

This PDF is a selection from a published volume from  
the National Bureau of Economic Research

Volume Title: NBER Macroeconomics Annual 2003,  
Volume 18

Volume Author/Editor: Mark Gertler and Kenneth  
Rogoff, editors

Volume Publisher: The MIT Press

Volume ISBN: 0-262-07253-X

Volume URL: <http://www.nber.org/books/gert04-1>

Conference Date: April 4-5, 2003

Publication Date: July 2004

Title: Optimal Monetary and Fiscal Policy: A Linear-Quadratic  
Approach

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URL: <http://www.nber.org/chapters/c11445>

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# *Optimal Monetary and Fiscal Policy: A Linear-Quadratic Approach*

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While substantial research literatures seek to characterize optimal monetary and fiscal policy, respectively, the two branches of the literature have largely developed in isolation, and on apparently contradictory foundations. The modern literature on dynamically optimal fiscal policy often abstracts from monetary aspects of the economy altogether and so implicitly allows no useful role for monetary policy. When monetary policy is considered within the theory of optimal fiscal policy, it is most often in the context of models with flexible prices. In these models, monetary policy matters only because (1) the level of nominal interest rates (and hence the opportunity cost of holding money) determines the size of certain distortions that result from the attempt to economize on money balances, and (2) the way the price level varies in response to real disturbances determines the state-contingent real payoffs on (riskless) nominally denominated government debt, which may facilitate tax-smoothing in the case that explicitly state-contingent debt is not available. The literature on optimal monetary policy has instead been mainly concerned with quite distinct objectives for monetary stabilization policy, namely, the minimization of the distortions that result from prices or wages that do not adjust quickly enough to clear markets. At the same time, this literature typically ignores the fiscal consequences of alternative monetary policies; the characterizations of optimal monetary policy obtained are thus strictly correct only for a world in which lump-sum taxes are available.

We would like to thank Stefania Albanesi, Marios Angeletos, Albert Marcet, and Ramon Marimon; seminar participants at New York University, Rutgers University, Universitat Pompeu Fabra, and the Macroeconomics Annual conference (2003), and the editors for helpful comments; Brad Strum and Vasco Curdia for research assistance; and the National Science Foundation for research support through a grant to the NBER.

Here we wish to consider the way in which the conclusions reached in each of these two familiar fields of study must be modified if one takes simultaneous account of the basic elements of the policy problems addressed in each. On the one hand, we wish to consider how conventional conclusions with regard to the nature of an optimal monetary policy rule must be modified if one recognizes that the government's only sources of revenue are distorting taxes, so that the fiscal consequences of monetary policy matter for welfare. And, on the other hand, we wish to consider how conventional conclusions with regard to optimal tax policy must be modified if one recognizes that prices do not instantaneously clear markets, so that output determination depends on aggregate demand, in addition to the supply-side factors stressed in the conventional theory of optimal taxation.

Several recent papers have also sought to consider optimal monetary and fiscal policy jointly, in the context of models with sticky prices; important examples include Correia et al. (2001), Schmitt-Grohé and Uribe (2001), and Siu (2001). Our approach differs from those taken in these papers, however, in several respects. First, we model price stickiness in a different way than in any of these papers, namely, by assuming staggered pricing of the kind introduced by Calvo (1983). This particular form of price stickiness has been widely used both in analyses of optimal monetary policy in models with explicit microfoundations (e.g., Goodfriend and King, 1997; Clarida et al., 1999; Woodford, 2003) and in the empirical literature on optimizing models of the monetary transmission mechanism (e.g., Rotemberg and Woodford, 1997; Gali and Gertler, 1999; Sbordone, 2002).

Perhaps more important, we obtain analytical results rather than purely numerical ones. To obtain these results, we propose a linear-quadratic approach to the characterization of optimal monetary and fiscal policy that allows us to nest both conventional analyses of optimal monetary policy, such as that of Clarida et al. (1999), and analyses of optimal tax-smoothing in the spirit of Barro (1979), Lucas and Stokey (1983), and Aiyagari et al. (2002) as special cases of our more general framework. We show how a linear-quadratic policy problem can be derived to yield a correct linear approximation to the optimal policy rules from the point of view of the maximization of expected discounted utility in a dynamic stochastic general-equilibrium model, building on our earlier work (Benigno and Woodford, 2003) for the case of optimal monetary policy when lump-sum taxes are available.

Finally, we do not content ourselves with merely characterizing the optimal dynamic responses of our policy instruments (and other state variables) to shocks under an optimal policy, given one assumption or another

about the nature and statistical properties of the exogenous disturbances to our model economy. Instead, we also wish to derive policy rules that the monetary and fiscal authorities may reasonably commit themselves to follow as a way of implementing the optimal equilibrium. In particular, we seek to characterize optimal policy in terms of optimal targeting rules for monetary and fiscal policy, of the kind proposed in the case of monetary policy by Svensson (1999), Svensson and Woodford (2003), and Giannoni and Woodford (2002, 2003). The rules are specified in terms of a target criterion for each authority; each authority commits itself to use its policy instrument each period in whatever way is necessary to allow it to project an evolution of the economy consistent with its target criterion. As discussed in Giannoni and Woodford (2002), we can derive rules of this form that are not merely consistent with the desired equilibrium responses to disturbances, but that in addition (1) imply a determinate rational-expectations equilibrium, so that there are not other equally possible (but less desirable) equilibria consistent with the same policy; and (2) bring about optimal responses to shocks regardless of the character of and statistical properties of the exogenous disturbances in the model.

### 1. The Policy Problem

Here we describe our assumptions about the economic environment and pose the optimization problem that joint optimal monetary and fiscal policies are intended to solve. The approximation method that we use to characterize the solution to this problem is then presented in the following section. Additional details of the derivation of the structural equations of our model of nominal price rigidity can be found in Woodford (2003, Chapter 3).

The goal of policy is assumed to be the maximization of the level of expected utility of a representative household. In our model, each household seeks to maximize:

$$U_{t_0} \equiv E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[ \tilde{u}(C_t; \xi_t) - \int_0^1 \tilde{v}(H_t(j); \xi_t) dj \right] \quad (1)$$

where  $C_t$  is a Dixit-Stiglitz aggregate of consumption of each of a continuum of differentiated goods:

$$C_t \equiv \left[ \int_0^1 c_t(i)^{(\theta-1)/\theta} di \right]^{\theta/(\theta-1)} \quad (2)$$

with an elasticity of substitution equal to  $\theta > 1$ , and  $H_t(j)$  is the quantity supplied of labor of type  $j$ . Each differentiated good is supplied by a single monopolistically competitive producer. There are assumed to be many

goods in each of an infinite number of industries; the goods in each industry  $j$  are produced using a type of labor that is specific to that industry, and they also change their prices at the same time. The representative household supplies all types of labor as well as consumes all types of goods.<sup>1</sup> To simplify the algebraic form of our results, we restrict attention in this paper to the case of isoelastic functional forms:

$$\bar{u}(C_t; \xi_t) \equiv \frac{C_t^{1-\bar{\sigma}^{-1}} \bar{C}_t^{\bar{\sigma}^{-1}}}{1-\bar{\sigma}^{-1}}$$

$$\bar{v}(H_t; \xi_t) \equiv \frac{\lambda}{1+\nu} H_t^{1+\nu} \bar{H}_t^{-\nu}$$

where  $\bar{\sigma}, \nu > 0$ , and  $\{\bar{C}_t, \bar{H}_t\}$  are bounded exogenous disturbance processes. (We use the notation  $\xi_t$  to refer to the complete vector of exogenous disturbances, including  $\bar{C}_t$  and  $\bar{H}_t$ .)

We assume a common technology for the production of all goods, in which (industry-specific) labor is the only variable input:

$$y_t(i) = A_t f(h_t(i)) = A_t h_t(i)^{1/\phi}$$

where  $A_t$  is an exogenously varying technology factor, and  $\phi > 1$ . Inverting the production function to write the demand for each type of labor as a function of the quantities produced of the various differentiated goods, and using the identity:

$$Y_t = C_t + G_t$$

to substitute for  $C_t$ , where  $G_t$  is exogenous government demand for the composite good, we can write the utility of the representative household as a function of the expected production plan  $\{y_t(i)\}$ .<sup>2</sup>

We can also express the relative quantities demanded of the differentiated goods each period as a function of their relative prices. This allows

1. We might alternatively assume specialization across households in the type of labor supplied; in the presence of perfect sharing of labor income risk across households, household decisions regarding consumption and labor supply would all be as assumed here.
2. The government is assumed to need to obtain an exogenously given quantity of the Dixit-Stiglitz aggregate each period and to obtain this in a cost-minimizing fashion. Hence, the government allocates its purchases across the suppliers of differentiated goods in the same proportion as do households, and the index of aggregate demand  $Y_t$  is the same function of the individual quantities  $\{y_t(i)\}$  as  $C_t$  is of the individual quantities consumed  $\{c_t(i)\}$ , defined in equation (2).

us to write the utility flow to the representative household in the form  $U(Y_t, \Delta_t; \xi_t)$ , where:

$$\Delta_t \equiv \int_0^1 \left( \frac{p_t(i)}{P_t} \right)^{-\theta(1+\omega)} di \geq 1 \tag{3}$$

is a measure of price dispersion at date  $t$ , in which  $P_t$  is the Dixit-Stiglitz price index:

$$P_t \equiv \left[ \int_0^1 p_t(i)^{1-\theta} di \right]^{1/(1-\theta)} \tag{4}$$

and the vector  $\xi_t$  now includes the exogenous disturbances  $G_t$  and  $A_t$  as well as the preference shocks. Hence, we can write equation (1) as:

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} U(Y_t, \Delta_t; \xi_t) \tag{5}$$

The producers in each industry fix the prices of their goods in monetary units for a random interval of time, as in the model of staggered pricing introduced by Calvo (1983). We let  $0 \leq \alpha < 1$  be the fraction of prices that remain unchanged in any period. A supplier that changes its price in period  $t$  chooses its new price  $p_t(i)$  to maximize:

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \Pi(p_t(i), p_T^j, P_T; Y_T, \tau_T, \xi_T) \right\} \tag{6}$$

where  $Q_{t,T}$  is the stochastic discount factor by which financial markets discount random nominal income in period  $T$  to determine the nominal value of a claim to such income in period  $t$ , and  $\alpha^{T-t}$  is the probability that a price chosen in period  $t$  will not have been revised by period  $T$ . In equilibrium, this discount factor is given by the following equation:

$$Q_{t,T} = \beta^{T-t} \frac{\tilde{u}_c(C_T; \xi_T) P_t}{\tilde{u}_c(C_t; \xi_t) P_T} \tag{7}$$

The function  $\Pi(p, p^j, P; Y, \tau, \xi)$ , defined in the appendix in Section 7, indicates the after-tax nominal profits of a supplier with price  $p$ , in an industry with common price  $p^j$ , when the aggregate price index is equal to  $P$ , aggregate demand is equal to  $Y$ , and sales revenues are taxed at rate  $\tau$ . Profits are equal to after-tax sales revenues net of the wage bill, and the real wage demanded for labor of type  $j$  is assumed to be given by:

$$w_t(j) = \mu_t^w \frac{\tilde{v}_h(H_t(j); \xi_t)}{\tilde{u}_c(C_t; \xi_t)} \tag{8}$$

where  $\mu_t^w \geq 1$  is an exogenous markup factor in the labor market (allowed to vary over time but assumed to be common to all labor markets),<sup>3</sup> and firms are assumed to be wage-takers. We allow for wage markup variations to include the possibility of a pure cost-push shock that affects equilibrium pricing behavior while implying no change in the efficient allocation of resources. Note that variation in the tax rate  $\tau_t$  has a similar effect on this pricing problem (and hence on supply behavior); this is the sole distortion associated with tax policy in the present model.

Each of the suppliers that revise their prices in period  $t$  choose the same new price  $p_t^*$ . Under our assumed functional forms, the optimal choice has a closed-form solution:

$$\frac{p_t^*}{P_t} = \left( \frac{K_t}{F_t} \right)^{1/(1 + \omega\theta)} \tag{9}$$

where  $\omega \equiv \phi(1 + \nu) - 1 > 0$  is the elasticity of real marginal cost in an industry with respect to industry output, and  $F_t$  and  $K_t$  are functions of current aggregate output  $Y_t$ ; the current tax rate  $\tau_t$ ; the current exogenous state  $\xi_t$ ; and the expected future evolution of inflation, output, taxes, and disturbances, defined in the appendix.<sup>4</sup>

The price index then evolves according to a law of motion:

$$P_t = \left[ (1 - \alpha)p_t^{*1-\theta} + \alpha P_{t-1}^{1-\theta} \right]^{1/(1-\theta)} \tag{10}$$

as a consequence of equation (4). Substitution of equation (9) into equation (10) implies that equilibrium inflation in any period is given by:

$$\frac{1 - \alpha \Pi_t^{\theta-1}}{1 - \alpha} = \left( \frac{F_t}{K_t} \right)^{(\theta-1)/(1 + \omega\theta)} \tag{11}$$

where  $\Pi_t \equiv P_t/P_{t-1}$ . This defines a short-run aggregate supply relation between inflation and output, given the current tax rate  $\tau_t$ ; current disturbances  $\xi_t$ ; and expectations regarding future inflation, output, taxes, and disturbances. Because the relative prices of the industries that do not change their prices in period  $t$  remain the same, we can also use equation (10) to derive a law of motion of the form:

$$\Delta_t = h(\Delta_{t-1}, \Pi_t) \tag{12}$$

3. In the case where we assume that  $\mu_t^w = 1$  at all times, our model is one in which both households and firms are wage-takers, or there is efficient contracting between them.

4. The disturbance vector  $\xi_t$  is now understood to include the current value of the wage markup  $\mu_t^w$ .

for the dispersion measure defined in equation (3). This is the source in our model of welfare losses from inflation or deflation.

We abstract here from any monetary frictions that would account for a demand for central-bank liabilities that earn a substandard rate of return. We nonetheless assume that the central bank can control the riskless short-term nominal interest rate  $i_t$ , which is in turn related to other financial asset prices through the arbitrage relation:<sup>5</sup>

$$1 + i_t = [E_t Q_{t,t+1}]^{-1}$$

We shall assume that the zero lower bound on nominal interest rates never binds under the optimal policies considered below.<sup>6</sup> Thus, we need not introduce any additional constraint on the possible paths of output and prices associated with a need for the chosen evolution of prices to be consistent with a nonnegative nominal interest rate.

Our abstraction from monetary frictions, and hence from the existence of seignorage revenues, does not mean that monetary policy has no fiscal consequences because interest-rate policy and the equilibrium inflation that results from it have implications for the real burden of government debt. For simplicity, we shall assume that all public debt consists of riskless nominal one-period bonds. The nominal value  $B_t$  of end-of-period public debt then evolves according to a law of motion:

$$B_t = (1 + i_{t-1})B_{t-1} - P_t s_t \quad (13)$$

where the real primary budget surplus is given by:

$$s_t \equiv \tau_t Y_t - G_t - \zeta_t \quad (14)$$

Here  $\tau_t$ , the share of the national product that is collected by the government as tax revenues in period  $t$ , is the key fiscal policy decision each period; the real value of (lump-sum) government transfers  $\zeta_t$  is treated as exogenously given, as are government purchases  $G_t$ . (We introduce the additional type of exogenously given fiscal needs to be able to analyze the consequences of a purely fiscal disturbance, with no implications for the real allocation of resources beyond those that follow from its effect on the government budget.)

5. For discussion of how this is possible even in a cashless economy of the kind assumed here, see Woodford (2003, Chapter 2).

6. This can be shown to be true in the case of small enough disturbances, given that the nominal interest rate is equal to  $\bar{r} = \beta^{-1} - 1 > 0$  under the optimal policy in the absence of disturbances.



Rational-expectations equilibrium requires that the expected path of government surpluses must satisfy an intertemporal solvency condition:

$$b_{t-1} \frac{P_{t-1}}{P_t} = E_t \sum_{T=t}^{\infty} R_{t,T} s_T \tag{15}$$

in each state of the world that may be realized at date  $t$ , where  $R_{t,T} \equiv Q_{t,T} P_T/P_t$  is the stochastic discount factor for a real income stream.<sup>7</sup> This condition restricts the possible paths that may be chosen for the tax rate  $\{\tau_t\}$ . Monetary policy can affect this constraint, however, both by affecting the period  $t$  inflation rate (which affects the left side) and (in the case of sticky prices) by affecting the discount factors  $\{R_{t,T}\}$ .

Under the standard (Ramsey) approach to the characterization of an optimal policy commitment, one chooses among state-contingent paths  $\{\Pi_t, Y_t, \tau_t, b_t, \Delta_t\}$  from some initial date  $t_0$  onward that satisfy equations (11), (12), and (15) for each  $t \geq t_0$ , given initial government debt  $b_{t_0-1}$  and price dispersion  $\Delta_{t_0-1}$ , to maximize equation (5). Such a  $t_0$ -optimal plan requires commitment, insofar as the corresponding  $t$ -optimal plan for some later date  $t$ , given the conditions  $b_{t-1}, \Delta_{t-1}$  obtaining at that date, will not involve a continuation of the  $t_0$ -optimal plan. This failure of time consistency occurs because the constraints on what can be achieved at date  $t_0$ , consistent with the existence of a rational-expectations equilibrium, depend on the expected paths of inflation, output, and taxes at later dates; but in the absence of a prior commitment, a planner would have no motive at those later dates to choose a policy consistent with the anticipations that it was desirable to create at date  $t_0$ .

However, the degree of advance commitment that is necessary to bring about an optimal equilibrium is only of a limited sort. Let:

$$W_t \equiv E_t \sum_{T=t}^{\infty} \beta^{T-t} \bar{u}_c(Y_T - G_T; \xi_T) s_T$$

and let  $\mathcal{F}$  be the set of values for  $(b_{t-1}, \Delta_{t-1}, F_t, K_t, W_t)$  such that there exist paths  $\{\Pi_T, Y_T, \tau_T, b_T, \Delta_T\}$  for dates  $T \geq t$  that satisfy equations (11), (12), and (15) for each  $T$ , that are consistent with the specified values for  $F_t, K_t$ , and  $W_t$ , and that imply a well-defined value for the objective  $U_t$  defined in equation (5). Furthermore, for any  $(b_{t-1}, \Delta_{t-1}, F_t, K_t, W_t) \in \mathcal{F}$ , let  $V(b_{t-1}, \Delta_{t-1}, X_t; \xi_t)$  denote the maximum attainable value of  $U_t$  among the state-contingent

7. See Woodford (2003, Chapter 2) for the derivation of this condition from household optimization together with market clearing. The condition should not be interpreted as an a priori constraint on possible government policy rules, as discussed in Woodford (2001). When we consider the problem of choosing an optimal plan from among the possible rational-expectations equilibria, however, this condition must be imposed among the constraints on the set of equilibria that one may hope to bring about.

paths that satisfy the constraints just mentioned, where  $X_t \equiv (F_t, K_t, W_t)$ .<sup>8</sup> Then the  $t_0$ -optimal plan can be obtained as the solution to a two-stage optimization problem, as shown in the appendix (Section 7).

In the first stage, values of the endogenous variables  $x_{t_0'}$  where  $x_t \equiv (\Pi_t, Y_t, \tau_t, b_t, \Delta_t)$ , and state-contingent commitments  $X_{t_0+1}(\xi_{t_0+1})$  for the following period, are chosen, subject to a set of constraints stated in the appendix, including the requirement that the choices  $(b_{t_0'}, \Delta_{t_0'}, X_{t_0+1}) \in \mathcal{F}$  for each possible state of the world  $\xi_{t_0+1}$ . These variables are chosen to maximize the objective  $\hat{J}[X_{t_0'}, X_{t_0+1}(\cdot)](\xi_{t_0})$ , where we define the functional:

$$\hat{J}[x_t, X_{t+1}(\cdot)](\xi_t) \equiv U(Y_t, \Delta_t; \xi_t) + \beta E_t V(b_t, \Delta_t, X_{t+1}; \xi_{t+1}) \quad (16)$$

In the second stage, the equilibrium evolution from period  $t_0 + 1$  onward is chosen to solve the maximization problem that defines the value function  $V(b_{t_0'}, \Delta_{t_0'}, X_{t_0+1}; \xi_{t_0+1})$ , given the state of the world  $\xi_{t_0+1}$  and the precommitted values for  $X_{t_0+1}$  associated with that state. The key to this result is a demonstration that there are no restrictions on the evolution of the economy from period  $t_0 + 1$  onward that are required for this expected evolution to be consistent with the values chosen for  $x_{t_0'}$  except consistency with the commitments  $X_{t_0+1}(\xi_{t_0+1})$  chosen in the first stage.

The optimization problem in stage two of this reformulation of the Ramsey problem is of the same form as the Ramsey problem itself, except that there are additional constraints associated with the precommitted values for the elements of  $X_{t_0+1}(\xi_{t_0+1})$ . Let us consider a problem like the Ramsey problem just defined, looking forward from some period  $t_0$ , except under the constraints that the quantities  $X_{t_0}$  must take certain given values, where  $(b_{t_0-1}, \Delta_{t_0-1}, X_{t_0}) \in \mathcal{F}$ . This constrained problem can similarly be expressed as a two-stage problem of the same form as above, with an identical stage-two problem to the one described above. Stage two of this constrained problem is thus of exactly the same form as the problem itself. Hence, the constrained problem has a recursive form. It can be decomposed into an infinite sequence of problems, in which in each period  $t$ ,  $(x_t, X_{t+1}(\cdot))$  are chosen to maximize  $\hat{J}[x_t, X_{t+1}(\cdot)](\xi_t)$ , subject to the constraints of the stage-one problem, given the predetermined state variables  $(b_{t-1}, \Delta_{t-1})$  and the precommitted values  $X_t$ .

Our aim here is to characterize policy that solves this constrained optimization problem (stage two of the original Ramsey problem), i.e., policy that is optimal from some date  $t$  onward given precommitted values for

8. In our notation for the value function  $V$ ,  $\xi_t$  denotes not simply the vector of disturbances in period  $t$ , but all information in period  $t$  about current and future disturbances. This corresponds to the disturbance vector  $\xi_t$  referred to earlier in the case that the disturbance vector follows a Markov process.

$X_t$ . Because of the recursive form of this problem, it is possible for a commitment to a time-invariant policy rule from date  $t$  onward to implement an equilibrium that solves the problem, for some specification of the initial commitments  $X_t$ . A time-invariant policy rule with this property is said by Woodford (2003, Chapter 7) to be "optimal from a timeless perspective."<sup>9</sup> Such a rule is one that a policymaker who solves a traditional Ramsey problem would be willing to commit to follow *eventually*, though the solution to the Ramsey problem involves different behavior initially because there is no need to internalize the effects of prior anticipation of the policy adopted for period  $t_0$ .<sup>10</sup> One might also argue that it is desirable to commit to follow such a rule immediately, even though such a policy would not solve the (unconstrained) Ramsey problem, as a way of demonstrating one's willingness to accept constraints that one wishes the public to believe that one will accept in the future.

## 2. A Linear-Quadratic Approximate Problem

In fact, we shall here characterize the solution to this problem (and similarly derive optimal time-invariant policy rules) only for initial conditions near certain steady-state values, allowing us to use local approximations in characterizing optimal policy.<sup>11</sup> We establish that these steady-state values have the property that if one starts from initial conditions close enough to the steady state, and exogenous disturbances thereafter are small enough, the optimal policy subject to the initial commitments remains forever near the steady state. Hence, our local characterization would describe the long-run character of Ramsey policy, in the event that disturbances are small enough, and that deterministic Ramsey policy would converge to the steady state.<sup>12</sup> Of greater interest here, it describes policy that is optimal from a timeless perspective in the event of small disturbances.

9. See also Woodford (1999) and Giannoni and Woodford (2002).

10. For example, in the case of positive initial nominal government debt, the  $t_0$ -optimal policy would involve a large inflation in period  $t_0$  to reduce the pre-existing debt burden, but a commitment not to respond similarly to the existence of nominal government debt in later periods.

11. Local approximations of the same sort are often used in the literature in numerical characterizations of Ramsey policy. Strictly speaking, however, such approximations are valid only in the case of initial commitments  $X_{t_0}$  near enough to the steady-state values of these variables, and the  $t_0$ -optimal (Ramsey) policy need not involve values of  $X_{t_0}$  near the steady-state values, even in the absence of random disturbances.

12. Our work (Benigno and Woodford, 2003) gives an example of an application in which Ramsey policy does converge asymptotically to the steady state, so that the solution to the approximate problem approximates the response to small shocks under the Ramsey policy, at dates long enough after  $t_0$ . We cannot make a similar claim in the present application, however, because of the unit root in the dynamics associated with optimal policy.

First, we must show the existence of a steady state, i.e., of an optimal policy (under appropriate initial conditions) that involves constant values of all variables. To this end, we consider the purely deterministic case, in which the exogenous disturbances  $\bar{C}_t, \bar{G}_t, \bar{H}_t, \bar{A}_t, \mu_t^w, \zeta_t$  each take constant values  $\bar{C}, \bar{G}, \bar{H}, \bar{A}, \bar{\mu}^w > 0$  and  $\bar{\zeta} \geq 0$  for all  $t \geq t_0$ , and assume an initial real public debt  $b_{t_0-1} = \bar{b} > 0$ . We wish to find an initial degree of price dispersion  $\Delta_{t_0-1}$  and initial commitments  $X_{t_0} = \bar{X}$  so that the solution to the stage-two problem defined above involves a constant policy  $x_t = \bar{x}, X_{t+1} = \bar{X}$  each period, in which  $\bar{b}$  is equal to the initial real debt and  $\bar{\Delta}$  is equal to the initial price dispersion. We show in the appendix (Section 7) that the first-order conditions for this problem admit a steady-state solution of this form, and we verify below that the second-order conditions for a local optimum are also satisfied.

Regardless of the initial public debt  $\bar{b}$ , we show that  $\bar{\Pi} = 1$  (zero inflation), and correspondingly that  $\bar{\Delta} = 1$  (zero price dispersion). Note that our conclusion that the optimal steady-state inflation rate is zero generalizes our result (Benigno and Woodford, 2003) for the case in which taxes are lump-sum at the margin. We may furthermore assume without loss of generality that the constant values of  $\bar{C}$  and  $\bar{H}$  are chosen (given the initial government debt  $\bar{b}$ ) so that in the optimal steady state,  $C_t = \bar{C}$  and  $H_t = \bar{H}$  each period.<sup>13</sup> The associated steady-state tax rate is given by:

$$\bar{\tau} = s_G + \frac{\bar{\zeta} + (1 - \beta)\bar{b}}{\bar{Y}}$$

where  $\bar{Y} = \bar{C} + \bar{G} > 0$  is the steady-state output level, and  $s_G \equiv \bar{G}/\bar{Y} < 1$  is the steady-state share of output purchased by the government. As shown in Section 7, this solution necessarily satisfies  $0 < \bar{\tau} < 1$ .

We next wish to characterize the optimal responses to small perturbations of the initial conditions and small fluctuations in the disturbance processes around the above values. To do this, we compute a linear-quadratic approximate problem, the solution to which represents a linear approximation to the solution to the stage-two policy problem, using the method we introduced in Benigno and Woodford (2003). An important advantage of this approach is that it allows direct comparison of our results with those obtained in other analyses of optimal monetary stabilization policy. Other advantages are that it makes it straightforward to verify whether the second-order conditions hold (the second-order conditions that are required for a solution to our first-order conditions to be at

13. Note that we may assign arbitrary positive values to  $\bar{C}, \bar{H}$  without changing the nature of the implied preferences as long as the value of  $\lambda$  is appropriately adjusted.

least a local optimum),<sup>14</sup> and that it provides us with a welfare measure with which to rank alternative suboptimal policies, in addition to allowing computation of the optimal policy.

We begin by computing a Taylor-series approximation to our welfare measure in equation (5), expanding around the steady-state allocation defined above, in which  $y_t(i) = \bar{Y}$  for each good at all times and  $\xi_t = 0$  at all times.<sup>15</sup> As a second-order (logarithmic) approximation to this measure, we obtain:

$$U_{t_0} = \bar{Y} \bar{u}_c \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \Phi \hat{Y}_t - \frac{1}{2} u_{yy} \hat{Y}_t^2 + \hat{Y}_t u_{\xi} \xi_t - u_{\Delta} \hat{\Delta}_t + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3) \tag{17}$$

where  $\hat{Y}_t \equiv \log(Y_t/\bar{Y})$  and  $\hat{\Delta}_t \equiv \log \Delta_t$  measure deviations of aggregate output and the price dispersion measure from their steady-state levels.<sup>16</sup> The term *t.i.p.* collects terms that are independent of policy (constants and functions of exogenous disturbances) and hence is irrelevant for ranking alternative policies;  $\|\xi\|$  is a bound on the amplitude of our perturbations of the steady state.<sup>17</sup> Here the coefficient:

$$\Phi \equiv 1 - \frac{\theta - 1}{\theta} \frac{1 - \bar{\tau}}{\bar{\mu}^w} < 1$$

measures the steady-state wedge between the marginal rate of substitution between consumption and leisure and the marginal product of labor, and hence the inefficiency of the steady-state output level  $\bar{Y}$ . Under the assumption that  $\bar{b} > 0$ , we necessarily have  $\Phi > 0$ , meaning that steady-state output is inefficiently low. The coefficients  $u_{yy}$ ,  $u_{\xi}$ , and  $u_{\Delta}$  are defined in the appendix (Section 7).

