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8 Monetary Influences on Nominal Income

Chapter 6 explores the hypothesis that the real quantity of money demanded can be regarded as a relatively stable function of a small number of variables. It isolated as the key variables entering into the demand function the degree of financial sophistication, real per capita income, the return on nominal-value assets, the return on physical assets, and two episodic sets of events handled by dummy variables—postwar readjustment and an upward shift in demand. A single function of these variables describes reasonably well the demand for real balances for the whole century our data cover and for both the United States and the United Kingdom, the one significant difference between the two countries being with respect to the income elasticity of demand for real balances. Chapter 7 explores the relations between the United States and the United Kingdom, documenting their roles as part of a unified monetary system. The major channel of influence during the gold standard system that prevailed before 1914 was through the stocks of money in each country. After 1914, when variable exchange rates came to prevail, this channel of influence was supplemented by others operating through exchange rates, interest rates, and perhaps other variables. These chapters give empirical content to "the generalization that changes in desired real balances (in the demand for money) tend to proceed slowly and gradually or to be the result of events set in train by prior changes in supply" (sec. 2.1).

Given a stable demand function and a variable nominal quantity of money, there remains for investigation the way in which changes in the nominal quantity of money work their way through the relatively stable demand function to alter nominal income, prices, interest rates, and output—how, in short, demand and supply interact, the theoretical issue with which much of chapter 2 deals.

For a hypothetical simple quantity theory world in which velocity is a numerical constant, prices are completely flexible, and reactions are instantaneous, the analysis of the interaction of supply and demand is trivial. Prices mirror instantly and perfectly the changes in the quantity of money. Output and the real yield on capital are unaffected, determined by other variables. Yet even in this hypothetical case, the effect of changes in the quantity of money on nominal interest rates depends on anticipated rates of price change and cannot be described without specifying how anticipations are formed.

Alter this hypothetical case in almost any respect and the analysis becomes far from trivial. For example, make velocity a function of nominal interest rates and let interest rates be connected with anticipated rates of price change, and the purely theoretical analysis becomes extremely complex—indeed, there is currently no satisfactory accepted theoretical analysis of this case, though with other specifications intact, output and real interest rates would still be unaffected by monetary changes.[1] Introduce lags in reaction, which is to say price or other rigidities, and output and real interest rates enter into the reaction process.

In principle, all the variables are simultaneously determined, and the effect of a monetary change on any one—say, nominal income—is linked to the way it affects the others—say, prices, output, and interest rates. However, to reduce the empirical problem to manageable scope, we propose to start with a simplified monetary theory of nominal income, which separates the effect of monetary change on nominal income from the effect on prices and output separately, and which treats interest rate changes as themselves traceable to earlier monetary change.[2] In chapter 9, we shall then investigate explicitly the division of changes in nominal income between prices and output, and, in chapter 10, the relation between monetary changes and interest rates.

We rely on the simplified model with some hesitancy, since it was designed to interpret short-term movements, and it is not clear therefore that it is appropriate for our phase-average data. However, that is one of the questions we shall explore.

1. See Fischer Black, "Active and Passive Monetary Policy in a Neo-classical Model," *Journal of Finance* 27 (September 1972): 801–4; Thomas Sargent and Neil Wallace, "The Stability of Models of Money and Growth with Perfect Foresight," *Econometrica* 41 (November 1973): 1043–48; idem, "Rational Expectations and the Dynamics of Hyperinflation," *International Economic Review* 14 (June 1973): 328–50; Thomas Sargent, "The Demand for Money during Hyperinflations under Rational Expectations: I," *International Economic Review* 18 (February 1977): 59–82; B. M. Friedman, "Stability and Rationality in Models of Hyperinflation," *International Economic Review* 19 (February 1978): 45–64; W. A. Brock, "Money and Growth: The Case of Long-Run Perfect Foresight," *International Economic Review* 14 (October 1974): 750–77.

2. See M. Friedman, "A Theoretical Framework," in *Milton Friedman's Monetary Framework*, ed. Gordon, pp. 34–43.

8.1 From the Demand for Balances to the Behavior of Nominal Income

To show how the demand function for real balances can be converted into a relation between nominal income and nominal money supply, let us start with the demand equations at the end of chapter 6 for the United States and the United Kingdom combined and for the full period.

$$\text{(1)} \qquad \log m = -1.47 + 1.64Z + (1.14 - 0.25Z)\log y - 9.3R_N - 0.47g_Y + 0.019W + 0.193S,$$

$$\text{(2)} \qquad g_m = (1.03 - 0.04Z)g_y - 8.7DR_N - 0.41\,Dg_Y + 0.030W_g + 0.004S_g.$$

Subtract log y from both sides of equation (1) and then reverse signs. Since $\log V = \log y - \log m$ (recall that y is per capita real income, m per capita real balances), we have

$$\text{(3)} \qquad \log V = 1.47 - 1.64Z - (0.14 - 0.25Z)\log y + 9.3R_N + 0.47g_Y - 0.019W - 0.193S.$$

Replace log V by its equivalent expression, $\log Y - \log M$, and transfer log M to the right-hand side. This gives

$$\text{(4)} \qquad \log Y = 1.47 - 1.64Z + \log M - (0.14 - 0.25Z)\log y + 9.3R_N + 0.47g_Y - 0.019W - 0.193S.$$

This expression now gives the level of nominal income consistent, for various values of other variables, with equality between actual and desired quantity of money if the desired level is determined by the demand equation defined by equation (1).

Similarly, subtract g_y from both sides of equation (2), reverse signs, replace g_V by $g_Y - g_M$, and we have

$$\text{(5)} \qquad g_Y = g_M - (0.03 - 0.04Z)g_y + 8.7\,DR_N + 0.41\,Dg_Y - 0.030W_g - 0.004S_g.$$

This expression gives the rates of change of nominal income consistent, for various other variables, with equality between the actual and desired rate of change of money if the desired rate of change is determined by the demand equation defined by equation (2).

Of course, if we were dealing with exact relationships observed without error and properly specified, equation (5) would be the time derivative of equation (4), and the coefficients of corresponding variables would be equal. In fact, we concluded in chapter 6 that the differences were not statistically significant. In both equation (1) and equation (2), the coefficients are lower limits (in absolute value) because of the regression effect, which is more important in equation (2) dealing with rates of change than in equation (1) dealing with levels. This explains why all coefficients (other than that for W_g) are lower (in absolute value) in equation (2) than in equation (1). In addition, of course, the lower limits in both equations are themselves subject to statistical errors of measurement, and the coefficients of $\log y$ in equation (1) and of g_y in equation (2) are biased toward unity by errors of measurement of population and prices.

Equations (4) and (5) are still not in the form corresponding to the monetary theory of nominal income because they include real per capita income. The deviance is small, however, since the coefficients of the real per capita income variable are close to zero (-0.14 and -0.03 for the United States; $+0.11$ and $+0.01$ for the United Kingdom). It looks as if this variable can be omitted with little loss, though this conclusion must be qualified somewhat because of the statistical bias toward zero of these coefficients.

Rather than proceed further with these transformed demand equations, we can estimate directly general equations of the form of equations (4) and (5), say,

$$\begin{aligned} \log Y = & -\log k - \lambda_3 Z + \zeta \log M + (1 - \alpha - \lambda_4 Z) \log y \\ & - \delta R_N - \epsilon g_Y - \lambda_1 W - \lambda_2 S \ , \end{aligned} \tag{6}$$

$$\begin{aligned} g_Y = & \zeta g_M + (1 - \alpha - \lambda_4 Z) g_y - \delta D R_N - \epsilon D g_Y \\ & - \lambda'_1 W_g - \lambda'_2 S_g \ , \end{aligned} \tag{7}$$

where $\log k$ is the constant term of demand equation (1) for the United States, λ_3 is the excess of the constant term for the United Kingdom over that for the United States, ζ is the elasticity of nominal income with respect to nominal money for given values of the other variables, α is the real per capita income elasticity of the demand for money in the United States, λ_4 is the excess of United Kingdom income elasticity over United States income elasticity, δ and ϵ are the semilogarithmic slopes of the demand for money with respect to the differential yield on nominal assets and the proxy yield on physical assets, respectively, λ_1 and λ'_1 are the coefficients of the postwar adjustment dummy for levels and rates of change, respectively, and λ_2 and λ'_2 of the upward demand shift dummy.

Equations (6) and (7) are special cases of equation (15) of chapter 2, equation (6) for $\phi = \infty$, which assures that M^S, or money supplied, always equals M^D, or money demanded; equation (7) for $\psi = \infty$, which assures that g_{MD} always equals g_{MS}.

If equations (1) and (2) were exact relations observed without error and properly specified, direct estimates of equations (6) and (7) would be identical with equations (4) and (5). In practice they will not be, for three reasons: (*a*) The coefficients of $\log M$ and g_M are free to take any value in equations (6) and (7), not restricted to unity, as they are in equations (4) and (5). (*b*) The changed dependent variable alters the bias resulting from errors of measurement. In equations (1) and (2), errors of measurement in population and prices bias the coefficients of log y and of g_y in equations (4) and (5) toward zero. In equations (6) and (7), the errors in the measurement of Y and of population and prices introduce a very different bias. They bias $(1-\alpha)$ in these equations upward toward a positive number less than or equal to unity rather than toward zero.[3] In

3. Recall that

(a) $$\log Y = \log y + \log NP$$

or, as actually computed,

(b) $$\log y = \log Y - \log NP.$$

Using the notation of note 18 of chapter 6,

(c) $$\log Y = (\log Y)^* + e_Y = (\log y)^* + (\log NP)^* + e_Y$$

(d) $$\log NP = (\log NP)^* + e_{NP}$$

(e) $$\log y = (\log y)^* + e_y = (\log y)^* + e_Y - e_{NP}.$$

Assume as in that footnote that e_Y and e_{NP} are independent of the asterisked values and of one another and that all variables are expressed as deviations from their means.

Consider now estimates of $(1-\alpha)$ in equation (6), for a fixed value of M^*, and neglecting the other variables. The estimate of $(1-\alpha)$ from the regression of log Y on log y is

(f) $$\begin{aligned}(1-\alpha)_{Yy} &= \frac{E \log Y \log y}{\sigma^2_{(\log y)}} \\ &= \frac{E[(\log Y)^* + e_Y][(\log y)^* + e_Y - e_{NP}]}{\sigma^2_{(\log y)^*} + \sigma^2_{e_Y} + \sigma^2_{e_{NP}}} \\ &= \frac{E(\log Y)^*(\log y)^* + \sigma^2_{e_Y}}{\sigma^2_{(\log y)^*} + \sigma^2_{e_Y} + \sigma^2_{e_{NP}}}.\end{aligned}$$

From the basic demand equation we have that

(g) $$\log k + \alpha^* (\log y)^* + (\log NP)^* = (\log M)^*,$$

where α^* is the "true" value of α.

But the fixed value of $(\log M)^*$ is zero, given that we are dealing with deviations from means, so that

(h) $$(\log k) + (\alpha^* - 1)(\log y)^* + (\log Y)^* = 0$$

Multiply through by $(\log y)^*$ and take expected values. The result is

(i) $$E(\log Y)^*(\log y)^* = (1-\alpha^*)\,\sigma^2_{(\log y)^*}.$$

Substitute in (f) and we have

(j) $$\begin{aligned}(1-\alpha)_{Yy} &= \frac{(1-\alpha^*)\,\sigma^2_{(\log y)^*} + \sigma^2_{e_Y}}{\sigma^2_{(\log y)^*} + \sigma^2_{e_Y} + \sigma^2_{e_{NP}}} \\ &= \frac{(1-\alpha^*)\,\sigma^2_{(\log y)^*} + \left(\sigma^2_{e_Y} + \sigma^2_{e_{NP}}\right)\left(\dfrac{\sigma^2_{e_Y}}{\sigma^2_{e_Y} + \sigma^2_{e_{NP}}}\right)}{\sigma^2_{(\log y)^*} + \sigma^2_{e_Y} + \sigma^2_{e_{NP}}}\end{aligned}$$

or a weighted average of $(1-\alpha^*)$ and $\dfrac{\sigma^2_{e_Y}}{\sigma^2_{e_Y} + \sigma^2_{e_{NP}}}$.

For an income elasticity of unity or above, the average is necessarily greater than $(1-\alpha)^*$, so terms of α, the income elasticity, the bias is downward, whereas in equations (4) and (5) it is toward unity. (*c*) In equation (6) for the United States and the United Kingdom combined, a single dummy variable will no longer suffice to allow fully for the difference between pounds and dollars. One dummy served in equation (1), despite the changes in the market exchange rate during the period the equation covers, because both real balances and real income are expressed in terms of 1929 prices, hence only the 1929 exchange rate is relevant. In equation (6), log Y and log M are in current prices. For $\zeta = 1$, this raises no problem because only the difference between log Y and log M (i.e., the logarithm of Y/M) is relevant, and this difference is not affected by the exchange rate. So this item is linked to item 1. For $\zeta \neq 1$, a real problem arises. We have dealt with it by converting the United Kingdom values of Y and M to dollars at the ruling exchange rate, and of y to dollars at the 1929 exchange rate. The coefficient of the dummy variable Z can then be regarded as reflecting any difference in the level of the demand function plus a constant percentage difference between the market exchange rate and the exchange rate that is relevant in comparing the services rendered by cash balances. In this way, for the United States and the United Kingdom combined, the form of the equation is given by equation (6) but the variables log Y, log M, and log y have a different meaning for the United Kingdom than they do in equation (4), or than they do in equation (6) estimated for the United Kingdom alone.[4]

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the bias is definitely upward. For an income elasticity of less than unity, the same result is possible but not necessary unless $\sigma^2_{e_{NP}} = 0$. Since α^* according to our empirical results is higher than unity for the United States and not much below unity for the United Kingdom, there is a presumption that the bias is definitely upward.

