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Volume Author/Editor: Charles L. Ballard, Don Fullerton, John B. Shoven, and John Whalley

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Chapter Author: Charles L. Ballard, Don Fullerton, John B. Shoven, John Whalley

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2 General Equilibrium Analysis of Tax Policies

In this chapter we describe the general equilibrium approach to the analysis of the impact of taxes which underlies our later evaluation of U.S. tax policies. We begin with a discussion of the pathbreaking work of Arnold Harberger (1959, 1962, 1966, 1974). Harberger's analyses represent a great advance over partial equilibrium models, but they have their own shortcomings. In particular, the Harberger model quickly becomes intractable in dealing with more than two sectors or two factors. Also, the model is not suited for considering large policy changes. We use a model that overcomes these shortcomings. After dealing with the Harberger model in section 2.1, we lay the foundations for our own model. In section 2.2 the essential elements of general equilibrium models are presented. Then we discuss the incorporation of taxes into the model in section 2.3. Section 2.4 covers computational techniques. Section 2.5 follows with a discussion of computations in which the size of government is held constant, even though the configuration of taxes is changed. Finally, we discuss a method by which the number of prices that must be computed can be reduced greatly. The details of our model of the United States and the data follow in chapters 3 through 7.

2.1 Harberger's General Equilibrium Tax Analyses

In the general equilibrium tax literature, Harberger's analyses of the incidence and efficiency effects of taxes have been a major stimulus to subsequent work. The Harberger approach enabled general equilibrium effects to be quantified for the first time. Although our work in this book goes further than that of Harberger in many ways, our dependence on Harberger's work will be obvious to anyone familiar with the literature. The 1962 Harberger general equilibrium tax model uses standard neo-

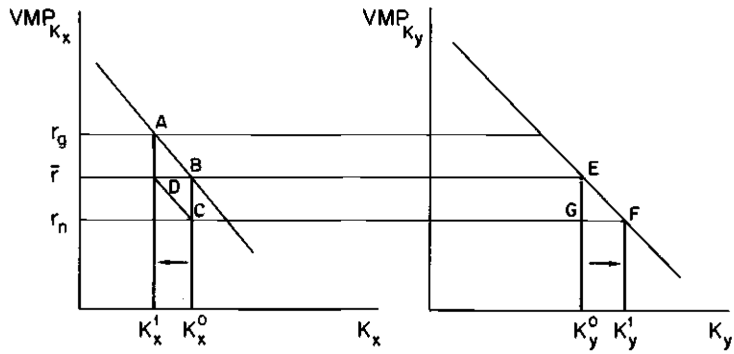


Fig. 2.1 The intersectoral allocation of capital in the Harberger general equilibrium model.

classical assumptions. It is assumed that the aggregate supplies of capital and labor are fixed, that factors are perfectly mobile among industries, and that perfect competition exists in factor and product markets. Production takes place under constant returns to scale. There are two sectors, each of which produces a single, homogeneous output. Harberger assumes a closed economy (i.e., there is no foreign trade). The model considers the effects of a distorting tax on capital in one of the sectors.[1]

The model is represented by a series of differential equations. For discrete changes, the analysis is only a local approximation. Using estimates of elasticities of substitution in production and consumption that are based on econometric literature, Harberger generates estimates of the incidence of particular taxes. Perhaps the most famous finding from this model is that the corporate tax is borne by all capital owners, regardless of whether their capital is used in incorporated enterprises.

Harberger also develops a procedure for estimating the welfare cost of a distortionary factor tax. In the case of capital tax distortions, he considers the economy to be represented ty two sectors, "heavily taxed" and "lightly taxed." These are represented as sectors X and Y in figure 2.1. Each sector uses capital in production, and the marginal revenue product schedules are assumed to be linear. The economy has a fixed capital endowment. In the absence of any taxes, market forces will ensure that capital is allocated between the two sectors such that the rate of return, $\bar{r}$, is equal. If, instead, a tax operates on capital income in sector X, the gross

1. A number of these assumptions have been relaxed in subsequent work. McLure has extended the model to cover interregional incidence (1969), and has introduced immobile factors (1971). Thirsk (1972) has extended the analysis to the case of three goods, and Mieszkowski (1972) has considered the case of three factors. Anderson and Ballentine (1976) have extended the analysis to incorporate the case of monopoly. Finally, Vandendorpe and Friedlaender (1976) extend the Harberger formulation to encompass an initial situation with a number of distortionary taxes.

rate of return, r_g, in that sector must be such that the net rate of return, r_n, is equalized across the sectors. The difference between r_g and r_n is the tax on each unit of capital used in sector X.

The situations with and without taxes are characterized by the capital allocations (K_x^1, K_y^1) and (K_x^0, K_y^0), respectively. In figure 2.1, the area $ABK_x^0K_x^1$ represents the value of the lost output in sector X when K_x decreases from K_x^0 to K_x^1 as the tax is imposed. $EFK_y^1K_y^0$ represents the value of the increased output of sector Y. Full employment guarantees that $K_x^0 - K_x^1 = K_y^1 - K_y^0$. The area $ABCD$ (= $ABK_x^0K_x^1 - EFK_y^1K_y^0$) represents the welfare cost of the tax, L, and is given by:

$$L = \tfrac{1}{2}(r_g - r_n)\,\Delta\,K_x = \tfrac{1}{2}\,T\,\Delta\,K_x, \tag{2.1}$$

where t is the tax rate on capital in sector X. The tax rate can be determined from readily available data. Harberger calculates $\Delta\,K_x$ by solving his two-sector general equilibrium model for this variable.