14. We (Benigno and Woodford, 2003) show that these conditions can fail to hold, so that a small amount of arbitrary randomization of policy is welfare-improving, but we argue that the conditions under which this occurs in our model are not empirically plausible.
15. Here the elements of  $\xi_t$  are assumed to be  $\bar{c}_t \equiv \log(\bar{C}_t/\bar{C})$ ,  $\bar{h}_t \equiv \log(\bar{H}_t/\bar{H})$ ,  $\bar{a}_t \equiv \log(\bar{A}_t/\bar{A})$ ,  $\bar{\mu}_t^w \equiv \log(\bar{\mu}_t^w/\bar{\mu}^w)$ ,  $\bar{G}_t \equiv (\bar{G}_t - \bar{G})/\bar{Y}$ , and  $\bar{\zeta}_t \equiv (\bar{\zeta}_t - \bar{\zeta})/\bar{Y}$ , so that a value of zero for this vector corresponds to the steady-state values of all disturbances. The perturbations  $\bar{C}_t$  and  $\bar{\zeta}_t$  are not defined to be logarithmic so that we do not have to assume positive steady-state values for these variables.
16. See the appendix (Section 7) for details. Our calculations here follow closely those of our earlier work (Woodford, 2003, Chapter 6; Benigno and Woodford, 2003).
17. Specifically, we use the notation  $\mathcal{O}(\|\xi\|)$  as shorthand for  $\mathcal{O}(\|\xi, \bar{b}_{t-1}, \bar{\Delta}_{t-1}^{1/2}, \bar{X}_t\|)$ , where in each case circumflexes refer to log deviations from the steady-state values of the various parameters of the policy problem. We treat  $\bar{\Delta}_{t-1}^{1/2}$  as an expansion parameter, rather than  $\bar{\Delta}_{t-1}$  because equation (12) implies that deviations of the inflation rate from zero of order  $\varepsilon$  only result in deviations in the dispersion measure  $\Delta_t$  from one of order  $\varepsilon^2$ . We are thus entitled to treat the fluctuations in  $\Delta_t$  as being only of second order in our bound on the amplitude of disturbances because, if this is true at some initial date, it will remain true thereafter.

Under the Calvo assumption about the distribution of intervals between price changes, we can relate the dispersion of prices to the overall rate of inflation, allowing us to rewrite equation (17) as:

$$U_{t_0} = \bar{Y}\bar{u}_c \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[ \Phi \hat{Y}_t - \frac{1}{2} u_{yy} \hat{Y}_t^2 + \hat{Y}_t u_{y\xi} \xi_t - \frac{1}{2} u_{\pi\pi} \pi_t^2 \right] + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3) \tag{18}$$

for a certain coefficient  $u_{\pi} > 0$  defined in the appendix, where  $\pi_t \equiv \log \Pi_t$  is the inflation rate. Thus, we can write our stabilization objective purely in terms of the evolution of the aggregate variables  $\{\hat{Y}_t, \pi_t\}$  and the exogenous disturbances.

We note that when  $\Phi > 0$ , there is a nonzero linear term in equation (18), which means that we cannot expect to evaluate this expression to second order using only an approximate solution for the path of aggregate output that is accurate only to first order. Thus, we cannot determine optimal policy, even up to first order, using this approximate objective together with approximations to the structural equations that are accurate only to first order. Rotemberg and Woodford (1997) avoid this problem by assuming an output subsidy (i.e., a value  $\bar{\tau} < 0$ ) of the size needed to ensure that  $\Phi = 0$ . Here, we do not wish to make this assumption because we assume that lump-sum taxes are unavailable, in which case  $\Phi = 0$  would be possible only in the case of a particular initial level of government assets  $\bar{b} < 0$ . Furthermore, we are more interested in the case in which government revenue needs are more acute than that would imply.

We (Benigno and Woodford, 2003) propose an alternative way of dealing with this problem; we use a second-order approximation to the aggregate-supply relation to eliminate the linear terms in the quadratic welfare measure. In the model that we consider, where taxes are lump-sum (and so do not affect the aggregate supply relation), a forward-integrated second-order approximation to this relation allows one to express the expected discounted value of output terms  $\Phi \hat{Y}_t$  as a function of purely quadratic terms (except for certain transitory terms that do not affect the stage-two policy problem). In the present case, the level of distorting taxes has a first-order effect on the aggregate-supply relation (see equation [22] below), so that the forward-integrated relation involves the expected discounted value of the tax rate as well as the expected discounted value of output. As shown in the appendix, however, a second-order approximation to the intertemporal solvency condition in equation (15) provides another relation between the expected discounted values of output and the tax rate and a set of purely quadratic

terms.<sup>18</sup> These two second-order approximations to the structural equations that appear as constraints in our policy problem can then be used to express the expected discounted value of output terms in equation (18) in terms of purely quadratic terms.

In this manner, we can rewrite equation (18) as:

$$U_{t_0} = -\Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{1}{2} q_y (\hat{Y}_t - \hat{Y}_t^*)^2 + \frac{1}{2} q_\pi \pi_t^2 \right\} + T_{t_0} + t.i.p. + o(\|\xi\|^3) \quad (19)$$

where again the coefficients are defined in the appendix (Section 7). The expression  $\hat{Y}_t^*$  indicates a function of the vector of exogenous disturbances  $\xi_t$  defined in the appendix, while  $T_{t_0}$  is a transitory component. When the alternative policies from date  $t_0$  onward must be evaluated and must be consistent with a vector of prior commitments  $X_{t_0}$ , one can show that the value of the term  $T_{t_0}$  is implied (to a second-order approximation) by the value of  $X_{t_0}$ . Hence, for purposes of characterizing optimal policy from a timeless perspective, it suffices that we rank policies according to the value that they imply for the loss function:

$$E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{1}{2} q_y (\hat{Y}_t - \hat{Y}_t^*)^2 + \frac{1}{2} q_\pi \pi_t^2 \right\} \quad (20)$$

where a lower value of expression (20) implies a higher value of expression (19). Because this loss function is purely quadratic (i.e., lacking linear terms), it is possible to evaluate it to second order using only a first-order approximation to the equilibrium evolution of inflation and output under a given policy. Hence, log-linear approximations to the structural relations of our model suffice, yielding a standard linear-quadratic policy problem.

For this linear-quadratic problem to have a bounded solution (which then approximates the solution to the exact problem), we must verify that the quadratic objective in equation (20) is convex. We show in the appendix (Section 7) that  $q_y, q_\pi > 0$ , so that the objective is convex, as long as the steady-state tax rate  $\bar{\tau}$  and share of government purchases  $s_G$  in the national product are below certain positive bounds. We shall here assume that these conditions are satisfied, i.e., that the government's fiscal needs are not too severe. Note that, in this case, our quadratic objective turns out to be of a form commonly assumed in the literature on monetary policy evaluation; that is, policy should seek to minimize the discounted value of a weighted sum of squared deviations of inflation from an optimal

18. Since we are interested in providing an approximate characterization of the stage-two policy problem, in which a precommitted value of  $W_t$  appears as a constraint, it is actually a second-order approximation to that constraint that we need. This latter constraint has the same form as equation (15), however; the only difference is that the quantities in the relation are taken to have predetermined values.

level (here, zero) and squared fluctuations in an output gap  $y_t \equiv \hat{Y}_t - \hat{Y}_t^*$ , where the target output level  $\hat{Y}_t^*$  depends on the various exogenous disturbances in a way discussed in the appendix. It is also perhaps of interest to note that a tax-smoothing objective of the kind postulated by Barro (1979) and Bohn (1990) does not appear in our welfare measure as a separate objective. Instead, tax distortions are relevant only insofar as they result in output gaps of the same sort that monetary stabilization policy aims to minimize.

We turn next to the form of the log-linear constraints in the approximate policy problem. A first-order Taylor series expansion of equation (11) around the zero-inflation steady state yields the log-linear aggregate-supply relation:

$$\pi_t = \kappa [\hat{Y}_t + \psi \hat{\tau}_t + c'_\xi \xi_t] + \beta E_t \pi_{t+1} \tag{21}$$

for certain coefficients  $\kappa, \psi > 0$ . This is the familiar new Keynesian Phillips curve relation.<sup>19</sup> It is extended here to account for the effects of variations in the level of distorting taxes on supply costs.

It is useful to write this approximate aggregate-supply relation in terms of the welfare-relevant output gap  $y_t$ . Equation (21) can be written as:

$$\pi_t = \kappa [y_t + \psi \hat{\tau}_t + u_t] + \beta E_t \pi_{t+1} \tag{22}$$

where  $u_t$  is composite cost-push disturbance, indicating the degree to which the various exogenous disturbances included in  $\xi_t$  preclude simultaneous stabilization of inflation, the welfare-relevant output gap, and the tax rate. Alternatively we can write:

$$\pi_t = \kappa [y_t + \psi (\hat{\tau}_t - \hat{\tau}_t^*)] + \beta E_t \pi_{t+1} \tag{23}$$

where  $\hat{\tau}_t^* \equiv -\psi^{-1} u_t$  indicates the tax change needed at any time to offset the cost-push shock, thus to allow simultaneous stabilization of inflation and the output gap (the two stabilization objectives reflected in equation [20]).

The effects of the various exogenous disturbances in  $\xi_t$  on the cost-push term  $u_t$  are explained in the appendix (Section 7). It is worth noting that under certain conditions  $u_t$  is unaffected by some disturbances. In the case that  $\Phi = 0$ , the cost-push term is given by:

$$u_t = u_{\xi 5} \hat{\mu}_t^w \tag{24}$$

where in this case,  $u_{\xi 5} = q_V^{-1} > 0$ . Thus, the cost-push term is affected only by variations in the wage markup  $\hat{\mu}_t$ ; it does not vary in response to taste shocks, technology shocks, government purchases, or variations in

19. See, e.g., Clarida et al. (1999) or Woodford (2003, Chapter 3).



government transfers. The reason is that when  $\Phi = 0$  and neither taxes nor the wage markup vary from their steady-state values, the flexible-price equilibrium is efficient; it follows that the level of output consistent with zero inflation is also the one that maximizes welfare, as discussed in Woodford (2003, Chapter 6).

Even when  $\Phi > 0$ , if there are no government purchases (so that  $s_G = 0$ ) and no fiscal shocks (meaning that  $\hat{G}_t = 0$  and  $\hat{\zeta}_t = 0$ ), then the  $u_t$  term is again of the form in equation (24), but with  $u_{\xi 5} = (1 - \Phi) q_y^{-1} t$ , as we discussed in Benigno and Woodford (2003). Hence, in this case, neither taste nor technology shocks have cost-push effects. The reason is that in this isoelastic case, if taxes and the wage markup never vary, the flexible-price equilibrium value of output and the efficient level vary in exactly the same proportion in response to each of the other types of shocks; hence, inflation stabilization also stabilizes the gap between actual output and the efficient level. Another special case is the limiting case of linear utility of consumption ( $\sigma^{-1} = 0$ ); in this case,  $u_t$  is again of the form in equation (24) for a different value of  $u_{\xi 5}$ . In general, however, when  $\Phi > 0$  and  $s_G > 0$ , all of the disturbances shift the flexible-price equilibrium level of output (under a constant tax rate) and the efficient level of output to differing extents, resulting in cost-push contributions from all of these shocks.

The other constraint on possible equilibrium paths is the intertemporal government solvency condition. A log-linear approximation to equation (15) can be written in the form:

$$\hat{b}_{t-1} - \pi_t - \sigma^{-1} y_t = -f_t + (1 - \beta) E_t \sum_{T=t}^{\infty} \beta^{T-t} [b_y y_T + b_\tau (\hat{\tau}_T - \hat{\tau}_T^*)] \quad (25)$$

where  $\sigma > 0$  is the intertemporal elasticity of substitution of private expenditure, and the coefficients  $b_y, b_\tau$  are defined in the appendix, as is  $f_t$ , a composite measure of exogenous fiscal stress. Here, we have written the solvency condition in terms of the same output gap and tax gap as equation (23) to make clear the extent to which complete stabilization of the variables appearing in the loss function of equation (20) is possible. The constraint can also be written in a flow form:

$$\hat{b}_{t-1} - \pi_t - \sigma^{-1} y_t + f_t = (1 - \beta)[b_y y_t + b_\tau (\hat{\tau}_t - \hat{\tau}_t^*)] + \beta E_t [\hat{b}_t - \pi_{t+1} - \sigma^{-1} y_{t+1} + f_{t+1}], \quad (26)$$

together with a transversality condition.<sup>20</sup>

20. If we restrict attention to bounded paths for the endogenous variables, then a path satisfies equation (25) in each period  $t \geq 0$  if and only if it satisfies the flow budget constraint in equation (26) in each period.

We note that the only reason why it should *not* be possible to stabilize both inflation and the output gap completely from some date  $t$  onward is if the sum  $\hat{b}_{i-1} + f_i$  is nonzero. The composite disturbance  $f_i$  therefore completely summarizes the information at date  $t$  about the exogenous disturbances that determines the degree to which stabilization of inflation and output is not possible; under an optimal policy, the state-contingent evolution of the inflation rate, the output gap, and the real public debt depend solely on the evolution of the single composite disturbance process  $\{f_i\}$ .

This result contrasts with the standard literature on optimal monetary stabilization policy, in which (in the absence of a motive for interest-rate stabilization, as here) it is instead the cost-push term  $u_i$  that summarizes the extent to which exogenous disturbances require that fluctuations in inflation and in the output gap should occur. Note that in the case when there are no government purchases and no fiscal shocks,  $u_i$  corresponds simply to equation (24). Thus, for example, it is concluded (in a model with lump-sum taxes) that there should be no variation in inflation in response to a technology shock (Khan et al., 2002; Benigno and Woodford, 2003). But even in this simple case, the fiscal stress is given by an expression of the form:

$$f_t \equiv h'_\xi \xi_t - (1 - \beta) E_t \sum_{T=t}^{\infty} \beta^{T-t} f'_\xi \xi_T \quad (27)$$

where the expressions  $h'_\xi \xi_t$  and  $f'_\xi \xi_t$  both generally include nonzero coefficients on preference and technology shocks, in addition to the markup shock, as shown in the appendix. Hence, many disturbances that do not have cost-push effects nonetheless result in optimal variations in both inflation and the output gap.

Finally, we wish to consider optimal policy subject to the constraints that  $F_{t_0}$ ,  $K_{t_0}$  and  $W_{t_0}$  take given (pre-committed) values. Again, only log-linear approximations to these constraints matter for a log-linear approximate characterization of optimal policy. As discussed in the appendix, the corresponding constraints in our approximate model are pre-commitments regarding the state-contingent values of  $\pi_{t_0}$  and  $y_{t_0}$ .

To summarize, our approximate policy problem involves the choice of state-contingent paths for the endogenous variables  $\{\pi_t, y_t, \hat{\tau}_t, \hat{b}_t\}$  from some date  $t_0$  onward to minimize the quadratic loss function in equation (20), subject to the constraint that the conditions in equations (23) and (25) be satisfied each period, given an initial value  $\hat{b}_{t_0-1}$ , and subject also to the constraints that  $\pi_{t_0}$  and  $y_{t_0}$  equal certain pre-committed values (that may depend on the state of the world in period  $t_0$ ). We shall first characterize the state-contingent evolution of the endogenous variables in response to exogenous shocks, in the rational-expectations equilibrium that solves

this problem. We then turn to the derivation of optimal policy rules, commitment to which should implement an equilibrium of this kind.

### 3. *Optimal Responses to Shocks: The Case of Flexible Prices*

In considering the solution to the problem of stabilization policy just posed, it may be useful first to consider the simple case in which prices are fully flexible. This is the limiting case of our model in which  $\alpha = 0$ , with the consequence that  $q_\pi = 0$  in equation (20), and that  $\kappa^{-1} = 0$  in equation (23). Hence, our optimization problem reduces to the minimization of:

$$\frac{1}{2} q_y E_{t_0} \sum_{i=t_0}^{\infty} \beta^{t-t_0} y_i^2 \quad (28)$$

subject to the constraints:

$$y_t + \psi(\hat{\tau}_t - \hat{\tau}_t^*) = 0 \quad (29)$$

and equation (25). It is easy to see that in this case, the optimal policy is one that achieves  $y_t = 0$  at all times. Because of equation (29), this requires that  $\hat{\tau}_t = \hat{\tau}_t^*$  at all times. The inflation rate is then determined by the requirement of government intertemporal solvency:

$$\pi_t = \hat{b}_{t-1} + f_t$$

This last equation implies that unexpected inflation must equal the innovation in the fiscal stress:

$$\pi_t - E_{t-1} \pi_t = f_t - E_{t-1} f_t$$

Expected inflation and hence the evolution of nominal government debt are indeterminate. If we add to our assumed policy objective a small preference for inflation stabilization, when this has no cost in terms of other objectives, then the optimal policy will be one that involves  $E_t \pi_{t+1} = 0$  each period.<sup>21</sup> Thus, the nominal public debt must evolve according to:

$$\hat{b}_t = -E_t f_{t+1}$$

21. Note that this preference can be justified in terms of our model, in the case that  $\alpha$  is positive though extremely small. Then there will be a very small positive value for  $q_\pi$ , implying that reduction of the expected discounted value of inflation is preferred to the extent that this does not require any increase in the expected discounted value of squared output gaps.

If, instead, we were to assume the existence of small monetary frictions (and zero interest on money), the tie would be broken by the requirement that the nominal interest rate equal zero each period.<sup>22</sup> The required expected rate of inflation (and hence the required evolution of the nominal public debt) would then be determined by the variation in the equilibrium real rate of return implied by a real allocation in which  $\hat{Y}_t = \hat{Y}_t^*$  each period. That is, one would have  $E_t \pi_{t+1} = -r_t^*$ , where  $r_t^*$  is the (exogenous) real rate of interest associated output at the target level each period, and so:

$$\hat{b}_t = -r_t^* - E_t f_{t+1}$$

We thus obtain simple conclusions about the determinants of fluctuations in inflation, output, and the tax rate under optimal policy. Unexpected inflation variations occur as needed to prevent taxes from ever having to be varied to respond to variations in fiscal stress, as in the analyses of Bohn (1990) and Chari and Kehoe (1999). This allows a model with only riskless nominal government debt to achieve the same state-contingent allocation of resources as the government would choose to bring about if it could issue state-contingent debt, as in the model of Lucas and Stokey (1983).

Because taxes do not have to adjust in response to variations in fiscal stress, as in the tax-smoothing model of Barro (1979), it is possible to smooth them across states as well as over time. However, the sense in which it is desirable to smooth tax rates is that of minimizing variation in the gap  $\hat{\tau}_t - \hat{\tau}_t^*$ , rather than variation in the tax rate itself.<sup>23</sup> In other words, it is really the tax gap  $\hat{\tau}_t - \hat{\tau}_t^*$  that should be smoothed. Under certain special circumstances, it will not be optimal for tax rates to vary in response

22. The result relies on the fact that the distortions created by the monetary frictions are minimized in the case of a zero opportunity cost of holding money each period, as argued by Friedman (1969). Neither the existence of effects of nominal interest rates on supply costs (so that an interest-rate term should appear in the aggregate-supply relation in equation [29]) nor the contribution of seignorage revenues to the government budget constraint make any difference to the result because unexpected changes in revenue needs can always be costlessly obtained through unexpected inflation, while any desired shifts in the aggregate-supply relation to offset cost-push shocks can be achieved by varying the tax rate.

23. Several authors (e.g., Chari et al., 1991, 1994; Hall and Krieger, 2000; Aiyagari et al., 2002) have found that in calibrated flexible-price models with state-contingent government debt, the optimal variation in labor tax rates is quite small. Our results indicate this as well, in the case that real disturbances have only small cost-push effects, and we have listed earlier various conditions under which this will be the case. But under some circumstances, optimal policy may involve substantial volatility of the tax rate and indeed more volatility of the tax rate than of inflation. This would be the case if shocks have large cost-push effects while having relatively little effect on fiscal stress.

to shocks; these are the conditions, discussed above, under which shocks have no cost-push effects, so that there is no change in  $\hat{\tau}_t^*$ . For example, if there are no government purchases and there is no variation in the wage markup, this will be the case. But more generally, all disturbances will have some cost-push effect and will result in variations in  $\hat{\tau}_t^*$ . Then there will be variations in the tax rate in response to these shocks under an optimal policy. There will be no unit root in the tax rate, however, as in the Barro (1979) model of optimal tax policy. Instead, as in the analysis of Lucas and Stokey (1983), the optimal fluctuations in the tax rate will be stationary and will have the same persistence properties as the real disturbances (specifically, the persistence properties of the composite cost-push shock).

Variations in fiscal stress will instead require changes in the tax rate, as in the analysis of Barro (1979), if we suppose that the government issues only riskless indexed debt rather than the riskless nominal debt assumed in our baseline model. (Again, for simplicity we assume that only one-period riskless debt is issued.) In this case the objective function in equation (20) and the constraints in equations (25) and (29) remain the same, but  $\underline{b}_{t-1} \equiv \hat{b}_{t-1} - \pi_t$ , the real value of private claims on the government at the beginning of period  $t$ , is now a predetermined variable. This means that unexpected inflation variations can no longer relax the intertemporal government solvency condition. In fact, rewriting the constraint in equation (25) in terms of  $\underline{b}_{t-1}$ , we see that the path of inflation is now completely irrelevant to welfare.

The solution to this optimization problem is now less trivial because complete stabilization of the output gap is not generally possible. The optimal state-contingent evolution of output and taxes can be determined using a Lagrangian method, as in Woodford (2003, Chapter 7). The Lagrangian for the present problem can be written as:

$$L_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{1}{2} q_y y_t^2 + \varphi_{1t} [y_t + \psi \hat{\tau}_t] + \varphi_{2t} [\underline{b}_{t-1} - \sigma^{-1} y_t] - (1 - \beta)(b_y y_t + b_\tau \hat{\tau}_t) - \beta(\underline{b}_t - \sigma^{-1} y_{t+1}) \right\} + \sigma \varphi_{2,t_0-1} y_{t_0} \quad (30)$$

where  $\varphi_{1t}$ ,  $\varphi_{2t}$  are Lagrange multipliers associated with the constraints in equations (29) and (26), respectively,<sup>24</sup> for each  $t \geq t_0$ , and  $\sigma \varphi_{2,t_0-1}$  is the notation used for the multiplier associated with the additional constraint that  $y_{t_0} = \hat{Y}_{t_0}$ . The latter constraint is added to characterize optimal policy from a timeless perspective, as discussed at the end of Section 2; the

24. Alternatively,  $\varphi_{2t}$  is the multiplier associated with the constraint in equation (25).

particular notation used for the multiplier on this constraint results in a time-invariant form for the first-order conditions, as seen below.<sup>25</sup> We have dropped terms from the Lagrangian that are not functions of the endogenous variables  $y_t$  and  $\hat{\tau}_t$ , i.e., products of multipliers and exogenous disturbances, because these do not affect our calculation of the implied first-order conditions.

The resulting first-order condition with respect to  $y_t$  is:

$$q_y y_t = -\phi_{1t} + [(1 - \beta)b_y + \sigma^{-1}]\phi_{2t} - \sigma^{-1}\phi_{2,t-1} \quad (31)$$

that with respect to  $\hat{\tau}_t$  is:

$$\psi\phi_{1t} = (1 - \beta)b_\tau\phi_{2t} \quad (32)$$

and that with respect to  $b_t$  is:

$$\phi_{2t} = E_t\phi_{2,t+1} \quad (33)$$

Each of these conditions must be satisfied for each  $t \geq t_0$ , along with the structural equations (29) and (25) for each  $t \geq t_0$ , for given initial values  $\underline{b}_{t_0-1}$  and  $y_{t_0}$ . We look for a bounded solution to these equations so that (in the event of small enough disturbances) none of the state variables leave a neighborhood of the steady-state values, in which our local approximation to the equilibrium conditions and our welfare objective remain accurate.<sup>26</sup> Given the existence of such a bounded solution, the transversality condition is necessarily satisfied so that the solution to these first-order conditions represents an optimal plan.

25. It should be recalled that, for policy to be optimal from a timeless perspective, the state-contingent initial commitment  $\hat{y}_{t_0}$  must be chosen so it conforms to the state-contingent commitment regarding  $y_t$  that will be chosen in all later periods, so that the optimal policy can be implemented by a time-invariant rule. Hence, it is convenient to present the first-order conditions in a time-invariant form.

26. In the only such solution, the variables  $\hat{\tau}_t$ ,  $\underline{b}_t$ , and  $y_t$  are all permanently affected by shocks, even when the disturbances are all assumed to be stationary (and bounded) processes. Hence, a bounded solution exists only under the assumption that random disturbances occur only in a finite number of periods. However, our characterization of optimal policy does not depend on a particular bound on the number of periods in which there are disturbances, or which periods these are; to allow disturbances in a larger number of periods, we must assume a tighter bound on the amplitude of disturbances for the optimal paths of the endogenous variables to remain within a given neighborhood of the steady-state values. Aiyagari et al. (2002) discuss the asymptotic behavior of the optimal plan in the exact nonlinear version of a problem similar to this one, in the case that disturbances occur indefinitely.

An analytical solution to these equations is easily given. Using equation (29) to substitute for  $\hat{\tau}_t$  in the forward-integrated version of equation (25), then equations (31) and (32) to substitute for  $y_t$  as a function of the path of  $\varphi_{2t}$ , and finally using equation (33) to replace all terms of the form  $E_t \varphi_{2,t+j}$  (for  $j \geq 0$ ) by  $\varphi_{2t}$ , we obtain an equation that can be solved for  $\varphi_{2t}$ . The solution is of the form:

$$\varphi_{2t} = \frac{m_b}{m_b + n_b} \varphi_{2,t-1} - \frac{1}{m_b + n_b} [f_t + \underline{b}_{t-1}]$$

Coefficients  $m_b, n_b$  are defined in the appendix (Section 7). The implied dynamics of the government debt are then given by:

$$\underline{b}_t = -E_t f_{t+1} - n_b \varphi_{2t}$$

This allows a complete solution for the evolution of government debt and the multiplier, given the composite exogenous disturbance process  $\{f_t\}$ , starting from initial conditions  $\underline{b}_{t_0-1}$  and  $\varphi_{2,t_0-1}$ .<sup>27</sup> Given these solutions, the optimal evolution of the output gap and tax rate are given by:

$$y_t = m_\varphi \varphi_{2t} + n_\varphi \varphi_{2,t-1}$$

$$\hat{\tau}_t = \hat{\tau}_t^* - \varphi^{-1} y_t$$

where  $m_\varphi, n_\varphi$  are again defined in the appendix (Section 7). The evolution of inflation remains indeterminate. If we again assume a preference for inflation stabilization when it is costless, optimal policy involves  $\pi_t = 0$  at all times.

In this case, unlike that of nominal debt, inflation is not affected by a pure fiscal shock (or indeed any other shock) under the optimal policy, but instead the output gap and the tax rate are. Note also that in the above solution, the multiplier  $\varphi_{2t}$ , the output gap, and the tax rate all follow unit root processes: a temporary disturbance to the fiscal stress permanently changes the level of each of these variables, as in the analysis of the optimal dynamics of the tax rate in Barro (1979) and Bohn (1990). However, the optimal evolution of the tax rate is not in general a pure random walk, as in the analysis of Barro and Bohn. Instead, the tax gap is an IMA(1,1) process, as in the local analysis of Aiyagari et al. (2002); the optimal tax

27. The initial condition for  $\varphi_{2,t_0-1}$  is chosen in turn so that the solution obtained is consistent with the initial constraint  $y_{t_0} = \bar{y}_{t_0}$ . Under policy that is optimal from a timeless perspective, this initial commitment is chosen in turn in a self-consistent fashion, as discussed further in Section 5. Note that the specification of  $\varphi_{2,t_0-1}$  does not affect our conclusions in this section about the optimal responses to shocks.

rate  $\hat{\tau}_t$ , may have more complex dynamics, in the case that  $\hat{\tau}_t^*$  exhibits stationary fluctuations. In the special case of linear utility ( $\sigma^{-1} = 0$ ),  $n_\varphi = 0$ , and both the output gap and the tax gap follow random walks (both co-move with  $\varphi_{2t}$ ). If the only disturbances are fiscal disturbances ( $\hat{C}_t$  and  $\hat{\xi}_t$ ), then there are also no fluctuations in  $\hat{\tau}_t^*$  in this case so that the optimal tax rate follows a random walk.

More generally, we observe that optimal policy smooths  $\varphi_{2t}$ , the value (in units of marginal utility) of additional government revenue in period  $t$  so that it follows a random walk. This is the proper generalization of the Barro tax-smoothing result, although it implies smoothing of tax rates in only fairly special cases. We find a similar result in the case that prices are sticky, even when government debt is not indexed, as we now show.

#### 4. Optimal Responses to Shocks: The Case of Sticky Prices

We turn now to the characterization of the optimal responses to shocks in the case that prices are sticky ( $\alpha > 0$ ). The optimization problem that provides a first-order characterization of optimal responses in this case is that of choosing processes  $\{\pi_t, y_t, \hat{\tau}_t, \hat{b}_t\}$  from date  $t_0$  onward to minimize equation (20), subject to the constraints in equations (23) and (25) for each  $t \geq t_0$ , together with initial constraints of the form:

$$\pi_{t_0} = \bar{\pi}_{t_0} \quad y_{t_0} = \bar{y}_{t_0}$$

given the initial condition  $\hat{b}_{t_0-1}$  and the exogenous evolution of the composite disturbances  $\{\hat{\tau}_t^*, f_t\}$ . The Lagrangian for this problem can be written as:

$$\begin{aligned} L_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{1}{2} q_y y_t^2 + \frac{1}{2} q_\pi \pi_t^2 + \varphi_{1t} [-\kappa^{-1} \pi_t + y_t + \psi \hat{\tau}_t + \kappa^{-1} \beta \pi_{t+1}] \right. \\ \left. + \varphi_{2t} [\hat{b}_{t-1} - \pi_t - \sigma^{-1} y_t - (1-\beta)(b_y y_t + b_\tau \hat{\tau}_t) - \beta(\hat{b}_t - \pi_{t+1} - \sigma^{-1} y_{t+1})] \right\} \\ + [\kappa^{-1} \varphi_{1,t_0-1} + \varphi_{2,t_0-1}] \pi_{t_0} + \sigma^{-1} \varphi_{2,t_0-1} y_{t_0} \end{aligned}$$

by analogy with equation (30).