For the regression of y on Y, the result is unambiguously an upward bias. That estimate is

(k) $$(1-\alpha)_{yY} = \frac{\sigma^2 \log Y}{E \log_Y \log y} = \frac{\sigma^2 (\log Y)^* + \sigma^2_{e_Y}}{E(\log Y)^* (\log y)^* + \sigma^2_{e_Y}}.$$

Multiplying equation (h) through by $(\log Y)^*$ and taking expectations, we have

(l) $$\sigma^2_{(\log Y)^*} = (1-\alpha^*)\, E (\log Y)^* (\log y)^*.$$

Substituting in equation (k) gives

(m) $$(1-\alpha)_{yY} = \frac{(1-\alpha^*)E(\log Y)^* (\log y)^* + \sigma^2_{e_Y}}{E (\log Y)^* (\log y)^* + \sigma^2_{e_Y}},$$

or a weighted average of $(1-\alpha)^*$ and 1, which is necessarily greater than $(1-\alpha)^*$ for any positive income elasticity.

4. A simpler procedure would be to use the United Kingdom figures converted into dollars for both the United Kingdom alone and the United States and United Kingdom combined. We have not done so because that introduces a thoroughly extraneous element into the data for the United Kingdom alone. These data were generated in pounds not dollars; the holders of pounds received them as pounds, not as dollar equivalents. If the

The direct estimates are given in table 8.1 for equations (6) and (7).[5] Because of the bias in the coefficients of $\log y$ and g_y, and because of our interest in omitting these variables to correspond with the monetary theory of nominal income, we also estimated equations without these terms:

(6a) $$\log Y = -\log k - \lambda_3 Z + \zeta \log M - \delta R_N - \epsilon g_Y - \lambda_1 W - \lambda_2 S\ ,$$

(7a) $$g_Y = \zeta g_M - \delta D(R_N) - \epsilon D g_Y - \lambda_1' W_g - \lambda_2' S_g\ .$$

For the two countries combined, the indirect and direct estimates compare as follows for levels:

(4) $$\log Y = 1.47 - 1.64Z + \log M - (0.14 - 0.25Z)\log y + 9.3R_N + 0.47g_Y - 0.019W - 0.193S,$$

(6) $$\log Y = -0.39 - 0.92Z + 0.83 \log M + (0.42 + 0.13Z)\log y + 11.4R_N + 0.16g_Y - 0.020W - 0.11S,$$

(6a) $$\log Y = 0.76 - 0.08Z + 0.97 \log M + 15.1R_N + 0.71g_Y - 0.014W - 0.15S,$$

and as follows for rates of change:

(5) $$g_Y = g_M - (0.03 - 0.04Z)g_y + 8.7\,DR_N + 0.41\,Dg_Y - 0.030\,W_g - 0.004\,S_g\ ,$$

(7) $$g_Y = 0.77g_M + (0.46 + 0.02Z)g_y + 14.0\,DR_N + 0.28\,Dg_Y - 0.020\,W_g - 0.003\,S_g\ ,$$

(7a) $$g_Y = 0.87g_M + 10.5\,DR_N + 0.53\,Dg_Y - 0.027\,W_g - 0.004\,S_g.$$

The most striking difference between the direct and indirect estimates is in the coefficient of the real income term, which, for the level equations, changes from −0.14 to +0.42 for the United States and from +0.11 to +0.55 for the United Kingdom and, for the rate-of-change equations, from −0.03 to +0.46 for the United States and from +0.01 to +0.48 for the United Kingdom. It is difficult to interpret these coef-

exchange rate had remained the same throughout, it would make no difference which procedure was used. However, with a variable exchange rate, it does make a difference.

One other parenthetical point. It may seem as if there should be a fourth effect, namely, a difference in the regression effect. However, this is not so. The direction of minimization of the sums of squares is the same since $\log y$ or g_y remains the independent variable.

5. We also made parallel calculations for equation (7) modified by adding a constant term. The constant term was statistically insignificant for each country separately, and at the margin of significance at a .05 level for the combined equation; it affected only slightly the standard error of estimate or the other coefficients. Hence, we do not present these calculations.

Table 8.1 **Regressions of Nominal Income on Nominal Quantity of Money and Other Variables**

Country	Equation[a]	Intercept	Coefficients: Log M or g_M	Log y or g_y	R_N or DR_N	g_Y or Dg_Y	S or S_g	W or W_g	Standard Error of Estimate	R^2
					Levels					
United States	(6)	−0.24	0.83	0.40	8.60	0.24	−0.14	−0.024	.0369	.9994
	(6a)	1.10	0.95		7.99	0.51	−0.17	−0.024	.0445	.9991
United Kingdom	(6)	−1.02	0.84	0.60	18.80	−0.08	−0.10	−0.009	.0394	.9990
	(6a)	0.38	1.01		14.17	0.53	−0.19	−0.011	.0599	.9977
United States and		U.S. U.K.		U.S. U.K.						
United Kingdom[b]	(6)	−0.39 −1.31	0.83	0.42 0.55	11.45	0.16	−0.11	−0.020	.0389	.9992
	(6a)	0.76 0.68	0.97		15.11	0.71	−0.15	−0.014	.0602	.9981
					Rates of Change					
United States	(7)		0.74	0.49	12.51	0.29	−0.003	−0.025	.0108	.9804
	(7a)		0.88		10.29	0.58	−0.005	−0.031	.0131	.9704
United Kingdom	(7)		0.78	0.49	15.54	0.22	−0.002	−0.017	.0103	.9660
	(7a)		0.86		12.43	0.41	−0.003	−0.026	.0118	.9534
United States and				U.S. U.K.						
United Kingdom[b]	(7)		0.77	0.46 0.48	14.03	0.28	−0.003	−0.020	.0102	.9732
	(7a)		0.87		10.54	0.53	−0.004	−0.027	.0121	.9616

[a]For definition of equations, see text.
[b]The intercept for the United Kingdom is computed by adding the coefficient of Z to the regression intercept shown for the United States. Similarly, the entry for United Kingdom for the coefficient of log y or g_y is the regression coefficient shown for the United States plus the coefficient of Z log y.

ficients from equations (6) and (7) as reflecting an economic phenomenon. If they did, they would imply an income elasticity of demand for real balances of 0.58 or 0.54 for the United States and of 0.45 or 0.52 for the United Kingdom; yet our earlier evidence has led us to conclude that the income elasticity exceeds unity for the United States and is only a little less than unity for the United Kingdom.[6] A more likely explanation is the bias in these coefficients arising from the measurement errors common to Y and y. As noted, these errors bias the coefficients in equations (6) and (7) upward, whereas the errors in population and prices bias the corresponding coefficients of equations (4) and (5) toward zero.

The equations without the real income terms are consistent with this explanation. With only three exceptions out of sixteen comparisons (the coefficients of Z, R_N, and W), the coefficients of equations (6a) and (7a) for the two countries combined are closer to the coefficients of equations (4) and (5) than the corresponding coefficients of equations (6) and (7).[7] That is also the case for the equations for the separate countries, with only three exceptions out of twenty-six comparisons. In addition, the standard errors of estimate for equations (6) and (7) are uniformly lower than for equations (6a) and (7a). While this might be the effect of either common measurement errors or an economically meaningful influence of real per capita income on nominal income for given money stock and other variables, the economically implausible values of the coefficients argue in favor of common measurement errors.

On that interpretation, we can use the standard errors of estimate to construct estimates of the size of the common measurement errors. These turn out to range from 2.5 to 4.6 percent for the levels of nominal income; from 0.5 to 0.75 of a percentage point for the rate of change of nominal income.[8] These values are certainly not implausible as estimates of pure measurement error.

6. Note that the regression bias cannot explain the conflict with earlier evidence. The corresponding coefficients computed from regressions of log y (g_y) on log Y (g_Y) are higher, implying even lower estimates of elasticity than those cited in the text, and hence an even greater conflict with earlier evidence.

7. Note that this includes comparing the imposed values of zero for the coefficients of log y and g_y in equations (6a) and (7a) with the computed coefficients in equations (4) and (7), as well as the imposed value of unity for the coefficients of log M and g_M in equations (4) and (5) with the computed coefficients in the other equations.

8. Assume that measurement errors account for the difference between the standard errors of estimate of equations (6) and (6a), and of equations (7) and (7a), and that measurement errors can be treated as independent of the "true" values of the independent variables. It then follows that

$$\sigma_{e_Y} = \sqrt{[SEE\,(6a)]^2 - [SEE\,(6)]^2}$$

is an estimate of the standard deviation of the percentage measurement error in Y, where SEE ( ) stands for the standard error of estimate of the equations in the parentheses, and

On the basis of this evidence, we shall proceed for most of this chapter to regard equations (6a) and (7a) as a valid representation of the relation between the nominal quantity of money and nominal income and shall treat the exclusion of real per capita income as a valid first approximation. We shall leave the division of the change in nominal income between prices and output for separate consideration in chapter 9.

How much better are equations (6a) and (7a) than simpler quantity theory relations that do not allow for the effect of asset yields as measured by R_N and g_Y? The final three columns of table 8.2 give one answer. Column 5 shows the standard deviation of log Y and the root mean square value of g_Y, which is to say, the coefficient of variation of nominal income itself and an estimate of the standard deviation of the rate of change of nominal income on the assumption that its mean value is zero. The next column shows the deviation that remains after allowing solely for log M or g_M (plus, for the United States, the changing degree of financial sophistication before 1903, and, for both countries, the postwar readjustment and upward demand shifts)—that is, standard errors of estimate from the equations

$$(8) \qquad \log Y = -\log k - \lambda_3 Z + \zeta \log M - \lambda_1 W - \lambda_2 S\,,$$

$$(9) \qquad g_Y \quad = \qquad \zeta\, g_M - \lambda' W_g - \lambda' S_g.$$

The final column shows the deviation that remains, after allowing also for the effect of asset yields, that is, the deviations from equations (6a) and (7a). Clearly, the simple quantity theory effect is far more important than the further refinement of allowing for changing yields, and this is true even for the rates of change, which largely eliminate the effects of common trends. But, equally clearly, the allowance for yields is important, always reducing the residual standard deviation appreciably—indeed, in some cases to not much above our estimate of pure measurement error.

One striking result in table 8.1 is that the coefficient of log M or g_M—the estimated elasticity of nominal income with respect to nominal money—is generally less than unity. Columns 1 to 4 of table 8.2 show that this result cannot be attributed to the regression effect. The upper limits are above unity only twice—for the United Kingdom level regressions for

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$$\sigma_{e_{g_Y}} = \sqrt{[SEE\,(7a)]^2 - [SEE\,(7)]^2}$$

is an estimate of the standard deviation of the percentage point measurement error in g_Y. The values calculated this way are as follows:

	σ_{e_Y}	$\sigma_{e_{g_Y}}$
United States	.025	.0074
United Kingdom	.045	.0058
United States and United Kingdom	.046	.0065

Table 8.2 **Effect of Allowing for Asset Yields on Relation between Nominal Quantity of Money and Nominal Income**

Variable and Country	Elasticity of Nominal Income with Respect to Nominal Quantity of Money: Equation (8) or (9)[a] Lower Limit (1)	Equation (8) or (9)[a] Upper Limit (2)	Equation (6a) or (7a)[a] Lower Limit (3)	Equation (6a) or (7a)[a] Upper Limit (4)	Standard Deviation of Log Y[b] or Root Mean Square Value of g_Y[b] (5)	Standard Error of Estimate[b]: Equation (8) or (9) (6)	Standard Error of Estimate[b]: Equation (6a) or (7a) (7)
Level of income (log Y)							
United States	0.935	0.937	0.950	0.951	141.37	5.72	4.45
United Kingdom	1.061	1.065	1.012	1.020	114.93	7.01	5.99
United States and United Kingdom	0.966	0.972	0.974	0.977	134.06	9.63	6.02
Rate of change of income (g_Y)							
United States	0.925	0.986	0.884	0.921	7.20	1.74	1.31
United Kingdom	0.908	0.985	0.858	0.918	5.08	1.40	1.18
United States and United Kingdom	0.918	0.987	0.873	0.922	5.97	1.51	1.21

Note: For definition of equations, see text.
[a]Lower limit is value of coefficient of log M or g_M in relevant regressions; upper limit is lower limit divided by square of corresponding partial correlation coefficient.
[b]Percentage (or percentage points).

which the lower limits are also above unity—and for the rest are below unity. A priori, one might expect allowance for yields to give an elasticity closer to unity than the simple regressions. That is the result for the level equations, but not for the rate-of-change equations. For them, allowance for yields appreciably improves the relation between nominal money and nominal income but generally lowers the elasticity.[9]

If all variables affecting nominal income other than the nominal quantity of money were allowed for, the elasticity of nominal income with respect to nominal money would be unity. The consistent tendency for the elasticity calculated from equations (6a) and (7a) to be less than unity means that some variables are omitted that are related in the opposite direction to nominal income for given quantities of money than they are to the quantity of money.

We have deliberately excluded one such variable, namely, real per capita income. If the income elasticity of demand for money is greater than unity, as we have concluded it is for the United States, then an increase in real income for a given money stock will tend to lower velocity, which is to say, to lower the level of nominal income corresponding to that quantity of money; and it will act conversely if the income elasticity is less than unity, as we have concluded it is for the United Kingdom. If, in addition, increases in nominal money tend to be associated with increases in real per capita income, as they have been in levels for both countries and in rates of change for the United States,[10]

9. The effect on the relation of allowing for yields can be shown by comparing the partial correlation coefficients between log Y or g_Y and log M or g_M, with and without allowance for yields.

Table 8.N.1 **Effect of Allowing for Yields on Relation between Nominal Income and Nominal Money**

	Partial Correlation			
	Log Y and Log M		g_Y and g_M	
	Yields Not Allowed for	Yields Allowed for	Yields Not Allowed for	Yields Allowed for
United States	.9992	.9992	.9684	.9794
United Kingdom	.9982	.9960	.9602	.9663
United States and United Kingdom	.9967	.9985	.9646	.9731

Four out of six of the correlations are higher with allowance for yields than without, one is the same, the final one is lower.

10. The correlations, after allowing for dummy variables by the technique explained below in note 13, are as follows:

then the exclusion of real per capita income will mean that nominal money serves partly as a proxy for real per capita income. For the United States, that would tend to lower the elasticity of nominal income with respect to nominal money; for the United Kingdom it would tend to raise it. However, in view of the closeness of the income elasticity of demand for money to unity in both countries, it is hard to believe that this effect is of major importance.