In his 1964 paper, Harberger applies a similar form of local analysis in an examination of the welfare costs of several key distortions in the tax system. In this paper, he presents the generalized triangle formula for the welfare cost of a distorting tax, upon which so much intuition for the size of the distorting costs of taxes has subsequently been based.

This intuition is most easily seen in the simple case of an output tax on a single product where the supply function is perfectly elastic. This is shown in figure 2.2 where DD' is the compensated demand function.

The tax is assumed to be paid by suppliers of the product, so that the supply curve in figure 2.2 shifts by the amount of the tax, and the quantity bought in the presence of the tax falls from q_0 to q_1. At q_1, the gross-of-tax price represents the demand-side evaluation of the welfare gain from an extra unit of production. The net-of-tax price represents the social opportunity cost of extra production. As long as the demand price exceeds the net-of-tax supply price, a gain is thus possible from extra production, and the shaded triangle in figure 2.2 is the welfare cost of the distorting tax.

Defining the difference between the gross-of-tax and net-of-tax prices as Δp, and $q_0 - q_1$ as Δq, the area of this triangle is the deadweight loss, L:

$$L = \tfrac{1}{2}\,\Delta q\,\Delta p. \tag{2.2}$$

If units for q are chosen such that $p = 1$ in the no-tax equilibrium, then $\Delta p = t$, where t is the ad valorem tax rate. Defining the demand elasticity as ϵ gives $\Delta q = \epsilon \Delta p \cdot q/p$. Evaluated at q_0 where $p = 1$ and $\Delta p = t$, we have $\Delta q = \epsilon t q_0$. Thus,

$$L = \tfrac{1}{2}\,t^2\,\epsilon\,q_0. \tag{2.3}$$

The deadweight loss increases with the square of the tax rate, and linearly with elasticities.

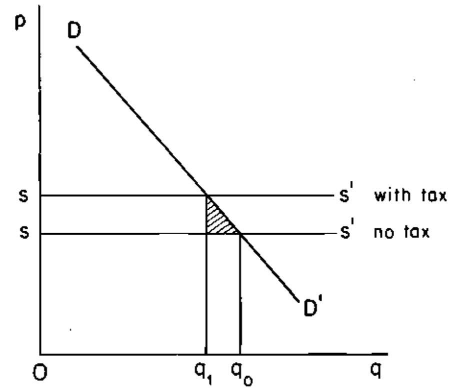


Fig. 2.2 The excess burden of an output tax in partial equilibrium.

To evaluate the overall efficiency of this tax as a source of revenue, it is sometimes useful to look at the deadweight loss per dollar of revenue R. Since $R = tq_0$ initially (where $p = 1$), we can rewrite the deadweight loss formula as $L = ½\, t\epsilon R$. The deadweight loss per dollar of revenue is $L/R = ½\, t\epsilon$. Thus the deadweight loss per dollar of revenue is linear in t and ϵ.

For comparison purposes, however, we may want to know the additional excess burden associated with a marginal dollar of revenue from each tax instrument. The numerator of this ratio is

$$dL/dt = \epsilon t q_0. \tag{2.4}$$

The denominator, dR/dt, is q_0. Therefore, the marginal excess burden is

$$\frac{\text{Marginal Loss}}{\text{Marginal Revenue}} = \frac{\epsilon t q_0}{q_0} = \epsilon t. \tag{2.5}$$

As noted by Browning (1976), Stuart (1984), and Ballard, Shoven, and Whalley (1982), the marginal cost of an extra dollar raised from an existing distorting tax exceeds the average welfare cost of all revenue raised by the tax. In this simple model, marginal deadweight loss is twice as great as average deadweight loss.

2.2 General Equilibrium Models

General equilibrium models have four essential ingredients—the endowments of consumers, their demand functions, the production technology, and the conditions for equilibrium.

In general, consumers may possess endowments of any or all of the commodities in the economy. The nonnegative economy-wide endowments are given by the vector W; $W_i \geq 0$, $i = 1, 2, \ldots, N$, where N is the number of commodities. In our model, consumers only possess endowments of capital and labor. Market demand functions are specified for each commodity. Demands are nonnegative and depend in a continuous manner on all prices. The market demand functions are denoted by $D_i(P)$, $i = 1, \ldots, N$, where the vector P represents the N market prices. The demands are homogeneous of degree zero in all prices, i.e., if all prices were to double, the quantities demanded would not change. Because of the homogeneity property, an arbitrary normalization of prices can be chosen; a common treatment is to consider nonnegative prices that sum to unity, i.e., prices that lie on the unit price simplex.

$$\left(P_i \geq 0; \sum_{i=1}^{N} P_i = 1\right).$$

Commodity demands are also assumed to obey Walras's law, which states that the value of market demands equals the value of endowments at *any* set of prices, regardless of whether they are equilibrium prices. Market demand functions are typically represented as the sum of the individual household demand functions, each of which may or may not be derived from utility maximization subject to a budget constraint. In our model, each representative consumer group has commodity demands derived from constrained utility maximization.