The first-order condition with respect to  $\pi_t$  is given by:

$$q_\pi \pi_t = \kappa^{-1}(\varphi_{1t} - \varphi_{1,t-1}) + (\varphi_{2t} - \varphi_{2,t-1}) \tag{34}$$

that with respect to  $y_t$  is given by:

$$q_y y_t = -\varphi_{1t} + [(1-\beta)b_y + \sigma^{-1}] \varphi_{2t} - \sigma^{-1} \varphi_{2,t-1} \tag{35}$$



and that with respect to  $\hat{\tau}_t$  is given by:

$$\psi\phi_{1t} = (1 - \beta)b_t\phi_{2t} \quad (36)$$

and finally that with respect to  $\hat{b}_t$  is given by:

$$\phi_{2t} = E_t\phi_{2,t+1} \quad (37)$$

These together with the two structural equations and the initial conditions are to be solved for the state-contingent paths of  $\{\pi_t, \hat{Y}_t, \tau_t, \hat{b}_t, \phi_{1t}, \phi_{2t}\}$ . Note that the last three first-order conditions are the same as for the flexible-price model with indexed debt; the first condition in equation (34) replaces the previous requirement that  $\pi_t = 0$ . Hence, the solution obtained in the previous section corresponds to a limiting case of this problem, in which  $q_\pi$  is made unboundedly large; for this reason the discussion above of the more familiar case with flexible prices and riskless indexed government debt also provides insight into the character of optimal policy in the present case.

In the unique bounded solution to these equations, the dynamics of government debt and of the shadow value of government revenue  $\phi_{2t}$  are again of the form:

$$\phi_{2t} = \frac{\tilde{m}_b}{\tilde{m}_b + n_b} \phi_{2,t-1} - \frac{1}{\tilde{m}_b + n_b} [f_t + \hat{b}_{t-1}]$$

$$\hat{b}_t = -E_t f_{t+1} - n_b \phi_{2t}$$

although the coefficient  $\tilde{m}_b$  now differs from  $m_b$  in a way also described in the appendix (Section 7). The implied dynamics of inflation and the output gap are then given by:

$$\pi_t = -\omega_\phi(\phi_{2t} - \phi_{2,t-1}) \quad (38)$$

$$y_t = m_\phi\phi_{2t} + n_\phi\phi_{2,t-1} \quad (39)$$

where  $m_\phi, n_\phi$  are defined as before, and  $\omega_\phi$  is defined in the appendix. The optimal dynamics of the tax rate are those required to make these inflation and output-gap dynamics consistent with the aggregate-supply relation in equation (23). Once again, the optimal dynamics of inflation, the output gap, and the public debt depend only on the evolution of the fiscal stress variable  $\{f_t\}$ ; the dynamics of the tax rate also depend on the evolution of  $\{\hat{\tau}_t^*\}$ .

We now discuss the optimal response of the variables to a disturbance in the level of fiscal stress. The laws of motion just derived for govern-

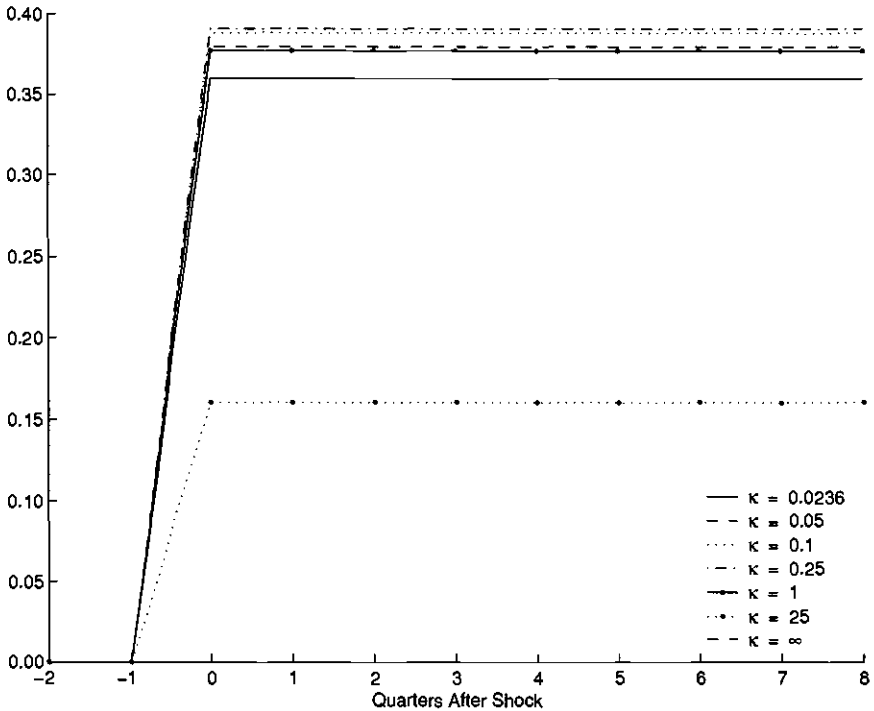
ment debt and the Lagrange multiplier imply that temporary disturbances in the level of fiscal stress cause a permanent change in the level of both the Lagrange multiplier and the public debt. This then implies a permanent change in the level of output, which in turn requires (because inflation is stationary) a permanent change in the level of the tax rate. Since inflation is proportional to the change in the Lagrange multiplier, the price level moves in proportion to the multiplier, which means a temporary disturbance to the fiscal stress results in a permanent change in the price level, as in the flexible-price case analyzed in the previous section. Thus, in this case, the price level, output gap, government debt, and tax rate all have unit roots, combining features of the two special cases considered in the previous section.<sup>28</sup> Both price level and  $\phi_{2t}$  are random walks. They jump immediately to a new permanent level in response to a change in fiscal stress. In the case of purely transitory (white noise) disturbances, government debt also jumps immediately to a new permanent level. Given the dynamics of the price level and government debt, the dynamics of output and tax rate then are jointly determined by the aggregate-supply relation and the government budget constraint.

We also find that the degree to which fiscal stress is relieved by a price-level jump (as in the flexible-price, nominal-debt case) as opposed to an increase in government debt and hence a permanently higher tax rate (as in the flexible-price, indexed-debt case) depends on the degree of price stickiness. We illustrate this with a numerical example. We calibrate a quarterly model by assuming that  $\beta = 0.99$ ,  $\omega = 0.473$ ,  $\sigma^{-1} = 0.157$ , and  $\kappa = 0.0236$ , in accordance with the estimates of Rotemberg and Woodford (1997). We also assume an elasticity of substitution among alternative goods of  $\theta = 10$ , an overall level of steady-state distortions  $\Phi = \frac{1}{2}$ , a steady-state tax rate of  $\hat{\tau} = 0.2$ , and a steady-state debt level  $\bar{b}/\bar{Y} = 2.4$  (debt equal to 60% of a year's gross domestic product (GDP)). Given the assumed degree of market power of producers (a steady-state gross price markup of 1.11) and the assumed size of the tax wedge, the value  $\Phi = \frac{1}{2}$  corresponds to a steady-state wage markup of  $\bar{\mu}^w = 1.08$ . If we assume that there are no government transfers in the steady state, then the assumed level of tax revenues net of debt service would finance steady-state government purchases equal to a share  $s_G = 0.176$  of output.

Let us suppose that the economy is disturbed by an exogenous increase in transfer programs  $\zeta_t$ , equal to 1% of aggregate output, and expected to last only for the current quarter. Figure 1 shows the optimal impulse response of the government debt  $\hat{b}$  to this shock (where quarter zero is the quarter of the

28. Schmitt-Grohé and Uribe (2001) similarly observe that in a model with sticky prices, the optimal response of the tax rate is similar to what would be optimal in a flexible-price model with riskless indexed government debt.

Figure 1 IMPULSE RESPONSE OF THE PUBLIC DEBT TO A PURE FISCAL SHOCK, FOR ALTERNATIVE DEGREES OF PRICE STICKINESS

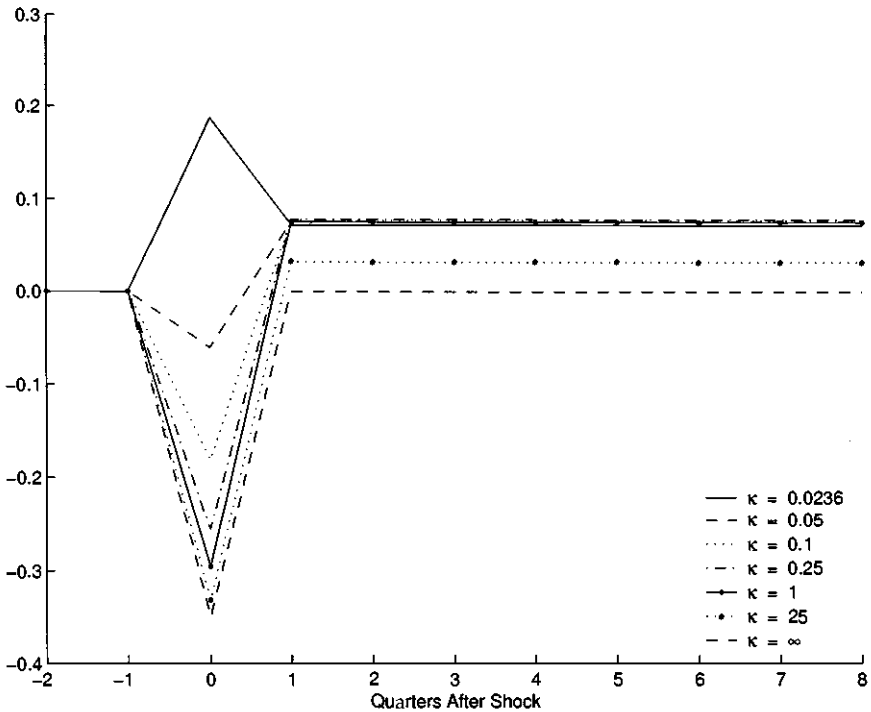


shock), for each of 7 different values for  $\kappa$ , the slope of the short-run aggregate-supply relation, maintaining the values just stated for the other parameters of the model. The solid line indicates the optimal response in the case of our baseline value for  $\kappa$ , based on the estimates of Rotemberg and Woodford; the other cases represent progressively greater degrees of price flexibility, up to the limiting case of fully flexible prices (the case  $\kappa = \infty$ ). Figures 2 and 3 also show the optimal responses of the tax rate and the inflation rate to the same disturbance, for each of the same seven cases.<sup>29</sup>

We see that the volatility of both inflation and tax rates under optimal policy depends greatly on the degree of stickiness of prices. Table 1 reports the initial quarter's response of the inflation rate, and the long-run response of the tax rate, for each of the seven cases. The table also indi-

29. In Figure 1, a response of 1 means a 1% increase in the value of  $b_t$ , from 60% to 60.6% of a year's GDP. In Figure 2, a response of 1 means a 1% decrease in  $\tau_t$ , from 20% to 20.2%. In Figure 3, a response of 1 means a 1% per annum increase in the inflation rate, or an increase of the price level from 1 to 1.0025 over the course of a quarter (given that our model is quarterly). The responses reported in Table 1 are measured in the same way.

Figure 2 IMPULSE RESPONSE OF THE TAX RATE TO A PURE FISCAL SHOCK

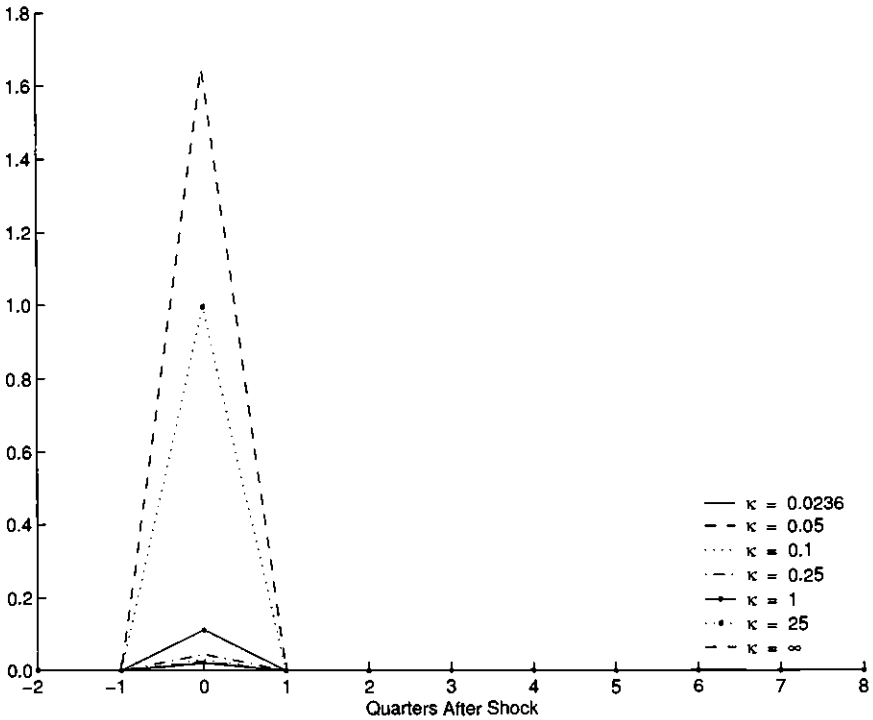


ates for each case the implied average time (in weeks) between price changes,  $T \equiv (-\log \alpha)^{-1}$ , where  $0 < \alpha < 1$  is the fraction of prices unchanged for an entire quarter implied by the assumed value of  $\kappa$ .<sup>30</sup> We first note that our baseline calibration implies that price changes occur only slightly less frequently than twice per year, which is consistent with survey evidence.<sup>31</sup> Next, we observe that even were we to assume an aggregate-supply relation several times as steep as the one estimated using U.S. data, our conclusions with regard to the size of the optimal responses of the (long-run) tax rate and the inflation rate would be fairly similar. At the same time, the optimal responses with fully flexible prices are quite different:

30. We have used the relation between  $\alpha$  and  $T$  for a continuous-time version of the Calvo model to express the degree of price stickiness in terms of an average time between price changes.

31. The indicated average time between price changes for the baseline case is shorter than that reported in Rotemberg and Woodford (1997), both because here we assume a slightly larger value of  $\theta$ , implying a smaller value of  $\alpha$ , and because of the continuous-time method used here to convert  $\alpha$  into an implied average time interval.

Figure 3 IMPULSE RESPONSE OF THE INFLATION RATE TO A PURE FISCAL SHOCK



the response of inflation is 80 times as large as under the baseline sticky-price calibration (implying a variance of inflation 6400 times as large), while the long-run tax rate does not respond at all in the flexible-price case.<sup>32</sup> But even a small degree of stickiness of prices makes a dramatic difference in the optimal responses; for example, if prices are revised only every five weeks on average, the variance of inflation is reduced by a factor of more than 200, while the optimal response of the long-run tax rate to the increased revenue need is nearly the same size as under the baseline degree of price stickiness. Thus, we find, as do Schmitt-Grohé and Uribe (2001) in the context of a calibrated model with convex costs of price adjustment, that the conclusions of the flexible-price analysis are

32. The tax rate does respond in the quarter of the shock in the case of flexible prices, but with the opposite sign to that associated with optimal policy under our baseline calibration. Under flexible prices, as discussed above, the tax rate does not respond to variations in fiscal stress at all. Because the increase in government transfers raises the optimal level of output  $\bar{Y}_0^*$ , for reasons explained in the appendix (Section 7), the optimal tax rate  $\bar{\tau}_0^*$  actually falls to induce equilibrium output to increase; under flexible prices, this is the optimal response of  $\bar{\tau}_0^*$ .

Table 1 IMMEDIATE RESPONSES FOR ALTERNATIVE DEGREES OF PRICE STICKINESS

$\kappa$	$T$	$\hat{\tau}_\infty$	$\pi_0$
.024	29	.072	.021
.05	20	.076	.024
.10	14	.077	.030
.25	9	.078	.044
1.0	5.4	.075	.113
25	2.4	.032	.998
$\infty$	0	0	1.651

quite misleading if prices are even slightly sticky. Under a realistic calibration of the degree of price stickiness, inflation should be quite stable, even in response to disturbances with substantial consequences for the government's budget constraint, while tax rates should instead respond substantially (and with a unit root) to variations in fiscal stress.

We can also compare our results with those that arise when taxes are lump-sum. In this case,  $\psi = 0$ , and the first-order condition in equation (36) requires that  $\varphi_{2t} = 0$ . The remaining first-order conditions reduce to:

$$q_\pi \pi_t = \kappa^{-1}(\varphi_{1t} - \varphi_{1,t-1})$$

$$q_y y_t = -\varphi_{1t}$$

for each  $t \geq t_0$ , as in Clarida et al. (1999) and Woodford (2003, Chapter 7). In this case the fiscal stress is no longer relevant for inflation or output-gap determination. Instead, only the cost-push shock  $u_t$  is responsible for incomplete stabilization. The determinants of the cost-push effects of underlying disturbances and of the target output level  $\hat{Y}_t^*$  are also somewhat different because in this case  $\vartheta_1 = 0$ . For example, a pure fiscal shock has no cost-push effect nor any effect on  $\hat{Y}_t^*$ , and hence no effect on the optimal evolution of either inflation or output.<sup>33</sup> Furthermore, as shown in the references just mentioned, the price level no longer follows a random walk; instead, it is a stationary variable. Increases in the price level due to a cost-push shock are subsequently undone by a period of deflation.

Note that the familiar case from the literature on monetary stabilization policy does not result simply from assuming that sources of revenue that do not shift the aggregate-supply (AS) relation are available; it is also important that the sort of tax that *does* shift the AS relation (like the sales tax here) is *not* available. We could nest both the standard model and our present baseline case within a single, more general framework by assuming that revenue can

33. See our work (Benigno and Woodford, 2003) for a detailed analysis of the determinants of  $u_t$  and  $\hat{Y}_t^*$  in this case.

be raised using either the sales tax or a lump-sum tax, but that there is an additional convex cost (perhaps representing collection costs, assumed to reduce the utility of the representative household but not using real resources) of increases in either tax rate. The standard case would then appear as the limiting case of this model in which the collection costs associated with the sales tax are infinite, while those associated with the lump-sum tax are zero; the baseline model here would correspond to an alternative limiting case in which the collection costs associated with the lump-sum tax are infinite, while those associated with the sales tax are zero. In intermediate cases, we would continue to find that fiscal stress affects the optimal evolution of both inflation and the output gap, as long as there is a positive collection cost for the lump-sum tax. At the same time, the result that the shadow value of additional government revenue follows a random walk under optimal policy (which would still be true) will not in general imply, as it does here, that the price level should also be a random walk; the perfect co-movement of  $\phi_{1t}$  and  $\phi_{2t}$  that characterizes optimal policy in our baseline case will not be implied by the first-order conditions except in the case that there are no collection costs associated with the sales tax. Nonetheless, the price level will generally contain a unit root under optimal policy, even if it will not generally follow a random walk.

We also obtain results more similar to those in the standard literature on monetary stabilization policy if we assume (realistically) that it is not possible to adjust tax rates on such short notice in response to shocks as can be done with monetary policy. As a simple way of introducing delays in the adjustment of tax policy, suppose that the tax rate  $\tau_t$  has to be fixed in period  $t-d$ . In this case, the first-order conditions characterizing optimal responses to shocks are the same as above, except that equation (36) is replaced by:

$$\psi E_t \phi_{1,t+d} = (1 - \beta) b_\tau E_t \phi_{2,t+d} \quad (40)$$

for each  $t \geq t_0$ . In this case, the first-order conditions imply that  $E_t \pi_{t+d+1} = 0$ , but they no longer imply that changes in the price level cannot be forecasted from one period to the next. As a result, price-level increases in response to disturbances are typically partially, but not completely, undone in subsequent periods. Yet there continues to be a unit root in the price level (of at least a small innovation variance), even in the case of an arbitrarily long delay  $d$  in the adjustment of tax rates.

### 5. *Optimal Targeting Rules for Monetary and Fiscal Policy*

We now wish to characterize the policy rules that the monetary and fiscal authorities can follow to bring about the state-contingent responses to

shocks described in the previous section. One might think that it suffices to solve for the optimal state-contingent paths for the policy instruments, but in general this is not a desirable approach to the specification of a policy rule, as discussed in Svensson (2003) and Woodford (2003, Chapter 7). A description of optimal policy in these terms would require enumeration of all of the types of shocks that might be encountered later, indefinitely far in the future, which is not feasible in practice. A commitment to a state-contingent instrument path, even when possible, also may not determine the optimal equilibrium as the locally unique rational-expectations equilibrium consistent with this policy; many other (much less desirable) equilibria may also be consistent with the same state-contingent instrument path.

Instead, we here specify targeting rules in the sense of Svensson (1999, 2003) and Giannoni and Woodford (2003). These targeting rules are commitments on the part of the policy authorities to adjust their respective instruments so as to ensure that the projected paths of the endogenous variables satisfy certain target criteria. We show that under an appropriate choice of these target criteria, a commitment to ensure that they hold at all times will determine a unique nonexplosive rational-expectations equilibrium in which the state-contingent evolution of inflation, output, and the tax rate solves the optimization problem discussed in the previous section. We also show that it is possible to obtain a specification of the policy rules that is robust to alternative specifications of the exogenous shock processes.

We apply the general approach of Giannoni and Woodford (2002), which allows the derivation of optimal target criteria with the properties just stated. In addition, Giannoni and Woodford show that such target criteria can be formulated that refer only to the projected paths of the target variables (the ones in terms of which the stabilization objectives of policy are defined—here, inflation and the output gap). Briefly, the method involves constructing the target criteria by eliminating the Lagrange multipliers from the system of first-order conditions that characterize the optimal state-contingent evolution, regardless of character of the (additive) disturbances. We are left with linear relations among the target variables that do not involve the disturbances and with coefficients independent of the specification of the disturbances that represent the desired target criteria.

Recall that the first-order conditions that characterize the optimal state-contingent paths in the problem considered in the previous section are given by subtracting equation (37) from equation (34). As explained in the previous section, the first three of these conditions imply that the evolution of inflation and of the output gap must satisfy the subtraction of equation (39) from equation (38) each period. We can solve this subtraction for the values of  $\varphi_{2t}$ ,  $\varphi_{2,t-1}$  implied by the values of  $\pi_t$ ,  $y_t$  that are observed in an optimal equilibrium. We can then replace  $\varphi_{2,t-1}$  in these two



relations by the multiplier implied in this way by observed values of  $\pi_{t-1}$ ,  $y_{t-1}$ . Finally, we can eliminate  $\varphi_2$  from these two relations to obtain a necessary relation between  $\pi_t$  and  $y_t$ , given  $\pi_{t-1}$  and  $y_{t-1}$ , given by:

$$\pi_t + \frac{n_\varphi}{m_\varphi} \pi_{t-1} + \frac{\omega_\varphi}{m_\varphi} (y_t - y_{t-1}) = 0 \quad (41)$$

This target criterion has the form of a flexible inflation target, similar to the optimal target criterion for monetary policy in the model with lump-sum taxation (Woodford, 2003, Chapter 7). It is interesting to note that, as in all of the examples of optimal target criteria for monetary policy derived under varying assumptions in Giannoni and Woodford (2003), it is only the projected rate of change of the output gap that matters for determining the appropriate adjustment of the near-term inflation target; the absolute level of the output gap is irrelevant.

The remaining first-order condition from the previous section, not used in the derivation of equation (41), is equation (37). By similarly using the solutions for  $\varphi_{2,t+1}$ ,  $\varphi_2$ , implied by observations of  $\pi_{t+1}$ ,  $y_{t+1}$  to substitute for the multipliers in this condition, one obtains a further target criterion:

$$E_t \pi_{t+1} = 0 \quad (42)$$

(The fact that this always holds in the optimal equilibrium—i.e., that the price level must follow a random walk—has already been noted in the previous section.) We show in the appendix that policies ensuring that the subtraction of equation (42) from equation (41) hold for all  $t \geq t_0$  determine a unique nonexplosive rational-expectations equilibrium.

This equilibrium solves the above first-order conditions for a particular specification of the initial lagged multipliers  $\varphi_{1,t_0-1}$ ,  $\varphi_{2,t_0-1}$ , which are inferred from the initial values  $\pi_{t_0-1}$ ,  $y_{t_0-1}$  in the way just explained. Hence, this equilibrium minimizes expected discounted losses from equation (20) given  $\hat{b}_{t_0-1}$  and subject to constraints on initial outcomes of the form:

$$\pi_{t_0} = \bar{\pi}(\pi_{t_0-1}, y_{t_0-1}) \quad (43)$$

$$y_{t_0} = \bar{y}(\pi_{t_0-1}, y_{t_0-1}) \quad (44)$$

Furthermore, these constraints are self-consistent in the sense that the equilibrium that solves this problem is one in which  $\pi_t$ ,  $y_t$  are chosen to satisfy equations of this form in all periods  $t > t_0$ . Hence, these time-invariant policy rules are optimal from a timeless perspective.<sup>34</sup> And they are optimal regardless of the specification of disturbance processes. Thus, we have obtained robustly optimal target criteria, as desired.

34. See Woodford (2003, Chapters 7 and 8) for additional discussion of the self-consistency condition that the initial constraints are required to satisfy.

We have established a pair of target criteria with the property that if they are expected to be jointly satisfied each period, the resulting equilibrium involves the optimal responses to shocks. This result in itself, however, does not establish which policy instrument should be used to ensure satisfaction of which criterion. Because the variables referred to in both criteria can be affected by both monetary and fiscal policy, there is no uniquely appropriate answer to that question. However, the following represents a relatively simple example of a way in which such a regime could be institutionalized through separate targeting procedures on the part of monetary and fiscal authorities.

Let the central bank be assigned the task of maximizing social welfare through its adjustment of the level of short-term interest rates, taking as given the state-contingent evolution of the public debt  $\{\hat{b}_t\}$ , which depends on the decisions of the fiscal authority. Thus, the central bank treats the evolution of the public debt as being outside its control, just like the exogenous disturbances  $\{\xi_t\}$ , and simply seeks to forecast its evolution to model correctly the constraints on its own policy. Here, we do not propose a regime under which it is actually true that the evolution of the public debt would be unaffected by a change in monetary policy. But there is no inconsistency in the central bank's assumption (because a given bounded process  $\{\hat{b}_t\}$  will continue to represent a feasible fiscal policy regardless of the policy adopted by the central bank), and we shall show that the conduct of policy under this assumption does not lead to a suboptimal outcome as long as the state-contingent evolution of the public debt is correctly forecasted by the central bank.

The central bank then seeks to bring about paths for  $\{\pi_t, y_t, \hat{\tau}_t\}$  from date  $t_0$  onward that minimize equation (20), subject to the constraints in equations (23) and (25) for each  $t \geq t_0$ , together with initial constraints of the form equation (44) to equation (43), given the evolution of the processes  $\{\hat{\tau}_t^*, f_t, \hat{b}_t\}$ . The first-order conditions for this optimization problem are given by equations (34), (35), and (37) each period, which in turn imply that equation (41) must hold each period, as shown above. One can further show that a commitment by the central bank to ensure that equation (41) holds each period determines the equilibrium evolution that solves this problem, in the case of an appropriate (self-consistent) choice of the initial constraints (43) to (44). Thus equation (41) is an optimal target criterion for a policy authority seeking to solve the kind of problem just posed, and since the problem takes as given the evolution of the public debt, it is obviously a more suitable assignment for the central bank than for the fiscal authority. The kind of interest-rate reaction function that can be used to implement a flexible inflation target of this kind is discussed in Svensson and Woodford (2003) and Woodford (2003, Chapter 7).

Correspondingly, let the fiscal authority be assigned the task of choosing the level of government revenue each period that will maximize social welfare, taking as given the state-contingent evolution of output  $\{y_t\}$ , which it regards as being determined by monetary policy. (Again, it need not really be the case that the central bank ensures a particular state-contingent path of output, regardless of what the fiscal authority does. But again, this assumption is not inconsistent with our model of the economy because it is possible for the central bank to bring about any bounded process  $\{y_t\}$  that it wishes, regardless of fiscal policy, in the case that prices are sticky.) If the fiscal authority regards the evolution of output as outside its control, its objective reduces to the minimization of:

$$E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \pi_t^2 \quad (45)$$

But this is a possible objective for fiscal policy, given the effects of tax policy on inflation dynamics (when taxes are not lump-sum) indicated by equation (23).

Forward integration of equation (23) implies that:

$$\pi_t = \kappa E_t \sum_{T=t}^{\infty} \beta^{T-t} y_T + \kappa \psi E_t \sum_{T=t}^{\infty} \beta^{T-t} (\hat{\tau}_T - \hat{\tau}_T^*) \quad (46)$$

Thus, what matters about fiscal policy for current inflation determination is the present value of expected tax rates, but this in turn is constrained by the intertemporal solvency condition in equation (25). Using equation (25) to substitute for the present value of taxes in equation (46), we obtain a relation of the form:

$$\pi_t = \mu_1 [\hat{b}_{t-1} - \sigma^{-1} y_t + f_t] + \mu_2 E_t \sum_{T=t}^{\infty} \beta^{T-t} y_T \quad (47)$$

for certain coefficients  $\mu_1, \mu_2 > 0$  defined in the appendix. If the fiscal authority takes the evolution of output as given, then this relation implies that its policy in period  $t$  can have no effect on  $\pi_t$ . However, it can affect inflation in the following period through the effects of the current budget on  $\hat{b}_t$  (implied by (27)), which then affects  $\pi_{t+1}$  (according to (47)). Furthermore, because the choice of  $\hat{b}_t$  has no effect on inflation in later periods (given that it places no constraint on the level of public debt that may be chosen in later periods),  $\hat{b}_t$  should be chosen to minimize  $E_t \pi_{t+1}^2$ .