Another set of variables that is excluded are yields other than those measured by R_N and g_Y. We know that R_N and g_Y are generally positively correlated with M, and DR_N and Dg_Y with g_M.[11] If other yields behave the same way, for fixed values of R_N and g_Y, their exclusion would tend to raise the computed elasticities, since higher yields tend to raise velocity and nominal income for a given quantity of money. The effect would be in the opposite direction from that required to explain elasticities less than unity and so would further raise the discrepancy to be explained. However, the correlations with M of the yields we use are generally very small, so they would not bias the results much, and we have no reason to suppose that other yields would show any higher correlation—or even be correlated in the same direction.

For quarterly or annual data, the observed elasticity of nominal income with respect to nominal money tends to be decidedly higher than unity, rather than lower as it is for our phase average data. We have interpreted the greater than unity cyclical elasticity as reflecting a difference between measured income—the observed variable used as the dependent variable—and permanent income—the unobserved variable that we regard as underlying the demand for money. Since measured income fluctuates cyclically more widely than permanent income, a unit elasticity with respect to permanent income would be, and a less than unit elasticity could be, converted into a greater than unit elasticity with respect to measured income.

This effect is presumably still present in our phase average data, though to a much smaller extent, since averaging over phases removes much of the difference between measured and permanent income.

	Levels *(M and y)*	*Rates of Change* *(g_M and g_y)*
United States	.990	.659
United Kingdom	.960	−.082

For a fuller discussion, see chapter 9.

11. The correlations, after allowing for dummies by the technique explained below in note 13 are:

	Levels		*Rates of Change*	
	United States	United Kingdom	United States	United Kingdom
M and R_N	−.68	.44	.50	.36
M and g_Y	.26	.48	.22	.24

However, to whatever extent it is present, it also is in the wrong direction to explain our results.

We have no fully satisfactory explanation of our finding that, for given yields on assets, a 1 percent increase in the nominal quantity of money tends to be associated with an increase of something less than one percent in nominal income—roughly, about 0.9 percent.

8.2 Replacing Yields hy Prior Income and Money

Aside from the dummies, equations (6a) and (7a) relate nominal income to three contemporary variables: quantity of money, the differential yield on nominal assets, and the proxy for the nominal yield on real assets. The simplified monetary theory of nominal income sketched in an earlier publication contains only a single yield, "the" nominal interest rate, which is taken to be the sum of a real interest rate plus an anticipated rate of change of prices. It further assumes that the difference between the anticipated or trend real rate of interest and the anticipated or trend real rate of growth of real income can be regarded as a constant. If we take "the" interest rate to be the short rate, this gives

$$R_S = k_o + (g_Y)^*. \tag{10}$$

It then interprets $(g_Y)^*$ as itself determined by current and prior movements of income, which in turn reflect current and prior movements of money, and so expresses current income as a functional of the prior course of money.

Equation (10) is in the same spirit as our use of g_Y as a measure of the yield on physical assets. However, our discussion of the observed coefficients of g_Y and R_S (or R_N) warns against regarding $(g_Y)^*$ and g_Y as equivalent. The observed interest rate in the market is a *direct* measure of *anticipated yield*, the observed rate of change of nominal income, an *indirect* measure of *actual yield*.

Nonetheless, as a first step in converting equations (6a) and (7a) into a relation between current income and current and prior money and income, it seems worth exploring the consequences of using equation (10) to eliminate R_S and assuming

$$(g_Y)^* = g_Y, \tag{11}$$

that is, that actual and anticipated rates of change of nominal income are equal. We can greatly simplify this exploration by modifying equations (6a) and (7a) by replacing R_N by R_S, which, as we saw in chapter 6, gives only slightly less satisfactory results.[12]

12. The effect on the standard errors of equations (6a) and (7a) of replacing R_N by R_S is as follows:

Making these substitutions into modified equation (6a), which we shall refer to as $(6a)'$, and omitting for simplicity the dummy variables,[13] gives

$$\text{(12)} \qquad \log Y = -(\log k + \delta k_0) + \zeta \log M - (\delta + \epsilon)\, g_Y.$$

If we interpret Y and M as continuous series, g_Y is the time derivative of Y. Equation (12) is then a first-order differential equation in nominal income, whose solution would give nominal income as an exponentially weighted average of past quantities of money, the weights declining the farther back in time. Equation (12) is of the same form as the one connecting permanent and measured income in the Cagan-Koyck version of adaptive expectations.[14] Hence the solution is the same, with nominal income bearing the same relation to money as permanent income does to measured income.

In order for equation (12) to give a stable solution, $-(\delta + \epsilon)$ must be greater than unity, otherwise the slightest disturbance drives $\log Y$ to plus or minus infinity. If $-(\delta + \epsilon)$ is greater than unity, the equation implies that nominal income will be stabler than nominal money, because it will be a weighted average of nominal money. As just noted, this result is sharply contrary to experience for quarterly or annual data, since a striking cyclical phenomenon is the wider amplitude of fluctuations in nominal income than in nominal money.[15] This phenomenon, which is muted but no doubt still present in our phase average data, conceivably

---

	Standard Error of Estimate			
	Level Equation with		*Rate of Change Equation with*	
	R_N	R_S	DR_N	DR_S
United States	.0445	.0471	.0131	.0128
United Kingdom	.0599	.0578	.0118	.0121
United States and United Kingdom	.0602	.0627	.0121	.0120

13. For the later empirical calculations, we simplify the allowance for the postwar readjustment and upward demand shift by using the device employed to allow for increasing financial sophistication in the United States, namely, adjusting the money series in advance for their effect. We have used for this purpose the estimated coefficients of the dummy variables from the final demand equations in chapter 6.

14. For example, let x be any variable and x^* its "permanent" or "anticipated" value. The Cagan-Koyck version assumes that

$$\text{(a)} \qquad \frac{dx^*}{dt} = b\,(x - x^*)\,,$$

if there is no secular trend,

$$\text{(b) or} \qquad \frac{dx^*}{dt} = b\,(x - x^*) + c$$

if there is a linear trend.

Equation (b) is of the same form as equation (12), with ($\log Y$) substituted for x^*, and $\zeta(\log M)$ for x, $b = 1/(\delta + \epsilon)$, and $c = -(\log k + \delta k_o) \,/\, (\delta + \epsilon)$.

15. See Friedman and Schwartz, "Money and Business Cycles," table 4.

could be consistent with equation (12) if it simply reflected a greater error of measurement in income than in money. However, in the absence of independent evidence on the relative size of the measurement errors, that seems an unsatisfactory way to reconcile the theoretical implications of equation (12) with the results of experience.

Our earlier results also indicate that equation (12) is unsatisfactory. Equation (12) is identical with equation (6a) with R_N omitted. In fitting that equation, and its companion (7a), the partial correlation of R_N with $\log Y$ (or of DR_N with g_Y) is almost invariably higher than the partial correlation of g_Y with $\log Y$ (or of Dg_Y with g_Y), indicating that if only one of the two variables R_N or g_Y is to be used, R_N is preferable. Moreover, the use of both rather than g_Y alone consistently produces a significant reduction in the standard error of estimate.[16]

Let us therefore drop assumption (11) while retaining assumption (10). This gives (see sec. 8.4):

$$\log Y = -[\log k + \delta k_o + \lambda_3 Z] + \zeta \log M - \delta g_Y^* - \epsilon g_Y. \tag{13}$$

If we now interpret $(g_Y)^*$ as determined by an adaptive expectations mechanism, that is, as determined by the past history of income, the effect will be to introduce higher derivatives of g_Y. The corresponding differential equation is of the second or higher order and no longer has the implication that led us to reject equation (12).

We could proceed to try to fit equation (13) by replacing $(g_Y)^*$ with a function of Dg_Y and perhaps still higher derivatives and then proceed to approximate g_Y and Dg_Y by the empirical counterpart of these variables we have used in earlier computations. However, for our present purpose of seeking to relate changes in nominal income to current and prior changes in money and income, this procedure has a serious defect. Our estimate of g_Y is based on a linear trend fitted to three successive phase average values of $\log Y$. The slope is then taken as an estimate of g_Y corresponding to the central phase. Dg_Y is calculated from first differences of g_Y, divided by the interval between them. As a result, using our statistical counterparts for g_Y and Dg_Y would in effect relate $\log Y$ to future as well as past levels of money and income.

To avoid this problem, let us rewrite equation (13) as a difference equation. If we assume the Cagan-Koyck type of adaptive expectations for the determination of $(g_Y)^*$ and treat all phases as equal in length, this difference equation (as shown in sec. 8.4) will be of the form:

$$\log Y(t) = a + b \log M(t) + c \log M(t-1) + d \log Y(t-1) + e \log Y(t-2) + fZ. \tag{14}$$

16. Equivalent statements hold if R_S is substituted for R_N in the regressions.

The corresponding rate of change equation, based on modified equation (7a) [(7a)′], has the form:

$$g_Y(t) = bg_M(t) + cg_M(t-1) + dg_Y(t-1) + eg_Y(t-2). \tag{15}$$

These equations correspond to equation (15) in chapter 2 for the special case of the monetary theory of nominal income.

The assumption that the phases are equal in length enters into the derivation of equations (14) and (15) in two rather different ways: (1) the weighting of past phases to estimate $(g_Y)^*$; (2) the calculation of g_Y from successive phase values. To use the equation as it stands with our phase data, we must (1) let the chronological time span over which expectations are assumed to be formed be a variable, depending on the length of the phases, and (2) compute rate of change per phase rather than per year.

Our initial view was that both item 1 and item 2 are defects. Further reflection, however, particularly in the light of some fundamental work by Maurice Allais, suggests that item 1 may be an advantage rather than a defect,[17] though we continue to regard item 2 as a defect and shall try to eliminate it. Allais argues that the rate at which people "forget" the past in judging the future—that is, the span of past time on which they base their anticipations—is variable and depends on the course of events themselves. If the relevant magnitude changes rapidly—for example, if prices change rapidly—then people also adapt their anticipations more rapidly, "forgetting" the past at a faster rate or using a smaller time span to form their anticipations, and conversely. Allais proposes a very specific and sophisticated hypothesis to connect "psychological" time, as he calls it, with chronological time. For our purpose, the general idea rather than its specific embodiment is relevant. A lengthy cycle phase means that economic events have been proceeding slowly; a brief cycle phase means they have been proceeding rapidly. Hence the chronological span over which anticipations are formed should, if Allais is correct, be longer when the phases are long than when they are short. The time period corresponding to the length of a phase, or to the interval between phases, might therefore come closer to representing a constant duration of psychological time than would a fixed chronological time interval.

And what is true of Allais's rate of forgetfulness is also plausible for the rate at which people eliminate discrepancies between desired and actual balances. If events move slowly, adjustments might well also move slowly.

17. Maurice Allais, "Réformulation de la théorie quantitative de la monnaie," *Bulletin Sedeis*, no. 928, suppl. (10 September 1965); "A Restatement of the Quantity Theory of Money," *American Economic Review* 56 (December 1966): 1123–57; "Forgetfulness and Interest," *Journal of Money, Credit and Banking* 4 (February 1972): 40–71.

But there is no corresponding justification for item 2, namely, using the rate of change of various variables per phase rather than per chronological time unit. The chronological rates of change are objective measurements. The change in the rate of forgetfulness in effect represents a changing evaluation of objective measurements; it affects the internal discount rate that is employed in choosing how much cash to hold, how much to save and spend, and so on.

Accepting item 1, but using chronological time in estimating g_Y from successive phase values, produces a more complex equation than equations (14) and (15), one that is a function of the same basic variables, but is not linear in its parameters.[18] It has the form

$$
\begin{aligned}
(16) \qquad \log Y(t) = \log Y(t-1) - (1/Q(t))\, \{ & w(\log k + \delta k_o + \lambda_3 Z) \\
& - \zeta[\log M(t) - \log M(t-1)] \\
& - w[\zeta \log M(t-1) - \log Y(t-1)] \\
& - [\epsilon(1-w)/\bar{n}(t-1)][\log Y(t-1) \\
& - \log Y(t-2)]\},
\end{aligned}
$$

where

$$n(t) = \text{length of phase } t$$

$$\bar{n}(t) = \frac{n(t) + n(t-1)}{2}$$

$$Q(t) = 1 + \frac{\delta w + \epsilon}{\bar{n}(t)}.$$

Although equation (16) seems at first glance linear in the parameters, it is not because $Q(t)$ is a function of δ, ϵ, and w. Accordingly, we have used an iterative nonlinear computer regression program to calculate equation (16), given initial values of the various parameters. The iterations converged in all cases, though in some it took a considerable number of iterations to arrive at a satisfactory result.

The corresponding rate of change equation has the form:

$$
\begin{aligned}
(17) \qquad g_Y(t) = g_Y(t-1) + (1/Q(t))\{ & \zeta[g_M(t) - g_M(t-1)] \\
& + w\,[\zeta g_M(t-1) - g_Y(t-1)] \\
& + [\epsilon(1-w)/\bar{n}(t-1)][g_Y(t-1) - g_Y(t-2)]\}.
\end{aligned}
$$

It was fitted by the same iterative program.

Table 8.3 compares the standard errors of the equations (14) to (17), which replace R_S and g_Y by earlier money and income, with the standard errors of the simple quantity theory equations (8) and (9) and equations (6a)′ and (7a)′ that incorporate R_S and g_Y directly. The table also in-

18. For derivation, see appendix to this chapter (sec. 8.4)

Table 8.3 **Alternative Relations between Nominal Income and Other Variables: Standard Errors of Estimate**

	Standard Error of Estimate: Income on Money				
			and Prior Phase's Money and Two Prior Phases' Income, with Phase Length		
	Only	and Yields	Assumed Constant	Allowed for	and Three Prior Phases' Money
	(8) and (9)[a]	(6a)′ and (7a)′	(14) and (15)	(16) and (17)	(18) and (19)
		Levels			
United States	.0599	.0463	.0520	.0513	.0601
United Kingdom	.0785	.0564	.0554	.0581	.0747
United States and United Kingdom	.0983	.0684	.0555	.0562	.0980
		Rates of Change (Zero Intercept)			
United States	.0178	.0126	.0155	.0156	.0155
United Kingdom	.0146	.0118	.0146	.0140	.0153
United States and United Kingdom	.0158	.0119	.0149	.0145	.0153
		Rates of Change (Nonzero Intercept)			
United States	.0179	.0123	.0156	.0158	.0143
United Kingdom	.0143	.0104	.0139	.0130	.0154
United States and United Kingdom	.0158	.0111	.0147	.0141	.0154

[a]Equations on which estimates are based.

cludes standard errors for rates of change for equations identical to equations (15) and (17) except that they include a constant term, and therefore allow for a trend, which may reflect the average effect of real income growth plus an $\alpha \neq 1$.