On the production side of a general equilibrium model, technology is usually described by a set of constant returns to scale activities, or by production functions that exhibit nonincreasing returns to scale. The advantage of the activity analysis approach is that the conditions for equilibrium are especially simple when production is modeled in this way. On the other hand, production functions are more convenient to use in applied work. Production functions are easy to parameterize, since most of the relevant econometric literature involves the estimation of production functions.

In this chapter we describe both activity analysis production and continuous production functions. In subsequent chapters we will only use production functions. With the activity analysis approach, each of the constant returns to scale activities which is available to the economy is represented as a vector of coefficients. Each coefficient, a_{ik}, represents the input or output of good i when activity k is operated at unit intensity. We adopt the convention that the a_{ik} are negative for inputs and positive for outputs. If there are K activities, the coefficients can be arranged in an $(N \times K)$ matrix, which we shall call A. The first N columns of this matrix are usually disposal activities, which allow for costless disposal of each commodity. Joint products are possible, i.e., an activity may have more than one output. However, activities are restricted to satisfy the boundedness condition that at any nonnegative set of activity levels x, $Ax + W$ is bounded. The interpretation of this condition is that it excludes infinite production of any of the commodities.

In the case of a continuous production function with constant return to scale, any set of prices leads to unique cost-minimizing input proportions. We can think of a continuous production function as an infinite set of activities.

Equilibrium in the activity analysis model is characterized by a nonnegative vector of prices and activity levels (P^*, x^*), where the * denotes the equilibrium level. At (P^*, x^*), demands equal supplies for all commodities:

$$(2.6) \qquad D_i(P^*) = \sum_{k=1}^{K} a_{ik}x_k^* + W_i \qquad \text{for } i = 1, \ldots, N.$$

Also, no activity makes positive profits, with those in use breaking even:

$$(2.7) \qquad \sum_{i=1}^{N} P_i^* a_{ik} \leqq 0 \; (=0 \text{ if } x_k^* > 0) \qquad \text{for } k = 1, \ldots, K.$$

A simplified numerical example of a model representative of those actually used in this book may help in understanding the general equilibrium structure (see table 2.1). For expositional purposes, we consider a model with two final goods (manufacturing and nonmanufacturing) and two factors of production (capital and labor). We consider two classes of consumers. Consumers have initial endowments of factors but have no initial endowments of goods. The "rich" consumer group owns capital, while the "poor" group owns labor. Production of each good takes place

Table 2.1 **Specification of Production Parameters, Demand Parameters, and Endowments for a Simple General Equilibrium Economy**

Production Parameters			
	Φ_i	δ_i	σ_i
Manufacturing	1.5	.6	2.0
Nonmanufacturing	2.0	.7	.5
Demand Parameters			
	α_{MFG}	α_{NonMFG}	σ
Rich consumers	0.5	0.5	1.5
Poor consumers	0.3	0.7	0.75
Endowments			
	K	L	
Rich households	25	0	
Poor households	0	60	

according to a constant elasticity of substitution (CES) production function, and each consumer class has demands derived from maximizing a CES utility function subject to its budget constraint.

The production functions are given by

$$(2.8)\qquad Q_i = \Phi_i \left(\delta_i^{\frac{1}{\sigma_i}} L_i^{\frac{\sigma_i - 1}{\sigma_i}} + (1-\delta_i)^{\frac{1}{\sigma_i}} K_i^{\frac{\sigma_i - 1}{\sigma_i}} \right)^{\frac{\sigma_i}{\sigma_i = 1}} \qquad i = \quad 1, 2, \ldots$$

where Q_i denotes output of the i^{th} industry, Φ_i is the scale or units parameter, δ_i is the distribution parameter, K_i and L_i are the factor inputs, and σ_i is the elasticity of factor substitution.

The CES utility functions are given by

$$(2.9)\qquad U_j = \left(\alpha_j^{\frac{1}{\sigma_j}} D_{1j}^{\frac{\sigma_j - 1}{\sigma_j}} + (1-\alpha_j)^{\frac{1}{\sigma_j}} D_{2j}^{\frac{\sigma_j - 1}{\sigma_j}} \right)^{\frac{\sigma_j}{\sigma_j - 1}},$$

where D_{ij} is the quantity of good i demanded by consumer j, α_j are share parameters, and σ_j is the substitution elasticity. If we maximize equation (2.9), subject to the constraint that the consumer cannot spend more than his income, $(P_1 D_1 + P_2 D_2 \leq P_L W_L + P_K W_K)$, we get the demand functions:

$$(2.10)\qquad D_{ij} = \frac{\alpha_j I}{P_i^{\sigma_j} (\alpha_j P_1^{1-\sigma_j} + (1-\alpha_j) P_2^{1-\sigma_j})} \qquad i = 1, 2, \ldots$$

where I is income and P_i are market prices.