The first-order condition for the optimal choice of  $\hat{b}_t$  is then simply equation (42), which we find is indeed a suitable target criterion for the fiscal authority. The decision rule implied by this target criterion is:

$$\hat{b}_t = - E_t f_{t+1} + \sigma^{-1} E_t y_{t+1} - (\mu_2 / \mu_1) E_t \sum_{T=t+1}^{\infty} \beta^{T-t-1} y_T$$

which expresses the optimal level of government borrowing as a function of the fiscal authority's projections of the exogenous determinants of fiscal stress and of future real activity. It is clearly possible for the fiscal authority to implement this target criterion and doing so leads to a determinate equilibrium path for inflation, given the path of output. We thus obtain a pair of targeting rules, one for the central bank and one for the fiscal authority, that if both pursued will implement an equilibrium that is optimal from a timeless perspective. Furthermore, each individual rule can be rationalized as a solution to a constrained optimization problem that the particular policy authority is assigned to solve.

## *6. Conclusion*

We have shown that it is possible to analyze optimal monetary and fiscal policy jointly within a single framework. The two problems, often considered in isolation, turn out to be more closely related than might have been expected. In particular, we find that variations in the level of distorting taxes should be chosen to serve the same objectives as those emphasized in the literature on monetary stabilization policy: stabilization of inflation and of a (properly defined) output gap. A single output gap can be defined that measures the total distortion of the level of economic activity, resulting both from the stickiness of prices (and the consequent variation in markups) and from the supply-side effects of tax distortions. This cumulative gap is what one wishes to stabilize, rather than the individual components resulting from the two sources; and both monetary policy and tax policy can be used to affect it. Both monetary policy and tax policy also matter for inflation determination in our model because of the effects of the tax rate on real marginal cost and hence on the aggregate-supply relation. Indeed, we have exhibited a pair of robustly optimal targeting rules for the monetary and fiscal authorities, respectively, under which both authorities consider the consequences of their actions for near-term inflation projections in determining how to adjust their instruments.

And not only should the fiscal authority use tax policy to serve the traditional goals of monetary stabilization policy; we also find that the monetary authority should take account of the consequences of its actions for the government budget. In the present model, which abstracts entirely from transactions frictions, these consequences have solely to do with the implications of alternative price-level and interest-rate paths for the real burden of interest payments on the public debt and not with any contribution of seignorage to government revenues. Nonetheless, under a calibration of our model that assumes a debt burden and a level of distorting taxes that would not be unusual for an advanced industrial

economy, taking account of the existence of a positive shadow value of additional government revenue (owing to the nonexistence of lump-sum taxes) makes a material difference for the quantitative characterization of optimal monetary policy. In fact, we have found that the crucial summary statistic that indicates the degree to which various types of real disturbances should be allowed to affect short-run projections for either inflation or the output gap is not the degree to which these disturbances shift the aggregate-supply curve for a given tax rate (i.e., the extent to which they represent cost-push shocks), but rather the degree to which they create fiscal stress (shift the intertemporal government solvency condition).

Our conclusion that monetary policy should account for the requirements for government solvency does not imply anything as strong as the result of Chari and Kehoe (1999) for a flexible-price economy with nominal government debt, according to which surprise variations in the inflation rate should be used to offset variations in fiscal stress completely so that tax rates need not vary (other than as necessary to stabilize the output gap). We find that in the case of even a modest degree of price stickiness—much less than what seems to be consistent with empirical evidence for the United States—it is not optimal for inflation to respond to variations in fiscal stress by more than a tiny fraction of the amount that would be required to eliminate the fiscal stress (and that would be optimal with fully flexible prices); instead, a substantial part of the adjustment should come through a change in the tax rate. But the way in which the acceptable short-run inflation projection should be affected by variations in the projected output gap is substantially different in an economy with only distorting taxes than would be the case in the presence of lump-sum taxation. With distorting taxes, the available trade-off between variations in inflation and in the output gap depends not only on the way these variables are related to one another through the aggregate-supply relation but also on the way that each of them affects the government budget.

## 7. Appendix

### 7.1 DERIVATION OF THE AGGREGATE-SUPPLY RELATION (EQUATION (11))

In this section, we derive equation (11) and we define the variables  $F_t$  and  $K_t$ . In the Calvo model, a supplier that changes its price in period  $t$  chooses a new price  $p_t(i)$  to maximize:

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \Pi(p_t(i), p_T^i, P_T; Y_T, \tau_T, \xi_T) \right\}$$

where  $\alpha^{T-t}$  is the probability that the price set at time  $t$  remains fixed in period  $T$ ,  $Q_{t,T}$  is the stochastic discount factor given by equation (7) and the profit function  $\Pi(\cdot)$  is defined as:

$$\Pi(p, p^j, P; T, \tau, \xi) \equiv (1 - \tau)pY(p/P)^{-\theta} - \mu^w \frac{\tilde{v}_h(f^{-1}(Y(p^j/P)^{-\theta}/A); \xi)}{\tilde{u}_c(Y - G; \xi)} P \cdot f^{-1}(Y(p/P)^{-\theta}/A) \quad (48)$$

Here Dixit-Stiglitz monopolistic competition implies that the individual supplier faces a demand curve each period of the form:

$$y_i(i) = Y_i(p_i(i)/P_i)^{-\theta}$$

so that after-tax sales revenues are the function of  $p$  given in the first term on the right side of equation (48). The second term indicates the nominal wage bill, obtained by inverting the production function to obtain the required labor input, and multiplying this by the industry wage for sector  $j$ . The industry wage is obtained from the labor supply equation (8), under the assumption that each of the firms in industry  $j$  (other than  $i$ , assumed to have a negligible effect on industry labor demand) charges the common price  $p^j$ . (Because all firms in a given industry are assumed to adjust their prices at the same time, in equilibrium the prices of firms in a given industry are always identical. We must nonetheless define the profit function for the case in which firm  $i$  deviates from the industry price so we can determine whether the industry price is optimal for each individual firm.)

We note that supplier  $i$ 's profits are a concave function of the quantity sold  $y_i(i)$  because revenues are proportional to  $y_i^{(\theta-1)/\theta}(i)$  and hence concave in  $y_i(i)$ , while costs are convex in  $y_i(i)$ . Because  $y_i(i)$  is proportional to  $p_i(i)^{-\theta}$ , the profit function is also concave in  $p_i(i)^{-\theta}$ . The first-order condition for the optimal choice of the price  $p_i(i)$  is the same as the one with respect to  $p_i(i)^{-\theta}$ ; hence, the first-order condition with respect to  $p_i(i)$  is both necessary and sufficient for an optimum.

For this first-order condition, we obtain:

$$E_i \left\{ \sum_{T=i}^{\infty} \alpha^{T-t} Q_{t,T} \left( \frac{p_i(i)}{P_T} \right)^{-\theta} Y_T \Psi_T(p_i(i), p^j) \right\} = 0$$

with

$$\Psi_T(p, p^j) \equiv \left[ (1 - \tau_T) - \frac{\theta}{\theta - 1} \mu_T^w \frac{\tilde{v}_h(f^{-1}(Y_T(p^j/P_T)^{-\theta}/A_T); \xi_T)}{\tilde{u}_c(Y_T - G_T; \xi_T)} \cdot A_T f^{-1}(Y_T(p/P_T)^{-\theta}/A_T) \right] \frac{P_T}{P}$$

Using the definitions:

$$u(Y_t; \xi_t) \equiv \tilde{u}(Y_t - G_t; \xi_t)$$

$$v(y_t(i); \xi_t) \equiv \tilde{v}(f^{-1}(y_t(i)/A_t); \xi_t) = \tilde{v}(H_t(i); \xi_t)$$

and noting that each firm in an industry will set the same price, so that  $p_t(i) = p_t^i = p_t^*$ , the common price of all goods with prices revised at date  $t$ , we can rewrite the above first-order condition as:

$$E_t \left\{ \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} Q_{t,T} \left( \frac{p_t^*}{P_T} \right)^{-\theta} Y_T \left[ (1 - \tau_T) - \frac{\theta}{\theta-1} \mu_T^w \frac{v_y(Y_T (p_t^*/P_T)^{-\theta}; \xi_T)}{u_c(Y_T; \xi_T)} P_T \right] \right\} = 0$$

Substituting the equilibrium value for the discount factor, we finally obtain:

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} u_c(Y_T; \xi_T) \left( \frac{p_t^*}{P_T} \right)^{-\theta} Y_T \left[ \frac{p_t^*}{P_T} (1 - \tau_T) - \frac{\theta}{\theta-1} \mu_T^w \frac{v_y(Y_T (p_t^*/P_T)^{-\theta}; \xi_T)}{u_c(Y_T; \xi_T)} \right] \right\} = 0 \quad (49)$$

Using the isoelastic functional forms given in previous sections, we obtain a closed-form solution to equation (49), given by:

$$\frac{p_t^*}{P_t} = \left( \frac{K_t}{F_t} \right)^{1/(1+\omega\theta)} \quad (50)$$

where  $F_t$  and  $K_t$  are aggregate variables of the form:

$$F_t \equiv E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} (1 - \tau_T) f(Y_T; \xi_T) \left( \frac{P_T}{P_t} \right)^{\theta-1} \quad (51)$$

$$K_t \equiv E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} k(Y_T; \xi_T) \left( \frac{P_T}{P_t} \right)^{\theta(1+\omega)} \quad (52)$$

in which expressions:

$$f(Y; \xi) \equiv u_c(Y; \xi) Y \quad (53)$$

$$k(Y; \xi) \equiv \frac{\theta}{\theta-1} \mu^w v_y(Y; \xi) Y \quad (54)$$

and where in the function  $k(\cdot)$ , the vector of shocks has been extended to include the shock  $\mu_t^w$ . Substitution of equation (50) into the law of motion for the Dixit-Stiglitz price index:

$$P_t = \left[ (1 - \alpha) p_t^{*1-\theta} + \alpha P_{t-1}^{1-\theta} \right]^{1/(1-\theta)} \quad (55)$$

yields a short-run aggregate-supply relation between inflation and output of the form in equation (11).

### 7.2 RECURSIVE FORMULATION OF THE POLICY PROBLEM

Under the standard (Ramsey) approach to the characterization of an optimal policy commitment, one chooses among state-contingent paths  $\{\Pi_t, Y_t, \tau_t, b_t, \Delta_t\}$  from some initial date  $t_0$  onward that satisfy:

$$\frac{1 - \alpha \Pi_t^{\theta-1}}{1 - \alpha} = \left( \frac{F_t}{K_t} \right)^{(\theta-1)/(1+\omega\theta)} \tag{56}$$

$$\Delta_t = h(\Delta_{t-1}, \Pi_t) \tag{57}$$

$$b_{t-1} \frac{P_{t-1}}{P_t} = E_t \sum_{T=t}^{\infty} R_{t,T} s_T \tag{58}$$

where:

$$s_t \equiv \tau_t Y_t - G_t - \zeta_t \tag{59}$$

for each  $t \geq t_0$ , given initial government debt  $b_{t_0-1}$  and price dispersion  $\Delta_{t_0-1}$ , to maximize:

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} U(Y_t, \Delta_t; \xi_t) \tag{60}$$

Here we note that the definition (3) of the index of price dispersion implies the law of motion:

$$\Delta_t = \alpha \Delta_{t-1} \Pi_t^{\theta(1+\omega)} + (1 - \alpha) \left( \frac{1 - \alpha \Pi_t^{\theta-1}}{1 - \alpha} \right)^{-\theta(1+\omega)/(1-\theta)} \tag{61}$$

which can be written in the form in equation (57); this is the origin of that constraint.

We now show that the  $t_0$ -optimal plan (Ramsey problem) can be obtained as the solution to a two-stage optimization problem. To this purpose, let:

$$W_t \equiv E_t \sum_{T=t}^{\infty} \beta^{T-t} \tilde{u}_c(Y_T - G_T; \xi_T) s_T$$

and let  $\mathcal{F}$  be the set of values for  $(b_{t-1}, \Delta_{t-1}, F_t, K_t, W_t)$  such that there exist paths  $\{\Pi_T, Y_T, \tau_T, b_T, \Delta_T\}$  for dates  $T \geq t$  that satisfy equations (56), (57), and (58), for each  $T$ , that are consistent with the specified values for  $F_t, K_t$ , defined in equations (63) and (64), and  $W_t$ , and that imply a well-defined value for the objective  $U_t$  defined in equation (60). Furthermore, for any  $(b_{t-1}, \Delta_{t-1}, F_t, K_t, W_t) \in \mathcal{F}$ , let  $V(b_{t-1}, \Delta_{t-1}, X_t; \xi_t)$  denote the maximum attainable



value of  $U_t$  among the state-contingent paths that satisfy the constraints just mentioned, where  $X_t \equiv (F_t, K_t, W_t)$ .<sup>35</sup> Among these constraints is the requirement that:

$$W_t = \frac{b_t - 1}{\Pi_t} \tilde{u}_c(Y_t - G_t; \xi_t) \tag{62}$$

for equation (58) to be satisfied. Thus, a specified value for  $W_t$  implies a restriction on the possible values of  $\Pi_t$  and  $Y_t$ , given the predetermined real debt  $b_{t-1}$  and the exogenous disturbances.

The two-stage optimization problem is the following. In the first stage, values of the endogenous variables  $x_{t_0}$ , where  $x_t \equiv (\Pi_t, Y_t, \tau_t, b_t, \Delta_t)$ , and state-contingent commitments  $X_{t_0+1}$  ( $\xi_{t_0+1}$ ) for the following period, are chosen to maximize an objective defined below. In the second stage, the equilibrium evolution from period  $t_0 + 1$  onward is chosen to solve the maximization problem that defines the value function  $V(b_{t_0}, \Delta_{t_0}, X_{t_0+1}; \xi_{t_0+1})$ , given the state of the world  $\xi_{t_0+1}$  and the precommitted values for  $X_{t_0+1}$  associated with that state.

In defining the objective for the first stage of this equivalent formulation of the Ramsey problem, it is useful to let  $\Pi(F, K)$  denote the value of  $\Pi_t$  that solves equation (56) for given values of  $F_t$  and  $K_t$ , and to let  $s(x; \xi)$  denote the real primary surplus  $s_t$  defined by equation (59) in the case of given values of  $x_t$  and  $\xi_t$ . We also define the functional relationships:

$$\begin{aligned} \hat{J}[x, X(\cdot)](\xi_t) &\equiv U(Y_t, \Delta_t; \xi_t) + \beta E_t V(b_t, \Delta_t, X_{t+1}; \xi_{t+1}) \\ \hat{F}[x, X(\cdot)](\xi_t) &\equiv (1 - \tau_t)f(Y_t; \xi_t) + \alpha \beta E_t \{\Pi(F_{t+1}, K_{t+1})^{\theta-1} F_{t+1}\} \\ \hat{K}[x, X(\cdot)](\xi_t) &\equiv k(Y_t; \xi_t) + \alpha \beta E_t \{\Pi(F_{t+1}, K_{t+1})^{\theta(1+\omega)} K_{t+1}\} \\ \hat{W}[x, X(\cdot)](\xi_t) &\equiv \tilde{u}_c(Y_t - G_t; \xi_t) s(x_t; \xi_t) + \beta E_t W_{t+1} \end{aligned}$$

where  $f(Y; \xi)$  and  $k(Y; \xi)$  are defined in equations (53) and (54).

Then in the first stage,  $x_{t_0}$  and  $X_{t_0+1}(\cdot)$  are chosen to maximize:

$$\hat{J}[x_{t_0}, X_{t_0+1}(\cdot)](\xi_{t_0}) \tag{63}$$

over values of  $x_{t_0}$  and  $X_{t_0+1}(\cdot)$  such that:

1.  $\Pi_{t_0}$  and  $\Delta_{t_0}$  satisfy equation (57);
2. the values:

35. As stated previously, in our notation for the value function  $V$ ,  $\xi_t$  denotes not simply the vector of disturbances in period  $t$  but all information in period  $t$  about current and future disturbances.

$$F_{t_0} = \hat{F}[x_{t_0}, X_{t_0+1}(\cdot)](\xi_{t_0}) \tag{64}$$

$$K_{t_0} = \hat{K}[x_{t_0}, X_{t_0+1}(\cdot)](\xi_{t_0}) \tag{65}$$

satisfy:

$$\Pi_{t_0} = \Pi(F_{t_0}, K_{t_0}) \tag{66}$$

3. the value:

$$W_{t_0} = \hat{W}[x_{t_0}, X_{t_0+1}(\cdot)](\xi_{t_0}) \tag{67}$$

satisfies equation (62) for  $t = t_0$ ; and

4. the choices  $(b_{t_0}, \Delta_{t_0}, X_{t_0+1}) \in \mathcal{F}$  for each possible state of the world  $\xi_{t_0+1}$ .

These constraints imply that the objective  $\hat{J}[x_{t_0}, X_{t_0+1}(\cdot)](\xi_{t_0})$  is well-defined and that values  $(x_{t_0}, X_{t_0+1}(\cdot))$  are chosen for which the stage-two problem will be well defined, whichever state of the world is realized in period  $t_0 + 1$ . Furthermore, in the case of any stage-one choices consistent with the above constraints, and any subsequent evolution consistent with the constraints of the stage-two problem, equation (66) implies that equation (56) is satisfied in period  $t_0$ , while equation (62) implies that equation (58) is satisfied in period  $t_0$ . Constraint 1 above implies that equation (57) is also satisfied in period  $t_0$ . Finally, the constraints of the stage-two problem imply that equations (56), (57) and (58) are satisfied in each period  $t \geq t_0 + 1$ ; thus, the state-contingent evolution that solves the two-stage problem is a rational-expectations equilibrium. Conversely, one can show that any possible rational-expectations equilibrium satisfies all these constraints.

One can then reformulate the Ramsey problem, replacing the set of requirements for rational-expectations equilibrium by the stage-one constraints plus the stage-two constraints. Because no aspect of the evolution from period  $t_0 + 1$  onward, other than the specification of  $X_{t_0+1}(\cdot)$ , affects the stage-one constraints, the optimization problem decomposes into the two stages defined above, where the objective in equation (63) corresponds to the maximization of  $U_{t_0}$  in the first stage.

The optimization problem in stage two of this reformulation of the Ramsey problem is of the same form as the Ramsey problem itself, except that there are additional constraints associated with the pre-committed values for the elements of  $X_{t_0+1}(\xi_{t_0+1})$ . Let us consider a problem like the Ramsey problem just defined, looking forward from some period  $t_0$ ,

except under the constraints that the quantities  $X_{t_0}$  must take certain given values, where  $(b_{t_0-1}, \Delta_{t_0-1}, X_{t_0}) \in \mathcal{F}$ . This constrained problem can also be expressed as a two-stage problem of the same form as above, with an identical stage-two problem to the one described above. The stage-one problem is also identical to stage one of the Ramsey problem, except that now the plan chosen in stage one must be consistent with the given values  $X_{t_0}$ , so that the conditions in equations (64), (65), and (67) are now added to the constraints on the possible choices of  $(x_{t_0}, X_{t_0+1}(\cdot))$  in stage one. [The stipulation that  $(b_{t_0-1}, \Delta_{t_0-1}, X_{t_0}) \in \mathcal{F}$  implies that the constraint set remains non-empty despite these additional restrictions.]

Stage two of this constrained problem is thus of exactly the same form as the problem itself. Hence, the constrained problem has a recursive form. It can be decomposed into an infinite sequence of problems, in which in each period  $t$ ,  $(x_t, X_{t+1}(\cdot))$  is chosen to maximize  $\hat{J}[x_t, X_{t+1}(\cdot)](\xi_t)$ , given the predetermined state variables  $(b_{t-1}, \Delta_{t-1})$  and the precommitted values  $X_t$ , subject to the constraints that:

1.  $\Pi_t$  is given by equation (66),  $Y_t$  is then given by equation (62), and  $\Delta_t$  is given by equation 57;
2. the pre-committed values  $X_t$  are fulfilled, i.e.:

$$\hat{F}[x_t, X_{t+1}(\cdot)](\xi_t) = F_t \quad (68)$$

$$\hat{K}[x_t, X_{t+1}(\cdot)](\xi_t) = K_t \quad (69)$$

$$\hat{W}[x_t, X_{t+1}(\cdot)](\xi_t) = W_t \quad (70)$$

and

3. the choices  $(b_t, \Delta_t, X_{t+1}) \in \mathcal{F}$  for each possible state of the world  $\xi_{t+1}$ .

Our aim in the paper is to provide a local characterization of policy that solves this recursive optimization, in the event of small enough disturbances, and initial conditions  $(b_{t_0-1}, \Delta_{t_0-1}, X_{t_0}) \in \mathcal{F}$  that are close enough to consistency with the steady state characterized in the next part of this appendix.

### 7.3 THE DETERMINISTIC STEADY STATE

Here we show the existence of a steady state, i.e., of an optimal policy (under appropriate initial conditions) of the recursive policy problem just defined that involves constant values of all variables. We now consider a deterministic problem in which the exogenous disturbances  $\tilde{C}_t, G_t, \tilde{H}_t, A_t, \mu_t^w, \zeta_t$  each take constant values  $\tilde{C}, \tilde{H}, \tilde{A}, \bar{\mu}^w > 0$  and  $\tilde{G}, \zeta \geq 0$  for all  $t \geq t_0$ , and

we start from initial conditions  $b_{t_0-1} = \bar{b} > 0$ . (The value of  $\bar{b}$  is arbitrary, subject to an upper bound discussed below.) We wish to find an initial degree of price dispersion  $\Delta_{t_0-1}$  and initial commitments  $X_{t_0} = \bar{X}$  so that the recursive (or stage-two) problem involves a constant policy  $x_{t_0} = \bar{x}$ ,  $X_{t+1} = \bar{X}$  each period, in which  $\bar{b}$  is equal to the initial real debt and  $\bar{\Delta}$  is equal to the initial price dispersion.

We thus consider the problem of maximizing:

$$U_{t_0} = \sum_{t=t_0}^{\infty} \beta^{t-t_0} U(Y_t, \Delta_t) \tag{71}$$

subject to the constraints:

$$K_t p(\Pi_t)^{(1+\omega)\theta/(1-\theta)} = F_t \tag{72}$$

$$F_t = (1 - \tau_t) f(Y_t) + \alpha \beta \Pi_{t+1}^{\theta-1} F_{t+1} \tag{73}$$

$$K_t = k(Y_t) + \alpha \beta \Pi_{t+1}^{\theta(1+\omega)} K_{t+1} \tag{74}$$

$$W_t = u_c(Y_t)(\tau_t Y_t - \bar{G} - \bar{\zeta}) + \beta W_{t+1} \tag{75}$$

$$W_t = \frac{u_c(Y_t) b_{t-1}}{\Pi_t} \tag{76}$$

$$\Delta_t = \alpha \Delta_{t-1} \Pi_t^{\theta(1+\omega)} + (1-\alpha) p(\Pi_t)^{-\theta(1+\omega)/(1-\theta)} \tag{77}$$

and given the specified initial conditions  $b_{t_0-1}, \Delta_{t_0-1}, X_{t_0}$ , where we have defined:

$$p(\Pi_t) \equiv \left( \frac{1 - \alpha \Pi_t^{\theta-1}}{1 - \alpha} \right)$$

we introduce Lagrange multipliers  $\phi_{2t}$  through  $\phi_{7t}$  corresponding to the constraints in equations (72) through (77), respectively. We also introduce multipliers dated  $t_0$  corresponding to the constraints implied by the initial conditions  $X_{t_0} = \bar{X}$ ; the latter multipliers are normalized so that the first-order conditions take the same form at date  $t_0$  as at all later dates. The first-order conditions of the maximization problem are then the following. The one with respect to  $Y_t$  is:

$$U_y(Y_t, \Delta_t) - (1 - \tau_t) f_y(Y_t) \phi_{2t} - K_y(Y_t) \phi_{3t} - \tau_t f_y(Y_t) \phi_{4t} + u_{cc}(Y_t)(\bar{G} + \bar{\zeta}) \phi_{4t} - u_{cc}(Y_t) b_{t-1} \Pi_t^{-1} \phi_{5t} = 0 \tag{78}$$

The one with respect to  $\Delta_t$  is:

$$U_{\Delta}(Y_t, \Delta_t) + \phi_{6t} - \alpha \beta \Pi_{t+1}^{\theta(1+\omega)} \phi_{6,t+1} = 0 \tag{79}$$

The one with respect to  $\Pi_t$  is:

$$\begin{aligned} & \frac{1 + \omega\theta}{\theta - 1} p(\Pi_t)^{(1 + \omega\theta)/(\theta - 1) - 1} p_\pi(\Pi_t) K_t \phi_{1,t} - \alpha(\theta - 1) \Pi_t^{\theta - 2} F_t \phi_{2,t-1} \\ & - \theta(1 + \omega)\alpha \Pi_t^{\theta(1 + \omega) - 1} K_t \phi_{3,t-1} + u_c(Y_t) b_{t-1} \Pi_t^{-2} \phi_{5t} \\ & - \theta(1 + \omega)\alpha \Delta_{t-1} \Pi_t^{\theta(1 + \omega) - 1} \phi_{6t} \\ & - \frac{\theta(1 + \omega)}{\theta - 1} (1 - \alpha) p(\Pi_t)^{(1 + \omega\theta)/(\theta - 1)} p_\pi(\Pi_t) \phi_{6t} = 0 \end{aligned} \tag{80}$$

The one with respect to  $\tau_t$  is:

$$\phi_{2t} - \phi_{4t} = 0 \tag{81}$$

The one with respect to  $F_t$  is:

$$-\phi_{1t} + \phi_{2t} - \alpha \Pi_t^{\theta - 1} \phi_{2,t-1} = 0 \tag{82}$$

The one with respect to  $K_t$  is:

$$p(\Pi_t)^{(1 + \omega\theta)/(\theta - 1)} \phi_{1t} + \phi_{3t} - \alpha \Pi_t^{\theta(1 + \omega)} \phi_{3,t-1} = 0 \tag{83}$$

The one with respect to  $W_t$  is:

$$\phi_{4t} - \phi_{4,t-1} + \phi_{5t} = 0 \tag{84}$$

And finally, the one with respect to  $b_t$  is:

$$\phi_{5t} = 0 \tag{85}$$

We search for a solution to these first-order conditions in which  $\Pi_t = \bar{\Pi}$ ,  $\Delta_t = \bar{\Delta}$ ,  $Y_t = \bar{Y}$ ,  $\tau_t = \bar{\tau}$ , and  $b_t = \bar{b}$  at all times. A steady-state solution of this kind also requires that the Lagrange multipliers take constant values. We also conjecture the existence of a solution in which  $\bar{\Pi} = 1$ , as stated previously. Note that such a solution implies that  $\bar{\Delta} = 1$ ,  $p(\bar{\Pi}) = 1$ ,  $p_\pi(\bar{\Pi}) = -(\theta - 1)\alpha/(1 - \alpha)$ , and  $\bar{K} = \bar{F}$ . Using these substitutions, we find that (the steady-state version of) each of the first-order conditions in equations (78) to (85) is satisfied if the steady-state values satisfy:

$$\phi_1 = (1 - \alpha)\phi_2$$

$$\left[ f_y(\bar{Y}) - k_y(\bar{Y}) - u_{cc}(\bar{Y} - \bar{G})(\bar{G} + \bar{\zeta}) \right] \phi_2 = U_y(\bar{Y}, 1)$$

$$\phi_3 = -\phi_2$$

$$\phi_4 = \phi_2$$

$$\phi_5 = 0$$

$$(1 - \alpha\beta)\phi_6 = -U_\Delta(\bar{Y}, 1)$$

These equations can obviously be solved (uniquely) for the steady-state multipliers given any value  $\bar{Y} > 0$ .

Similarly, (the steady-state versions of) the constraints in equations (72) to (77) are satisfied if:

$$(1 - \bar{\tau})u_c(\bar{Y} - \bar{G}) = \frac{\theta}{\theta - 1} \bar{\mu}^w v_y(\bar{Y}) \quad (86)$$

$$\bar{\tau}\bar{Y} = \bar{G} + \bar{\zeta} + (1 + \beta)\bar{b} \quad (87)$$

$$\bar{K} = \bar{F} = (1 - \alpha\beta)^{-1} k(\bar{Y})$$

$$\bar{W} = u_c(\bar{Y} - \bar{G})\bar{b}$$

Equations (86) and (87) provide two equations to solve for the steady-state values  $\bar{Y}$  and  $\bar{\tau}$ . Under standard (Inada-type) boundary conditions on preferences, equation (86) has a unique solution  $Y_1(\tau) > \bar{G}$  for each possible value of  $0 \leq \tau < 1$ .<sup>36</sup> This value is a decreasing function of  $\tau$  and approaches  $\bar{G}$  as  $\tau$  approaches 1. We note that, at least in the case of all small enough values of  $\bar{G}$ , there exists a range of tax rates  $0 < \tau_1 < \tau < \tau_2 \leq 1$  over which  $Y_1(\tau) > \bar{G} / \tau$ .<sup>37</sup> Given our assumption that  $\bar{b} > 0$  and that  $\bar{G} \bar{\zeta} \geq 0$ , equation (87) is satisfied only by positive values of  $\bar{\tau}$ ; for each  $\bar{\tau} > 0$ , this equation has a unique solution  $Y_2(\tau)$ . We also note that the locus  $Y_1(\tau)$  is independent of the values of  $\bar{\zeta}$  and  $\bar{b}$ , while  $Y_2(\tau)$  approaches  $\bar{G}/\tau$  as  $\bar{\zeta}$  and  $\bar{b}$  approach zero. Fixing the value of  $\bar{G}$  (at a value small enough for the interval  $(\tau_1, \tau_2)$  to exist), we then observe that for any small enough values of  $\bar{b} > 0$  and  $\bar{\zeta} \geq 0$ , there exist values  $0 < \tau < 1$  at which  $Y_2(\tau) < Y_1(\tau)$ . On the other hand, for all small enough values of  $\tau > 0$ ,  $Y_2(\tau) > Y_1(\tau)$ . Thus, by continuity, there must exist a value  $0 < \bar{\tau} < 1$  at which  $Y_1(\bar{\tau}) = Y_2(\bar{\tau})$ .<sup>38</sup> This allows us to obtain a solution for  $0 < \bar{\tau} < 1$  and  $\bar{Y} > 0$ , in the case of any small enough values of  $\bar{G}$ ,  $\bar{\zeta} \geq 0$  and  $\bar{b} > 0$ . The remaining equations can then be solved (uniquely) for  $\bar{K} = \bar{F}$  and for  $\bar{W}$ .