Equations (14) and (15), which implicitly neglect variability in phase length, give appreciably lower standard errors than the simple quantity equation, with only one exception (no change for United Kingdom rates of change with zero intercept). However, by comparison with equations (6a)′ and (7a)′, equation (14) for levels gives a lower standard error for only two out of three comparisons, and equation (15), for rates of change, gives a decidedly higher standard error for all six comparisons. The level equations presumably show an improvement because past values of income approximate the trend of income better than does the current value of money alone. The rate-of-change equations show a worsening because the trend factor is of negligible importance—though of enough importance to make the equations with an intercept generally slightly superior to those without. Either the failure to allow for differences in length of phases or the replacement of R_S and g_Y by earlier money and income introduces a significant source of error.

Equations(16) and (17) allow for differences in length of phase and so eliminate one of these sources of error. While allowing for differences in length of phases reduces the standard error in five out of nine comparisons, the reduction is trivial, as are the increases in the remaining four comparisons. The minor improvement is in line with theoretical expectations, but hardly worth the extra complication and, in any event, leaves the regressions that replace current yields by prior money and income decidedly less satisfactory than those that allow for yields directly.

Table 8.4 gives estimates of the structural parameters yielded by the various approaches: ζ, the elasticity of nominal income with respect to nominal money; δ, the slope of the logarithm of real per capita money with respect to the estimated yield (R_S) on nominal assets; ϵ, the slope of the logarithm of real per capita money with respect to the estimated yield (g_Y) on physical assets; and w, the weight given the current phase in estimating anticipated yields on real assets.

Like the comparisons of standard errors, these results are disappointing. The equations replacing current yields by earlier money and income give roughly the same estimates for ζ as the equations including current yields, but generally very different and more erratic estimates for δ and ϵ. For the United States, the estimates of δ and ϵ are at least mostly of the right sign (ten out of twelve); but for the United Kingdom alone most of the estimates are of the wrong sign—positive rather than negative.

The estimates of w are the only additional information provided by the table. The equations that do and do not allow for the length of phases give roughly the same results, though those for the equations that do not allow

Table 8.4 **Estimates of Structural Parameters from Alternative Equations of Relations between Nominal Income and Other Variables**

	Elasticity of Income with Respect to Money (ζ)			Slope of Log of Real per Capita Money with Respect to: Nominal Yield on Nominal Assets (δ)			Slope of Log of Real per Capita Money with Respect to: Proxy Nominal Yield on Physical Assets (ε)			Weight of Current Phase in Forming Expectations (w)	
	(6a)' and (7a)'[a]	(14) and (15)	(16) and (17)	(6a)' and (7a)'	(14) and (15)	(16) and (17)	(6a)' and (7a)'	(14) and (15)	(16) and (17)	(14) and (15)	(16) and (17)
Levels											
United States	0.93	0.93	0.92	−2.05	−0.46	−0.98	−0.53	0.14	0.40	0.45	0.50
United Kingdom	1.02	1.08	1.09	−3.28	2.44	1.97	−0.54	0.11	0.18	0.32	0.34
United States and United Kingdom	0.95	0.95	0.94	−3.62	0.81	0.24	−0.78	0.20	0.34	0.15	0.14
Rates of Change (Zero Intercept)											
United States	0.86	0.84	0.88	−2.75	−0.26	−0.53	−0.64	−0.28	−0.10	0.48	0.58
United Kingdom	0.86	0.91	0.84	−2.96	0.48	−0.50	−0.45	−0.26	−0.10	0.45	0.39
United States and United Kingdom	0.86	0.86	0.87	−2.78	0.34	−0.50	−0.57	−0.27	−0.12	0.46	0.49
Rates of Change (Nonzero Intercept)											
United States	0.79	0.90	0.88	−3.56	−0.20	−0.56	−0.67	−0.41	−0.09	0.47	0.58
United Kingdom	0.73	0.65	0.66	−3.35	1.68	−1.19	−0.48	0.16	0.09	0.42	0.43
United States and United Kingdom	0.77	0.74	0.77	−3.37	−1.13	−0.89	−0.59	−0.03	0.06	0.45	0.52

[a]Equations on which estimates are based.

for length of phases are somewhat more homogeneous. In general, the value of w is lower for levels than for rates of change, particularly for the United Kingdom, a result to be expected from the greater importance of trends for levels. For rates of change and equation (15), all the estimates are close together, ranging from .42 to .48. The reciprocal of w is the *average* number of phases over which expectations are formed.[19] For rates of change and equation (15), that average varies only from 2.1 to 2.4 phases. In terms of years, the estimated average period over which expectations are formed is slightly over four years for the United States, about six years for the United Kingdom—periods that are eminently reasonable by comparison with similar estimates for different phenomena.

8.3 Replacing Prior Income by Prior Money

Equations (14) and (16) express current income as a function of the current and prior phases' money and the two preceding phases' income. Income for the two preceding phases can be replaced by the relevant similar expression, and so on, so that these equations can in principle be converted into equations expressing current income as a function solely of current and past levels of money. Similarly, equations (15) and (17) can be converted into equations expressing the current rate of change of income as a function solely of present and past rates of change of money.

This conversion is straightforward for equations (14) and (15), which treat the phases as equal in length, and, as we have seen, little if anything is gained by allowing for the variable length of phase. Repeated substitution converts these equations into equations of the form:

$$\text{(18)} \qquad \log Y(t) = a' + f'Z + b_0 M(t) + b_1 M(t-1) + b_2 M(t-2) + \dots + b_i M(t-i) + \dots$$

$$\text{(19)} \qquad g_Y(t) = b_o g_M(t) + b_1 g_M(t-1) + b_2 g_M(t-2) + \dots + b_i g_M(t-i) + \dots,$$

where the coefficients are functions of the coefficients of equations (14) and (15).[20]

19. The average is a weighted average of the time gap between prior phases and the current phase, the weights being the exponential weights assumed to be used in forming expectations.

20. The relation between the sets of coefficients is as follows:

$$a' = \frac{a}{1-d-e}$$

$$f' = \frac{f}{1-d-e}$$

$$b_0 = b$$

We have estimated equations (18) and (19) directly by cutting off the series at $M(t-3)$; that is, including the current and three prior phases for money. The final column of table 8.3 gives the standard errors of estimate of the resulting equations. The results are most unsatisfactory. Not only are the standard errors higher than those for the equations that allow for yields directly, and higher than or roughly the same as those for equations including prior phases' income, but also, for three of the nine comparisons, the standard errors are higher than those from the simple regression of income on current money alone—that is, adding the prior phases' money simply uses up degrees of freedom.

We have also estimated the coefficients in equations (18) and (19) indirectly for an indefinite number of phases from the computed coefficients of equations (14) and (15) by using the expressions in footnote 20 above.[21] The direct and indirect estimates are given in table 8.5 for both levels and rates of change, with a zero intercept. Three items are of interest in this table.

1. There is only a family resemblance between direct and indirect coefficients, somewhat closer for rates of change than for levels, but not really close for any pair. For the indirect estimates, coefficients are given separately for the current and six prior phases, but only the sum of the coefficients, always trivial, is given for the remaining phases. The greatest similarity is in the coefficient of current money, which is to be expected, since this term dominates most of the correlations.

2. Nine of the twelve reaction patterns—three for levels and all six for rates of change—conform to the patterns predicted in chapter 2 from

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$$\begin{aligned} b_1 &= db + c \\ b_2 &= db_1 + eb_0 \\ b_3 &= db_2 + eb_1 \\ &\cdots\cdots\cdots\cdots \\ b_i &= db_{i-1} + eb_{i-2} \end{aligned}$$

For derivation, see appendix to this chapter (sec. 8.4.).

A similar conversion is extremely complex for equations (16) and (17), which allow for differences in lengths of phases. We have not been able to express the result in a compact form suitable for presentation, much less for computation. Accordingly we have had to restrict ourselves to the simpler expressions (18) and (19). However, from the estimates of the structural parameters based on equations (16) and (17) it is possible to estimate coefficients of hypothetical equations (14) and (15) for equal length phases, and thence of hypothetical equations (18) and (19) for equal length phases. Equations (18) and (19) are specific versions of equation (36) in Gordon, *Milton Friedman's Monetary Framework*, p. 41.

21. We have also estimated them indirectly from the structural parameters derived from equations (16) and (17), which allow for the variable length of phases. However, we do not present those estimates because they add little or nothing to the results from equations (14) and (15).

Table 8.5 Relation between Nominal Income and Money in Current and Prior Phases

Country and Method of Estimation[a]	Coefficient of Money in Phase							Sum of Coefficients	
	t	t-1	t-2	t-3	t-4	t-5	t-6	Remaining Phases	All Phases
				Levels					
United States									
Direct	1.024	−0.085	−0.011	0.000					0.928
Indirect	0.958	0.020	−0.024	−0.014	−0.007	−0.004	−0.002	−0.001	0.926
United Kingdom									
Direct	1.027	−0.201	0.096	0.144					1.066
Indirect	0.806	0.084	0.048	0.036	0.027	0.020	0.015	0.047	1.083
United States and United Kingdom									
Direct	0.994	−0.110	0.055	0.016					0.955
Indirect	0.836	0.070	0.010	0.005	0.004	0.003	0.003	0.023	0.954
				Rates of Change (Zero Intercept)					
United States									
Direct	1.105	−0.214	−0.094	0.143					0.940
Indirect	1.144	−0.305	0.033	−0.021	−0.002	−0.003	−0.001	−0.001	0.844
United Kingdom									
Direct	1.070	−0.351	0.238	0.036					0.993
Indirect	0.926	−0.059	0.081	0.036	0.022	0.013	0.008	−0.116	0.911
United States and United Kingdom									
Direct	1.112	−0.282	0.026	0.105					0.961
Indirect	1.066	−0.201	0.015	−0.011	−0.003	−0.002	−0.001	−0.001	0.862

[a]Indirect estimates are based on coefficients of equations (14) and (15).

theoretical considerations. The six rate-of-change patterns plus two of the level patterns (direct estimates for the United States and for the United States plus the United Kingdom) correspond to the solid and dotted lines in chart 2.3. That is, there is an initial overshoot, in the sense of an initial percentage increase in nominal income in response to a 1 percent increase in the rate of monetary growth larger than the ultimate increase as measured by the sum of the coefficients, followed by a damped cyclical return. The pattern is highly damped, so that generally only the first two or three coefficients are appreciable in size. One additional level pattern (indirect for the United States) shows an initial overshoot followed by an asymptotic return. The remaining three level patterns do not overshoot. Two show a steady unidirectional approach to the final position. The assumptions we have used to convert the demand curve into a relation between current income and current and prior money permit a wide variety of patterns of adjustment, but the empirical results concentrate on only a few—especially an initial overshoot followed by cyclical return.

It is not surprising that the rate-of-change results correspond to theoretical expectations better than the level results. We are here exploring dynamic patterns, and these are likely to be registered more faithfully in the rate-of-change observations than in the trend-dominated level observations. The implication is that the rate-of-change results deserve greater weight for the present purpose than the level results.

3. The sum of coefficients is less than unity in ten out of twelve cases. The two exceptions are for the United Kingdom estimates from levels. The differences from unity are relatively small and may not be statistically significant. Nonetheless, the consistency of the tendency is intriguing.

The assumptions we have used to translate the demand curve for money balances into a relation between income and prior money imply that the sum of the coefficients should be unity. A real income elasticity of demand of unity implies a rate of growth of nominal income equal to the rate of growth of nominal money, even though real income changes over time. Suppose the rate of monetary growth rises from, say, 5 percent to 6 percent a year. The equilibrium rate of growth of nominal income would then, on our assumptions, also rise from 5 percent to 6 percent a year. True, at the higher rate of monetary growth prices would rise at a 1 percent higher rate, and hence nominal yields on nominal and physical assets would be 1 percent higher, which would affect the amount of money demanded for given income, but this would be a transient effect. Once nominal income adjusted to the new yields, income would again rise at the same rate as money.

The tendency for the sum of the coefficients to be less than one requires a closer examination of the effect of an income elasticity not equal to unity, and of transient effects. We shall discuss the effect of each separately, then present the results of our computations to test these effects for the two together.

8.3.1 Income Elasticity

If real per capita income grows, as it has in fact in the United States and the United Kingdom during the past century, an income elasticity different from unity would make nominal income grow at a different rate than nominal money—at a lower rate for the greater than unity elasticity that we have found to characterize the United States, and at a higher rate for the less than unity elasticity that we have found to characterize the United Kingdom.[22] However, so long as changes in the rate of monetary growth did not affect the *rate* of real growth, a one percentage point increase in the rate of monetary growth would still imply a one percentage point increase in the rate of income growth. In the example of the next to last paragraph, a 5 percent rate of monetary growth might imply a 4 percent rate of income growth—one percentage point of the monetary growth being absorbed, say, by a 1 percent per year rise in population, one percentage point directly by a 1 percent per year rise in per capita real income, one percentage point indirectly via higher real cash balances, and the remaining two percentage points by rising prices, so that the 4 percent rate of income growth would be half in prices, half in total output. A rise in the rate of monetary growth from 5 percent to 6 percent a year would, transitional effects aside, raise the rate of income growth from 4 to 5 percent, the extra one percentage point taking the form of a higher rate of price rise.

In our calculations, this effect would bias our estimated sum of coefficients. For the level equations, it means that the long-term trends of income and money would differ. Because of the strong trend component in money, its coefficients would reflect the trend effect. For rate-of-change equations, it means that a nonzero constant term equal to $(1 - \alpha)g_y$ is required, whereas we have used a zero constant term.[23] The differences in table 8.5 between the United States and the United Kingdom with respect to the sum of the coefficients are consistent with such a bias: The sum is uniformly higher for the United Kingdom than for the United States, precisely the result to be expected from a greater than unity elasticity for the United States and a less than unity elasticity for the United Kingdom. Similarly, the less than unity sums for the United States, and the greater than unity sums for levels for the United Kingdom

22. Nominal income growth (g_Y) would differ from nominal money growth (g_M) by $(1-\alpha)g_y$, where α is the income elasticity of demand for real balances per capita and y is per capita real income.