Once we have specified the parameters of these production and utility functions, plus the individual endowments, we have a complete general equilibrium model such as the one in table 2.1. Market-clearing conditions require that supply equals demand for each good and factor, with no excess profits. This model is much less complicated than the general equilibrium model to which most of this book is devoted. Nevertheless, the two models bear distinct formal similarities. Each can be solved with any of several computational algorithms. In computing the equilibrium prices for the system of table 2.1, we use an algorithm that searches across the unit simplex. However, in reporting our results, we find it more convenient to adopt the normalization that the price of labor is unity. (We continue to use this normalization in later chapters.) The approximate solution for this illustrative model is shown in table 2.2. The solution shows that, at these prices, the total demand for each output exactly matches the amount produced. It follows that producer revenues equal consumer expenditures. It also is true, to a high degree of approximation, that the labor and capital endowments are fully employed and that consumer factor incomes equal producer factor costs. The cost per

Table 2.2 Equilibrium Solution for Illustrative Simple General Equilibrium Model (specified in table 2.1)

		Equilibrium Prices		
Manufacturing output			1.399	
Nonmanufacturing output			1.093	
Capital			1.373	
Labor			1.000	
		Production		
	Quantity	Revenue	Capital	Capital Cost
Manufacturing	24.942	34.894	6.212	8.529
Nonmanufacturing	54.379	59.436	18.789	25.797
TOTAL		94.330	25.001	34.326
	Labor	Labor Cost	Total Cost	Cost per Unit Output
Manufacturing	26.364	26.364	34.893	1.399
Nonmanufacturing	33.634	33.634	59.431	1.093
TOTAL	59.998	59.998	94.324	
		Demands		
	Manufacturing		Nonmanufacturing	Expenditure
Rich households	11.514		16.674	34.333
Poor households	13.428		37.705	59.997
TOTAL	24.942		54.379	94.330
	Labor Income		Capital Income	Total Income
Rich households	0		34.325	34.325
Poor households	60		0	60.000
TOTAL	60		34.325	94.325

unit of output in each sector matches the price, meaning that economic profits are zero. The expenditure of each household exhausts its income. Thus the solution closely approximates all of the properties of an equilibrium for this economy. The degree of approximation can be improved by increasing the amount of computation time allowed for the solution algorithm.

Our general equilibrium analysis of the U.S. tax system follows closely the approach of the example shown above. We construct a numerical general equilibrium model for policy analysis, assuming that the data and analysis are representative of conditions in the U.S. economy. Tax distortions are introduced and equilibria are computed under different policy regimes. Finally, since utility functions are explicitly specified, changes in consumer welfare can easily be computed.

This set of exercises would not be particularly instructive if the equilibrium were not unique for any particular tax policy. There is no theoretical argument that guarantees uniqueness in models of this type. Fortunately, uniqueness has been demonstrated for the model used here in a paper by Kehoe and Whalley (1982).

2.3 Incorporating Taxes in a General Equilibrium Model

We can incorporate a system of taxes and government expenditures into our general equilibrium model. The taxes may apply to the purchases of goods and services by consumers, the use of factors and intermediate inputs by producers, and the final outputs of the various production sectors. The tax rates may differ for each good, consumer, and producer. The government uses the tax proceeds to finance transfer payments to consumers and to purchase final goods and services. In our model, in equilibrium, the government budget must be balanced, i.e., any government income remaining after transfers to persons must be spent on commodity purchases.[2]

The methods for incorporating taxes into a formal general equilibrium model are presented in Shoven and Whalley (1973). In order to get the flavor of that analysis, consider the very simple case in which the only taxes are consumer purchase taxes and the government makes no exhaustive expenditures, i.e., all revenues are returned to consumers as transfer payments. Each of the J consumers is assigned a vector of nonnegative ad valorem tax rates, $e_j = (e_{1j}, \ldots, e_{Nj})$, which must be paid on expenditures on each of the N commodities. The total tax revenue, R, is used to distribute among consumers according to a proportional distribution scheme;

$$R_j = r_j R; \; r_j \geq 0, \; \sum_j r_j = 1.$$

This scheme for tax revenue distribution leads to a fundamental problem that did not exist in a model with no government. The problem is one of simultaneity. Consumers cannot determine their demands until they know their incomes, and they do not know their incomes until they know the amount of their transfer revenues. At the same time, the transfer payments cannot be determined until the total revenue is known, and revenue will not be known until consumers make their purchases.

2. The government could run deficits or surpluses if the model included a government bond market. The model described in this book does not include bonds. For an attempt to integrate a bond market into a general equilibrium model, see Feltenstein 1984.

This problem of simultaneity can be solved if we model consumers and producers as reacting to revenue as well as prices. That is, demand functions depend on (P,R), where (P,R) denotes the vector $(P_1, \ldots, P_N, R)$. The market demand functions $D_i(P,R)$ are (as before) assumed to be nonnegative, continuous functions. However, they are now assumed to be homogeneous of degree zero in the vector (P,R), which means that a doubling of all prices and tax revenues will double both incomes and consumer prices, so that physical quantities demanded are unchanged. Whereas we previously normalized prices so that they fell on the unit simplex in prices, $(\Sigma_{i=1}^{N} P_i = 1)$, we now use an augmented simplex in prices and revenue $(\Sigma_{i=1}^{N} P_i + R = 1)$.

