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AN OPTIMIZATION APPROACH IN MULTIPLE TARGET PROBLEMS USING INEQUALITY CONSTRAINTS: THE CASE AGAINST WEIGHTED CRITERION FUNCTIONS

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The conventional approach in economic policy optimization using a weighted (normally quadratic) criterion function is criticized for its conceptual inadequacy and arbitrariness. An alternative approach is developed, representing the entire problem in a rigorous inequality framework. The computational realization is based on an alternating sequence of optimization steps and evaluation of dual variables. The algorithm for the solution of the implied NLP problem is outlined and the whole procedure illustrated with a numerical example. To indicate the method's flexibility and usefulness some modifications and extensions of the basic idea are pointed out.

Introduction

During the last few years the discussion on optimal control of economic systems has primarily focused on the development of adequate optimization techniques. Until recently, however, comparably less effort has been devoted to the aspects of an adequate formal representation of the objectives of economic policy. The standard quadratic criterion function, originally advocated by Simon, Theil and Holt,[1] has been applied almost uniformly, only sometimes accompanied by a dissociative phrase. This paper sets forth an alternative approach applying inequality constraints instead of quadratic or otherwise penalized deviations. In ch. 1 we start with a short critical discussion of the conventional approach. The basic ideas of our formulation are to be found in ch. 2; while ch. 3 deals with the problem of its computational implementation, the approach is applied to a medium-sized nonlinear economic model in ch. 4, the following ch. 5 being devoted to the discussion of some modifications and extensions of the basic concept. The final ch. 6 summarizes what we think to be the main advantages of the proposal.

1. The Conventional Approach

Consider the standard macroeconomic model formulation

$$f(x, x_{-}; u, u_{-}; \epsilon) = 0 \tag{1}$$

*We are indebted to two anonymous referees for valuable criticism of an earlier draft of this paper.

[1]Cf. Simon (1956), Theil (1957, 1968), Holt (1962).

where x is the vector of contemporaneous endogenous variables
x_- is a vector of lagged endogenous variables
u is the vector of contemporaneous instruments
u_- is a vector of lagged instruments
e is a vector of exogenous variables, which, in the case of a stochastic systems, contains the error terms as well
f is a vector of nonlinear, interdependent, and implicitly defined functions.

In general the optimization of models of this kind proceeds along the following lines:

- Take the preferred values of targets and instruments and put them together in a vector z^*. Note that the term "values" in this context is meant in the most comprehensive sense; it includes e.g. ratios and changes of variables as well.[2]
- Specify a criterion function suited, say, to stabilize the economic path around some a priori track and/or to reduce the period-to-period fluctuations in the economic path:

(2) $$(z - z^*)'A(z - z^*)$$

  Hence the coefficients of the weighting matrix are depending on the interpretation of the respective elements of z and z^* intended, first, to describe the politicians' preferences, second, to account for penalty term effects and, third, to represent certain smoothness requirements on the solution paths of targets and instruments.
- Finally, to derive the optimal policy minimize the (expected) welfare loss, i.e.

(3) $$\min_u \; (z - z^*)'A(z - z^*)$$
$$\text{s.t. } f(.) = 0$$

This approach is widespread in application primarily due to its operational convenience. One of the most frequent justifications for using the quadratic formula has been that this is probably the simplest form which allows for decreasing marginal rates of substitution;[3] in the absence of inequality constraints this is in general necessary for the existence of finite solutions (cf. Friedman (1975), p. 3). The standard approach has, however, been subject to growing criticism; some of the most significant items shall be discussed briefly.

[2] For a thorough discussion of its interpretation in general cf. Chow (1975), ch. 9, Garbade (1975), ch. 5, Holbrook (1973 ch. 6, 1974, 1975) or Norman, Norman-Palash (1975).

[3] Cf. Theil (1968), pp. 3-5, where some additional reference is given to related principles in statistics and engineering.