23. See appendix to this chapter (sec. 8.4).

are consistent with such a bias, but the below unity sums for the United Kingdom for rates of change are not. This exception presumably reflects the absence of a positive correlation for the United Kingdom between rates of monetary change and rates of real income change.

8.3.2 Transient Effects

We pointed out in chapter 2 that a movement from one rate of monetary growth to another rate of monetary growth would have transient effects—a once-for-all shift to bring real cash balances to the new level appropriate to the new rate of price change associated with the new rate of monetary growth. In the example of a rise in the rate of monetary growth depicted in chart 2.3, we described the transient effects by saying that the area between the adjustment path and the ultimate equilibrium rate of nominal income growth must be positive. In terms of the sum of the b's, this means that the observed sum would be greater than unity for a shift to a higher rate of monetary growth, less than unity for a shift to a lower rate of monetary growth. The equations we have so far used obviously do not embody this effect, since for a constant rate of monetary growth there is no way of knowing whether it came from a higher or a lower prior rate. Put differently, to allow for transient effects we must go to higher derivatives of monetary growth.

If the rate of monetary growth had fluctuated around the same level throughout the period, above-average rates of monetary growth would have offset below-average rates, so that no bias would have been introduced, though the transient effect would obviously introduce error into the statistical estimates. However, in practice, the rate of monetary growth on the average rose over the period as a whole. By itself, this should have tended to bias the sum of the b's upward.

We show in section 8.4 that one way to allow for the transient effects is to include terms in g_M in the level equations and terms in Dg_M in the rate-of-change equations. To keep down the number of parameters estimated, we have added only two terms—$g_M(t-1)$ and $g_M(t-2)$ to the level equations, $Dg_M(t)$ and $Dg_M(t-1)$ to the rate-of-change equations.[24]

8.3.3 The Effect of Nonunit Elasticity and Transient Effects on the Sum of the b's

Table 8.6 summarizes the sum of the b coefficients for both direct estimates of equations (18) and (19) and indirect estimates, both allowing for and not allowing for trend and transient effects. Because of the difference in income elasticity between the United States and the United Kingdom, results are given only for the two countries separately. For the

24. We have used different lags for the level and rate of change equations because our method of calculating $g_M(t)$ uses $M(t+1)$, $M(t)$ and $M(t-1)$, while our method of calculating $Dg_M(t)$ uses only $g_M(t)$ and $g_M(t-1)$.

Table 8.6 **Sum of *b* Coefficients: Effect of Allowing for Trend and Transient Effects; Significance of Difference from Unity**

	Sum of Coefficients, Allowing for			*F* Ratio for Significance of Difference between Unity and Sum of *b* Coefficients, Allowing for			
Country and Method of Estimation[a]	Neither Trend nor Transient Effects (1)	Trend Only (2)	Both Trend and Transient Effects (3)	Neither Trend nor Transient Effects (4)	Trend Only (5)	Both Trend and Transient Effects (6)	.05 Value[b] (7)
			Levels				
United States							
Direct	0.93	1.06	1.07	136.21	2.02	2.57	4.1
Indirect	0.93	1.03	1.09	9.27	0.17	1.60	
United Kingdom							
Direct	1.07	1.20	1.24	23.33	7.83	10.23	4.2
Indirect	1.08	1.14	1.27	2.64	0.33	2.48	
			Rates of Change				
United States							
Direct	0.94	1.15	1.11	1.54	3.04	1.52	4.1
Indirect	0.84	0.91	0.88	5.48	0.66	1.06	
United Kingdom							
Direct	0.99	0.90	1.03	0.01	0.59	0.08	4.2
Indirect	0.91	0.65	0.78	0.48	3.84	1.51	

[a]Indirect method based on equations (14) and (15).
[b]Differences in degrees of freedom between columns 4, 5, and 6 do not affect *F* value to the one decimal given here.

United States, the results are in the direction to be expected from above unit income elasticity. The sum of the b's is raised by allowing for trend alone or trend and transient effects, in all eight comparisons. The effect is to make the sum higher than the theoretical unity rather than lower in six out of eight comparisons. For the United Kingdom the results are more mixed. The sum is raised, rather than lowered as would be expected from a below unit income elasticity, in all four level comparisons and in one of the four rate-of-change comparisons, though by a trivial amount. Three of the eight sums are below unity, five above.

With respect to the deviation from unity, allowing for trends or trend and transient effects increases the deviation from unity for eleven out of sixteen comparisons, a rather disappointing result. However, the F ratios indicate that this result is somewhat misleading. Even for some of the comparisons in which the deviation from unity is increased, the statistical significance of the deviation is sharply reduced. This effect is particularly marked for the level equations, and particularly for the direct estimates. The reason is that if trend is not allowed for explicitly, the deviation of the sum of b's from unity implicitly allows for trend: both income on the left-hand side of the equation and the values of M on the right-hand side have an upward trend. For the United States, the secular decline in velocity (after allowance for dummies) means that the trend in nominal money is steeper than the trend in nominal income; a sum of b's less than unity allows for this difference in trend. Similarly for the United Kingdom, a sum of b's greater than unity allows for a steeper trend in nominal income than in nominal money. If the sum of b's is forced to unity there is no way to allow for trend, which is why the difference from unity is so significant statistically. Once trend is explicitly allowed for, it takes up the variance previously accounted for by the difference between the sum of b's and unity, so the difference of the sum of the b's from unity is less important.

The same effect is present to a much lesser extent for the indirect estimates. The reason is that two of the variables on the right-hand side are prior incomes, which also have different trends from money, and thereby implicitly allow for the trend on the right-hand side.

Once trend is allowed for, only two F ratios are clearly significant (both for levels and for the United Kingdom), and one additional one is on the borderline, also for the United Kingdom but for rates of change. It is notable that a deviation of a given size yields a lower F ratio for the indirect than for the direct estimates. The reason is that the statistical error is compounded in computing the indirect estimates.[25] As earlier, the rate-of-change results are more satisfactory than the level results.

25. The condition for the sum of the coefficients to be greater than unity is that $b + c + d + e \lesseqgtr 1$; but the estimated sum of the coefficients is $\frac{b + c}{1 - d - e}$, which is less stable statistically than $b + c + d + e$. This sum alone is much less erratic but necessarily, of course, on the same side of unity as the numbers entered in the table.

Taken as a whole, and allowing crudely for the nonindependence of the various comparisons, restricting the sum of the b's to unity, which saves a degree of freedom and conforms to theoretical expectations, does not seem inconsistent with the data.

Another way of judging the effect of allowing for trend, and for transient effects, and also of restricting the sum of the b's to unity, is to compare the estimates of α obtained from the trend terms of the level equations and the constant terms of the rate-of-change equations. These estimates, summarized in table 8.7, suggest that restricting the sum of the b's to unity provides the more reasonable estimates of α. All such estimates are above unity for the United States, below unity and positive for the United Kingdom, whereas four of the estimates for the United Kingdom from the unrestricted equations are above unity and two are negative. All of the estimates from the restricted equations are either within or close to the limits on the elasticity in table 6.15, whereas many of the estimates for the unrestricted equations are not. These elasticity

Table 8.7 **Estimates of Income Elasticity Based on Trend Terms**

Country and Method of Estimation	Estimated Value of Income Elasticity (α)[a]			
	Transient Effects Not Allowed for, Sum of b's		Transient Effects Allowed for, Sum of b's	
	Not Restricted	Restricted to Unity	Not Restricted	Restricted to Unity
	Levels			
United States				
Direct	1.20	1.11	1.45	1.24
Indirect	1.17	1.12	1.52	1.24
United Kingdom				
Direct	1.18	0.92	1.63	0.80
Indirect	1.08	0.89	1.66	0.71
	Rates of Change			
United States				
Direct	1.77	1.38	1.56	1.25
Indirect	1.29	1.62	1.10	1.52
United Kingdom				
Direct	0.60	0.89	0.66	0.58
Indirect	−0.40	0.65	−0.05	0.64

[a]For direct method, calculated from slope of trend term added to equation (18) or constant term added to equation (19), both truncated at t-3. These terms were set equal to $(1-\alpha)\bar{g}_y$, where $\bar{g}_y$ is the slope of a straight-line trend fitted to per capita real income for the period 1873–1973.

For indirect method, calculated from coefficients of equation (14) with trend term added and equation (15) with constant term added by setting these terms equal to $(1-\alpha)(1-d-e)\bar{g}_y$.

See appendix equations (A71) and (A72).

estimates reinforce the desirability of restricting the sum of the b's to unity.

8.3.4 Estimates of the Transient Effect

The equations we have used to estimate the transient effect are as follows:

For levels,

(20) $$\log Y = a + k\,T + b_0 M(t) + b_1 M(t-1) + b_2 M(t-2) + b_3 M(t-3) + j_1 g_M(t-1) + j_2 g_M(t-2) + fZ\,,$$

(21) $$\log Y = a' + k'T + b\,M(t) + c\,M(t-1) + d\,Y(t-1) + e\,Y(t-2) + j_1 g_M(t-1) + j_2 g_M(t-2) + fZ\,,$$

where T is chronological time.

For rates of change,

(22) $$g_Y = k + b_0 g_M(t) + b_1 g_M(t-1) + b_2\, g_M(t-2) + b_3\, g_M(t-3) + j_0\, Dg_M(t) + j_1 Dg_M(t-1)\,,$$

(23) $$g_Y = k' + b\, g_M(t) + c\, g_M(t-1) + d\, g_Y(t-1) + e g_Y(t-2) + j_0 Dg_M(t) + j_1 D\, g_M(t-1).$$

Equations (20) and (22) are the ones we have referred to as the direct estimates, (21) and (23) are the indirect estimates.

We show in section 8.4 that, if j_1 and j_2 or j_0 and j_1 fully captured all transient effects, their sum would theoretically equal $-(\delta + \epsilon)$, that is, the negative of the sum of the logarithmic slopes of the demand curve for real money balances with respect to nominal yields on nominal and physical assets. Accordingly, we give our direct estimates of $-(\delta + \epsilon)$ in the first column of table 8.8. The second column gives the sum of the j terms of equations (20) and (22) for the direct estimates and equations (21) and (23) for the indirect estimates, with the sum of the b's restricted to unity.[26] The next two columns give the individual coefficients.

Clearly the two terms we include have not captured anything like the whole of the theoretical transient effect. The largest sum is less than half of the theoretical estimate. For six of the eight sets of estimates, the initial calculated transient effect is positive and the second negative, indicating a cyclical reaction; for the other two, both are positive. Our failure to confirm a larger fraction of the theoretical effect may reflect our including only the first cycle in the reaction and not a later rebound, implying a rather long-lasting transient effect, given that our time unit is a phase. Some experimental calculations for annual data are not inconsistent with this interpretation.

26. For equations (21) and (23), this is equivalent to restricting $b + c + d + e$ to unity. We have also estimated similar equations without the restrictions with essentially similar results.

Table 8.8 **Estimates of Transient Effects and Tests of Their Statistical Significance (Sum of *b*'s Restricted to Unity)**

Country and Method of Estimation	Sum of Transient Effects: Hypothetical (1)	Sum of Transient Effects: Calculated (2)	Estimates of Separate Effects: Initial (3)	Estimates of Separate Effects: Second (4)	*F* Ratio: Calculated (5)	*F* Ratio: .05 Value (6)
Levels						
United States						
Direct	2.58	1.17	1.23	−0.06	2.6	3.2
Indirect		0.27	0.77	−0.51	1.4	
United Kingdom						
Direct	3.82	1.29	2.30	−1.01	1.9	3.4
Indirect		−0.17	0.62	−0.79	1.1	
Rates of Change						
United States						
Direct	4.23	0.72	0.20	0.51	2.2	3.2
Indirect		0.17	0.49	−0.32	1.0	
United Kingdom						
Direct	3.83	1.48	0.42	1.06	9.6	3.4
Indirect		0.52	1.12	−0.60	4.0	

All in all, these estimates are not statistically very firmly established, as the F ratios in column 5, as well as t values for individual coefficients, indicate. Only the United Kingdom estimates from rates of change are clearly statistically significant. Yet it is encouraging that, as theory suggests, seven out of the eight signs of the calculated effects in column 2 are positive, and that none of the sums exceed the theoretical total, indicating that we are capturing the transient effect, but only part of the whole.

For rates of change, the estimated United Kingdom transient effects are both larger in absolute value than those for the United States and statistically far more significant. Nonetheless, there does not appear to be a statistically significant difference between the rate-of-change equations for the United States and the United Kingdom.[27] However, there is no necessary contradiction. If Dg_M varied more relative to its mean value for the United Kingdom than for the United States, the transient effects would be more important for the United Kingdom than for the United States, they would be estimated more accurately, and they would appear statistically more significant; yet the coefficients estimated for the United Kingdom might give fairly good estimates of the effect of the narrower movements of Dg_M in the United States, and so the equations for the two countries might not differ significantly. And indeed that seems to be the case.

8.3.5 Summary

We summarize in table 8.9 and in chart 8.1 our final estimates of the reaction pattern of nominal income to nominal money. In all cases the sum of the b's is restricted to unity and the reaction patterns are based on the indirect equations including current money and past money and income, in order to have a full reaction pattern, and including terms to allow for transient effects. We have included not only patterns for the United States and the United Kingdom separately but also a pattern for the two combined that allows for a difference in income elasticity.[28]

The table gives (and the graph plots) the cumulated effect of a sustained increase in the rate of monetary growth by one percentage point—

27. The F ratio for the difference between the equations with the sum of the b's restricted to unity is 1.9 for the direct equations, 0.75 for the indirect, compared with a .05 F value of 2.3. The results are essentially the same for equations for which the sum of the b's is not restricted. Such tests are not readily available for the level equations because the data used in computing a United States plus United Kingdom level equation are not simply the combination of the data used in computing the United States and United Kingdom equations separately. As explained earlier (see footnote 4 above), for the combined equation, United Kingdom money and income figures were first converted to dollars, whereas for the separate United Kingdom equations they were kept in pounds.

28. The combined pattern for the two allows for the difference in income elasticity by including in the level equations not only a country dummy but also a dummy times the trend variable, and in the rate of change equations, a country dummy.