We still assume that the demand functions satisfy Walras's law. Now, however, Walras's law states that the value of market demands, gross of expenditure taxes, is equal to total consumer incomes:

$$(2.11) \qquad \sum_{j=1}^{J} \sum_{i=1}^{N} P_i D_{ij}(P,R)(1 + e_{ij}) = \sum_{j=1}^{J} \left(\sum_{i=1}^{N} P_i W_{ij} + R_j \right),$$

where D_{ij} is the quantity of good i consumed by consumer j, and e_{ij} is the ad valorem rate of expenditure tax on good i faced by consumer j. Equation (2.11) will hold at any vector (P,R). In addition to vectors of expenditure tax rates, each of the J consumers may be assigned a nonnegative income tax rate τ_j which is charged on taxable income.[3] We can also model taxes which are levied on the production side of the economy. In the model with activity analysis production, vectors of ad valorem tax rates T^i can be assigned to each of the productive activities, where $T^i = (t_{1k}, \ldots, t_{Nk})$. We adopt the convention that t_{ik} are nonnegative for outputs and nonpositive for inputs, such that the taxes paid will be nonnegative. The producer who operates activity k at unit intensity will incur a tax liability of $\Sigma_{i=1}^{N} \alpha_{ij} t_{ij} P_i$. As before, competitive equilibria can be examined, but now these equilibria reflect the presence of taxation. Market prices have to be defined with care when taxation is introduced into the economy. The prices included in the vector (P,R) are those equalized across all the traders on any market. Thus, prices P_i are sellers' prices for inputs (prices net-of-producer input taxes) and wholesale prices for outputs (prices net-of-consumer purchase taxes, but gross of any producer output taxes). An equilibrium in the presence of taxation is thus a vector $((P^*, R^*),(x^*))$ such that:

$$(2.12) \qquad \sum_{i=1}^{N} P_i^* + R^* = 1;\ P_i^*,\ R^* \geq 0;\ P_i^* > 0 \text{ for at least one } i.$$

3. Characteristics of income taxation systems, such as an annually exempt amount of income or deductions for expenditures on particular commodities, can be incorporated into the model, although these are ignored here to simplify the notation.

Once again, the * denotes the equilibrium level. In equilibrium, demands equal supplies for all commodities:

$$(2.13) \qquad D_i(P^*, R^*) = W_i + \sum_{k=1}^{K} a_{ik} x_k^* \qquad (i = 1, \ldots, N).$$

In equilibrium, no activity yields any producer the possibility of positive profit after payment of producer taxes, with those activities in use just breaking even:

$$(2.14) \quad \sum_{i=1}^{N} P_i^* a_{ik}(1 - t_{ik}) \leq 0 \qquad (= 0 \text{ if } x_k^* > 0)\ (k = 1, \ldots, K).$$

In equilibrium the budget will be balanced, so that

$$(2.15) \qquad R^* = \sum_{i=1}^{N} \sum_{k=1}^{K} a_{ik} t_{ik} P_i^* x_K^* + \sum_{i=1}^{N} \sum_{j=1}^{J} e_{ij} P_i^* D_{ij}(P^*, R^*) + \sum_{j=1}^{J} \tau_j \left(\sum_{i=1}^{n} P_i^* W_j + R_j^* \right).$$

The three terms on the right hand side of equation (2.15) are the revenue from producer output taxes, consumer purchase taxes, and income taxes.

This approach to incorporating taxes in a general equilibrium model can also be followed in a model with continuous production functions. Production taxes would then apply to the capital and labor inputs, as well as to outputs, consumer purchases, and incomes. The simultaneity involving tax revenues in the evaluation of market functions remains, and it necessitates characterizing the equilibrium in terms of prices and revenues, as shown. Our model of the U.S. economy and tax system uses this production function approach.

2.4 Computing General Equilibria with Taxes

Our general equilibrium model is sufficiently complicated that equilibrium prices cannot be determined analytically. Instead, we use a grid search algorithm developed by Orin Merrill (1972). Merrill's algorithm is based on an algorithm developed by Herbert Scarf (1967, 1973). Since the focus of this book is on the tax model of the United States and its policy applications, we devote only a short section to computational method here. For descriptions of several such algorithms, the interested reader is referred to Scarf (1973, 1981, 1984).

The algorithms of Scarf and Merrill are examples of methods that are guaranteed to find fixed points of certain mappings. The algorithms can be used to solve many types of computational problems, including many noneconomic ones. For the models discussed here, we are able to formulate the problem in such a way that fixed points and economic equilibria are one and the same.

An intuitive grasp of Scarf's algorithm can be obtained by considering a simple case of a mapping that meets the conditions necessary for the Brouwer (1910) fixed-point theorem. The theorem states that, if S is a nonnull, closed, bounded, convex set mapped into itself by the continuous mapping $X \rightarrow F(X)$, then there exists a fixed point, i.e., a point $\hat{X} \epsilon S$ for which $F(\hat{X}) = \hat{X}$.

Suppose that S is taken to be the unit interval (0,1), and the continuous mapping is the function $F(X)$, whose values all lie on the same unit interval. A fixed point in this case is given by the intersection of the correspondence $(X, F(X))$ and 45° line, as drawn in figure 2.3.

It is easy to find the fixed point $\hat{X}$ in figure 2.3, since the problem is so simple. This rapidly becomes more difficult as the dimension of the problem increases. Scarf's algorithm exploits a property of graph theory to provide an ingenious way of always finding a fixed point, $\hat{X}$, irrespective of the dimensionality involved.

The intuition underlying this method can be illustrated with the same diagram as in figure 2.3, redrawn with the unit interval on the X axis subdivided into a number of line segments of equal length (see figure 2.4). Each end point of these line segments has associated with it a label that is either 0 or 1. The label is calculated using the following rule: if $F(X) \geq X$ the label is 0, otherwise the label is 1.