a) First, it is to be noted that the theoretically rather flexible approach has lost some of its attractiveness in application, since frequently a diagonal weighting matrix is used.[4]

b) Furthermore, it seems almost unlikely that the policymaker's preferences fit into such an artificially limiting framework as the quadratic function. In this context the most serious problem seems to be the implied symmetric reaction. It is well known that symmetric functions incorporate a potential to bias policy behavior if the numerical values of the z^*-elements are not chosen appropriately (cf. Palash (1977), Shupp (1977)). This can be overcome by the use of truncated or exponential criterion functions (cf. Palash (1977)). Apart from this, however, as Friedman (1975, p. 183) points out, "...often policy makers see certain variables more as constraints, in the sense of bearing an implicit preference loss only for values outside of a particular range." This requires an asymmetrical and possibly piecewise formulation.[5]

c) Regardless of the degree of sophisticatedness of the functional form of the criterion function a numerical weighting is indispensible. As numerous examples indicate, however, the optimization results in general turn out to be rather sensitive as to the choice of the coefficients in the weighting matrix.[6] Hence a meaningful application of this approach requires a precise knowledge of the coefficients' numerical values. For obvious reasons it seems fairly improbable that the policymaker is able to specify in an appropriate numerical form his preferences concerning the relative importance of concurrent targets and, even more tedious, in addition to that to fix the weights of the cross products between targets and instruments.[7,8] The coincidence of both these facts seems to give the entire approach a somewhat arbitrary touch. There have been some proposals to integrate the iterative process of optimization and preference revelation into a unified ap-

[4]On the other hand the further restrictive assumption of positive definiteness of A often to be found in earlier studies is no more significant, since nowadays models generally are neither convex nor concave.

[5]Cf. Friedman (1972; 1975, pp. 183–196), Fair (1974), Poirier (1976).

[6]Analytical derivations of consequences of a misspecification of A are to be found in Theil (1968, chs. 2, 3 & 5), Zellner/Geisel (1968), Zellner (1974) or Hallett (1977).

[7]Even if the model builder tries to figure out the policymakers' preferences by an adequate procedure it is to ask too much to expect a consistent preference scheme on an a priori basis. For some of these a priori procedures to derive a functional representation of the politicians' preferences cf. e.g. Johansen (1974) Bray (1974, 1975). Additional information concerning the penalization of instrument variations can, however, under certain circumstances be drawn according to Gordon (1976).

[8]The problem of relative weighting is generally aggravated by the fact that the variables use to be unnormalized. Hence one cannot discriminate between that part of the weighting scheme serving for normalization and that one expressing preference ordering.

proach.[9] As far as the application of these algorithms to large nonlinear systems is concerned there seems, however, to be no convincing evidence up to now.

d) An integrated welfare loss function of the standard type obviously lacks the facility to discriminate efficiently between targets and restrictions. Although restrictions can be made effective simply by an arbitrarily high weighting term, this causes numerical problems in the case of more than that particular argument in the criterion function since the value of the criterion function is dominated by penalty terms.[10] If there are more than one "restriction" of that type taken into account the weights may in effect cancel out. Furthermore, it seems possible that the policymaker is not indifferent concerning the order of activation of certain instruments, possibly due to decentralized or hierarchical decision processes. in the standard approach this has to be expressed within the framework of the general weighting scheme too.

e) Evaluating the optimization results by means of just a single welfare index may be insufficient under various aspects: first, there is no obvious economic interpretation of the results: second, this implies among other things that the performance of different runs (employing different z^+-values) cannot be evaluated in terms of ultimate targets; third, due to the fact that a lot of simulations is necessary to evaluate the local properties of a particular solution, a systematic sensitivity analysis turns out to be somewhat tedious.

As far as the first three items are concerned Livesey's (1973a, p. 11) conclusion seems inevitable: "Hoping to come up with the unique social welfare function is a fruitless task. For this reason it would be desirable to keep the welfare function as simple as possible and to incorporate policy objectives... in the model as inequality constraints." This is the way which is to be pursued in the present paper. A consequent application of nonlinear programming (NLP)- techniques will—as can be shown throughout the presentation—in addition surmount at least in part the shortcomings of the traditional approach indicated under d) and e) above.

2. An Alternative Approach

In the previous chapter we tried to bring out some of the main problems associated with the application of a weighted loss function in eco-

[9]Cf. Rustem/Velupillai/Westcott (1977) and earlier work by Zeleny/Cochrane (1973), followed by Wallenius/Wallenius/Vartia (1976) or Donckels (1977). All these approaches are, however, developed within the standard quadratic framework.

[10]Cf. Luenberger (1969), p. 302. The significance of this point depends upon the relative magnitude of the weighting terms as compared with the numerical preciseness of the computer. For an alternative penalizing scheme handling limited discretion cf. Garbade (1977).

nomic policy optimization. In what follows we outline the features of our alternative approach. In order to keep to the essentials and to clarify our position we start with the description of a rather rigorous version of the basic procedure. In ch. 5 below some of the stronger assumption will be relaxed in order to point out some potential modifications.