Table 8.9 **Cumulative Effect on the Rate of Growth of Nominal Income of a Sustained One Percentage Point Increase in the Rate of Monetary Growth Initiated in Phase *t*, Eliminating and Including Transient Effects**

Country	Trend (Annual Percentage)	Cumulative Effect at Phase t	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	∞
	Levels, Eliminating Transient Effects								
United States	−0.20	0.884	0.912	0.956	0.977	0.988	0.994	0.998	1.000
United Kingdom	0.09	0.650	0.829	0.900	0.937	0.959	0.973	0.982	1.000
United States and United Kingdom	U.S. −0.12	0.811	0.942	0.972	0.984	0.991	0.995	0.997	1.000
	U.K. 0.07								
	Levels, Including Transient Effects								
United States	−0.20	0.884	1.284	0.712	0.977	0.988	0.994	0.998	1.000
United Kingdom	0.09	0.650	1.064	0.600	0.937	0.959	0.973	0.982	1.000
United States and United Kingdom	U.S. −0.12	0.811	1.093	0.724	0.984	0.991	0.995	0.997	1.000
	U.K. 0.08								
	Rates of Change, Eliminating Transient Effects								
United States	−0.52	1.020	0.968	0.990	0.996	0.999	1.000	1.000	1.000
United Kingdom	0.18	0.321	0.701	0.925	1.012	1.028	1.020	1.010	1.000
United States and United Kingdom	U.S. −0.38	0.702	0.941	0.999	1.006	1.004	1.002	1.001	1.000
	U.K. 0.03								
	Rates of Change, Including Transient Effects								
United States	−0.52	1.253	0.816	0.990	0.996	0.999	1.000	1.000	1.000
United Kingdom	0.18	0.746	0.474	0.925	1.012	1.028	1.020	1.010	1.000
United States and United Kingdom	U.S. −0.25	1.047	0.733	0.999	1.006	1.004	1.002	1.001	1.000
	U.K. 0.05								

Note: Computed indirectly from equations (21) and (23) with sum of b's restricted to unity. Patterns including transient effects calculated by using equations (A89) and (A90) of sec. 8.4. United States plus United Kingdom equations allow for difference in income elasticity by including in level equations not only a country dummy but a country dummy times T (chronological time) and in rate-of-change equations a country dummy.

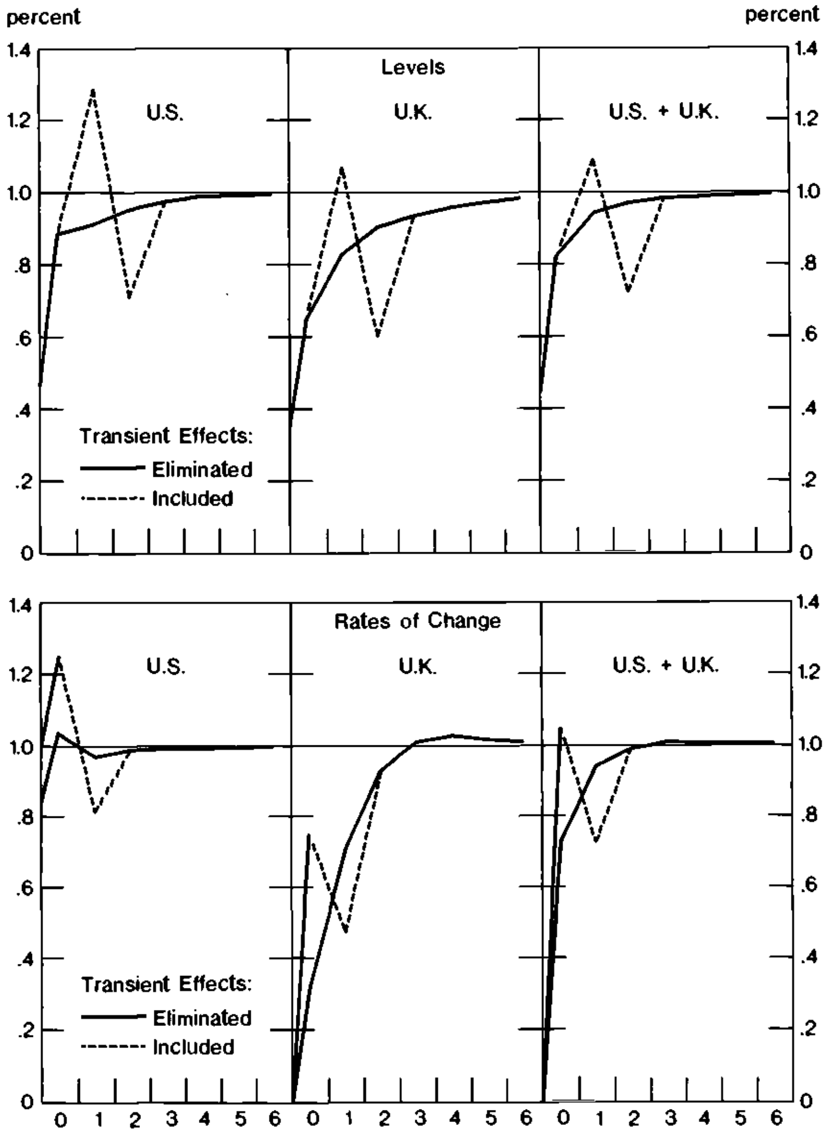

Chart 8.1 Reaction patterns of nominal income to monetary change.

that is, the values entered in the table and plotted on the chart are the cumulative sums of the *b*'s. This makes the chart strictly comparable with chart 2.3, except that we have not shown explicitly the zero axis.[29] The solid lines beginning at t_o eliminate the transient effects, the dashed lines add them in.

29. The initial solid line segments in chart 8.1, if extended, would go through the zero horizontal axis at the midpoint of the prior phase.

For levels, the patterns for the United States and the United Kingdom are highly similar. Both show a gradual, roughly asymptotic movement to the new equilibrium position, somewhat slower for the United Kingdom than for the United States, but in neither case involving any overshooting. The failure to adjust immediately reflects a lag in adjustment, which is in a way a transient effect, but a very different one from that involved in the once-and-for-all shift to a higher level of velocity because of a higher rate of price rise and hence of nominal interest rates. That effect is embodied in the dashed line. The dashed lines, which incorporate the transient effect and thus show the predicted actual reaction pattern, involve an overshoot and then a cyclical reaction.

For rates of change there is more of a difference between the solid lines. For the United States there is almost immediate adjustment, for the United Kingdom, a long-delayed adjustment. But the dashed lines again show the same pattern for the two countries. Despite the apparent large difference in the solid lines, our earlier results indicate that the difference between them or between the dashed lines is not statistically significant.

For the United States plus the United Kingdom, the solid line for the levels, eliminating transient effects, repeats the gradual roughly asymptotic movement to the new equilibrium observed for the separate countries. The dashed line for the levels, including transient effects, shows an overshoot and then a cyclical reaction. For rates of change for the two countries combined, the solid line shows a delayed adjustment, less marked than in the case of the United Kingdom alone. The dashed line, as for the levels, shows an overshoot and a cyclical reaction thereafter.

We cannot claim to have pinned down the reaction pattern in any precise quantitative terms, yet the similarity among the patterns for the two countries and for levels and rates of change, and the concordance with theoretical expectations is encouraging, especially given the compounding of substantial errors of estimation by the indirect derivation of the patterns.

8.4 Appendix: Derivation of Relations between Nominal Income and Other Variables

8.4.1 Relation of Nominal Income to Real per Capita Income and Yields

Levels

We start with the demand equations of chapter 6, omitting for simplicity the postwar readjustment and demand shift dummies.[30]

30. In section 8.4.1, the omission is readily repaired by recognizing that it is only necessary to add terms for W and S or W_g and S_g duplicating those for Z. In section 8.4.2 the omission cannot be repaired, but in the computations based on the equations in that section we have taken account of these effects by adjusting the money series at the outset. See note 13.

(A1)
$$\log m(t) = \log k + \lambda_3 Z + (\alpha + \lambda_4 Z) \log y(t) + \delta R_N(t) + \epsilon g_{Y(t)},$$

where

$m(t)$ = per capita real money balances at time t

$y(t)$ = per capita real income at time t

$R_N(t)$ = differential interest yield at time t on nominal assets

$g_Y(t)$ = rate of change of nominal income at time t as proxy for nominal yield on physical assets

Z = dummy variable equal to 1 if the observations are for the United Kingdom, to 0 if they are for the United States. Note that Z is not a function of time, but only of country,

and

$\log k$ = constant term of demand function

α = elasticity of demand for real per capita money balances with respect to real per capita income

δ = percentage change in m for a one percentage point change in R_N, given y and g_Y, or semilogarithmic money demand slope with respect to differential yield on nominal assets

ϵ = percentage change in m for a one percentage point change in g_Y, given y and R_N, or semilogarithmic money demand slope with respect to nominal yield on physical assets

λ_3 = excess of level of log m in United Kingdom over that in United States for given y, R_N, and g_y. If y for United Kingdom is in pounds, for United States in dollars, includes adjustment for exchange rate

λ_4 = excess of income elasticity in United Kingdom over that in United States.

Given that the same price index and population are used to deflate aggregate nominal income and nominal money to per capita real income and per capita real money, we have

(A2)
$$\log y - \log m = \log Y - \log M,$$

where

$$Y = \text{aggregate nominal income}$$
$$M = \text{aggregate nominal money.}$$

Solve equation (A2) for

$$\text{(A3)} \qquad \log m = \log y - \log Y + \log M,$$

substitute the right-hand side for $\log m$ in equation (A1), and solve for log Y. This gives

$$\text{(A4)} \qquad \begin{aligned} \log Y(t) = & -[\log k + \lambda_3 Z] + \log M(t) \\ & + (1 - \alpha - \lambda_4 Z) \log y(t) \\ & - \delta R_N(t) - \epsilon g_Y(t). \end{aligned}$$

Insert a free coefficient, ζ, before $\log M(t)$, and we have

$$\text{(A5)} \qquad \begin{aligned} \log Y(t) = & -[\log k + \lambda_3 Z] + \zeta \log M(t) \\ & + (1 - \alpha - \lambda_4 Z) \log y(t) - \delta R_N(t) - \epsilon g_Y(t), \end{aligned}$$

which, except for dummies, is equation (6) of the text.

Let $\alpha = 1$, $\lambda_4 = 0$, and we have

$$\text{(A6)} \qquad \begin{aligned} \log Y(t) = & -[\log k + \lambda_3 Z] + \zeta \log M(t) \\ & - \delta R_N(t) - \epsilon g_Y(t), \end{aligned}$$

which is equation (6a) of the text.

Rates of Change

The rate-of-change demand equation:

$$\text{(A7)} \qquad g_m(t) = [\alpha + \lambda_4 Z] g_y(t) + \delta\, D[R_N(t)] + \epsilon D[g_Y(t)],$$

by precisely similar steps can be expressed as

$$\text{(A8)} \qquad \begin{aligned} g_Y(t) = \zeta g_M(t) + (1 - \alpha - \lambda_4 Z) g_y(t) - \delta D[R_N(t)] \\ - \epsilon D[g_Y(t)], \end{aligned}$$

and

$$\text{(A9)} \qquad g_Y(t) = \zeta g_M(t) - \delta D[R_N(t)] - \epsilon D[g_Y(t)],$$

which are the counterparts of equations (7) and (7a) of the text.

8.4.2 Relation of Nominal Income to Current and Prior Money and Income

Levels

Assume that

$$\text{(A10)} \qquad R_S(t) = k_o + g_Y^*(t)\ ,$$

$$\text{(A11)} \qquad g_Y^*(t) = w\, g_Y(t) + (1 - w)\, g_Y^*(t-1)\ ,$$

the appropriate adaptive expectations equation if g_Y has no anticipated trend,[31]
and that

$$\text{(A12)} \qquad g_Y(t) = \frac{\log Y(t) - \log Y(t-1)}{\bar{n}(t)},$$

where

(A13) $n(t)$ = length of phase t, with t interpreted as number of phase, not chronological date,

(A14) $\bar{n}(t) = \dfrac{n(t) + n(t)}{2}$ = average length of phase t and $t-1$, or period between central points of two successive phases.

In the rest of this book we have estimated $g_Y(t)$ by the slope of a straight line trend fitted to three successive phase values of log Y, but we shift to this form at this point for reasons outlined in the text.

Substitute from equation (A10) into equation (A5), modified to substitute R_S for R_N, in order to simplify the subsequent analysis,

$$\text{(A15)} \qquad \begin{aligned} \log Y(t) = & -[\log k + \delta k_o + \lambda_3 Z] \\ & + \zeta \log M(t) + (1 - \alpha - \lambda_4 Z) \log y(t) \\ & - \delta g_Y^*(t) - \epsilon g_Y(t), \end{aligned}$$

which for $\alpha = 1$, $\lambda_4 = 0$, is the counterpart of equation (13) of the text.

Write equation (A15) for $(t-1)$, and subtract $(1-w)$ times that rewritten equation from (A15), modified by replacing $g_Y{}^*(t)$ by the right-hand side of equation (A11). The result is

$$\text{(A16)} \qquad \begin{aligned} \log Y(t) = & (1-w) \log Y(t-1) - w [\log k + \delta k_o + \lambda_3 Z] \\ & + \zeta[\log M(t) - (1-w) \log M(t-1)] + (1 \\ & - \alpha - \lambda_4 Z) [\log y(t) - (1-w) \log y(t-1)] \\ & - \delta w g_Y(t) - \epsilon[g_Y(t) - (1-w)\, g_Y(t-1)]. \end{aligned}$$

Replacing the g_Y terms by the relevant expressions from equation (A12) and rearranging terms yields

31. If g_Y has an anticipated exponential trend, the relevant adaptive expectations equation is

$$\text{(11)}' \qquad g_Y^*(t) = w\, g_Y(t) + (1-w)(1+k)\, g_Y^*(t-1).$$

If it has an anticipated linear trend, the relevant equation is

$$\text{(11)}'' \qquad g_Y^*(t) = w g_Y(t) + (1-w)\, [g_Y^*(t-1) + k].$$

We have assumed k to be zero because, in fact, g_Y has had no unambiguous trend.