By construction, the end points of the unit interval have the labels 0 and 1. Since all other points have either a label 1 or 0, there must exist a line segment that has both labels. For this line segment, $F(X) \geq X$ at one end point, and $F(X) < X$ at the other. Thus, since the function is

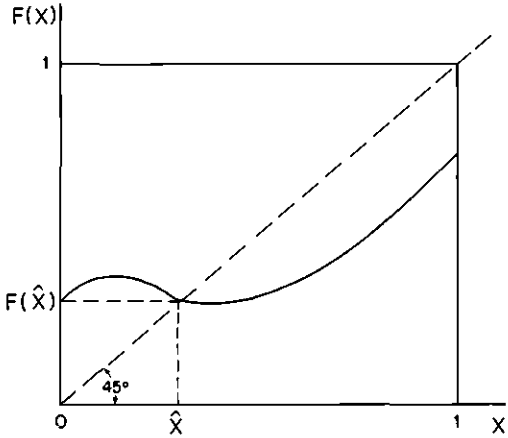


Fig. 2.3 A fixed point of a continuous mapping.

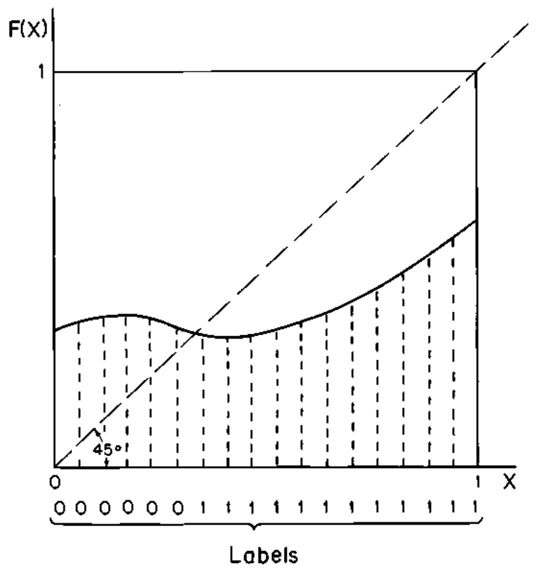


Fig. 2.4 Using labels to approximate a fixed point.

continuous, a fixed point where $F(\hat{X}) = \hat{X}$ must exist within this line segment. If the number of these line segments becomes large, the approximation to a fixed point represented by the completely labeled line segment will be more accurate. In the limit, where a dense grid of points on the line segment is approached, the calculation of the fixed point will be exact.

Scarf's algorithm appeals to this idea in higher dimensional space. In the case of an economic problem in which three prices are to be determined, we would deal with the unit simplex shown in figure 2.5.

A major accomplishment in general equilibrium economics was the proof of the existence of an equilibrium (see Arrow and Debreu 1954; Debreu 1959). This was achieved by finding a mapping that met the conditions of Brouwer's theorem and whose fixed point could be interpreted as an economic equilibrium. The simplest such mapping is for a model of pure exchange. Let $g_i(P) = D_i(P) - W_i$ be the excess demand for commodity i. These excess demand functions are continuous since the demand functions are continuous, and note that Walras's law states that $\Sigma_{i=1}^{N} P_i g_i(P) = 0$. Consider the following mapping of prices in the simplex into image prices P':

$$P_i' = \frac{P_i + \text{Max}\,(0, g_i(P))}{1 + \sum_{i=1}^{N} \text{Max}\,(0, g_i(P))} \qquad \text{for } i = 1, \ldots, N. \tag{2.16}$$

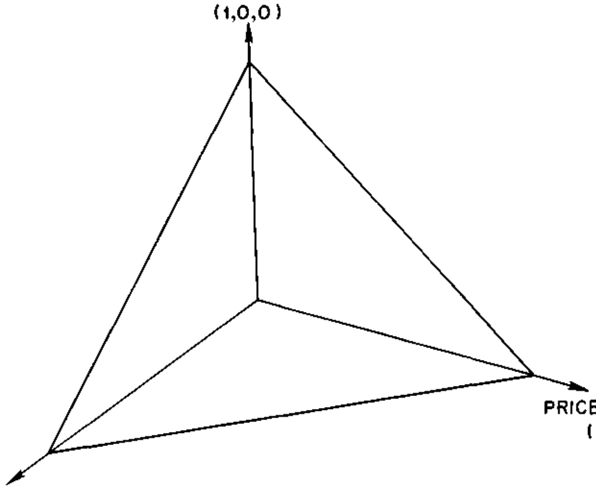


Fig. 2.5 The unit simplex.

Note that

$$\sum_{i=1}^{N} P_i' = \frac{1 + \sum_{i=1}^{N} \text{Max}\,(0, g_i(P))}{1 + \sum_{i=1}^{N} \text{Max}\,(0, g_i(P))} = 1, \tag{2.17}$$

and that $P_i' \geq 0$ for all i. Thus P' is on the unit simplex and a continuous function of P. Therefore, the conditions of Brouwer's theorem apply, and a fixed point of this mapping must exist. At this fixed point where $P_i' = P_i$, all $g_i \leq 0$, which is the condition for equilibrium in a pure exchange model.

