectly appropriate in the context of the single agent optimization model analysed by Driffill. But as we have noted above in my general remarks, the specification of policy is important in the context of multiagent optimization problems. Miller and Salmon treat the real interest rate as being the monetary instrument. Since this is not the usual monetary policy instrument, it would be of interest to consider alternatives. Indeed, in the context of multiagent games (either static or dynamic) one can consider a monetary instrument problem analogous to that initially considered by Poole (1970) in a stochastic context. Should the monetary authorities target on the interest rate or some monetary aggregate? Or is some combination of the two preferable? These seem to be interesting questions to consider in the context of this model.

In their earlier paper, Miller and Salmon emphasize the problem of *time consistent* solutions within an intertemporal optimization context. This issue again arises in the present paper. This phenomenon frequently arises in intertemporal optimization models involving forward looking behavior. As a result, certain state variables are not tied to the past, but instead are able to respond freely to unanticipated disturbances. They are often referred to as 'jump' variables and their associated costate variables become zero at the initial point of optimization. The ability to re-optimize at each point of time implies that these costate variables, together with their derivatives, must be zero at all times. Denoting the vector of relevant costate variables by $p(t)$, this leads to an equation

$$\dot{p}(t) = p(t) = 0 \tag{4}$$

In effect, we get an additional dynamic equation and the question of time consistency revolves around whether or not this additional equation conflicts with the other dynamic equations of the system. Note that while conditions such as (4) constrain the dynamics a lot, time inconsistency need not always arise.

Miller and Salmon eliminate the potential problem of time inconsistency by assuming that the time path of the real exchange rate is given, when the policy is designed. That is, they do not allow any jumps in the exchange rate to occur, but instead constrain it to move continuously everywhere. I am somewhat uneasy about this procedure, since much of the current work on exchange rate dynamics emphasizes its role as an information variable and its ability to respond instantaneously to previously unanticipated shocks. I agree with the authors that their paradoxical ranking that increased coordination leads to higher rather than lower joint welfare costs is probably a consequence of their neglect of the jump in the exchange rate.

A simple way of capturing this possibility would be to augment the cost function to

$$\text{Min } C(E(0)-E_0)+\frac{1}{2}\int_0^\infty (\beta\pi^2+y^2)\,dt \tag{5}$$

where E_0 is the inherited exchange rate, $E(0)$ is the endogenously determined exchange rate, following the jump, and $C(\cdot)$ is the cost function attached to the jump. The idea is that when the policy is first introduced, the fact that it is unanticipated will cause the exchange rate to jump. This in turn generates jumps in the relative price and output, which in turn impose real adjustment costs on the economy. These are captured by the cost function C.

For certain classes of cost functions, the problem of time inconsistency may be eliminated. This can be shown to be true, for example, if the cost associated with the initial jump is proportional to its absolute magnitude, although not if it is quadratic.[6] Basically, if such costs are sufficiently high, the costs of recomputing the optimal policy and the associated adjustment costs these impose on the economy, are sufficiently large to eliminate the incentive for the government to revise its previously announced plan and to cheat on the private sector.

The introduction of a cost on unanticipated jumps in the exchange rate could I believe reverse the paradoxical ordering noted earlier, regarding the welfare costs and increased coordination. With more coordination one would expect the jump in the exchange rate to be mitigated, so that this element in the overall cost function would decline with increased cooperation between the policy makers.

Of course, the specification of the cost function $C(\cdot)$ is open to the change that it is arbitrary and that it is not motivated by deep economic considerations. While this may or may not be the case, one can argue that this criterion is just a special case of the more general criterion proposed in the control theory literature, from which the approach adopted in this paper is derived. A general statement of the intertemporal objective function introduces costs on the initial and terminal states of the system, as well as on the states and controls during the transition. In traditional applications of control theory, the initial state is given, so the costs associated with those states are effectively bygones and are given as far as any subsequent optimization is concerned. But when some of the state variables, most notably in the present context the nominal exchange rate, are allowed to undergo initial jumps, a subvector of the initial state vector becomes endogenous and the costs associated with those initial variables become an integral part of the overall optimization.

We may note further that the balancing of costs associated with initial jumps in the system, together with those from having it deviate during the transitional path from some desired target, was the motivation used to justify some of the early distributed lag models; see the survey by Griliches (1967). The modification of the cost function being proposed is along these lines. Furthermore, we may argue that the entire linear-quadratic optimal policy approach, although a convenient representation of the policy maker's problem, is open to the charge of being arbitrary. The specification of an initial cost function C is no less so than the specification of the rest of the cost functional.

We conclude with two further comments. First, the problem of time consistency can be solved by seeking alternative solution concepts. This is the procedure adopted by Oudiz and Sachs (1984) and is superior to either the Miller–Salmon procedure or the modified cost function suggested above. Secondly, instead of working with ad hoc macro models, it may be desirable to work with models based on private sector optimization. In this case, the government's objective is to maximize the utility of the representative individual in the economy. In such models the derivation of equilibrium often eliminates much of the dynamics, thereby possibly eliminating the problem of time inconsistency. To a large degree whether such models are characterized by time inconsistency depends upon how the government policy variables impact on the private sector's optimization.[7]

In summary, the methods of dynamic game theory provide a powerful and fruitful approach to the study of macroeconomic policy making in interdependent economies. The Miller–Salmon paper presents an interesting application of these techniques and serves as a promising start for further research in this area.

NOTES

1 See Miller and Salmon (1983).
2 Further discussion of these issues is given by Basar and Olsder (1982, Chapter 3).
3 A simple static example illustrating this is given by Basar and Olsder (1982, p. 193).
4 The typical specification of the expectations-augmented Phillips curve for an open economy under rational expectations is

$$p = \phi y + i$$

where y, i are as defined in the Miller–Salmon paper, p is the rate of inflation of domestic goods. Defining p^* analogously and letting e denote the rate of exchange depreciation of the domestic currency, we have

$$i = \delta p + (1-\delta)(p^* + e)$$

where δ is the share of the domestic good in domestic consumption. Writing this equation as

$$i = p + (1-\delta)(p^* + e - p) = p + (1-\delta)\,Dc$$

it is seen that the expectations augmented Phillips curve reduces to

$$\phi y + (1-\delta)\,Dc = 0$$

which is equivalent to (1).

5 See p. 10 of the Miller and Salmon text.

6 This result is shown formally by Stemp and Turnovsky (1984).

7 This is one of the conclusions of Turnovsky and Brock (1980).

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