(A17)
$$\log Y(t)\left[1+\frac{\delta w+\epsilon}{\bar{n}(t)}\right]$$
$$\begin{aligned}&=\left[1+\frac{\delta w+\epsilon}{\bar{n}(t)}\right]\log Y(t-1)\\&\quad-w(\log k+\delta k_o-\lambda_3 Z)+\zeta[\log M(t)\\&\quad-\log M(t-1)]+(1-\alpha-\lambda_4 Z)[\log y(t)\\&\quad-(1-w)\log y(t-1)]+w[\zeta\log M(t-1)\\&\quad-\log Y(t-1)]\\&\quad+\epsilon(1-w)\frac{[\log Y(t-1)-\log Y(t-2)]}{\bar{n}(t-1)}.\end{aligned}$$

Let

(A18)
$$Q(t)=1+\frac{\delta w+\epsilon}{\bar{n}(t)},$$

and divide both sides of equation (A17) by $Q(t)$. The result is

(A19)
$$\begin{aligned}\log Y(t)&=\log Y(t-1)-w\frac{[\log k+\delta k_o+\lambda_3 Z]}{Q(t)}\\&\quad+\zeta\frac{[\log M(t)-\log M(t-1)]}{Q(t)}\\&\quad+(1-\alpha-\lambda_4 Z)\frac{[\log y(t)-(1-w)\log y(t-1)]}{Q(t)}\\&\quad+w\frac{[\zeta\log M(t-1)-\log Y(t-1)]}{Q(t)}\\&\quad+\epsilon(1-w)\frac{[\log Y(t-1)-\log Y(t-2)]}{\bar{n}(t-1)\,Q(t)}.\end{aligned}$$

For $\alpha = 1$, $\lambda_4 = 0$, this is equation (16) of the text.

As explained in the text, we used an iterative nonlinear computer regression program to calculate this equation, given initial values of the various parameters. The iterations converged in all cases.

To get the equivalent of equation (14) of the text, assume

(A20)
$$n(t) = n = \text{constant independent of } t$$

so that

(A21)
$$\begin{aligned}Q(t)&=1+\frac{\delta w+\epsilon}{\bar{n}(t)}\\&=Q \text{ a constant independent of } t.\end{aligned}$$

Rearrange the terms in equation (A19) to get:

$$\text{(A22)} \qquad \log Y(t) = -w\frac{[\log k + \delta k_o]}{Q} - \left[\frac{w\lambda_3}{Q}\right]Z$$
$$+ \frac{1-\alpha-\lambda_4 Z}{Q}[\log y(t) - (1-w)\log y(t-1)]$$
$$+ \frac{\zeta}{Q}\log M(t) - \frac{(1-w)\zeta}{Q}\log M(t-1)$$
$$+ \left[1 - \frac{w}{Q} + \frac{\epsilon(1-w)}{nQ}\right]\log Y(t-1)$$
$$- \frac{\epsilon(1-w)}{nQ}\log Y(t-2).$$

For $\alpha = 1, \lambda_4 = 0$, this is now precisely in the form of equation (14) of the text:

$$\text{(A23)} \qquad \log Y(t) = a + b\log M(t) + c\log M(t-1) + d\log Y(t-1) + e\log Y(t-2) + fZ,$$

where

$$\text{(A24)} \qquad a = \frac{-w(\log k + \delta k_o)}{Q},$$

$$\text{(A25)} \qquad b = \frac{\zeta}{Q},$$

$$\text{(A26)} \qquad c = \frac{-\zeta(1-w)}{Q},$$

$$\text{(A27)} \qquad d = 1 - \frac{w}{Q} + \frac{\epsilon(1-w)}{nQ},$$

$$\text{(A28)} \qquad e = \frac{-\epsilon(1-w)}{nQ},$$

$$\text{(A29)} \qquad f = \frac{-\lambda_3 w}{Q},$$

$$\text{(A30)} \qquad Q = 1 + \frac{\delta w + \epsilon}{n},$$

and n is a number that we may set equal to the average length of a phase.

We can solve these equations for the structural parameters, except that we can estimate only $\log k + \delta k_o$, not $\log k$ and k_o separately. Divide equation (A26) by equation (A25) and solve for w:

(A31) $$w = 1 + \frac{c}{b}.$$

Replace the final term of the right-hand side of equation (A27) by its equivalent, $-e$, from equation (A28), then replace w by its equivalent from equation (A31) and solve for Q:

(A32) $$Q = \frac{1 + \frac{c}{b}}{1-d-e}.$$

Substitute from equations (A31) and (A32) into equations (A24), (A25), (A28), and (A29):

(A33) $$\log k + \delta k_o = - \frac{a}{1-d-e}$$

(A34) $$\zeta = \frac{b+c}{1-d-e}$$

(A35) $$\epsilon = \frac{e\,n(b+c)}{c(1-d-e)}$$

(A36) $$\lambda_3 = - \frac{f}{1-d-e}$$

From equation (A30),

(A37) $$\delta = \frac{(Q-1)n-\epsilon}{w}.$$

Substitute into equation (A37) from equations (A31), (A32), and (A35) and simplify:

(A38) $$\delta = -n\,b\left[\frac{1}{b+c} + \frac{\frac{e}{c} - \frac{1}{b}}{1-d-e}\right].$$

Equations (A31), (A34), (A35), (A36), and (A38) were used to calculate the estimates of structural parameters in table 8.4.

Though the equations in this and the next section for rates of change are expressed in a general form in which they apply to the United States plus the United Kingdom as well as to each country separately, it should be noted that such an application requires assuming not only that all phases are the same length in each country separately but also that they are of the same length in the United States and the United Kingdom; yet we know that, as we have defined the phases, they are longer on the average in the United Kingdom than in the United States. This difference could in principle be allowed for in equation (A22), but only at the cost of adding inordinate complexity to an already unduly complex equation. Accordingly, we have not tried to do so. As a result, any computations in the text based on this and related equations for the two countries combined deserve less confidence than the computations for the separate countries.

Rates of Change

The rate-of-change equation corresponding to equation (A19) is derived starting from equation (A8) in the same way as equation (A19) is derived from equation (A5). In doing so, we have not used equation (A12), as the basis for computing g_Y and g_M but have continued, as in the rest of the work, to use as g_Y and g_M the slopes of straight lines fitted to successive triplets of values of log Y and log M, because this raises no problem at this level about different time references on the two sides of the final equation. Since we have throughout calculated $D(g_Y)$ and $D(g_M)$ from first differences of successive values of g_Y and g_M, no problem arises for them similar to the problem that arose for g_Y and g_M themselves, so $D(g_Y)$ and $D(g_M)$ are calculated by the equivalent of equation (A12). The counterparts to equations (A10) and (A11) are simply the derivatives of those equations, or

(A39a) $$D[R_S(t)] = D[g^*_Y(t)]$$

(A39b) $$D[g^*_Y(t)] = wD[g_Y(t)] + (1-w)\, D\, [g^*_Y(t-1)].$$

The final equation is then

(A40) $$\begin{aligned} g_Y(t) = {} & g_Y(t-1) \\ & + (1-\alpha-\lambda_4 Z)[\frac{g_Y(t)-(1-w)g_Y(t-1)}{Q(t)}] \\ & + \zeta[\frac{g_M(t)-g_M(t-1)}{Q(t)}] \\ & + w[\frac{\zeta g_M(t-1)-g_Y(t-1)}{Q(t)}] \\ & + \epsilon(1-w)[\frac{g_Y(t-1)-g_Y(t-2)}{\bar{n}(t-1)Q(t)}]. \end{aligned}$$

If we assume that all phases are of equal length, the counterpart to equation (A22) is

(A41)
$$g_Y(t) = \frac{1-\alpha-\lambda_4 Z}{Q}[g_y(t)-(1-w)g_y(t-1)]+\frac{\zeta}{Q}g_M(t)$$
$$-\frac{(1-w)\zeta}{Q}g_M(t-1)+[(1-\frac{w}{Q}\epsilon+\frac{(1-w)}{nQ}]g_Y(t-1)$$
$$-\epsilon\frac{(1-w)}{nQ}g_Y(t-2),$$

and, for $\alpha = 1$, $\lambda_4 = 0$, the counterpart to equation (A23) is

(A42)
$$g_Y(t) = bg_M(t) + cg_M(t-1) + dg_Y(t-1) + eg_Y(t-2).$$

Equations (A31), (A34), (A35), and (A38), as they stand, express the structural coefficients w, ζ, ϵ, and δ in terms of these coefficients b, c, d, and e, which is why we have used these letters interchangeably as coefficients of both the level equations and the rate of change equations.

8.4.3 Relation of Nominal Income to Current and Prior Money Only

Levels

On the most general level, we can replace log $Y(t-1)$ and log $Y(t-2)$ in equation (A19) by their equivalents obtained by writing equation (A19) for $t-1$ and $t-2$; replace log $Y(t-2)$ and log $Y(t-3)$ in the result by their equivalents obtained by writing equation (A19) for $t-2$ and $t-3$; and so on; and in this way, for $\alpha = 1$ and $\lambda_4 = 0$, end up with an equation expressing log $Y(t)$ as a function solely of earlier monetary history. However, the resulting expressions quickly become unmanageably complicated because of the presence of terms in $Q(t)$ and $\bar{n}(t)$, and so we finally gave this transformation up as a bad job.

Instead we have restricted ourselves to the case in which we can treat phases as equal in length, that is, to equation (A23).

Using the lag operator L, where $L\ X(t) = X(t-1)$, we can write equation (A23) as

(A43)
$$\log Y(t) = a + (b + cL) \log M(t) + (dL + eL^2) \log Y(t) + fZ.$$

Solve for log $Y(t)$:

(A44)
$$\log Y(t) = \frac{a}{1-dL-eL^2} + \frac{b+cL}{1-dL-eL^2}\log M(t) + \frac{f}{1-dL-eL^2}Z.$$

Since a and fZ are the same for all time units, the constant term and the coefficient of Z can be obtained by setting $L = 1$, giving

(A45) $$\log Y(t) = \frac{a}{1-d-e} + \frac{b+cL}{1-dL-eL^2}\log M(t) + \frac{f}{1-d-e}Z.$$

Write the middle term on the right hand side as

(A46) $$\begin{aligned} f(L)\log M(t) &= \frac{b+cL}{1-dL-eL^2}\log M(t) \\ &= (b_0 + b_1L + b_2L^2 + \ldots + b_iL^i \\ &\quad + \ldots)\log M(t) = b_0 \log M(t) \\ &\quad + b_1 \log M(t-1) + b_2 \log M(t-2) \\ &\quad + \ldots + b_i \log M(t-1) + \ldots . \end{aligned}$$

The sum of the b's is obtained very simply, since that is the value of $f(L)$ if $\log M(t) = \log M(t-i)$ for all i, which is again equivalent to setting $L = 1$. This gives

(A47) $$\sum_{i=0}^{\infty} b_i = \frac{b+c}{1-d-e}.$$

The values of the individual b's are given by:

(A48) $$\begin{aligned} &b_0 = b \\ &b_1 = c + d\,b \\ &b_i = d\,b_{i-1} + e\,b_{i-2} \text{ for } i > 1. \end{aligned}$$

To prove that these are the correct values of b, write

(A49) $$\begin{aligned} f(L) &= \sum_{i=0}^{\infty} b_iL^i = b_0 + b_1L + \sum_{i=2}^{\infty}(d\,b_{i-1} + e\,b_{i-2})L^i \\ &= b_0 + b_1L + dL\sum_{i=2}^{\infty} b_{i-1}L^{i-1} + eL^2\sum_{i=2}^{\infty} b_{i-2}L^{i-2} \\ &= b_0 + b_1L + dL\left[\sum_{i=0}^{\infty} b_iL^i - b_o\right] + eL^2\left[\sum_{i=0}^{\infty} b_iL^i\right] \\ &= b_0 + b_1L - b_0dL + dLf(L) + eL^2f(L). \end{aligned}$$

Solve for $f(L)$:

(A50) $$f(L) = \frac{b_0 + (b_1 - b_0d)L}{1-dL-eL^2}.$$

Replace b_0 and b_1 by their values from equation (A48):

(A51) $$f(L) = \frac{b+cL}{1-dL-eL^2}. \quad \text{Q.E.D.}$$

We have used equation (A48) to estimate in the text the coefficients of equation (A45) from the coefficients of equation (A23).

Rates of Change

Equation (A42) is identical in form to equation (A23) except for $a = f = 0$. Hence, it yields equations identical to (A45), (A46), and (A48), except for setting $a = f = 0$, or

(A52)
$$g_Y(t) = b_0 g_M(t) + b_1 g_M(t-1) + b_2 g_M(t-2) \ldots + b_i g_M(t-i) + \ldots$$

with

(A53) (Identical to A48)
$$b_0 = b$$
$$b_1 = c + d\,b$$
$$b_i = d\,b_{i-1} + e\,b_{i-2} \text{ for } i > 1.$$

We have used equations (A53) to estimate in the text the coefficients of equation (A52) from the coefficients of equation (A42).

Effect of Nonunit Income Elasticity

Equation (A52) is homogeneous of zero degree in g_Y and g_M because of our assumption that the income elasticity of demand for real balances per capita (α) is unity. As noted in the text, this assumption introduces a bias into the computed b's that seems greater than any comparable bias in earlier equations, so it is desirable to generalize equations (A45) and (A52) to include an $\alpha \neq 1$.

At this point we shall drop any attempt to keep our analysis sufficiently general to apply to the United States plus the United Kingdom as well as to each country separately. Our earlier conclusion that the real income elasticity is different in the two countries would require us to allow not only for the difference in level measured by the parameter λ_3, but also for the difference in elasticities measured by the parameter λ_4. That alone would render what follows extremely complex. But in addition either we would have to neglect the difference in the average length of phases in the two countries as well as in the average rate of real income growth, or else we would have to introduce additional parameters to allow for such a difference. The added complexity would be so great that we have decided that the better part of wisdom is to proceed from here on to develop our equations only for each country separately—which is equivalent to setting $Z = 0$ in the earlier equations and interpreting the parameters as applying only to the country in question.