36. There is plainly no possibility of positive supply of output by producers in the case that  $\tau_i \geq 1$  in any period; hence, the steady state must involve  $\bar{\tau} < 1$ .

37. This is true for any tax rate at which  $(1 - \tau) u_c(\bar{G}(\tau^{-1} - 1))$  exceeds  $(\theta/(\theta - 1)) \bar{\mu}^w v_y(\bar{G}/\tau)$ . Fixing any value  $0 < \tau < 1$ , our Inada conditions imply that this inequality holds for all small enough values of  $\bar{G}$ . And if the inequality holds for some  $0 < \tau < 1$ , then by continuity it must hold for an open interval of values of  $\tau$ .

38. In fact, there must exist at least two such solutions because the Inada conditions also imply that  $Y_2(\tau) > Y_1(\tau)$  for all  $\tau$  close enough to 1. These multiple solutions correspond to a Laffer curve result under which two distinct tax rates result in the same equilibrium level of government revenues. We select the lower-tax, higher-output solution as the one around which we compute our Taylor-series expansions; this is clearly the higher-utility solution.

We have thus verified that a constant solution to the first-order conditions exists. With a method to be explained below, we check that this solution is indeed at least a local optimum. Note that, as asserted previously, this deterministic steady state involves zero inflation and a steady-state tax rate  $0 < \bar{\tau} < 1$ .

7.4 A SECOND-ORDER APPROXIMATION TO UTILITY (EQUATIONS [17] AND [18])

We derive here equations (17) and (18), taking a second-order approximation to equation (60) following the treatment in Woodford (2003, Chapter 6). We start by approximating the expected discounted value of the utility of the representative household:

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[ u(Y_t; \xi_t) - \int_0^1 v(y_t(i); \xi_t) di \right] \tag{88}$$

First, we note that:

$$\int_0^1 v(y_t(i); \xi_t) di = \frac{\lambda}{1 + v} \frac{Y_t^{1+\omega}}{A_t^{1+\omega} \bar{H}_t^v} \Delta_t = v(Y_t; \xi_t) \Delta_t$$

where  $\Delta_t$  is the measure of price dispersion defined previously. We can then write equation (88) as:

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[ u(Y_t; \xi_t) - v(Y_t; \xi_t) \Delta_t \right] \tag{89}$$

The first term in equation (89) can be approximated using a second-order Taylor expansion around the steady state defined in the previous section as:

$$\begin{aligned} u(Y_t; \xi_t) &= \bar{u} + \bar{u}_c \bar{Y}_t + \bar{u}_\xi \bar{\xi}_t + \frac{1}{2} \bar{u}_{cc} \bar{Y}_t^2 + \bar{u}_{c\xi} \bar{\xi}_t \bar{Y}_t + \frac{1}{2} \bar{\xi}'_t \bar{u}_{\xi\xi} \bar{\xi}_t \\ &\quad + \mathcal{O}(\|\xi\|^3) \\ &= \bar{u} + \bar{Y} \bar{u}_c \cdot (\hat{Y}_t + \frac{1}{2} \hat{Y}_t^2) + \bar{u}_\xi \bar{\xi}_t + \frac{1}{2} \bar{Y} \bar{u}_{cc} \hat{Y}_t^2 + \bar{Y} \bar{u}_{c\xi} \bar{\xi}_t \hat{Y}_t \\ &\quad + \frac{1}{2} \bar{\xi}'_t \bar{u}_{\xi\xi} \bar{\xi}_t + \mathcal{O}(\|\xi\|^3) \\ &= \bar{Y} \bar{u}_c \hat{Y}_t + \frac{1}{2} [\bar{Y} \bar{u}_c + \bar{Y}^2 \bar{u}_{cc}] \hat{Y}_t^2 - \bar{Y}^2 \bar{u}_{cc} g_t \hat{Y}_t + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3) \\ &= \bar{Y} \bar{u}_c \left\{ \hat{Y}_t + \frac{1}{2} (1 - \sigma^{-1}) \hat{Y}_t^2 + \sigma^{-1} g_t \hat{Y}_t \right\} + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3) \end{aligned} \tag{90}$$

where a bar denotes the steady-state value for each variable, a tilde denotes the deviation of the variable from its steady-state value (e.g.,  $\tilde{Y}_t \equiv Y_t - \bar{Y}$ ), and a circumflex refers to the log deviation of the variable from its steady-state value (e.g.,  $\hat{Y}_t \equiv \ln Y_t / \bar{Y}$ ). We use  $\xi_t$  to refer to the entire vector of exogenous shocks:

$$\xi_t \equiv [\hat{\zeta}_t \ \hat{G}_t \ g_t \ q_t \ \hat{\mu}_t^w]$$

in which  $\hat{\zeta}_t \equiv (\zeta_t - \bar{\zeta}) / \bar{Y}$ ,  $\hat{G}_t \equiv (G_t - \bar{G}) / \bar{Y}$ ,  $g_t \equiv \hat{G}_t + s_C \bar{c}_t$ ,  $\omega q_t \equiv v \bar{h}_t + \phi (1 + v) a_{lv}$ ,  $\hat{\mu}_t^w \equiv \ln \mu_t^w / \bar{\mu}^w$ ,  $\bar{c}_t \equiv \ln \bar{C}_t / \bar{C}$ ,  $a_{lv} \equiv \ln A_t / \bar{A}$ ,  $\bar{h}_t \equiv \ln H_t / \bar{H}$ . We use the definitions  $\sigma^{-1} \equiv \bar{\sigma}^{-1} s_C^{-1}$  with  $s_C \equiv \bar{C} / \bar{Y}$  and  $s_C + s_C = 1$ . We have used the Taylor expansion:

$$\frac{Y_t}{\bar{Y}} = 1 + \hat{Y}_t + \frac{1}{2} \hat{Y}_t^2 + \mathcal{O}(\|\xi\|^3)$$

to get a relation for  $\tilde{Y}_t$  in terms of  $\hat{Y}_t$ . Finally the term *t.i.p.* denotes terms that are independent of policy and may accordingly be suppressed as far as the welfare ranking of alternative policies is concerned.

We may similarly approximate  $v(Y_t; \xi_t) \Delta_t$  by:

$$\begin{aligned} v(Y_t; \xi_t) \Delta_t &= \bar{v} + \bar{v}(\Delta_t - 1) + \bar{v}_y(Y_t - \bar{Y}) + \bar{v}_y(\Delta_t - 1)(Y_t - \bar{Y}) + (\Delta_t - 1) \bar{v}_\xi \xi_t \\ &\quad + \frac{1}{2} \bar{v}_{yy}(Y_t - \bar{Y})^2 + (Y_t - \bar{Y}) \bar{v}_{y\xi} \xi_t + \mathcal{O}(\|\xi\|^3) \\ &= \bar{v}(\Delta_t - 1) + \bar{v}_y \bar{Y} \left( \hat{Y}_t + \frac{1}{2} \hat{Y}_t^2 \right) + \bar{v}_y(\Delta_t - 1) \bar{Y} \hat{Y}_t + (\Delta_t - 1) \bar{v}_\xi \xi_t \\ &\quad + \frac{1}{2} \bar{v}_{yy} \bar{Y}^2 \hat{Y}_t^2 + \bar{Y} \hat{Y}_t \bar{v}_{y\xi} \xi_t + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3) \\ &= \bar{v}_y \bar{Y} \left[ \frac{\Delta_t - 1}{1 + \omega} + \hat{Y}_t + \frac{1}{2} (1 + \omega) \hat{Y}_t^2 + (\Delta_t - 1) \hat{Y}_t - \omega \hat{Y}_t q_t \right. \\ &\quad \left. - \frac{\Delta_t - 1}{1 + \omega} \omega q_t \right] + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3) \end{aligned}$$

We take a second-order expansion of equation (61) to obtain:

$$\hat{\Delta}_t = \alpha \hat{\Delta}_{t-1} + \frac{\alpha}{1 - \alpha} \theta (1 + \omega) \frac{\pi_t^2}{2} + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3) \tag{91}$$

This in turn allows us to approximate  $v(Y_t; \xi_t) \Delta_t$  as:

$$\begin{aligned} v(Y_t; \xi_t) \Delta_t &= (1 - \Phi) \bar{Y} u_c \left\{ -\frac{\hat{\Delta}_t}{1 + \omega} + \hat{Y}_t + \frac{1}{2} (1 + \omega) \hat{Y}_t^2 - \omega \hat{Y}_t q_t \right\} \\ &\quad + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3) \end{aligned} \tag{92}$$



where we have used the steady state relation  $\bar{v}_y = (1 - \Phi) \bar{u}_c$  to replace  $\bar{v}_y$  by  $(1 - \Phi) \bar{u}_c$ , and where:

$$\Phi \equiv 1 - \left(\frac{\theta - 1}{\theta}\right) \left(\frac{1 - \bar{\tau}}{\bar{\mu}^\omega}\right) < 1$$

measures the inefficiency of steady-state output  $\bar{Y}$ .

Combining equations (90) and (92), we finally obtain equation (17):

$$U_{t_0} = \bar{Y} \bar{u}_c \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} [\Phi \hat{Y}_t - \frac{1}{2} u_{yy} \hat{Y}_t^2 + \hat{Y}_t u_{y\xi} \xi_t - u_\Delta \hat{\Delta}_t] + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3) \tag{93}$$

where:

$$u_{yy} \equiv (\omega + \sigma^{-1}) - \Phi(1 + \omega)$$

$$u_{\xi} \xi_t \equiv [\sigma^{-1} g_t + (1 - \Phi)\omega q_t]$$

$$u_\Delta \equiv \frac{(1 - \Phi)}{1 + \omega}$$

We finally observe that equation (91) can be integrated to obtain:

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0} \hat{\Delta}_t = \frac{\alpha}{(1 - \alpha)(1 - \alpha\beta)} \theta(1 + \omega)(1 - \omega\theta) \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{\pi_t^2}{2} + \text{t.i.p} + \mathcal{O}(\|\xi\|) \tag{94}$$

By substituting equation (94) into equation (93), we obtain:

$$U_{t_0} = \bar{Y} \bar{u}_c \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} [\Phi \hat{Y}_t - \frac{1}{2} u_{yy} \hat{Y}_t^2 + \hat{Y}_t u_{y\xi} \xi_t - \frac{1}{2} u_\pi \pi_t^2] + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3)$$

This coincides with equation (18), where we have further defined:

$$\kappa \equiv \frac{(1 - \alpha\beta)(1 - \alpha)}{\alpha} \frac{(\omega + \sigma^{-1})}{1 + \theta\omega}$$

$$u_\pi \equiv \frac{\theta(\omega + \sigma^{-1})(1 - \Phi)}{\kappa}$$

### 7.5 A SECOND-ORDER APPROXIMATION TO THE AGGREGATE SUPPLY EQUATION (EQUATION [11])

We now compute a second-order approximation to the aggregate supply (AS) equation (56), or equation (11). We start from equation (50), which can be written as:

$$\tilde{p}_t = \left(\frac{K_t}{F_t}\right)^{1/(1 + \omega\theta)}$$

where  $\hat{p}_t \equiv p_t^*/P_t$ . As we have shown (Benigno and Woodford, 2003), a second-order expansion of this can be expressed in the form:

$$\begin{aligned} \frac{(1 + \omega\theta)}{(1 - \alpha\beta)} \hat{p}_t &= z_t + \alpha\beta \frac{(1 + \omega\theta)}{(1 - \alpha\beta)} E_t(\hat{p}_{t+1} - \hat{P}_{t,t+1}) + \frac{1}{2} z_t X_t \\ &\quad - \frac{1}{2} (1 + \omega\theta) \hat{p}_t Z_t + \frac{1}{2} \alpha\beta (1 + \omega\theta) E_t\{(\hat{p}_{t+1} - \hat{P}_{t,t+1}) Z_{t+1}\} \\ &\quad + \frac{\alpha\beta}{2(1 - \alpha\beta)} (1 - 2\theta - \omega\theta) (1 + \omega\theta) E_t\{(\hat{p}_{t+1} - \hat{P}_{t,t+1}) \hat{P}_{t,t+1}\} \\ &\quad + \text{s.o.t.i.p.} + \mathcal{O}(\|\xi\|^3) \end{aligned} \tag{95}$$

where we define:

$$\begin{aligned} \hat{P}_{t,T} &\equiv \log(P_t/P_T) \\ z_t &\equiv \omega(\hat{Y}_t - q_t) + \bar{\sigma}^{-1}(\hat{C}_t - \bar{c}_t) - \hat{S}_t + \hat{\mu}_t^w \\ Z_t &\equiv E_t \left\{ \sum_{T=t}^{+\infty} (\alpha\beta)^{T-t} [X_T + (1 - 2\theta - \omega\theta) \hat{P}_{t,T}] \right\} \end{aligned}$$

and in this last expression:

$$X_T \equiv (2 + \omega) \hat{Y}_T - \omega q_T + \hat{\mu}_T^w + \hat{S}_T - \bar{\sigma}^{-1}(\hat{C}_T - \bar{c}_T)$$

where  $\hat{S}_t = \ln(1 - \tau_t)/(1 - \bar{\tau})$ . Here, s.o.t.i.p. refers to second-order (or higher) terms independent of policy; the first-order terms have been kept because these will matter for the log-linear aggregate-supply relation that appears as a constraint in our policy problem.

We next take a second-order expansion of the law of motion in equation (55) for the price index, obtaining:

$$\hat{p}_t = \frac{\alpha}{1 - \alpha} \pi_t - \frac{1 - \theta}{2} \frac{\alpha}{(1 - \alpha)^2} \pi_t^2 + \mathcal{O}(\|\xi\|^3) \tag{96}$$

where we have used the fact that:

$$\hat{\hat{p}}_t = \frac{\alpha}{1 - \alpha} \pi_t + \mathcal{O}(\|\xi\|^2)$$

and  $\hat{P}_{t-1,t} = -\pi_t$ . We can then plug equation (96) into equation (95) to obtain:

$$\begin{aligned} \pi_t &= \frac{1 - \theta}{2} \frac{1}{(1 - \alpha)} \pi_t^2 + \frac{\kappa}{(\omega + \sigma^{-1})} z_t + \beta E_t \pi_{t+1} - \frac{1 - \theta}{2} \frac{\alpha\beta}{(1 - \alpha)} E_t \pi_{t+1}^2 \\ &\quad + \frac{1}{2} \frac{\kappa}{(\omega + \sigma^{-1})} z_t X_t - \frac{1}{2} (1 - \alpha\beta) \pi_t Z_t + \frac{\beta}{2} (1 - \alpha\beta) E_t\{\pi_{t+1} Z_{t+1}\} \\ &\quad - \frac{\beta}{2} (1 - 2\theta - \omega\theta) E_t\{\pi_{t+1}^2\} + \text{s.o.t.i.p.} + \mathcal{O}(\|\xi\|^3) \end{aligned} \tag{97}$$

We note that a second-order approximation to the identity  $C_t = Y_t - G_t$  yields:

$$\hat{C}_t = s_c^{-1} \hat{Y}_t - s_c^{-1} \hat{G}_t + \frac{s_c^{-1}(1 - s_c^{-1})}{2} \hat{Y}_t^2 + s_c^{-2} \hat{Y}_t \hat{G}_t + \text{s.o.t.i.p.} + \mathcal{O}(\|\xi\|^3) \quad (98)$$

and that:

$$\hat{S}_t = -\omega_\tau \hat{\tau}_t - \frac{\omega_\tau}{(1 - \bar{\tau})} \bar{\tau}_t^2 + \mathcal{O}(\|\xi\|^3) \quad (99)$$

where  $\omega_\tau \equiv \bar{\tau} / (1 - \bar{\tau})$ . By substituting equations (98) and (99) into the definition of  $z_t$  in equation (97), we finally obtain a quadratic approximation to the AS relation.

This can be expressed compactly in the following form:

$$V_t = \kappa(c'_x x_t + c'_\xi \xi_t + \frac{1}{2} x'_t C_x x_t + x'_t C_\xi \xi_t + \frac{1}{2} c_\pi \pi_t^2) + \beta E_t V_{t+1} + \text{s.o.t.i.p.} + \mathcal{O}(\|\xi\|^3) \quad (100)$$

where we have defined:

$$x_t \equiv \begin{bmatrix} \hat{\tau}_t \\ \hat{Y}_t \end{bmatrix}$$

$$c'_x = [\Psi \quad 1]$$

$$c'_\xi = [0 \quad 0 \quad -\sigma^{-1}(\omega + \sigma^{-1})^{-1} \quad -\omega(\omega + \sigma^{-1})^{-1} \quad (\omega + \sigma^{-1})^{-1}]$$

$$C_x = \begin{bmatrix} \Psi & (1 - \sigma^{-1})\Psi \\ (1 - \sigma^{-1})\Psi & (2 + \omega - \sigma^{-1}) + \sigma^{-1}(1 - s_c^{-1})(\omega + \sigma^{-1})^{-1} \end{bmatrix}$$

$$C_\xi = \begin{bmatrix} 0 & 0 & \Psi\sigma^{-1} & 0 & 0 \\ 0 & \frac{\sigma^{-1}s_c^{-1}}{(\omega + \sigma^{-1})} & -\frac{\sigma^{-1}(1 - \sigma^{-1})}{(\omega + \sigma^{-1})} & -\frac{\omega(1 + \omega)}{(\omega + \sigma^{-1})} & \frac{(1 + \omega)}{(\omega + \sigma^{-1})} \end{bmatrix}$$

$$c_\pi = \frac{\theta(1 + \omega)}{\kappa}$$

$$V_t = \pi_t + \frac{1}{2} v_\pi \pi_t^2 + v_z \pi_t Z_t$$

$$Z_t = z'_x x_t + z_\pi \pi_t + z'_\xi \xi_t + \alpha \beta E_t Z_{t+1}$$

in which the coefficients:

$$\Psi \equiv \frac{\omega_\tau}{(\omega + \sigma^{-1})}$$

and:

$$v_\pi \equiv \theta(1 + \omega) - \frac{1 - \theta}{(1 - \alpha)}$$

$$v_z \equiv \frac{(1 - \alpha\beta)}{2}$$

$$v_k \equiv \frac{\kappa}{(\omega + \sigma^{-1})} \frac{\alpha\beta}{1 - \alpha\beta} (1 - 2\theta - \omega\theta)$$

$$z'_x \equiv [(2 + \omega - \sigma^{-1}) + v_\kappa(\omega + \sigma^{-1}) - \omega_\tau(1 - v_\kappa)]$$

$$z'_\xi \equiv [0 \quad 0 \quad \sigma^{-1}(1 - v_k) \quad -\omega(1 + v_k) \quad (1 + v_k)]$$

$$z_\pi \equiv -\frac{\omega + \sigma^{-1}}{k} v_k$$

Note that in a first-order approximation, equation (100) can be written as simply:

$$\pi_t = \kappa[\hat{Y}_t + \psi \hat{v}_t + c'_\xi \xi_t] + \beta E_t \pi_{t+1} \tag{101}$$

where:

$$c'_\xi \xi_t \equiv (\omega + \sigma^{-1})^{-1} [-\sigma^{-1} g_t - \omega \eta_t + \hat{\mu}_t^w]$$

We can also integrate equation (100) forward from time  $t_0$  to obtain:

$$V_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \kappa (c'_x x_t + \frac{1}{2} x'_t C_x x_t + x'_t C_\xi \xi_t + \frac{1}{2} c_\pi \pi_t^2) + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3) \tag{102}$$

where the term  $c'_\xi \xi_t$  is now included in terms independent of policy. Such terms matter when they are part of the log-linear constraints, as in the case of equation (61), but not when they are part of the quadratic objective.

### 7.6 A SECOND-ORDER APPROXIMATION TO THE INTERTEMPORAL GOVERNMENT SOLVENCY CONDITION (EQUATION [15])

We now derive a second-order approximation to the intertemporal government solvency condition. We use the definition:

$$W_t \equiv E_t \sum_{T=t}^{\infty} \beta^{T-t} \tilde{u}_c(Y_T; \xi_T) s_T \tag{103}$$

where:

$$s_t \equiv \tau_t Y_t - G_t - \zeta_t \tag{104}$$

and:

$$W_t = \frac{b_t - 1}{\Pi_t} \tilde{u}_c(C_t; \xi_t) \tag{105}$$

First, we take a second-order approximation of the term  $\tilde{u}_c(C_t; \xi_t) s_t$  to obtain:

$$\begin{aligned} \tilde{u}_c(C_t; \xi_t) s_t &= \bar{s} \bar{\tilde{u}}_c + \bar{\tilde{u}}_{cc} \bar{s} \tilde{C}_t + \bar{\tilde{u}}_c \tilde{s}_t + \bar{s} \bar{\tilde{u}}_{c\xi} \xi_t \\ &\quad + \frac{1}{2} \bar{s} \bar{\tilde{u}}_{ccc} \tilde{C}_t^2 + \bar{\tilde{u}}_{cc} \tilde{C}_t \tilde{s}_t + \bar{s} \tilde{C}_t \bar{\tilde{u}}_{cc\xi} \xi_t \\ &\quad + \tilde{s}_t \bar{\tilde{u}}_{c\xi} \xi_t + \text{s.o.t.i.p.} + \mathcal{O}(\|\xi\|^3) \\ &= \bar{s} \bar{\tilde{u}}_c + \bar{\tilde{u}}_{cc} \bar{s} \tilde{C} \hat{C}_t + \bar{\tilde{u}}_c \tilde{s}_t + \bar{s} \bar{\tilde{u}}_{c\xi} \xi_t \\ &\quad + \frac{1}{2} \bar{s} (\bar{\tilde{u}}_{cc} \tilde{C} + \bar{\tilde{u}}_{cc} \tilde{C}^2) \tilde{C}_t^2 + \bar{C} \bar{\tilde{u}}_{cc} \hat{C}_t \tilde{s}_t \\ &\quad + \bar{s} \tilde{C} \bar{\tilde{u}}_{c\xi} \xi_t \hat{C}_t + \bar{\tilde{u}}_{c\xi} \xi_t \tilde{s}_t + \text{s.o.t.i.p.} + \mathcal{O}(\|\xi\|^3) \\ &= \bar{s} \bar{\tilde{u}}_c + \bar{\tilde{u}}_c [-\bar{\sigma}^{-1} \bar{s} \hat{C}_t + \tilde{s}_t + \bar{s} \bar{\tilde{u}}_c^{-1} \bar{\tilde{u}}_{c\xi} \xi_t \\ &\quad + \frac{1}{2} \bar{s} \bar{\sigma}^{-2} \tilde{C}_t^2 - \bar{\sigma}^{-1} \tilde{s}_t \hat{C}_t \\ &\quad + \bar{s} \tilde{C} \bar{\tilde{u}}_c^{-1} \bar{\tilde{u}}_{c\xi} \xi_t \hat{C}_t + \bar{\tilde{u}}_c^{-1} \bar{\tilde{u}}_{c\xi} \xi_t \tilde{s}_t] \\ &\quad + \text{s.o.t.i.p.} + \mathcal{O}(\|\xi\|^3) \\ &= \bar{s} \bar{\tilde{u}}_c + \bar{\tilde{u}}_c [-\bar{\sigma}^{-1} \bar{s} (\hat{C}_t - \bar{c}_t) + \tilde{s}_t \\ &\quad + \frac{1}{2} \bar{s} \bar{\sigma}^{-2} \tilde{C}_t^2 - \bar{\sigma}^{-1} \tilde{s}_t (\hat{C}_t - \bar{c}_t) - \bar{\sigma}^{-2} \bar{s} \bar{c}_t \hat{C}_t] \\ &\quad + \text{s.o.t.i.p.} + \mathcal{O}(\|\xi\|^3) \end{aligned} \tag{106}$$

where we have followed previous definitions and use the isoelastic functional forms assumed. Note that we can write  $\bar{\tilde{u}}_c^{-1} \bar{\tilde{u}}_{c\xi} \xi_t = \bar{\sigma}^{-1} \bar{c}_t$  and  $\bar{C} \bar{\tilde{u}}_c^{-1} \bar{\tilde{u}}_{c\xi} \xi_t = -\bar{\sigma}^{-2} \bar{c}_t$ . Plugging equation (98) into equation (106), we obtain:

$$\begin{aligned} \tilde{u}_c(C_t; \xi_t) s_t &= \bar{s} \bar{\tilde{u}}_c [1 - \sigma^{-1} \hat{Y}_t + \sigma^{-1} g_t + \bar{s}^{-1} \tilde{s}_t + \frac{1}{2} [\sigma^{-1} (s_c^{-1} - 1) + \sigma^{-2}] \hat{Y}_t^2 \\ &\quad - \sigma^{-1} \bar{s}^{-1} (\hat{Y}_t - g_t) \tilde{s}_t - \sigma^{-1} (s_c^{-1} \hat{G}_t + \sigma^{-1} g_t) \hat{Y}_t] \\ &\quad + \text{s.o.t.i.p.} + \mathcal{O}(\|\xi\|^3) \end{aligned} \tag{107}$$

by using previous definitions.