This existence proof made it possible to contemplate the problem of computing an economic equilibrium. Unfortunately, however, the Brouwer mapping demonstration of existence does not readily suggest a computational procedure. The computational methods use labels as demonstrated above and an ingenious numerical analysis technique of Lemke and Howson (1964). For the three-dimensional pure exchange problem of figure 2.5, one could examine a large number of candidate equilibrium price vectors on the simplex. The label for any point could be the number associated with the commodity with the largest excess demand. Along each side of the simplex, where one commodity price is zero, the label would be that commodity number. We might expect that the simplex would be labeled in a manner somewhat like figure 2.6.

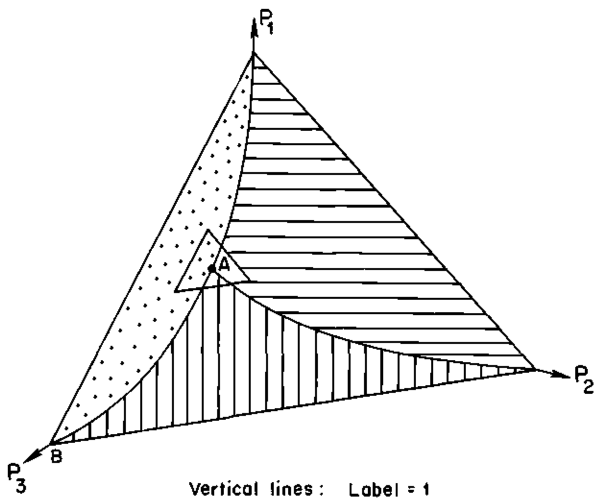


Fig. 2.6 Economic equilibrium in the three-good case.

In figure 2.6, equilibrium will be at point A. At any one time, Scarf's algorithm examines N price vectors that are close together in a sense that he defines precisely. The algorithm can be shown to converge to a set of N candidate vectors that are very close together, each of which has a different label. If each price vector is labeled with the number of the commodity with the largest excess demand and if each vector has a different label, then each of these excess demands must be nearly zero. This is true because Walras's law states that at any specific price vector, the price-weighted sum of the excess demands is zero:

$$\sum_{i=1}^{N} P_i g_i(P) = 0. \tag{2.18}$$

The excess demands are continuous, and if within a very small region of the price simplex each commodity has the largest excess demand, then the excess demands must all be close to zero. Thus, this labeling technique results in the computation of a fixed point that represents an economic equilibrium.

For a general equilibrium model with taxes, the necessary labels are described in Shoven and Whalley (1973). This involves using the traditional price simplex used in most general equilibrium models, but augmented by one additional dimension for tax revenue. The labeling procedure outlined by Shoven and Whalley is for a general equilibrium model with production, with taxes on both producers and consumers. Their labeling rule guarantees that a subsimplex that is completely labeled provides the required approximation to a general equilibrium with taxes.

The problem with Scarf's algorithm is that it uses a relatively large amount of computational time. It starts at a corner of the simplex and therefore usually evaluates the excess demands at many points before approaching an approximate equilibrium. Even if we have a good guess about where the equilibrium might be, we cannot use that information. Also, since the algorithm uses a fixed grid of candidate price vectors, it is not possible to improve the accuracy of the solution once we find an approximate equilibrium. Consequently, there is a stringent trade-off between accuracy and computational time. For the calculations used in this book, we use Merrill's algorithm, because it overcomes these problems while still guaranteeing convergence. Other algorithms of this type which allow for fast computations are those of Kuhn and MacKinnon (1975), Eaves (1972), and van der Laan and Talman (1979).

2.5 Equal Yield Equilibria

In this section we describe an important extension to the general equilibrium approach to tax policy. Through the use of an equal-tax-yield equilibrium concept, we are able to undertake "differential" analysis in the tradition of Musgrave (1959). Such analysis allows an existing tax to be replaced by an alternative tax system that raises equivalent revenue. This change in procedure allows us to maintain the size of government when the effects of an existing tax are evaluated. This is important, since a changing size of government can contaminate model findings. Because we are interested here in the effects of changes in the structure rather than the level of taxes, we want to be able to interpret our results without worrying about changes in the pattern of total demands that are caused by changes in the amount of government spending.

The exact meaning of equal revenue yield is somewhat unclear because, in general, the adoption of a new tax regime will lead to different equilibrium prices. It is not satisfactory merely to give the government the same number of dollars as it had before the tax change, since the goods that government buys with those dollars will have changed in price. Shoven and Whalley (1977) discuss a variety of price indexes that might be used to correct for these price changes, so as to preserve equal "real" revenue.

In the present model we do not use price indexes as such. Instead, we give the government a utility function, and then use the corresponding expenditure function to calculate the revenue required for the government to achieve constant utility at any set of prices. The expenditure function expresses the amount of money necessary to attain a given level of utility at a given set of prices. When we calculate a *base-case equilib-*

rium, we calculate the government's utility. In equilibrium calculations for changes in tax regime, we give the government enough revenue so that it reaches the same level of utility. This is the exact sense in which we preserve "real" government revenue.

Most of the simulations reported in this book involve reductions in revenues. This is the case for corporate tax integration and for the adoption of a consumption tax. In these cases, equal yield calculations involve increasing taxes somewhere else in the economy.

In general, a number of replacement tax schemes could be used. However, it is simplest to consider only those schemes where the replacement tax can be expressed in terms of a single scalar. In using our model, we employ four such replacement schemes.