For $\alpha \neq 1$ and $Z = 0$, the counterpart to equation (A23) can be written as

(A54)
$$\log Y(t) = a + b \log M(t) + h \log y(t) + c \log M(t-1) + h(\tfrac{c}{b}) \log y(t-1) + d \log Y(t-1) + e \log Y(t-2),$$

where we have written $h(c/b)$ as the coefficient of $\log y(t-1)$ instead of a free coefficient, to incorporate the restriction that the coefficients of $\log y(t)$ and $\log y(t-1)$ must be in the same ratio as the coefficients of $\log M(t)$ and $\log M(t-1)$, and where

(A55)
$$h = \frac{1-\alpha}{Q}, \text{ so that}$$

(A56)
$$\alpha = 1 - hQ = 1 - \frac{h}{b}\frac{b+c}{1-d-e}.$$

In the same way, for rates of change, the counterpart to equation (A42) is

(A57)
$$g_Y(t) = b\, g_M(t) + h\, g_y(t) + c\, g_M(t-1) + h(\tfrac{c}{b}) g_y(t-1) + d\, g_Y(t-1) + e\, g_Y(t-2).$$

If we proceeded, as before, to eliminate from equations (A54) and (A57) the earlier nominal income terms, we could end up with equations like equations (A45) and (A52) except that each would also include a series of current and past values of $y(t)$, for equation (A54), and $g_y(t)$ for equation (A57), with coefficients of the same form as in equations (A48) and (A53).[32]

We do not propose to explore this in detail, partly because it introduces the serious problem of spurious correlation arising from the fact that $\log y(t)$ is a component of $\log Y(t)$, and partly because of the number of degrees of freedom that would have to be used up. However, to free the b's from bias, we can allow for the *average* rate of growth of real income per capita by assuming that

(A58)
$$\log y\,(t) = A + \bar{g}_y\, T(t),$$

where $\bar{g}_y$ is the average per year rate of growth, and $T(t)$ is the chronological date corresponding to phase t.

This would convert equation (A54) into the expression:

(A59)
$$\log Y(t) = [a + h\, A\, (1 + \tfrac{c}{b}) - h \tfrac{c}{b}\, \bar{g}_y n] + b \log M(t) + c \log M(t-1) + d \log Y(t-1) + e \log Y(t-2) + h\bar{g}_y(1 + \tfrac{c}{b})\, T\,(t),$$

32. The only difference is that b in equations (A48) and (A53) would be replaced by h and c would be replaced by $h\,(\frac{c}{b})$.

and equation (A57) into

(A60)
$$g_Y(t) = h\bar{g}_y(1 + \frac{c}{b}) + b\, g_M(t) + c\, g_M(t-1) + d\, g_Y(t-1) + e\, g_Y(t-2),$$

that is, a constant term, say $k' = h\bar{g}_y(1 + \frac{c}{b})$, would be added to equation (A42).[33]

If we now proceed to eliminate from equation (A59) the earlier values of log Y, the result will be to modify equation (A44) by adding an additional constant term and a term in T, that is, a trend term. In order to have only the phase number t as a variable and not also chronological time T, we can replace T by nt, on the assumption that the phases are all equal in length. The final result is

(A61)
$$\log Y(t) = \frac{a'}{1 - dL - eL^2} + \frac{k'n}{1 - dL - eL^2}t + \frac{b + cL}{1 - dL - eL^2} M(t),$$

where

(A62)
$$a' = a + hA(1 + \frac{c}{b}) - h\frac{c}{b}\bar{g}_y n$$

and

(A63)
$$k' = h\bar{g}_y(1 + \frac{c}{b}).$$

As before, we can get the constant term by setting $L = 1$, and the successive coefficients of $M(t-1)$, namely b_i, will be given by equation (A48). To get the coefficient of the t term, write

(A64)
$$\frac{k'n}{1 - dL - eL^2}t = k'\left[\sum_{i=0}^{\infty} b_i' L^i\right] nt = n\,k' \sum_{i=0}^{\infty} b_i'(t-i) = n\,k'\left[t \sum_{i=0}^{\infty} b_i' - \sum_{i=0}^{\infty} i b_i'\right].$$

33. It may offhand seem as if the constant term should be

$$k' = h\bar{g}_y(1 + \frac{c}{b})\, n$$

instead of

$$k' = h\bar{g}_y(1 + \frac{c}{b}).$$

That would be correct if g_Y represented the rate of change per phase. But we have used it to designate the rate of change per year, so the differentiation involved in passing from level to rate of change equations is with respect to T or (nt), not (t). We use k' to distinguish it from the k which enters into the constant term of the demand function for money.

By setting $L = 1$, we know that

$$\text{(A65)} \qquad \Sigma b_i' = \frac{1}{1-d-e}.$$

To estimate $\Sigma_{i=0}^{\infty} i\, b_i'$, we can show, by a proof comparable to that used to demonstrate that equation (A48) is correct, that

$$\text{(A66)} \qquad \begin{cases} b_0' = 1 \\ b_1' = d \\ b_i' = d\, b_{i-1}' + e\, b_{i-2}' \text{ for } i > 1. \end{cases}$$

$$\begin{aligned} \text{(A67)} \qquad \sum_{i=0}^{\infty} i\, b_i' = \sum_{i=1}^{\infty} i\, b_i' &= d + \sum_{i=2}^{\infty} [d\, b_{i-1}' + e\, b_{i-2}']i \\ &= d + d \sum_{i=2}^{\infty} [b_{i-1}'(i-1) + b_{i-1}'] \\ &+ e \sum_{i=2}^{\infty} [b_{i-2}'(i-2) + 2b_{i-2}'] \\ &= d + d \sum_{i=1}^{\infty} b_i' + 2e \sum_{i=0}^{\infty} b_i' \\ &+ d \sum_{i=1}^{\infty} b_i' i + e \sum_{i=0}^{\infty} b_i' i. \end{aligned}$$

From which

$$\begin{aligned} \text{(A68)} \qquad \sum_{i=0}^{\infty} i\, b_i'[1-d-e] &= d + d[\frac{1}{1-d-e} - 1] + 2e[\frac{1}{1-d-e}] \\ &= \frac{d-d^2-ed+d-d+d^2+de+2e}{1-d-e} \\ &= \frac{d+2e}{1-d-e}, \end{aligned}$$

so that

$$\text{(A69)} \qquad \sum_{i=0}^{\infty} i\, b_i' = \frac{d+2e}{(1-d-e)^2}.$$

We then have

$$\begin{aligned} \text{(A70)} \qquad \log Y(t) = \frac{a'}{1-d-e} - k'n \frac{d+2e}{(1-d-e)^2} + \frac{k'n}{1-d-e}t \\ + \sum_{i=0}^{\infty} b_i M(t-i), \end{aligned}$$

where the values of b_i are given by equation (A48).

If we proceed to eliminate from equation (A60) the earlier values of g_Y, we get, as the counterpart of equation (A52),

$$\begin{aligned} \text{(A71)} \qquad g_Y(t) = (1-\alpha)\bar{g}_y + b_0 g_M(t) + b_1 g_M(t-1) \\ + b_2 g_M(t-2) + \ \ldots \ + b_i g_M(t-i) + \ \ldots \end{aligned}$$

with the values of the b's again given by equations (A53).

The constant term of equation (A71) is intuitively obvious. If the income elasticity is unity, then real income growth does not affect the equality of g_Y and g_M for steady g_M. But if the income elasticity is higher than unity, then g_Y will tend to be less than g_M by the percentage rate of rise of real balances expressed as weeks of income (i.e., percentage rate of decline of velocity), which is precisely $(\alpha - 1)g$.

If we wish to derive the constant term of equation (A71) not by direct estimate but from equation (A60), we must divide the constant term of equation (A60) by 1-d-e, giving

$$\text{(A72)} \qquad (1-\alpha)\bar{g}_y = \frac{k'}{1-d-e} = k''.$$

There is no way of deriving $\bar{g}_y$ from the computed coefficients, but it can be estimated by the actual trend of real per capita income, which gives an estimate of α.

This is the expression we have used in table 8.7 to estimate the constant term of equation (A71).

8.4.4 Allowing for Transient Effects

Consider demand equation (A1) for $Z = 0$ and R_N replaced by R_S but for the moment assume $\epsilon = 0$, so that we have

$$\text{(A73)} \qquad \log m = \log k + \alpha \log y + \delta R_S.$$

Convert this equation into one in nominal income:

$$\text{(A74)} \qquad \log Y = -\log k + \log M + (1-\alpha)\log y - \delta R_S.$$

Assume that the real interest rate, the rate of growth of real per capita income, and the rate of growth of population are all constant equal to ρ_o, $\bar{g}_y$, and $\bar{g}_N$ respectively, and that the actual and expected rates of change are equal. We will then have:

$$\text{(A75)} \qquad R_S = \rho_o + g_P$$

and

$$\text{(A76)} \qquad g_P = g_M - \alpha\bar{g}_y - \bar{g}_N.$$

Substitute equation (A76) in equation (A75) and the result in equation (A4), with Z and $\epsilon = 0$ and R_N replaced by R_S, to get

$$\text{(A77)} \qquad \begin{aligned} \log Y &= -\log k + \log M + (1-\alpha)\log y - \delta[\rho_o \\ &\quad + g_M - \alpha\bar{g}_y - \bar{g}_N] \\ &= [\log k - \delta\rho_o + \delta\,\alpha\bar{g}_y \\ &\quad + \delta\bar{g}_N] + (1-\alpha)\log y + \log M - \delta g_M. \end{aligned}$$

If g_M were a constant over time, this reduces to our earlier results simply adding another term to the constant term. If either log y is constant, or $\alpha = 1$, log Y equals log M aside from a constant and $g_Y = g_M$. If g_M is not constant, we have the result described in chapter 2, that, with $\delta < 0$, a rise in the rate of monetary growth will raise income relative to money, that is, raise velocity; a fall in the rate of monetary growth will lower velocity.

If $\epsilon \neq 0$, the effect in equation (A77) is to replace δ by $\delta + \epsilon$.

Along the lines of equation (A58), assume that we can replace log y by a time trend:

$$\text{(A78)} \qquad \log y = A + \bar{g}_y T.$$

Substitute equation (A78) for log y in equation (A77) and denote the constant term by a. We then have

$$\text{(A79)} \qquad \log Y(t) = a + (1 - \alpha)\bar{g}_y T + \log M(t) - \delta g_M(t).$$

This equation is for full equilibrium. To get a corresponding equation that allows for a distributed lag in adjustment, we can replace equation (A79) by:

$$\text{(A80)} \qquad \log Y(t) = a + (1-\alpha)\bar{g}_y T + \sum_{i=0}^{\infty} b_i \log M(t-i) + \sum_{i=0}^{\infty} j_i g_M(t-i),$$

where the theory suggests that

$$\text{(A81)} \qquad \Sigma\, b_i = 1,$$

$$\text{(A82)} \qquad \Sigma\, j_i = -(\delta + \epsilon).$$

To use this in practice, we have estimated it directly, by setting $b_i = 0$ for $i > 3$, and $j_0 = j_i = 0$ for $i > 2$. The reason for setting $j_0 = 0$ is that our method of calculating $g_M(t)$ uses $M(t + 1)$, and we have wanted to use only current or past data.

We have also used the results of the earlier sections of this appendix to estimate equation (A80) indirectly from

$$\text{(A83)} \qquad \log Y(t) = a + k\, T + b \log M(t) + c \log M(t-1) + d \log Y(t-1) + e \log Y(t-2) + j_1 g_M(t-1) + j_2 g_M(t-2).$$

Take the time derivative of equation (A79):

$$\text{(A84)} \qquad g_Y(t) = (1 - \alpha)\bar{g}_y + g_M(t) - \delta D g_M(t).$$

Again in distributed lag form, we have estimated it by

$$\text{(A85)} \qquad g_Y(t) = (1-\alpha)\bar{g}_y + \sum_{i=0}^{\infty} b_i g_M(t-i) + \sum_{i=0}^{\infty} j_i D g_M(t-i),$$

this time setting $b_i = 0$ for $i > 3$ and $j_i = 0$ for $i > 2$, since our method of calculation does not raise the same problem for Dg_M as for g_M.

For an indirect estimate, we have computed

$$\text{(A86)} \qquad g_Y(t) = k' + b\,g_M(t) + cg_M(t-1) + d\,g_Y(t-1) + e\,g_Y(t-2) + j_0\,Dg_M(t) + j_1\,D\,g_M(t-1).$$

If all phases were in fact equal in length, the j terms in equations (A80) and (A85) would be redundant because $g_M(t)$ and $Dg_M(t)$ would be expressed as linear combinations of log $M(t-i)$, which would render the determinant of the least-squares equations zero. However, the phases are not equal in length, and differences in length are taken into account in computing $g_M(t)$ and $Dg_M(t)$. For equation (A86), the j_o terms would be redundant if phases were equal in length. However, the j terms in equation (A83) would not be redundant, and neither would the j_1 term in equation (A86), because they are based on values of $M(t-i)$ or $g_M(t-i)$ that do not otherwise enter the equations explicitly.

Given estimates for the j_i, an average reaction pattern excluding transient effects is given by the b_i coefficients estimated directly from equations (A80) and (A85), or the b_i coefficients estimated indirectly from equations (A83) and (A86). An average reaction pattern including transient effects can be obtained from equations (A80) and (A83) by setting

$$\text{(A87)} \qquad g_M(t-i) = \frac{\log M(t-1) - \log M(t-i-1)}{\bar{n}},$$

and in equations (A85) and (A86) by setting

$$\text{(A88)} \qquad Dg_M(t-i) = \frac{g_M(t-i) - g_M(t-i-1)}{\bar{n}},$$

and then collecting like terms, in equations (A83) and (A86) after replacing the mixed money and income terms by the equivalent terms in M alone.

For equations (A83) and (A86), as we have estimated them, the coefficients including the transient effects (designated by a prime) are:

$$\text{(A89)} \qquad b_1' = b_1 + \frac{j_1}{\bar{n}},$$

$$b_2' = b_2 - \frac{j_1}{\bar{n}} + \frac{j_2}{\bar{n}}$$

$$b_3' = b_3 - \frac{j_2}{\bar{n}},$$

and all other b's are the same primed and unprimed.

For rates of change,

$$b_o' = b_o + \frac{j_o}{\bar{n}}, \tag{A90}$$

$$b_1' = b_1 - \frac{j_o}{\bar{n}} + \frac{j_1}{\bar{n}},$$

$$b_2' = b_2 - \frac{j_1}{\bar{n}},$$

and all other b's are the same primed and unprimed.