We recall now that the primary surplus is defined as:

$$s_t = \tau_t Y_t - G_t - \zeta_t$$

which can be expanded in a second-order expansion to get:

$$\begin{aligned} \bar{s}^{-1} \tilde{s}_t &= (1 + \omega_g)(\hat{Y}_t + \hat{\tau}_t) - s_d^{-1}(\hat{G}_t + \hat{\zeta}_t) + \frac{(1 + \omega_g)}{2}(\hat{Y}_t + \hat{\tau}_t)^2 \\ &+ \text{s.o.t.i.p.} + \mathcal{O}(\|\xi\|^3) \end{aligned} \tag{108}$$

where we have defined  $s_d \equiv \bar{s}/\bar{Y}$ ,  $\omega_g = (\bar{G} + \bar{\zeta})/\bar{s}$  and  $\hat{\zeta}_t = (\zeta_t - \bar{\zeta})/\bar{Y}$ . Using equation (108) to substitute for  $\tilde{s}_t$  in equation (107), we obtain:

$$\begin{aligned} \bar{u}_c(C_t; \tilde{\xi}_t) s_t &= \bar{s} \bar{u}_c [1 - \sigma^{-1} \hat{Y}_t + (1 + \omega_g)(\hat{Y}_t + \hat{\tau}_t) + \sigma^{-1} g_t - s_d^{-1}(\hat{G}_t + \hat{\zeta}_t)] \\ &+ \frac{(1 + \omega_g)}{2} \hat{\tau}_t^2 + (1 + \omega_g)(1 - \sigma^{-1}) \hat{\tau}_t \hat{Y}_t \\ &+ \frac{1}{2} [1 + \omega_g + \sigma^{-1}(s_c^{-1} - 1) + \sigma^{-2} - 2\sigma^{-1}(1 + \omega_g)] \hat{Y}_t^2 \\ &- \sigma^{-1} [s_c^{-1} \hat{G}_t + (\sigma^{-1} - 1 - \omega_g) g_t - s_d^{-1}(\hat{G}_t + \hat{\zeta}_t)] \hat{Y}_t \\ &+ \sigma^{-1}(1 + \omega_g) g_t \hat{\tau}_t + \text{s.o.t.i.p.} + \mathcal{O}(\|\xi\|^3) \end{aligned} \tag{109}$$

Substituting equation (109) in equation (103), we obtain:

$$\begin{aligned} \tilde{W}_t &= (1 - \beta) \left[ b'_x x_t + b'_\xi \xi_t + \frac{1}{2} x'_t B_x x_t + x'_t B_\xi \xi_t \right] + \beta E_t \tilde{W}_{t+1} \\ &+ \text{s.o.t.i.p.} + \mathcal{O}(\|\xi\|^3) \end{aligned} \tag{110}$$

where  $\tilde{W}_t \equiv (W_t - \bar{W})/\bar{W}$  and:

$$b'_x = \left[ (1 + \omega_g) \quad (1 + \omega_g) - \sigma^{-1} \right]$$

$$b'_\xi = \left[ -s_d^{-1} \quad -s_d^{-1} \quad \sigma^{-1} \quad 0 \quad 0 \right]$$

$$B_x = \begin{bmatrix} (1 + \omega_g) & (1 - \sigma^{-1})(1 + \omega_g) \\ (1 - \sigma^{-1})(1 + \omega_g) & (1 + \omega_g) + (s_c^{-1} - 1)\sigma^{-1} + \sigma^{-2} - 2\sigma^{-1}(1 + \omega_g) \end{bmatrix}$$

$$B_\xi = \begin{bmatrix} 0 & 0 & \sigma^{-1}(1 + \omega_g) & 0 & 0 \\ s_d^{-1}\sigma^{-1} & s_d^{-1}\sigma^{-1} - s_c^{-1}\sigma^{-1} & -\sigma^{-1}(\sigma^{-1} - 1 - \omega_g) & 0 & 0 \end{bmatrix}$$

We note from equation (110) that:

$$\tilde{W}_t \equiv (\hat{b}_{t-1} - \pi_t - \tilde{\sigma}^{-1} \hat{C}_t + \bar{c}_t) + \frac{1}{2} (\hat{b}_{t-1} - \pi_t - \tilde{\sigma}^{-1} \hat{C}_t + \bar{c}_t)^2 + \mathcal{O}(\|\xi\|^3)$$

Substituting in equation (98), we obtain:

$$\tilde{W}_t \equiv \hat{b}_{t-1} - \pi_t - \sigma^{-1} (\hat{Y}_t - g_t) - \frac{\sigma^{-1} (1 - s_c^{-1})}{2} \hat{Y}_t^2 - \sigma^{-1} s_c^{-1} \hat{Y}_t \hat{G}_t + \frac{1}{2} (\hat{b}_{t-1} - \pi_t - \sigma^{-1} (\hat{Y}_t - g_t))^2 + \text{s.o.t.i.p.} + \mathcal{O}(\|\xi\|^3)$$

which can be written as:

$$\tilde{W}_t = \hat{b}_{t-1} - \pi_t + \omega'_x x_t + \omega'_\xi \xi_t + \frac{1}{2} x'_t W_x x_t + x'_t W_\xi \xi_t + \frac{1}{2} \left[ \hat{b}_{t-1} - \pi_t + \omega'_x x_t + \omega'_\xi \xi_t \right]^2 + \text{s.o.t.i.p.} + \mathcal{O}(\|\xi\|^3)$$

where

$$w'_x = [0 \quad -\sigma^{-1}]$$

$$w'_\xi = [0 \quad 0 \quad \sigma^{-1} \quad 0 \quad 0]$$

$$W_x = \begin{bmatrix} 0 & 0 \\ 0 & (s_c^{-1} - 1) \sigma^{-1} \end{bmatrix}$$

$$W_\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -s_c^{-1} \sigma^{-1} & 0 & 0 & 0 \end{bmatrix}$$

Note that in the first-order approximation, we can simply write equation (110) as:

$$\hat{b}_{t-1} - \pi_t + w'_x x_t + w'_\xi \xi_t = (1 - \beta) \left[ b'_x x_t + b'_\xi \xi_t \right] + \beta E_t \left[ \hat{b}_t - \pi_{t+1} + w'_x x_{t+1} + w'_\xi \xi_{t+1} \right] \tag{111}$$

Integrating equation (110) forward, we obtain:

$$\tilde{W}_{t_0} = (1 - \beta) E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[ b'_x x_t + \frac{1}{2} x'_t B_x x_t + x'_t B_\xi \xi_t \right] + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3) \tag{112}$$

where we have included  $b'_\xi \xi_t$  in *t.i.p.*

7.7 A QUADRATIC POLICY OBJECTIVE (EQUATIONS [19] AND [20])

We now derive a quadratic approximation to the policy objective function. To this end, we combine equations (102) and (112) in a way that eliminates the linear term in equation (93). Indeed, we find  $\vartheta_1, \vartheta_2$  so that:

$$\vartheta_1 b'_x + \vartheta_2 c'_x = a'_x \equiv [0 \quad \Phi]$$

The solution is given by:

$$\vartheta_1 = - \frac{\Phi \omega_\tau}{\Gamma}$$

$$\vartheta_2 = \frac{\Phi(1 + \omega_g)}{\Gamma}$$

where:

$$\Gamma = (\omega + \sigma^{-1})(1 + \omega_g) - \omega_\tau(1 + \omega_g) + \omega_\tau \sigma^{-1}$$

We can write:

$$\begin{aligned} E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \Phi \hat{Y}_t &= E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} [\vartheta_1 b'_x + \vartheta_2 c'_x] x_t \\ &= - E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[ \frac{1}{2} x'_t D_x x_t + x'_t D_\xi \xi_t + \frac{1}{2} d_\pi \pi_t^2 \right] \\ &\quad + \vartheta_1 \bar{W}_{t_0} + \vartheta_2 \kappa^{-1} V_{t_0} + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3) \end{aligned}$$

where:

$$D_x \equiv \vartheta_1 B_x + \vartheta_2 C_x$$

and so on. Hence:

$$\begin{aligned} U_{t_0} &= \Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ a'_t x_t - \frac{1}{2} x'_t A_x x_t - x'_t A_\xi \xi_t - \frac{1}{2} a_\pi \pi_t^2 \right\} + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3) \\ &= - \Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{1}{2} x'_t Q_x x_t + x'_t Q_\xi \xi_t + \frac{1}{2} q_\pi \pi_t^2 \right\} + T_{t_0} + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3) \\ &= - \Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{1}{2} q_y (\hat{Y}_t - \hat{Y}_t^*)^2 + \frac{1}{2} q_\pi \pi_t^2 \right\} + T_{t_0} + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3) \end{aligned} \tag{113}$$



In these expressions,  $\Omega = \bar{u}_c \bar{Y}$  and

$$Q_x = \begin{bmatrix} 0 & 0 \\ 0 & q_y \end{bmatrix}$$

with:

$$q_y \equiv (1 - \Phi)(\omega + \sigma^{-1}) + \Phi(\omega + \sigma^{-1}) \frac{(1 + \omega_g)(1 + \omega)}{\Gamma} \\ + \Phi\sigma^{-1} \frac{(1 + \omega_\tau)(1 + \omega_g)}{\Gamma} - \Phi\sigma^{-1} s_c^{-1} \frac{1 + \omega_g + \omega_\tau}{\Gamma}$$

We have defined:

$$Q_\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ q_{\xi 1} & q_{\xi 2} & q_{\xi 3} & q_{\xi 4} & q_{\xi 5} \end{bmatrix}$$

with:

$$q_{\xi 1} = - \frac{\Phi\omega_\tau}{\Gamma} s_d^{-1} \sigma^{-1},$$

$$q_{\xi 2} = - \frac{\Phi\sigma^{-1} s_d^{-1} \omega_\tau}{\Gamma} + \frac{\sigma^{-1} s_c^{-1} \Phi(1 + \omega_g + \omega_\tau)}{\Gamma}$$

$$q_{\xi 3} = - (1 - \Phi)\sigma^{-1} - \frac{\sigma^{-1} \Phi(1 + \omega)(1 + \omega_g)}{\Gamma}$$

$$q_{\xi 4} = - (1 - \Phi)\omega - \frac{\omega\Phi(1 + \omega)(1 + \omega_g)}{\Gamma}$$

$$q_{\xi 5} = \Phi \frac{1 + \omega_g}{\Gamma} (1 + \omega)$$

and:

$$q_\pi = \frac{\Phi(1 + \omega_g)\theta(1 + \omega)(\omega + \sigma^{-1})}{\Phi\kappa} + \frac{(1 - \Phi)\theta(\omega + \sigma^{-1})}{\kappa}$$

We have defined  $\hat{Y}_t^*$ , the desired level of output, as:

$$\hat{Y}_t^* = - q_y^{-1} q_\xi' \xi_t$$

Finally:

$$T_{t_0} \equiv \bar{Y} \bar{u}_c [\partial_1 \bar{W}_{t_0} + \partial_2 \kappa^{-1} V_{t_0}]$$

is a transitory component.

Equation (113) corresponds to equation (19). In particular, given the commitments on the initial values of the vector  $X_{t_0}$ ,  $\bar{W}_{t_0}$  implies that  $\bar{W}_{t_0}$  is given when characterizing the optimal policy from a timeless perspective.  $\bar{F}_{t_0}$  and  $\bar{K}_{t_0}$  imply that  $V_{t_0}$  and  $Z_{t_0}$  are also given. It follows that the value of the transitory component  $T_{t_0}$  is predetermined under stage two of the Ramsey problem. Hence, over the set of admissible policies, higher values of equation (113) correspond to lower values of:

$$E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{1}{2} q_y (\hat{Y}_t - \hat{Y}_t^*)^2 + \frac{1}{2} q_\pi \pi_t^2 \right\} \quad (114)$$

It follows that we may rank policies in terms of the implied value of the discounted quadratic loss function in equation (114) which corresponds to equation (20). Because this loss function is purely quadratic (i.e., lacking linear terms), it is possible to evaluate it to second order using only a first-order approximation to the equilibrium evolution of inflation and output under a given policy. Hence, the log-linear approximate structural relations in equations (101) and (111) are sufficiently accurate for our purposes. Similarly, it suffices that we use log-linear approximations to the variables  $V_{t_0}$  and  $\hat{W}_{t_0}$  in describing the initial commitments, which are given by:

$$\begin{aligned} \hat{V}_{t_0} &= \pi_{t_0}, \\ \hat{W}_{t_0} &= \hat{b}_{t_0-1} - \pi_{t_0} + w'_x x_{t_0} + w'_\xi \xi_{t_0} \\ &= \hat{b}_{t_0-1} - \pi_{t_0} - \sigma^{-1}(\hat{Y}_{t_0} - g_{t_0}) \end{aligned}$$

Then an optimal policy from a timeless perspective is a policy from date  $t_0$  onward that minimizes the quadratic loss function in equation (114) subject to the constraints implied by the linear structural relations in equations (101) and (111) holding in each period  $t \geq t_0$ , given the initial values  $\hat{b}_{t_0-1}$ ,  $\hat{\Delta}_{t_0-1}$ , and subject also to the constraints that certain predetermined values for  $\hat{V}_{t_0}$  and  $\hat{W}$  (or alternatively, for  $\pi_{t_0}$  and for  $\hat{Y}_{t_0}$ ) be achieved.<sup>39</sup> We note that under the assumption  $\omega + \sigma^{-1} > \omega_\tau = \bar{\tau} / (1 - \bar{\tau})$ ,  $\Gamma > 0$ , which implies that  $q_\pi > 0$ . If:

$$s_C > \frac{\Phi \sigma^{-1} (1 + \omega_g + \omega_\tau)}{(1 - \Phi)(\omega + \sigma^{-1})\Gamma + \Phi(\omega + \sigma^{-1})(1 + \omega_g)(1 + \omega) + \Phi \sigma^{-1} (1 + \omega_g)(1 + \omega_\tau)}$$

then  $q_y > 0$  and the objective function is convex. Because the expression on the right side of this inequality is necessarily less than one (given that  $\Gamma > 0$ ), the inequality is satisfied for all values of  $s_C$  less than a positive upper bound.

### 7.8 THE LOG-LINEAR AGGREGATE-SUPPLY RELATION AND THE COST-PUSH DISTURBANCE TERM

The AS equation (101) can be written as:

$$\pi_t = \kappa [y_t + \psi \hat{\tau}_t + u_t] + \beta E_t \pi_{t+1} \quad (115)$$

39. The constraint associated with a predetermined value for  $Z_{t_0}$  can be neglected in a first-order characterization of optimal policy because the variable  $Z_t$  does not appear in the first-order approximation to the aggregate-supply relation.

where  $u_t$  is composite cost-push shock defined as  $u_t \equiv c'_\xi \xi_t + \hat{Y}_t^*$ . We can write equation (115) as:

$$\pi_t = \kappa[y_t + \psi(\hat{\tau}_t - \hat{\tau}_t^*)] + \beta E_t \pi_{t+1} \tag{116}$$

where we have further defined:

$$u_t = u'_\xi \xi_t \equiv \hat{Y}_t^* + c'_\xi \xi_t$$

where:

$$u_{\xi 1} \equiv \frac{\Phi \omega_\tau}{q_y \Gamma} s_d^{-1} \sigma^{-1},$$

$$u_{\xi 2} \equiv \frac{\Phi \sigma^{-1} s_d^{-1} \omega_\tau}{q_y \Gamma} - \frac{\sigma^{-1} s_c^{-1} \Phi (1 + \omega_g + \omega_\tau)}{q_y \Gamma}$$

$$u_{\xi 3} \equiv -\Phi \sigma^{-2} \frac{(1 + \omega_\tau)(1 + \omega_g)}{q_y \Gamma (\omega + \sigma^{-1})} + \Phi \sigma^{-2} s_c^{-1} \frac{1 + \omega_g + \omega_\tau}{q_y \Gamma (\omega + \sigma^{-1})}$$

$$u_{\xi 4} \equiv \omega \sigma u_{\xi 3},$$

$$u_{\xi 5} \equiv -\sigma u_{\xi 3} + \frac{(1 - \Phi)}{q_y}$$

we finally define:

$$\hat{\tau}_t^* \equiv -\psi^{-1} u_t$$

so that we can write equation (101) as:

$$\pi_t^* = \kappa[(\hat{Y}_t - \hat{Y}_t^*) + \psi(\hat{\tau}_t - \hat{\tau}_t^*)] + \beta E_t \pi_{t+1} \tag{117}$$

which is equation (23).

### 7.9 THE LOG-LINEAR INTERTEMPORAL SOLVENCY CONDITION AND THE FISCAL STRESS DISTURBANCE TERM

The flow budget constraint in equation (111) can be solved forward to yield the intertemporal solvency condition:

$$\hat{b}_{t-1} - \pi_t - \sigma^{-1} y_t = -f_t + (1 - \beta) E_t \sum_{T=t}^{\infty} [b_y y_T + b_\tau (\hat{\tau}_T - \hat{\tau}_T^*)] \tag{118}$$

where  $f_t$ , the fiscal stress disturbance term, is defined as:

$$f_t \equiv \sigma^{-1}(g_t - \hat{Y}_t^*) - (1 - \beta)E_t \sum_{T=t}^{\infty} \beta^{T-t} [b_y \hat{Y}_T^* + b_\tau \hat{\tau}_T^* + b'_\xi \xi_T]$$

$$= \sigma^{-1}(g_t - \hat{Y}_t^*) - (1 - \beta)E_t \sum_{T=t}^{\infty} \beta^{T-t} [\omega_\tau^{-1} \Gamma \hat{Y}_T^* - (b'_\xi - b_\tau \psi^{-1} c'_\xi) \xi_T]$$

This can be rewritten in a more compact way as:

$$f_t \equiv h'_\xi \xi_t + (1 - \beta)E_t \sum_{T=t}^{\infty} \beta^{T-t} f'_\xi \xi_T$$

where:

$$h_{\xi 1} \equiv - \frac{\Phi \omega_\tau}{q_y \Gamma} \frac{\sigma^{-2}}{s_d}$$

$$f_{\xi 1} \equiv \frac{\Phi}{q_y} \frac{\sigma^{-1}}{s_d} + \frac{1}{s_d}$$

$$h_{\xi 2} \equiv - \frac{\Phi \sigma^{-2} s_d^{-1} \omega_\tau}{\Gamma q_y} + \frac{\sigma^{-2} s_c^{-1} \Phi (1 + \omega_g + \omega_\tau)}{\Gamma q_y}$$

$$f_{\xi 2} \equiv \frac{\Phi \sigma^{-1} s_d^{-1}}{q_y} - \frac{\omega_\tau^{-1} \sigma^{-2} s_c^{-1} \Phi (1 + \omega_g + \omega_\tau)}{\omega_\tau q_y} + \frac{1}{s_d}$$

$$h_{\xi 3} \equiv \Phi \sigma^{-2} \frac{(1 + \omega_\tau)(1 + \omega_g)}{q_y \Gamma} - \Phi \sigma^{-2} s_c^{-1} \frac{1 + \omega_g + \omega_\tau}{q_y \Gamma}$$

$$+ \frac{(1 - \Phi) \omega \sigma^{-1}}{q_y} + \frac{\omega \sigma^{-1} \Phi (1 + \omega)(1 + \omega_g)}{\Gamma q_y}$$

$$f_{\xi 3} \equiv \frac{\omega_\tau^{-1} \Gamma (1 - \Phi) \sigma^{-1}}{q_y} + \frac{\omega_\tau^{-1} \sigma^{-1} \Phi (1 + \omega)(1 + \omega_g)}{q_y} - \sigma^{-1} (1 + \omega_\tau^{-1})(1 + \omega_g)$$

$$h_{\xi 4} \equiv - \frac{\sigma^{-1} (1 - \Phi) \omega}{q_y} - \frac{\sigma^{-1} \omega \Phi (1 + \omega)(1 + \omega_g)}{\Gamma q_y}$$

$$f_{\xi 4} \equiv \frac{\omega_\tau^{-1} \Gamma (1 - \Phi) \omega}{q_y} + \frac{\omega_\tau^{-1} \omega \Phi (1 + \omega)(1 + \omega_g)}{q_y} - \omega \omega_\tau^{-1} (1 + \omega_g)$$

$$h_{\xi 5} \equiv \sigma^{-1} \Phi \frac{(1 + \omega_g)(1 + \omega)}{q_y \Gamma}$$

$$f_{\xi 5} \equiv - \omega_\tau^{-1} \Phi \frac{(1 + \omega_g)(1 + \omega)}{q_y} + \omega_\tau^{-1} (1 + \omega_g)$$

#### 7.10 DEFINITION OF THE COEFFICIENTS IN SECTIONS 3, 4, AND 5

The coefficients  $m_\varphi$ ,  $n_\varphi$ ,  $n_b$ ,  $m_b$ ,  $\tilde{m}_b$ ,  $\omega_\varphi$  are defined as:

$$m_\varphi \equiv - q_y^{-1} \psi^{-1} (1 - \beta) b_\tau + q_y^{-1} [(1 - \beta) b_y + \sigma^{-1}]$$

$$n_\phi \equiv -q_y^{-1}\sigma^{-1}$$

$$n_b \equiv b_\tau(\psi^{-1} - 1)(m_\phi + n_\phi)$$

$$m_b \equiv -n_\phi[(1 - \beta)b_\tau\psi^{-1} - (1 - \beta)b_y - \sigma^{-1}]$$

$$\bar{m}_b \equiv \sigma^{-1}n_\phi + \omega_\phi - (1 - \beta)[b_\tau\psi^{-1} - b_y]n_\phi + (1 - \beta)\psi^{-1}\kappa^{-1}b_\tau\omega_\phi$$

$$\omega_\phi \equiv -q_\pi^{-1}(\kappa^{-1}(1 - \beta)b_\tau\psi^{-1} + 1)$$

$$\phi \equiv \kappa^{-1}q_\pi^{-1}q_y$$

$$\gamma_1 \equiv \kappa^{-1}q_\pi^{-1}[(1 - \beta)b_y + \sigma^{-1}]$$

$$\gamma_2 \equiv \kappa^{-1}q_\pi^{-1}\sigma^{-1}$$

The coefficients  $\mu_1$  and  $\mu_2$  of Section 5 are defined as:

$$\mu_1 \equiv \frac{\kappa\psi}{(1 - \beta)b_\tau + \kappa\psi}$$

$$\mu_2 \equiv \frac{\kappa(1 - \beta)(b_\tau - \psi\phi_y)}{(1 - \beta)b_\tau + \kappa\psi}$$

#### 7.11 PROOF OF DETERMINACY OF EQUILIBRIUM UNDER THE OPTIMAL TARGETING RULES

We now show that there is a determinate equilibrium if policy is conducted to ensure that the two target criteria:

$$E_t \pi_{t+1} = 0 \tag{119}$$

and:

$$\Delta y_t + \omega_\phi^{-1}(m_\phi + n_\phi)\pi_t - \omega_\phi^{-1}n_\phi \Delta \pi_t = 0 \tag{120}$$

are satisfied in each period  $t \geq t_0$ . Note that equation (120) can be written as:

$$\Delta y_t = \gamma_3 \pi_t + \gamma_4 \pi_{t-1} \tag{121}$$

where:

$$\gamma_3 \equiv -\omega_\phi^{-1}m_\phi$$

$$\gamma_4 \equiv -\omega_\phi^{-1}n_\phi$$

Use equation (119), combined with:

$$\tau_t - \hat{\tau}_t^* = \kappa^{-1}\pi_t - \psi^{-1}y_t - \kappa^{-1}\beta E_t \pi_{t+1} \tag{122}$$

and:

$$E_t \Delta y_{t+1} = -\omega_\phi^{-1} n_\phi \pi_t$$

to eliminate  $E_t \pi_{t+1}$ ,  $E_t y_{t+1}$  and  $\tau_t - \hat{\tau}_t^*$  from:

$$\begin{aligned} \hat{b}_{t-1} - \pi_t - \sigma^{-1} y_t + f_t &= (1 - \beta)[b_y y_t + b_\tau(\hat{\tau}_t - \hat{\tau}_t^*)] \\ &+ \beta E_t [\hat{b}_t - \pi_{t+1} - \sigma^{-1} \hat{y}_{t+1} + f_{t+1}] \end{aligned}$$

Then equation (120) can be used to eliminate  $y_t$  from the resulting expression to obtain an equation of the form:

$$\hat{b}_t = \beta^{-1} \hat{b}_{t-1} + m_{41} \pi_t + m_{42} \pi_{t-1} + m_{43} y_{t-1} + \varepsilon_t$$

where  $\varepsilon_t$  is an exogenous disturbance. The system consisting of this equation plus equations (120) and (119) can then be written as:

$$E_t z_{t+1} = Mz_t + N\varepsilon_t \tag{123}$$

where:

$$z_t \equiv \begin{bmatrix} \pi_t \\ \pi_{t-1} \\ y_{t-1} \\ \hat{b}_{t-1} \end{bmatrix} \quad M \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 \\ m_{41} & m_{42} & m_{43} & \beta^{-1} \end{bmatrix} \quad N \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ n_{41} \end{bmatrix}$$

Because  $M$  is lower triangular, its eigenvalues are the four diagonal elements: 0, 0, 1, and  $\beta^{-1}$ . Hence, there is exactly one eigenvalue outside the unit circle, and equilibrium is determinate (but possesses a unit root). Because of the triangular form of the matrix, one can also easily solve explicitly for the elements of the left eigenvector:

$$v' = [v_1 \quad v_2 \quad v_3 \quad 1]$$

associated with the eigenvalue  $\beta^{-1}$ , where:

$$\begin{aligned} v_1 &= (1 + \omega_g)[\psi^{-1} - 1] \beta \gamma_4 - (1 - \beta \sigma^{-1} \gamma_4) - (1 - \beta)(1 - \omega_g)(\kappa \psi)^{-1} \\ &+ (1 + \omega_g)[\psi^{-1} - 1] \gamma_3, \end{aligned}$$

$$v_2 = (1 + \omega_g)[\psi^{-1} - 1] \gamma_4$$

$$v_3 = (1 + \omega_g)[\psi^{-1} - 1]$$

By pre-multiplying the vector equation (123) by  $v'$ , one obtains a scalar equation with a unique nonexplosive solution of the form:

$$v'z_t = - \sum_{j=0}^{\infty} \beta^{j+1} E_t \varepsilon_{t+j}$$

If  $v_1 \neq 0$ , this can be solved for  $\pi_t$  as a linear function of  $\pi_{t-1}$ ,  $y_{t-1}$ ,  $\hat{b}_{t-1}$  and the exogenous state vector:

$$\pi_t = - \frac{1}{v_1} \hat{b}_{t-1} - \frac{v_2}{v_1} \pi_{t-1} - \frac{v_3}{v_1} y_{t-1} + \frac{1}{v_1} f_t \quad (124)$$

The solution for  $\pi_t$  can then be substituted into the above equations to obtain the equilibrium dynamics of  $y_t$  and  $\hat{b}_t$ , and hence of  $\tau_t$ .

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## Comment

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### 1. Introduction

Benigno and Woodford seek to offer an integrated analysis of optimal fiscal and monetary policy building on two branches of the literature. The

Prepared for the 2003 NBER Macroeconomics Annual Conference, April 4–5, 2003. I wish to thank Larry Christiano and Henry Siu for stimulating discussions.



first is the one on dynamic optimal taxation, stemming from the seminal contribution of Lucas and Stokey (1983).<sup>1</sup> The second part of the literature is on optimal monetary stabilization policy, for example, in Goodfriend and King (1997), Rotemberg and Woodford (1997) and Woodford (2000).<sup>2</sup> Both areas in the literature consider the problem of a benevolent government seeking to stabilize the response of economic outcomes to exogenous shocks with a combination of fiscal and monetary policies chosen once and for all at some previous date. The optimal taxation literature considers *fiscal* shocks, such as fluctuations in government expenditures, and rules out lump-sum taxes in the tradition of Ramsey (1927). Distortionary taxes generate wedges between marginal rates of transformation and marginal rates of substitution, and government policy becomes a source of frictions. The monetary stabilization literature, instead, considers environments where frictions are present even without government policy. These frictions are due to nominal rigidities and imperfect competition in product or labor markets. The corresponding wedges reduce the level of economic activity and may be subject to stochastic fluctuations, known as cost-push shocks. The government's only fiscal policy instrument is a lump-sum tax.

Both parts of the literature are characterized by an underlying tension. The fiscal shocks considered by the optimal taxation literature do not affect any wedges and should ideally be offset through lump-sum taxes. Yet the government has access only to distortionary fiscal instruments. The optimal stabilization literature considers fluctuations in wedges that could be offset with appropriate fiscal instruments acting on the same margins, but the government has access only to lump-sum taxes. Given this tension, monetary policy acquires an auxiliary role in responding to shocks. Lucas and Stokey show that it is optimal to respond to fiscal shocks by appropriately setting the state contingent returns on government debt. Taxes and real returns on government debt inherit the serial correlation structure of underlying shocks, and taxes are smooth, in the sense of having a small variance relative to fiscal shocks. Chari, Christiano, and Kehoe (1991, 1995) extend the analysis to monetary economies with risk-free debt and show that it is optimal to use state-contingent inflation as a fiscal shock absorber. They find that the standard deviation of optimal taxes is close to zero, while real returns on government debt are highly volatile for calibrated examples. In the monetary stabilization literature, rigidities in nominal prices and wages imply that

1. An excellent survey of this literature can be found in Chari and Kehoe (1999).

2. Additional important contributions in this literature are King and Wolman (1999); Kahn, King, and Wolman (2000); and Giannoni and Woodford (2002).

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innovations in inflation reduce the average markups and increase equilibrium output. At the same time, nominal rigidities imply that inflation generates relative price distortions. The resulting trade-off between inflation and output stabilization implies that the volatility and persistence of optimal inflation will depend on the stochastic properties of the cost-push disturbances and on the degree of nominal rigidity. Hence, the interdependence between fiscal and monetary policy is generated in both branches of the literature by a lack of appropriate fiscal instruments. Given appropriate instruments, the government would be indifferent to the stochastic path of inflation.

Recent contributions to the optimal taxation literature, such as Correia, Nicolini, and Teles (2001); Schmitt-Grohé and Uribe (2001); and Siu (2001), have incorporated monopolistic competition and nominal price rigidity. Correia, Nicolini, and Teles (2001) assume that state-contingent bonds are available. Their theoretical analysis allows for fiscal shocks as well as cost-push shocks, and shows that the same equilibrium outcomes as those in a flexible price economy can be achieved. In addition, they describe the assumptions about fiscal instruments required for the path of inflation to be neutral to equilibrium outcomes, thus remarking the auxiliary role of inflation in this class of policy problems. Schmitt-Grohé and Uribe (2001) and Siu (2001) focus on government consumption shocks and do not allow for state-contingent debt, thus reinstating the role of inflation as a fiscal shock-absorber. With sticky prices, however, the benefits of volatile inflation must be balanced against the resource misallocation resulting from the associated relative price distortions. They find that, for government consumption processes with similar volatility to the postwar United States, the departures from optimal policy with flexible prices are striking. Optimal inflation volatility is close to zero, even for very small degrees of price rigidity. Tax rates and the real value of government debt exhibit random walk behavior, as in Barro (1979), regardless of the degree of autocorrelation of the underlying shocks. Siu (2001) also considers large fiscal shocks, such as fluctuations in government expenditure that would arise in an economy alternating between war and peace. He finds that optimal inflation volatility is high regardless of the degree of price stickiness for large fiscal shocks. The intuition for this is that the benefits of using inflation as a shock-absorber outweigh the costs of the resulting misallocation in this case. Hence, the stochastic properties of taxes and inflation in a Ramsey equilibrium with monopolistic competition and nominal rigidities can be understood as the outcome of a struggle between the costs of volatile inflation and the benefits of smoothing government outlays in the face of fiscal shocks.

Benigno and Woodford's main contribution is to allow for both fiscal and cost-push shocks. Their analytical results demonstrate that the different time-series behavior of optimal policies in flexible and sticky-price environments do not depend on the nature of the underlying shocks. With flexible prices, state-contingent inflation is used to offset fiscal shocks, implying volatile real debt returns. Because taxes are set to offset cost-push shocks and stabilize output, however, their variance is not necessarily small. With sticky prices, volatile inflation is costly and taxes are used to respond to both fiscal and cost-push shocks. Taxes and real debt returns have a unit root behavior, regardless of the stochastic properties of underlying shocks, and output cannot be stabilized. Benigno and Woodford's quantitative exercise is limited, however, to fiscal shocks. They find that the optimal response of inflation to a one-time increase in government expenditure is inversely related to the degree of price rigidity. This is not surprising, given the findings in the previous studies.

The rest of this comment expands on the previous discussion. In Section 2, I relate the analytical results in Benigno and Woodford to those in the literature on optimal taxation. The section discusses the benefits and costs of state-contingent inflation as a function of the volatility of exogenous shocks and raises several concerns about the solution method. Section 3 illustrates the notion of optimal policy from a timeless perspective with a simple example and relates it to limited commitment. I conclude with some questions for further research.

## 2. *Optimal Policy with Nominal Rigidities*

Benigno and Woodford (BW) adopt a standard new Keynesian framework with monopolistic competition in product markets and Calvo pricing. They allow for labor market frictions by assuming that a wage markup as well as a price markup are present, and they abstract from monetary frictions. There are four types of shocks: government consumption shocks, government transfer shocks, preference shocks, and wage markup shocks. The first three are common to the optimal taxation literature, while wage markup shocks are typically considered by the optimal monetary stabilization literature. The government's objective is to maximize the representative agent's lifetime utility. In the linear-quadratic problem, the government has two policy instruments; the tax rate on sales,  $\hat{\tau}_t$ , and the inflation rate,  $\pi_t$ .<sup>3</sup> These instruments are set to respond to a cost-push shock,  $u_t$ , and a fiscal shock,  $f_t$ . The variables  $u_t$  and  $f_t$  are not primitive shocks but a complex convolution of those primitive distur-

3. All variables denote percentage deviations from steady-state values.

bances. In particular, the primitive shocks contributing to the cost-push or fiscal shock depend on the available policy instruments and on the degree of price stickiness. However, the shock  $u_t$  can arise only if wage markup shocks are present.