1. *Lump-Sum Taxes*. In this case, the reduction in revenues caused by a tax change is recovered by lump-sum taxes paid by each consumer group. The weights for the division of these taxes among the household groups are fixed, but the size of the lump-sum taxes in aggregate is determined endogenously.

2. *Additive Changes to Marginal Income Tax Rates*. Here revenues are recovered by adding the same number of percentage points to all household marginal income tax rates.

3. *Multiplicative Changes to Marginal Income Tax Rates*. In this case, revenues are recovered by multiplying the marginal tax rates of all households by the same scalar. (Thus the rates of the higher-bracket taxpayers are increased by higher absolute amounts.)

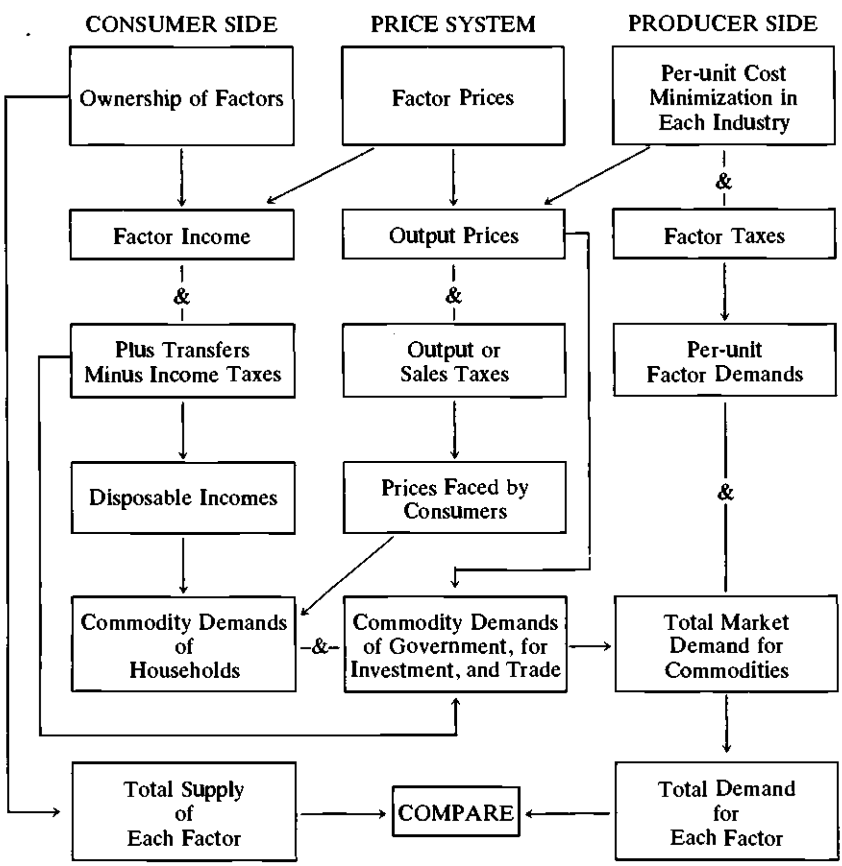
4. *Multiplicative Changes to Consumer Sales Tax Rates*. In this case, we raise the required revenue by multiplying the consumer sales tax rates facing all consumers by the same scalar.

Distributional effects clearly depend upon this choice of tax replacement. We also consider a few cases in which additional revenues are created by the tax change. Examples are the adoption of a value-added tax or reductions in deductions for saving. In these instances we use yield-preserving subsidies instead of taxes. Once again, we use subsidies of the lump-sum, additive, and multiplicative types.

We should note that there is no guarantee that equal yield equilibria will always exist. Obviously, we could not replace the entire revenue system of the United States with a tax on popcorn. We typically consider broadly based replacement taxes, in order to minimize the possibility of being unable to calculate an equal yield equilibrium.

2.6 Computations in the Space of Factor Prices

Even though we use Merrill's algorithm for our equilibrium computations, we are still concerned with reducing computational costs in every way possible. The amount of time required to compute an equilibrium at any level of accuracy increases rapidly with the number of dimensions. Until now, we have described general equilibrium tax models as if a separate price must be computed for every commodity in the model. A convenient procedure to use in computation is to reduce the number of prices by expressing some of the commodity prices as functions of other prices. In our model we separate commodities into goods and factors, and calculate goods prices from factor prices. Since we use a fixed-coefficient input-output matrix with no joint products, we are able to use the Samuelson (1951) nonsubstitution theorem to calculate goods prices directly from factor prices. This enables us to compute an equilibrium in the space of factor prices rather than the space of all commodity prices,



Competitive equilibrium achieved when:
1. Demands equal supplies for all goods and factors
2. Zero profits (net of taxes) prevail in all industries

Fig. 2.7 Flow diagram of factor space calculations in general equilibrium tax model.

reducing greatly the cost of computation. In our model we calculate equilibria in three dimensions—the factor prices, capital and labor—and an additional dimension for government revenues.

In figure 2.7 we present a stylized version of the way in which our factor space computations work. The starting point is factor prices. Using factor prices and the conditions for cost minimization by producers, we can evalutate the cost-covering prices of goods. This directly imposes the zero profit condition for each industry. When consumers know their incomes and the prices of goods, they can calculate their demands for goods. If producers meet demands, this gives derived demands for factors, which results in a system of two-factor excess demand correspondences (capital and labor), and a government budget imbalance. An equilibrium is attained when all three correspondences equal zero.