The evolution of equilibrium outcomes in response to the shocks and government policy is summarized by the expectational Phillips curve:

$$\pi_t = \kappa [y_t + \psi \hat{\tau}_t + u_t] \beta E_t \pi_{t+1} \tag{1}$$

The expression in equation (1) makes clear that by setting the tax rate according to  $\hat{\tau}_t^* = -u_t/\psi$ , the government can completely stabilize output in the face of cost-push shocks because cost-push shocks and the tax rate on sales act on the same margin. Hence, fluctuations in equilibrium output away from the steady state behave according to:

$$y_t \sim (\hat{\tau}_t - \hat{\tau}_t^*)$$

The evolution of equilibrium outcomes and policy must also satisfy the government's intertemporal budget constraint:

$$\hat{b}_{t-1} + f_t - \pi_t - \sigma^{-1} y_t - (1 - \beta) E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} (b_y y_t + b_\tau (\hat{\tau}_t - \hat{\tau}_t^*)) = 0$$

This equation clarifies that setting taxes equal to  $\hat{\tau}_t^*$  requires inflation to respond fully to the fiscal stress shock.

The government strives to achieve three goals (see equation [20] in BW). The first two, output and inflation stabilization, appear directly in the objective function. The third goal is to minimize the intertemporal cost of raising government revenues measured by  $\phi_{2,t}$ , the multiplier on the government's intertemporal budget constraint. These goals are traded off based on the available policy instruments. In the monetary stabilization literature, taxes are lump-sum and  $\phi_{2,t} = 0$ . Hence,  $f_t$  does not influence  $y_t$  or  $\pi_t$ . However,  $u_t$  cannot be offset and  $y_t \neq 0$ .

In the optimal taxation literature, the stochastic properties of optimal policy depend on whether the returns on government debt are state contingent. In a monetary economy with nominal risk-free debt, bond returns can be made state contingent by setting the process for inflation appropriately. If no distortions are associated with inflation, as in the case with fully flexible prices, it is optimal to use inflation as a fiscal shock absorber:

$$\begin{aligned} \pi_t - E_{t-1} \pi_t &= f_t - E_{t-1} f_t \\ \hat{b}_t &= -E_t f_{t+1} \end{aligned}$$

as in Chari, Christiano, and Kehoe (1991, 1995). Taxes can then be set to meet the output stabilization objective so that  $\hat{\tau}_t = \hat{\tau}_t^*$  and  $y_t = 0$ . This

implies that the cost of raising government revenues is equalized across states in each period:

$$\phi_{2,t} = \phi_{2,t-1}$$

Consequently,  $\hat{\tau}_t$ ,  $\pi_t$ , and  $\hat{b}_t$  inherit the stochastic properties of the underlying shocks. If the primitive shocks are stationary, tax rates, inflation, and the real value of government debt will also be stationary. The difference with a Ramsey model with only shocks to government spending is that smoothing the cost of raising fiscal revenues across states does not correspond to a smooth path of taxes. The volatility of the optimal taxes will depend on the volatility of the cost-push shocks.

With nominal risk-free debt and some degree of price rigidity, the properties of optimal fiscal and monetary policy resemble those in a real economy and risk-free debt, as in Barro (1979) and Aiyagari, Marcet, Sargent, and Seppala (2002). The optimal policy will smooth the cost of raising taxes over time, given the costs of fully smoothing it across states. This imparts a martingale behavior to the shadow cost of raising government revenues:

$$\phi_{2,t} = E_t \phi_{2,t+1} \quad (2)$$

Inflation does not fully respond to fiscal stress:

$$\pi_t = -\omega\phi(\phi_{2,t} - \phi_{2,t-1})$$

$$\hat{b}_t = -E_t f_{t+1} - \eta_b \phi_{2,t}$$

and taxes cannot be set to stabilize output fully:

$$y_t = m_\phi \phi_{2,t} + \eta_\phi \phi_{2,t-1} \neq 0$$

The unit-root behavior of the shadow cost of raising government revenues makes the equilibrium response of taxes, output, prices, and the real value of government debt to cost-push and fiscal shocks nonstationary, regardless of the autocorrelation properties of the primitive shocks.

## 2.1 DISCUSSION

While in a real economy with risk-free debt, the government has no alternative but to smooth the cost of raising distortionary revenues according to equation (2), in a monetary economy with nominal bonds, it is possible to make bond returns state-contingent in real terms by setting ex post inflation. If nominal rigidities are present, however, the government faces a trade-off between the costs of market incompleteness and the costs of volatile inflation. The properties of optimal policy will depend on the relative size of these costs. BW and previous studies focus on models in which the resource misallocation associated with volatile inflation

increases with the degree of price stickiness. The costs of market incompleteness, on the other hand, should depend on the size and persistence of the primitive shocks. To understand this issue, it is useful to review the findings in Siu (2001) for a similar environment because BW's linear quadratic approach cannot be used to explore this aspect.<sup>4</sup>

Siu studies Ramsey policy in a cash-credit good economy with monopolistic competition in which a fraction of firms in each period sets their prices before current exogenous shocks are realized. The remaining firms set prices after the realization of the current exogenous shocks.<sup>5</sup> Government purchases,  $g$ , follow a two-state first-order Markov process with support:  $[g, \bar{g}]$  and  $\underline{g} < \bar{g}$ . Siu characterizes optimal policies as a function of the unconditional standard deviation of government purchases with a nonlinear numerical procedure. He finds that the optimal inflation volatility decreases with the degree of price stickiness for business-cycle fluctuations in  $g$ , while for  $g$ -processes designed to model an economy fluctuating between war and peace—large fiscal shocks—optimal inflation volatility is high regardless of the degree of price stickiness, as illustrated in Figures 1 and 2, reproduced from Siu.<sup>6</sup>

The percentage loss in output that occurs in the economy with sticky prices under the Ramsey policy corresponding to the flexible-price economy is a measure of the misallocation caused by volatile inflation. Figure 3 illustrates the behavior of this measure.<sup>7</sup> The horizontal axis measures  $L_s/L_f$ , the labor demand from sticky-price firms relative to flexible-price firms, and the vertical axis measures the corresponding misallocation. In each graph, the star on the left corresponds to a sequence  $[g, g]$ ; the one on the right corresponds to a sequence  $[\bar{g}, \bar{g}]$ .<sup>8</sup> When the current value of  $g$  is low, sticky-price firms have higher prices than flexible-price firms and  $L_s/L_f < 1$ . Concavity in production implies that the cost in terms of foregone output is very large for a large misallocation and decreases at a decreasing rate. The graphs suggest that the misallocation cost is large

4. The approximation is valid only for stochastic processes with an absorbing state and a small range. See Section 2.2 for further discussion.

5. The impact effect of a shock on the nominal price index in Siu's model is the same as the one in a model with Calvo pricing if the fraction of prices that remains unchanged in any period is set equal across the two models. In Siu's model, all remaining price adjustment takes place in the subsequent period, while the adjustment is smoothed across several periods with Calvo pricing.

6. The calibration is based on U.S. data for the twentieth century. The small-shock case matches fluctuations in government purchases that occur in the postwar United States. The transition probability between the  $\bar{g}$  and  $\underline{g}$  states is  $\rho = 0.95$ , and the standard deviation of  $g$  is 6.7%. For the large-shock case, the standard deviation is 21% and all other parameters are kept constant.

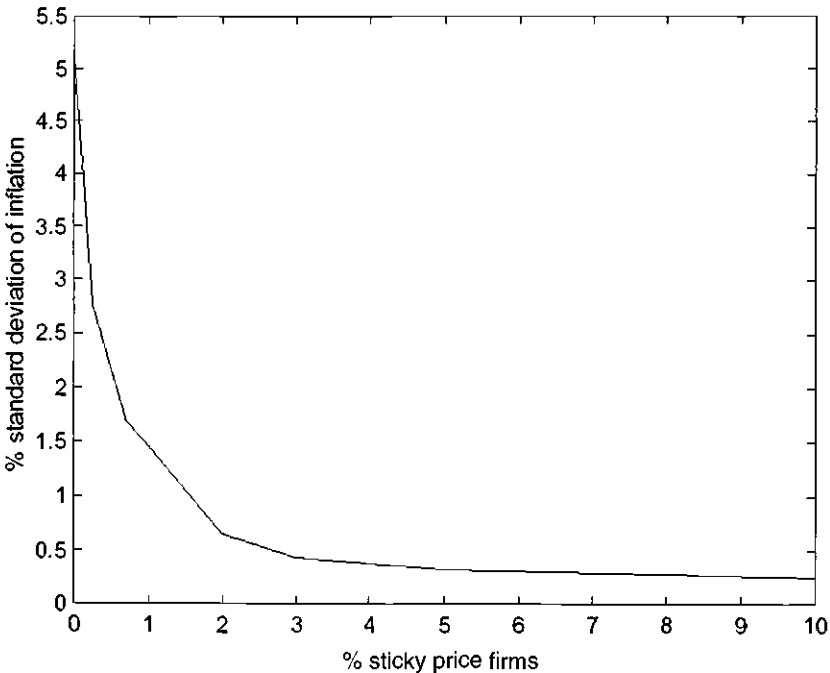
7. I thank Henry Siu for providing Figure 3 and Figure 4.

8. Given the assumed persistence of the government consumption process, these are the most likely sequences.

and increasing in the size of government spending shocks when the fluctuations in the spending shock are small, but it is small for large fiscal shocks. This pattern stems from the incentive for firms setting prices to frontload and set high prices to insure against the possibility of having negative profits. Because government consumption shocks are persistent, if the shock was high in the previous period, sticky-price firms will set high prices. If the shock in this period is high, inflation will be high under the Ramsey policy for the flexible-price economy, and the misallocation will be small. If the shock was low in the previous period, firms setting prices will still set them high. If the realized value of  $g$  is low, inflation will be low, which will give rise to a large misallocation. The tendency to frontload is a general feature of sticky-price models and is exacerbated when firms fix prices for longer periods of time.

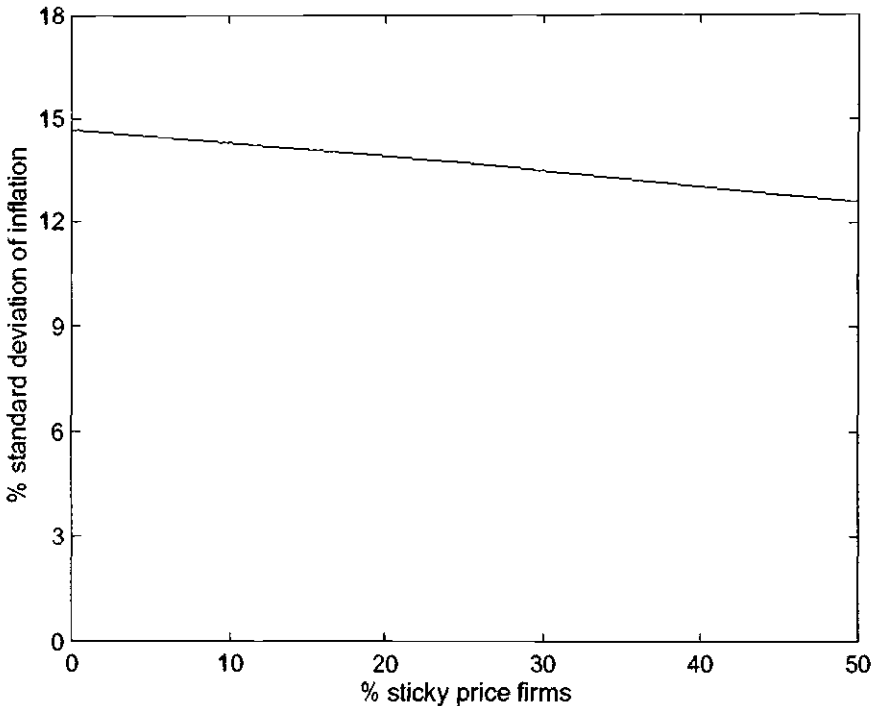
Figure 4 plots the misallocation cost in consumption equivalents against the volatility of government consumption when 10% of firms have sticky prices. It rises steeply initially but then flattens out. The cost in terms of foregone consumption of not being able to smooth government

Figure 1 OPTIMAL INFLATION VOLATILITY FOR SMALL SHOCKS



Reproduced from Siu (2001)

Figure 2 OPTIMAL INFLATION VOLATILITY FOR LARGE SHOCKS



Reproduced from Siu (2001)

revenues across states as a function of the variability of government consumption shocks is also shown.<sup>9</sup> Not surprisingly it is increasing in the variance of the shocks. This explains the finding that, for large government expenditure shocks, optimal inflation volatility is high even when a large fraction of prices are fixed, while for business-cycle type fluctuations in government consumption, optimal inflation volatility is close to 0. For very small expenditure shocks, the costs of not using inflation to make real bond returns respond to government consumption is low, as is inflation volatility in the Ramsey equilibrium for the flexible-price economy. Because the distortion caused by taxation is first order, it will be optimal to have a smooth path of taxes and volatile inflation.<sup>10</sup>

The role of the size of fiscal shocks for the stochastic behavior of taxes and policy with nominal price rigidities raises several questions for future

9. This is the welfare loss in average consumption equivalents of using the Ramsey policy for the sticky-price economy in the flexible-price economy.

10. This is true in BW's model. Recall that the steady-state wedge between the marginal rate of substitution between consumption and leisure and the marginal product of labor,  $\Phi$ , is positive if the initial level of public debt is positive.

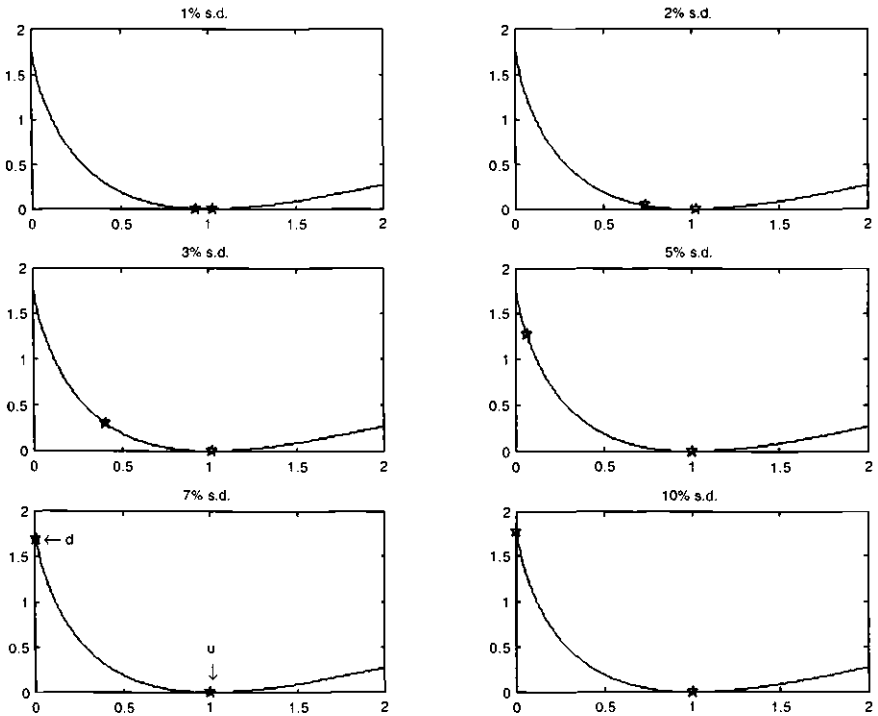


research. Do these results depend on the nature of the shock? Would they differ if government transfers rather than government purchases were considered? Would the findings change for wage markup shocks? This issue is of interest because the consideration of wedge-type shocks is the novelty in BW's analysis. And last, what is the welfare cost of the lack of state-contingency relative to nonfiscal shocks? These questions cannot be addressed within BW's linear-quadratic approach, as I explain below.

2.2 THE LINEAR-QUADRATIC APPROACH

BW solve the optimal policy problem by analyzing the exact solution to a linear-quadratic problem, which should coincide with the solution to a linear approximation of the policy problem for the original economy. This amounts to a local approximation around a nonstochastic steady state. Chari, Christiano, and Kehoe (1995) provide several examples of inaccurate linear approximations in a similar context. They show that the inaccuracy is particularly severe for the computation of policies. In one

Figure 3 THE OUTPUT COST OF RELATIVE PRICE DISTORTIONS

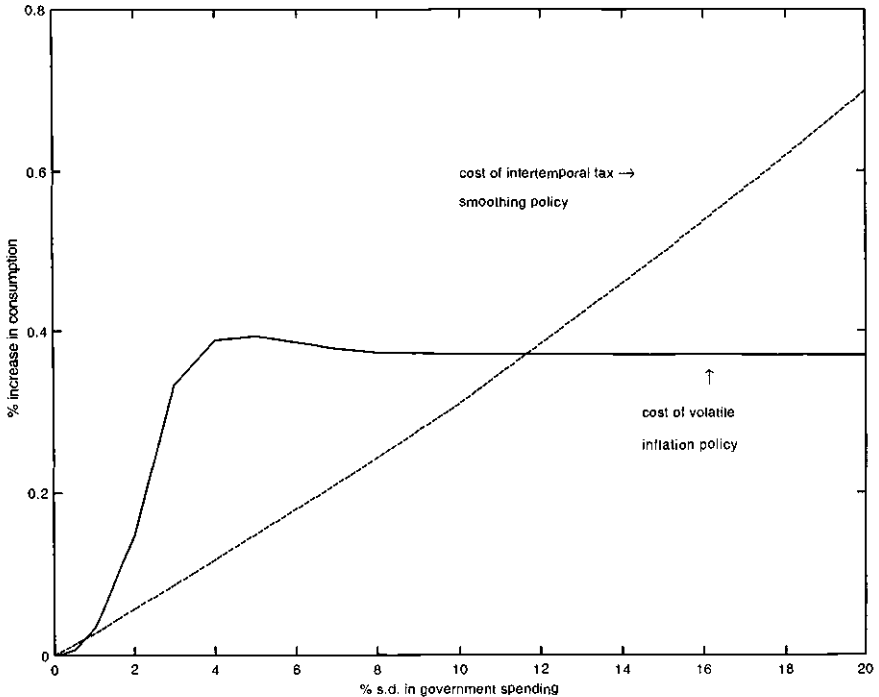


Reproduced from Siu (2001)

example, for which the analytical solution is available, they show that the linear approximation misses on basic statistics such as the mean and the standard deviation of tax rates. The degree of inaccuracy appears to increase with the curvature of preference and technology parameters and with the volatility of driving processes.

An additional and more severe concern arises in the model with sticky prices. Because equilibrium responses are nonstationary and the economy drifts away from the initial steady state permanently in response to shocks, the analysis must be limited to stochastic processes with a small range and with an absorbing state. This is a restrictive assumption for the purpose of studying stabilization policy from a quantitative standpoint. It rules out, for example, analyzing responses to business-cycle type fluctuations, which are naturally of interest in macroeconomics. More important, it raises the question of which steady state should be considered as a benchmark for the approximation. Aiyagari, Marcet, Sargent, and Seppala (2002) numerically characterize the Ramsey equilibrium for a real economy with risk-free debt, where the lack of state contingency of government

Figure 4 THE TRADE-OFF BETWEEN INFLATION AND TAX VOLATILITY



Reproduced from Siu (2001)

debt returns also imparts a unit root behavior to taxes and real variables.<sup>11</sup> They show that if the exogenous shock has an absorbing state, the analogue of  $\phi_{2,t}$  converges to a value that depends on the realization of the path for the exogenous shocks when the economy enters the absorbing state. The incomplete markets allocation coincides with the complete markets allocation that would have occurred under the same shocks but for a different initial debt. They also consider the case in which the government expenditure process does not have an absorbing state. In this case, the analogue of  $\phi_{2,t}$  converges to 0. Hence, the incomplete markets allocation converges to the first best allocation, and no distortionary taxes need to be raised without upper bounds on government asset accumulation.

### 3. *Timeless Perspective and Limited Commitment*

BW characterize the solution to the policy problem from a timeless perspective. This approach amounts to a particular recursive formulation of the optimal policy problem under commitment. As is well known, the Ramsey problem is not recursive in the natural state variables, which complicates the analysis substantially in the presence of stochastic shocks. It is possible, however, to formulate the Ramsey problem recursively by augmenting the set of natural state variables with a vector of costate variables, which depend on the specific problem. Solving this recursive problem gives rise to policy rules that are Markovian in the augmented set of states and the shocks for  $t \geq 1$ . This method was first suggested by Kydland and Prescott (1980) and was generalized by Marcet and Marimon (1999). Aiyagari, Marcet, Sargent, and Seppala (2002) and Siu (2001) also adopt a variant of this approach. The Ramsey equilibrium outcome depends on the values of exogenous state variables at time 0. There are different ways to deal with this dependence. The timeless perspective proceeds by suggesting that the Markovian policy rule, which is optimal from the standpoint of  $t \geq 1$ , is also optimal at time 0. This amounts to endogenizing the initial values of the exogenous states.

To see how this works in practice, it is useful to work through a simple example. Government policy is given by  $\Pi_t = \{\tau_t, R_t\}$ , where  $\tau_t$  is a linear tax on labor and  $R_t$  is the state contingent bond return. Government consumption,  $g_t$ , is exogenous. Consumers solve the problem:

$$\max_{\{c_t, n_t, b_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \text{ s.t.} \\ b_{t+1} \leq b_t R_t + (1 - \tau_t) n_t - c_t$$

11. Siu (2001) derives the constraint imposed on the set of attainable equilibria with sticky prices and shows that it is of the same nature as the one arising in the real economy with risk-free debt analyzed by Aiyagari, Marcet, Sargent, and Seppala (2002).

where  $b_t$  denotes holdings of government-issued bonds at time  $t$ . Their first-order conditions are:

$$\begin{aligned} u_{c,t} &= \lambda_t \\ -u_{n,t} &= \lambda_t(1 - \lambda\tau_t) \\ \lambda_t &= \beta\lambda_{t+1}R_{t+1} \end{aligned}$$

where  $\lambda_t$  is the multiplier on their budget constraint. A competitive equilibrium is a policy  $\{\tau_t, R_t, g_t\}$  and an allocation  $\{c_t, n_t, b_{t+1}\}_{t \geq 0}$  in which allocation solves the consumer's problem given the policy, and the government budget constraint:

$$b_{t+1} + \tau_t n_t = b_t R_t + g_t$$

is satisfied. A Ramsey equilibrium is a competitive equilibrium that maximizes the representative consumer's lifetime utility.

The solution to the household problem clearly displays the potential for time inconsistency in the Ramsey problem: the fact that households have to choose  $b_{t+1}$  based on expectations of  $R_{t+1}$ ; hence, the government might have an incentive to change  $R_{t+1}$  at  $t + 1$ . This time inconsistency makes the Ramsey problem nonrecursive in  $b_t$ . Despite this, it is possible to formulate the Ramsey problem recursively. For  $t > 0$ , the consumer's first-order conditions can be used to define a mapping from policy at time  $t$  to the competitive equilibrium allocation at time  $t$  and the shadow value of outstanding wealth,  $\lambda_t$ :

$$\begin{aligned} x_t &\equiv (c_t, n_t, b_{t+1}) = d(b_t, \Pi_t, \lambda_t) \\ \lambda_t &= h(\Pi_t, b_t, \lambda_{t-1}) \end{aligned}$$

The policy problem can then be rewritten as follows:

$$\begin{aligned} v(b, \lambda_{-1}) &= \max_{\phi, x, \lambda} \{u(c, n) + \beta v(b', \lambda)\} \\ \text{s.t.} \quad x &= d(b, \Pi, \lambda) \\ \lambda &= h(\Pi, b, \lambda_{-1}) \\ g &\leq \tau n + b' - Rb \end{aligned}$$

The solution to this problem is the function,  $\hat{\Pi}; (b, \lambda_{-1})$ , which represents a Markovian policy rule, in the state  $(b, \lambda_{-1})$ . The constraint  $\lambda = h(\phi, b, \lambda_{-1})$  embeds the assumption of commitment because it ties today's choices to decisions made in the past by linking them through the costate variable  $\lambda$ . It is unusual because it goes back in time. The government solution to this problem selects the value of  $\lambda$  that the government wants to commit to

because it induces future governments to choose the policy that is optimal from the standpoint of the current period.

Clearly, this procedure does not pin down the value of  $\lambda_0$ . This implies that the policy problem at time 0 is different from other periods. The policy problem at time 0 is:

$$\begin{aligned} \Pi_0(b_0) &= \arg \max_{\phi_0, x_0, \lambda_0} u(c_0, n_0) + \beta v(b_1, \lambda_0) \\ \text{s.t. } x_0 &= d(b_0, \Pi_0, \lambda_0) \\ g &\leq \tau_0 n_0 + b_1 - R_0 b_0 \end{aligned}$$

The policies chosen at at time 0 depend on initial conditions and influence the solution for all future periods. Adopting a timeless perspective involves removing this dependence on initial conditions by leave as is that the choice of  $\lambda_0$  and of policy at time 0 is governed by the same Markovian rule that is optimal from time 1 onward. The system:

$$\begin{aligned} \Pi_0(b_0) &= \hat{\Pi}(b_0, \lambda_{(-1)}) \\ \lambda_0 &= h(\Pi_0, b_0, \lambda_{(-1)}) \end{aligned}$$

defines an implicit equation for  $\lambda_{(-1)}$  and  $\lambda_0$ . The second constraint pins down  $\lambda_{(-1)}$  as a function of  $b_0$  and  $\lambda_0$ , and the first constraint, which imposes consistency between the solution to the time 0 problem and the Markovian decision rule, pins down  $b_0$ . This procedure affects only the average level of taxes and does not alter the stochastic properties of the optimal policy. It is important to note that the recursive formulation of Ramsey problems discussed here does not imply that the resulting optimal choices are time consistent, even if it gives rise to Markovian policy rules. A government choosing policies sequentially under discretion would not make these choices. The discretionary solution would generate Markovian policy rules in the natural state, in this case,  $b_t$ .

This approach is appealing not only because it provides a tractable algorithm for solving Ramsey equilibria but also because it is related to a notion of limited commitment. Time inconsistency may arise in Ramsey models because private agents take certain actions before the government chooses policy and therefore must base their decisions on expectations of government policy. To ensure that the Ramsey equilibrium is implemented, it is not required that the government commits at time 0 to the entire path of future policy. A limited one-period-ahead commitment to those policies that influence expectations is, in general, enough. A recursive formulation of the Ramsey problem naturally identifies the minimum set of variables that the government must commit to. A drawback is that these variables are not primitive. They are rather complex functions

related to the shadow value of government surpluses at the start of the following period. Hence, implementation of this solution with a simple strategy is incredibly valuable.

BW suggest that the optimal policy can be implemented with the flexible inflation target:

$$\pi_t + a\pi_{t-1} + b(y_t - y_{t-1}) = 0 \quad (3)$$

$$E_t \pi_{t+1} = 0 \quad (4)$$

They also propose a particular institutional arrangement associated with this rule. The monetary authority should have a mandate over both inflation and output stabilization. It should set interest rates so that equation (3) is met. The fiscal authority should have a mandate defined over inflation stabilization only and should set the path of debt so that equation (4) is met.

This institutional setup does relate to those proposed and implemented to tackle the potential time inconsistency problem in monetary policy.<sup>12</sup> It embeds a notion of independence because the monetary authority takes the path of government debt as given, and the fiscal authority takes the path of output as given. However, endowing the fiscal authority with a mandate over inflation stabilization seems rather unusual. One also wonders whether it would be viable because of the strong prevalence of political considerations in the debate over fiscal policy.

#### 4. Conclusion

The Ramsey literature and the optimal monetary stabilization literature have two important elements in common: the assumption of commitment and the auxiliary role of monetary policy. Woodford (2000) and Clarida, Gali, and Gertler (1999) have underlined how the optimal monetary stabilization policy under commitment differs in terms of stochastic responses from the policy under discretion. Woodford shows that inflation tends to overreact to cost-push shocks under discretion relative to commitment. This stabilization bias can arise even when no inflation bias is present. (By inflation bias, I mean a tendency for average inflation to be higher when the government cannot commit.) The stabilization bias arises due to the lack of alternative policy instruments (because lump-sum taxes cannot remove the distortions generated by cost-push shocks). Albanesi,

12. The central bank of New Zealand and most recently the central bank of Brazil follow a flexible inflation targeting scheme.

Chari, and Christiano (2002) show in a general equilibrium model with monetary frictions that the resource misallocation due to price dispersion may be large enough, with plausible parameters, to eliminate the inflation bias under discretion. They also show that multiple Markov equilibria are possible, giving rise to a potential for alternating high- and low-inflation regimes under discretion. It would be interesting to explore the joint role of monetary policies and distortionary taxation without commitment. Would the possibility of responding to cost-push shocks via fiscal policy remove the overreaction in inflation that occurs with lump-sum taxes? What is optimal inflation volatility in response to fiscal shocks under discretion? Is the misallocation resulting from the price dispersion associated with inflation sufficient to reduce the inflation bias in general?

The auxiliary role for monetary policy in these branches of the literature stems from the fact that, in these models, money is not essential. The optimal monetary stabilization literature often abstracts from money demand altogether. The Ramsey literature usually considers money in the utility function or cash-in-advance models. These assumptions are meant to stand in for some role for money that is not made explicit but ought to be. Instead, money is essential when spatial, temporal, and informational friction makes the use of money an efficient arrangement, as in the search-theoretic approach pioneered by Kiyotaki and Wright (1989, 1993).<sup>13</sup> The essential nature of money has implications for optimal monetary policy. For example, optimality of the Friedman rule generally occurs in Ramsey models. This result stems from the fact that money does not overcome any primitive friction, and agents use it for transactions because they are forced to. Hence, it is optimal to equate the return on money to that of other assets to minimize the distortions associated with this arrangement.<sup>14</sup> In environments where money is essential, the optimal monetary policy responds to changes in the distribution of liquidity needs. Levine (1991) and Kocherlakota (2003) show that, in this case, higher interest rates increase welfare.<sup>15</sup> Research on the properties of optimal monetary policy when money is essential is still in its infancy; however, this class of environments constitute the most natural laboratory for understanding the effect of monetary policy on the economy.

13. Additional contributions include Shi (1995), Trejos and Wright (1995), Kocherlakota (1998), and Wallace (2001).

14. I consider (in Albanesi, 2002) a costly nonmonetary transactions model with heterogeneous agents in which departures from the Friedman rule redistribute toward high-income households and the Friedman rule may not be optimal.

15. See also Woodford (1990) for some related examples. Lagos and Wright (2002) show that optimality of the Friedman rule occurs in an environment where money is essential. In their model, however, the distribution of currency is degenerate.

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## Comment

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### 1. Introduction

The paper by Benigno and Woodford makes an important contribution to the theory of cyclical fiscal and monetary policies.

Following the tradition of Ramsey (1927), Barro (1979), and Lucas and Stokey (1983), the neoclassical literature on optimal fiscal policy has emphasized that, when taxation is distortionary, welfare is maximized if the government smoothes taxes across different periods of time and different realizations of uncertainty. To what extent, however, such smoothing is possible depends on the ability of the government to transfer budget resources from one date and state to another. If the government can trade a complete set of Arrow securities (or state-contingent debt), perfect smoothing across all dates and states is possible, implying that the

Prepared for the NBER Macroeconomics Annual Conference. I thank Oliver Blanchard and Ivan Werning for helpful comments and stimulating discussions.

optimal tax rate is essentially invariant (Lucas and Stokey, 1983; Chari, Christiano, and Kehoe, 1991). If instead insurance is unavailable, any innovation in fiscal conditions needs to be spread over time, implying that the optimal tax rate follows essentially a random walk (Barro, 1979; Aiyagari et al., 2002).

When the government cannot trade state-contingent debt, there might be other ways to obtain insurance. Bohn (1990) and Chari, Christiano, and Kehoe (1991) have argued that, when the government trades nominal bonds, unexpected variation in inflation may generate all the desirable variation in the real value of the outstanding public debt and may therefore replicate state-contingent debt. A serious caveat with this argument, however, is that it considers a world where prices are perfectly flexible and price volatility has no welfare consequences.

But when nominal prices are sticky, unexpected variation in the aggregate level of prices creates distortions in the allocation of resources and reduces welfare. The new Keynesian literature on optimal monetary policy has therefore stressed the importance of minimizing price volatility to minimize inefficiencies in the cross-sectoral allocation of resources.<sup>1</sup> Recent work by Schmitt-Grohé and Uribe (2001) and Siu (2001) shows that the conflict between insurance and price stability is likely to be resolved overwhelmingly in favor of the latter.<sup>2</sup> At the same time, the new Keynesian literature has noted that fiscal policy could, in principle, help stabilize output by offsetting cyclical variation in monopolistic distortions (price or wage markups), but has bypassed this possibility and instead focused on monetary policy.

The paper by Pierpaolo Benigno and Michael Woodford merges the new Keynesian paradigm of optimal monetary policy with the neoclassical paradigm of optimal fiscal policy. It examines the joint determination of optimal fiscal and monetary policy in the presence of incomplete insurance and sticky prices. Furthermore, it shows how one can start from a full-fledged micro-founded model and, through a long series of approximations, end with a simple linear-quadratic framework similar to the ad-hoc specifications used in the early contributions to both fiscal and monetary policy.

The welfare costs of business cycles in economies with sticky prices and incomplete markets and the consequent stabilization role of fiscal and monetary policy are important questions. The paper by Pierpaolo Benigno and Michael Woodford makes an important contribution in this

1. See, for example, Clarida et al. (1999), the excellent textbook by Woodford (2003), and the references therein.

2. This result is also verified by the findings of Benigno and Woodford.

direction. I have one concern, however, regarding the strategy of the paper. It is important to know that ad-hoc representations of the policy problem can be backed up by proper micro-foundations, but in their paper this comes at the cost of numerous linear-quadratic approximations, which I find hard to follow. The reduced-form analytic representation is hard to interpret. For example, all impulse responses are found to depend critically on a composite exogenous variable that the authors call "fiscal stress," but what exactly this variable is remains a mystery.

In the present discussion, I will attempt a simpler route. I will set up an ad-hoc framework from the very beginning. This will permit us to derive the essential results with less effort and more clarity. We will see that, when the government has access to either lump-sum taxation or complete insurance, the inflation rate is always zero, the output gap is always constant, and output stabilization is obtained only via fiscal policy. When instead there is incomplete insurance, the output gap has a unit root, like the tax rate and the level of government debt, and monetary policy complements fiscal policy in stabilizing the economy. Innovations in the inflation rate, the output gap, or the tax rate are driven by innovations in an exogenous fiscal stress variable, which simply measures the annuity value of government spending plus the subsidy that would have been necessary to implement the first best.

## 2. *Optimal Fiscal and Monetary Policy: A Simple Model*

### 2.1 SOCIAL WELFARE

We can approximate social welfare around the first-best outcome as:

$$u = - \sum_{t=0}^{\infty} \beta^t E_t \left[ (y_t - y_t^*)^2 + \omega \cdot \pi_t^2 \right] \quad (1)$$

where  $y_t^*$  is an exogenous random variable representing the efficient (or first-best) level of output, whereas  $y_t$  and  $\pi_t$  are the endogenous actual levels of output and inflation. The last term in equation (1) reflects the welfare loss associated with the distortion in the cross-sectoral allocation of resources cause by a higher dispersion of prices.<sup>3</sup> The scalar  $\omega \geq 0$  depends on how flexible prices are. If  $1 - \alpha$  is the probability that a firm can adjust prices in any given period, so that  $\alpha$  measures the degree of price stickiness, then  $\omega = \omega(\alpha)$  is increasing in  $\alpha$ ; flexible prices correspond to  $\alpha = 0$  and  $\omega = 0$ .

3. The implicit assumption is that the first-best level of inflation is zero.

## 2.2 MARKET EQUILIBRIUM

We can similarly summarize the market equilibrium with the following condition characterizing the equilibrium level of output:

$$y_t = -\psi\tau_t + \chi(\pi_t - \beta E_t \pi_{t+1}) + \varepsilon_t \tag{2}$$

where  $\psi > 0$  and  $\chi \geq 0$ . The first term in equation (2) reflects the distortion of the tax on final output or the sale of intermediate goods. More generally, the first term can be interpreted as aggregate demand management via fiscal policy. The second term reflects the output effect of monetary policy when prices are sticky. The slope  $\chi = \chi(\alpha)$  is increasing in  $\alpha$ , the degree of price stickiness.<sup>4</sup> Provided  $\chi > 0$ , equation (2) gives the new Keynesian Phillips curve:

$$\pi_t = \beta E_t \pi_{t+1} + \frac{1}{\chi}(y_t - y_t^n)$$

where  $y_t^n \equiv -\psi\tau_t + \varepsilon_t$  represents the natural level of output. Finally, the exogenous random variable  $\varepsilon_t$  captures what the literature has called a cost-push shock (e.g., Clarida et al. 1999). As will become clear, variation in  $\varepsilon_t$  is isomorphic to variation in  $y_t^n$ ; either one reflects variation in markups and other shorts or distortions, namely, shocks that affect the natural level of output differently from the efficient level.

## 2.3 THE GOVERNMENT BUDGET

Suppose the government trades only one-period discount bonds, both real and nominal. For simplicity, I will ignore seigniorage and the fiscal effect of variations in either the growth rate of output or the real interest rate. I will also assume that the government freely adjusts the level of real bond issues but keeps the level of nominal bond issues constant at some level  $\bar{d}$  (as a fraction of gross domestic product [GDP]), and treats  $\bar{d}$  as a parameter. The government budget then reduces to the following equation:<sup>5</sup>

4. Provided  $\chi > 0$ , equation (2) gives the new Keynesian Phillips curve:

$$\pi_t = \beta E_t \pi_{t+1} + \frac{1}{\chi}(y_t - y_t^n),$$

where  $y_t^n \equiv -\psi\tau_t + \varepsilon_t$ , represents the natural level of output.

5. To see this, let  $v_t$  and  $D_t$  denote the quantity of real and nominal bonds issued in the end of period  $t$  (as a fraction of GDP) and write the government budget as:

$$\left(v_t - 1 + \frac{D_t - 1}{P_t}\right) + g_t = \tau_t + z_t + \left(\frac{1}{1 + r_t} v_t + \frac{1}{1 + R_t} \frac{D_t}{P_t}\right)$$

The first term represents the total real liabilities of the government in the beginning of period  $t$ , while the last term represents the revenue from the issue of new bonds.  $P_t$  is the price level,  $r_t$  is the real interest rate, and  $R_t$  is the nominal interest rate. Next, let  $1/P_t \approx (1 - \pi_t)/P_{t-1}$ ,  $(1 + r_t)^{-1} \approx \beta$ ,  $(1 + R_t)^{-1} \approx \beta(1 - E_t \pi_{t+1})$ , and  $D_{t-1}/P_{t-1} = D_t/P_t = \bar{d}$ ; and define  $b_t \equiv v_t + D_t/P_t$ . The budget constraint then reduces to equation (3).

$$b_{t-1} = [\tau_t + z_t + \bar{d}(\pi_t - \beta E_t \pi_{t+1}) - g_t] + \beta b_t \quad (3)$$

The term  $b_t$  denotes the total level of public debt (as a fraction of GDP), and  $\tau_t$  denotes the tax rate on aggregate income. The initial value of debt is  $b_{-1} = \bar{b}$ . The term  $g_t$  denotes the level of government spending (also as a fraction of GDP) and follows a stationary Markov process with mean  $E_t g_t = \bar{g}$ . The term  $\bar{d}(\pi_t - \beta E_t \pi_{t+1})$  captures the gains from unexpected deflation of nominal debt. Finally,  $z_t$  captures any state-contingent lump-sum transfers the government potentially receives from the private sector. These may reflect either direct lump-sum taxation or various explicit and implicit kinds of insurance (other than the inflation of nominal debt). I will later distinguish three cases: (1) unrestricted lump-sum taxation, in which case  $z_t$  is a free control variable; (2) no lump-sum taxation but complete insurance, in which case  $z_t$  has to satisfy only the constraint  $E_{t-1} z_t = 0$ ; (3) no lump-sum taxation and no insurance, in which case  $z_t = 0$  in all periods and events.

## 2.4 THE RAMSEY PROBLEM

The government seeks to maximize social welfare subject to its budget constraint and the equilibrium condition for aggregate economic activity. Hence, the Ramsey problem is given by:

$$\min E_0 \sum_{t=0}^{\infty} \beta^t \left[ (y_t - y_t^*)^2 + \omega \pi_t^2 \right] \quad (4)$$

$$\text{s.t. } y_t = -\psi \tau_t + \chi (\pi_t - \beta E_t \pi_{t+1}) + \varepsilon_t$$

$$b_{t-1} - \beta b_t = \tau_t + z_t - g_t + \bar{d}(\pi_t - \beta E_t \pi_{t+1})$$

The Lagrangian of this problem can be written as  $\mathcal{L} = \sum_{t=0}^{\infty} \beta^t E_t [L_t]$ , where:

$$L_t \equiv \frac{1}{2} \left[ (y_t - y_t^*)^2 + \omega \pi_t^2 \right] - \mu_t \left[ \chi (\pi_t - \beta \pi_{t+1}) - (y_t + \psi \tau_t + \varepsilon_t) \right] + \lambda_t \left[ (b_{t-1} - \beta b_t) - (\tau_t - g_t + z_t) - \bar{d}(\pi_t - \beta \pi_{t+1}) \right] \quad (5)$$

$\mu_t \geq 0$  represents the shadow value of real resources and  $\lambda_t \geq 0$  represents the shadow cost of the government budget.<sup>6</sup> Taking the First order conditions (FOCs) with respect to  $y_t$ ,  $\pi_t$ ,  $b_t$ , and  $\tau_t$ , and using the last one to substitute for  $\mu_t$ , we conclude to the following optimality conditions.<sup>7</sup>

6. Note that the exogenous disturbances of the economy are given by  $s_t = (y_t^*, \varepsilon_t, g_t)$  and the endogenous variables  $(\pi_t, \tau_t, y_t, b_t)$  are contingent on  $s^t = (s_0, \dots, s_t)$ . Along the optimal plan, however, the history in the beginning of period  $t$  can be summarized by  $(\mu_{t-1}, \lambda_{t-1})$ . See Marcet and Marimon (2001).

7. To be precise, equation (7) holds for  $t \geq 1$ . Period  $t = 0$  is special for the usual reason, namely, that expectations formed in the past are now sunk.

$$y_t - y_t^* = -\frac{1}{\bar{\psi}} \lambda_t \tag{6}$$

$$\pi_t = \frac{1}{\bar{\omega}} \left( \frac{\chi}{\bar{\psi}} + \bar{d} \right) (\lambda_t - \lambda_{t-1}) \tag{7}$$

$$\lambda_t = E_t \lambda_{t+1} \tag{8}$$

These conditions, together with the equilibrium output condition in equation (2), the budget constraint in equation (3), and the initial condition  $b_{-1} = \bar{b}$ , pin down the optimal policy plan.

Note that equations (7) and (8) imply  $E_t \pi_{t+1} = 0$ . Note also that both the optimal output gap and the optimal inflation rate are determined merely by the shadow cost of government budget resources (the multiplier  $\lambda_t$ ). Finally, equation (8) states that the shadow cost of government budget resources follows a random walk. This property reflects intertemporal smoothing, which is possible as long as the government can freely borrow and lend in riskless bonds. How large is the variance of the innovation in  $\lambda_t$  depends critically on how much insurance the government may obtain against the fiscal consequences of business cycles.

### 2.5 THE FIRST BEST

Suppose for a moment that the government had unlimited access to lump-sum taxation. This means that the government can freely choose  $z_t$ . The FOC, with respect to  $z_t$ , implies  $\lambda_t = 0$ , for all periods and events. That is, the shadow cost of the government budget is always zero, reflecting simply the fact that there is unrestricted lump-sum taxation. It follows that:

$$y_t - y_t^* = 0 \quad \text{and} \quad \pi_t = 0 \tag{9}$$

meaning that there is complete output and price stabilization exactly at the first-best levels.

The first-best outcome is implemented by setting the tax rate so that the aggregate supply condition is satisfied at the efficient level of output with zero inflation. This gives:

$$\tau_t = \tau_t^* \equiv -\frac{1}{\bar{\psi}} (y_t^* + \varepsilon_t) \tag{10}$$

The sum  $y_t^* + \varepsilon_t$  measures the overall distortion in the economy due to monopolistic competition or other market imperfections, and  $\tau_t^*$  represents the Pigou tax (or subsidy) that corrects any such distortion and implements the first-best outcome. (In the case of monopolistic distortions,

output is inefficiently low, and therefore  $\tau_t^* < 0$ , meaning that the government uses a subsidy to offset the monopolistic distortions.) Finally, to balance the government budget, we can pick the level of lump-sum taxes so that they are enough to finance the level of government spending, plus the interest payments on the initial public debt, plus the subsidy that implements the first-best level of output. That is, we let  $z_t = (g_t - \tau_t^*) + (1 - \beta)\bar{b}$ .

## 2.6 OPTIMAL POLICY WITH COMPLETE MARKETS

Suppose now that lump-sum taxation is not available, but the government can issue state-contingent debt (or otherwise replicate full insurance). The government chooses  $z_t$  subject to the constraint  $E_{t-1} z_t = 0$ . The FOCs with respect to  $z_t$ , together with equation (8), now imply  $\lambda_t = \bar{\lambda} > 0$  for all periods and events. That is, the shadow value of tax revenues is positive (because taxation is distortionary) but constant across all periods and events (because markets are complete). It follows that:

$$y_t - y_t^* = -\frac{1}{\Psi} \bar{\lambda} < 0 \quad \text{and} \quad \pi_t = 0 \quad (11)$$

This outcome is now obtained by setting:

$$\tau_t = \frac{1}{\Psi^2} \bar{\lambda} + \tau_t^* \quad (12)$$

where  $\tau_t^* \equiv -\frac{1}{\Psi} (y_t^* + \varepsilon_t)$  is again the Pigou tax that would implement the first best, and letting

$$z_t = (g_t - \tau_t) - (\bar{g} - \bar{\tau}) \quad (13)$$

where  $\bar{g} \equiv E g_t$  and  $\bar{\tau} \equiv E \tau_t$ . That is, variation in  $z_t$  absorbs any business-cycle variation in either the level of government spending or the subsidy that implements the first-best level of output. Finally, to compute  $\bar{\lambda}$ , note that the government budget clears if and only if  $\bar{\tau} = \bar{g} + (1 - \beta)\bar{b}$ , which together with equation (12) implies:

$$\bar{\lambda} = \Psi^2 [(1 - \beta)\bar{b} + (\bar{g} - \bar{\tau}^*)] \quad (14)$$

That is, the (constant) shadow cost of budget resources is proportional to the interest cost of public debt, plus the annuity value of government spending, plus the annuity value of the subsidy that would be necessary to implement the first best. It follows that the (constant) output gap is higher the higher the initial level of public debt, the higher the average level of government spending, or the higher the monopolistic distortion in the economy. Finally, substituting equation (14) in equation (12), we infer:

$$\tau_t = \bar{\tau} + (\tau_t^* - \bar{\tau}^*) \tag{15}$$

where  $\bar{\tau} = \bar{g} + (1 - \beta) \bar{b}$ . Note that  $\bar{\tau}$  corresponds to the optimal tax rate in a neoclassical economy, such as in Barro (1979) or Lucas and Stokey (1983). In the presence of a Keynesian business cycle, the optimal tax rate inherits in addition a cyclical component, the latter being the cyclical variation in the Pigou subsidy that would have implemented the first-best level of output. Fiscal policy can thus eliminate the inefficient business cycle by simply offsetting the cyclical variation in the monopolistic (or other) distortion.

To see how fiscal policy works under complete markets, consider a negative shock in the output gap (a shock that reduces the natural rate of output more than the first-best level). The government can offset this shock and fully stabilize the output gap by simply lowering the rate of taxation while keeping the price level constant. This policy leads to a primary deficit, but the latter is totally covered by an increase in state-contingent transfers. Hence, the government does not need to issue any new public debt, and the stabilization policy has no fiscal consequences for the future.

### 2.7 OPTIMAL POLICY WITH INCOMPLETE MARKETS

Finally, consider the case that the government cannot obtain any insurance. It is useful to define the variable:

$$f_t \equiv (1 - \beta) \sum_{j=0}^{\infty} \beta^j E_t (g_{t+j} - \tau_{t+j}^*) \tag{16}$$

which measures the annuity value of government spending plus the annuity value of the subsidy that is necessary to implement the first best, and let  $\xi_t \equiv f_t - E_{t-1} f_t$  denote the innovation in this variable. (The term  $f_t$  corresponds to the mysterious object that Benigno and Woodford call the fiscal stress variable.) After some tedious algebra, we can show that the shadow value of budget resources satisfies:

$$\lambda_{t-1} = \psi^2 [(1 - \beta) b_{t-1} + E_{t-1} f_t] \quad \text{and} \quad \lambda_t - \lambda_{t-1} = \eta \xi_t \tag{17}$$

for some constant  $\eta > 0$ . That is, the shadow cost of budget resources is proportional to the interest cost of public debt, plus the annuity value of government spending, plus the annuity value of the subsidy that would be necessary to implement the first best; the innovation in the shadow cost of budget resources is proportional to the innovation in the fiscal stress variable  $f_t$ . It follows that any transitory change in  $f_t$  results in a permanent change in  $\lambda_t$ , which manifests the effect of intertemporal tax



smoothing. Finally, using equation (17) together with the equations (6) to (8), we conclude with the following impulse-response functions:

$$\pi_t = \varphi_\pi \xi_t, \quad (18)$$

$$y_t - y_{t-1} = (y_t^* - y_{t-1}^*) - \varphi_y \xi_t \quad (19)$$

$$\tau_t - \tau_{t-1} = (\tau_t^* - \tau_{t-1}^*) + \varphi_\tau \xi_t \quad (20)$$

for some constant  $\varphi_\pi, \varphi_y, \varphi_\tau > 0$ . It follows that inflation is white noise.<sup>8</sup> The output gap and the tax rate, however, follow a martingale plus a stationary component, which is proportional to the change in the output gap (that is, the distance from the first best). This cyclical component of optimal fiscal and monetary policy is absent in the neoclassical paradigm (Barro, 1979; Lucas and Stokey, 1983) and arises here because cyclical variation exists in the extent of distortions in the economy. Finally, the coefficients  $\varphi_\pi$ ,  $\varphi_y$ , and  $\varphi_\tau$  are decreasing in  $\bar{d}$ , reflecting the fact that a higher level of nominal debt permits the government to obtain more insurance with less inflation volatility.

To see how fiscal policy works under incomplete markets, consider a negative shock in the output gap (a shock that reduces the natural rate of output more than the first-best rate). Contrary to what was the case with complete markets, the government cannot fully stabilize the output gap and keep the price level constant at the same time. Because complete insurance is no longer available, lowering the contemporaneous rate of taxation necessarily results in a primary deficit that has to be financed by an increase in public debt and thus an increase in future taxes. The government thus finds it optimal to lower the tax rate by less than what it would have done under complete markets, that is, by less than what is necessary to offset the negative cyclical shock. And because fiscal policy can no longer do it all, it becomes optimal to use monetary policy also for the purpose of output stabilization. Actually, an unexpected increase in inflation not only stimulates aggregate demand but also lowers the real value of nominal public debt and thus eases fiscal conditions. Nonetheless, monetary policy cannot do it all either. Because inflation surprises distort the cross-sectoral allocation of resources, the government finds it optimal to raise inflation by an amount less than what would be necessary to stabilize output fully and cover the primary deficit. Overall,

8. Note that the white-noise result for inflation is not robust to the introduction of lags in fiscal policy. The martingale property for fiscal policy and the output gap, however, is likely to be robust to more general frameworks.

both fiscal and monetary policy are now used to stabilize output, but the negative cyclical shock is only partly offset and results in a permanent increase in the level of public debt and thereby in a permanent increase in taxes and a permanent reduction in output.

### 3. Conclusion

I conclude with some important (at least in my view) open questions about fiscal policy over the business cycle.

First, the theory suggests that there are important welfare gains to be made if the government traded state-contingent debt (or at least gains of the same order of magnitude as the gains from eliminating business cycles). And if state-contingent debt is not available, our models predict that the government could obtain insurance by appropriately designing the maturity structure of public debt (Angeletos, 2002) or the cyclical properties of consumption taxes (Correia et al., 2002). Similarly, the government could replicate more insurance with less cyclical variation in inflation by issuing a lot of nominal debt and at the same time investing in real assets to keep the overall level of public debt at the desired level. Yet none of these forms of insurance appear to play an important role in practice. Why not?<sup>9</sup>

Second, the implications of incomplete insurance become even more interesting once we abandon the simple linear-quadratic framework. If the lack of insurance is due to exogenous reasons, then a precautionary motive dictates that the government should accumulate a large amount of assets to use it as a buffer stock against cyclical shocks (Aiyagari et al., 2002). If instead the lack of insurance is due to the government's own moral hazard, then a desire to minimize the costs of providing future governments with optimal incentives dictates that the government should accumulate a large amount of debt (Sleet, 2002). But which of the two opposite predictions should we follow?

Third, consider the comparison of fiscal and monetary policy as instruments for managing the business cycle. The simple model presented here, the more elaborate models of Correia et al. (2002) or Benigno and Woodford, and probably any model we teach our graduate students share the prediction that fiscal policy cannot do all (if markets are complete) or most (if markets are incomplete) of the job of stabilizing the economy. One could even argue that fiscal policy is superior to monetary policy in

9. Moral hazard in government behavior is only part of the answer: If it were severe enough to explain complete lack of insurance, one would also expect the government customarily to default on (domestic) public debt, which is not the case in reality.

stabilizing the economy because its effectiveness does not depend on the extent of nominal rigidities. In practice, however, fiscal policy is quickly dismissed on the basis that it takes time to implement changes in fiscal policy and even more time for these changes to have an effect on economic activity. But where is the hard proof for this? And even if there are important lags involved in discretionary cyclical fiscal policy, why don't we undertake the necessary reforms to reduce them, or why don't we redesign the existing automatic stabilizers to implement the optimal cyclical variation in fiscal policy? Or why should a systematic fiscal policy rule have a weaker and slower impact on market incentives than a systematic monetary policy rule? Similarly, cyclical fiscal policy may have a differential impact on different sectors of the economy, but this is equally true for monetary policy. I am not totally convinced that monetary policy is intrinsically more effective as an instrument for managing the business cycle, I believe that we should carefully investigate the alleged asymmetries between fiscal and monetary policies, and I wonder if it is mostly a historical coincidence that economists and policymakers alike have been obsessed with monetary policy.<sup>10</sup>

Finally, consider the nature of the shocks that justify policy intervention. The conventional wisdom is that we should try to stabilize the actual level of output, or the gap between the actual level and some smoother level (the empirically measured natural rate). The theory instead dictates that we should stabilize the gap between the actual and the first-best level of output because it minimizes welfare losses. What is more, the canonical model predicts that productivity and taste shocks move the actual and the first-best level of output proportionally, in which case there is no inefficient business cycle and thus no reason for countercyclical policy.<sup>11</sup> The resolution to this unappealing theoretical prediction has been to introduce ad hoc shocks that perturb directly the gap between the actual and the first-best level of output.<sup>12</sup> It remains an open question what exactly these shocks are, and why they may be highly correlated with the actual level of output, in which case only the conventional wisdom and the common policy practice would be justifiable.

10. For the possibility that sunspots (or self-fulfilling expectations) determine which policy instruments are effective and actively used in equilibrium, see Angeletos, Hellwig, and Pavan (2003).

11. See Woodford (2003) for an extensive analysis of this issue.

12. These shocks are commonly called cost-push shocks, although they may have little to do with real-life cost-push shocks, such as an increase in oil prices, which are bound to affect both the actual and the first-best level of output and may have an ambiguous effect on the level of the distortion in the economy. See Blanchard (2003) for a critical assessment.

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## Discussion

Participants as well as discussants were concerned by the generality of the authors' local linear-quadratic approximation approach. Mike Woodford responded to the discussion of Stefania Albanesi that the advantage of this approach over the approaches of Siu, Schmitt-Grohé, and other authors is that analytic solutions for optimal policies can be obtained, and optimal responses to shocks with general stochastic properties can be

calculated. He noted that the disadvantage of the linear-quadratic approach is that very large shocks cannot be analyzed. Stefania Albanesi noted that the linear-quadratic approximation is thus less likely to be appropriate for analyzing the problems facing developing countries where shocks are larger. Ken Rogoff said that, while on the one hand, analytic results can be useful for building intuition, on the other hand, it may not be possible to answer every important question analytically.

Robert Hall was worried that the results of the paper place a huge weight on the Calvo price adjustment mechanism and the Dixit-Stiglitz-Spence model of imperfect competition. He questioned whether this is how economies really work and whether the results are robust to more general assumptions. Mike Woodford agreed that the results are subject to assumptions about the form of distortions, including the form of sluggish price adjustment. He noted that the aim of the paper is to analyze the effect of sluggish price adjustment on optimal fiscal policy, something that the literature has not investigated before. He also remarked that the framework is quite flexible in allowing various distortions to be added or taken away.

Andrés Velasco was curious about the generality of the model result that debt indexation is undesirable. Mike Woodford responded that when prices are sticky, nominal debt is desirable, but the degree to which unexpected inflation is optimally used to achieve state contingency is limited. Stefania Albanesi elaborated that the cost of not having state-contingent debt depends on the size of shocks. When shocks are of the business-cycle variety, as in the authors' framework, the costs of not having state-contingent debt are small. Marios Angeletos remarked that it would be optimal to have nominal debt but real assets. To add to Stefania Albanesi's point, Angeletos noted that in the authors' exercise, the difference between nominal and indexed debt is small because the optimal volatility of inflation is, in any case, almost zero.

Several participants were curious about the ability of the authors' framework to nest important policy concerns. John Williams asked whether the linear-quadratic framework can allow for realistic distortions such as the nonneutrality of the U.S. tax system with respect to inflation. Mike Woodford responded that this can indeed be integrated into the framework. Mark Gertler remarked that the old-fashioned argument that lags in the implementation of fiscal policy make monetary policy a more appropriate stabilization instrument was not taken account of by the authors' framework. He was curious about the effect on optimal policy of allowing for such. Mike Woodford responded that delays in fiscal policy implementation are indeed realistic and can be included. He said that even with lags in the implementation of fiscal policy, optimal monetary

policy in the case of distortionary taxation is not the same as when all taxes are lump sum because, with distortionary taxation, there is a nonzero shadow value of additional resources to the government that the monetary authority should take into account. On this question, Marios Angeletos noted that in the linear-quadratic framework, thinking of fiscal and monetary policy as independent is actually not a bad approximation. Where shocks are big, however, this is much less likely to be an appropriate assumption.

Ken Rogoff pointed out that political economy concerns were absent from the authors' model, while in the real world, fiscal policy has effectively been dismissed as a tool of stabilization because the degree of commitment available to fiscal authorities is so limited. He was curious about what type of institutional framework the authors envisaged for implementing the optimal fiscal policy. Kjetil Storesletten suggested that the authors do more to confront the time-consistency issue and work out the time-consistent policy for their framework. Mike Woodford responded that this could be done but that, in reality, governments appear to have some ability to accept constraints on fiscal policy, as shown by their reluctance to tax existing capital.