Our starting point is a vector z^+ too. although we make a different use of the information contained therein. As has been outlined above, z^+ is a rather heterogeneous mixture of targets of economic policy, expressed in numerical values of certain endogenous variables (but of differing importance for the policy-maker). and "preferred" values of instruments and/or their respective paths. One should, however, recognize that some of the elements of z^+ have originally been incorporated into the criterion function in order to approximate some more or less technically determined restrictions on the time paths of those particular variables. In a first step. we try to separate these elements from the entire z^+-vector. This is done by direct formulation of inequality constraints on the respective variables: these constraints are combined into a vector z_2^* and added as an integral part to the model. As Kornai (1967. p. 398) points out this "...system of constraints expresses thus the compelling forces of outward circumstances," which are to be distinguished from the wishes of economic policy entering the objective function to express the preference of economic administration. For expository purposes we will assume that z_2^* contains only instruments recognizing that in short-term planning because of the inertia of legislative processes this vector might contain more elements than in the long-term framework.[11]

In view of the criticism raised in the previous chapter we further reformulate the whole bundle of remaining z^+-values in terms of suitably chosen inequality restrictions, the upper bounds of which are integrated into a vector z_1^*.[12] Finally. we require an initial ordering of these inequalities according to their relative importance. Actually, it is not necessary to assign to each z_1^*-value a cardinal preference number; all what is required is an ordering.[13]

Without having made it explicit up to now. the fundamental difference between this approach and the traditional formulation can be exemplified figuring out the basically different interpretations of z^+ und z^*. While the former is assigned to "desired levels," the latter is to indicate the maximum (or minimum) tolerable level of certain variables.

[11]For the moment we further leave out of consideration that there may be preferences concerning a sequential activation of instruments. In our application (cf. ch. 4) this idea is taken up again.

[12]Note that this general formulation may without any further complications imply upper and lower bounds on the range of instruments and/or endogenous variables. The separation of targets and instruments in z_1 and z_2 actually is nonessential. cf. ch. 5.

[13]Even if there is only a grouped ordering our approach – although somewhat more complicated – is still applicable: cf. ch. 5.

Actually, as far as upper and lower bounds are concerned, we consider this to be a more adequate representation of the real policy problem; furthermore, in most realistic situations we expect some ordering of targets to exist. If this is the case a formulation as outlined above turns out to be somewhat more straightforward than to try to catch up this ordering within a reasonably complex weighting scheme; apart from that we consider it to be intuitively more pausible to the policymaker. We are now in a position to identify the basic methodological differences. While the standard approach can be characterized as a minimization problem under equality constraints the essence of this method is to construct feasible solutions of a mixed equality – inequality system by an iterative procedure.

Before we now come to the exposition of our procedure, let us sum up: z_2^* is a vector containing upper bounds on instruments, the respective variables are contained in the vector z_2 (the elements of z_2 form a subset of u). $\bar{z}_1^*$ is the vector of upper bounds on endogenous variables, its elements being ordered according to their importance for the politician. Let us assume that the system's endogenous variables are ordered such that the i-th order restriction refers to the variable x_i. Let z_1 contain the first i_{max} endogenous variables, which are to be restricted. From what was explained above it follows that the politician wants to know a policy which fulfills the following equality/inequality system:

$$
\begin{aligned}
z_1 &\leq \bar{z}_1^* \\
f(.) &= 0 \\
z_2 &\leq z_2^*
\end{aligned}
\tag{4}
$$

But, as it was pointed out—among others—by Livesey (1976): "The formulation of economic policy is . . an iterative procedure, with the outcome of one policy evaluation influencing the formulation of the next planning exercise." So our aim is to give the policy-maker together with a solution of (4) as much information as possible for the evaluation of the derived policy under the aspects of the relative importance of his preferences and the implications of their modification.

With this model formulation we now come to the calculations. The basic idea of the solution method is, starting from a z_2^*-feasible point, in the procedure stepwise addition of the restrictions in $\bar{z}_1^*$ according to their order—either to construct a (4)-feasible point or region – or to show that (4) has no solution; in this case we compute the feasible value for that particular restriction that causes infeasibility.

In the simplest formulation the steps of computation are therefore:

0) (Construction of a feasible starting vector) Choose any (reasonable) z_2^*-feasible set of instrument values and compute via $f(.) = 0$

the respective endogenous variables. Set the $\bar{z}_1^*$-element counter $i := 0$.

1) Set $i := i + 1$. If $i > i_{max}$, the number of elements in $\bar{z}_1^*$, then stop.
2) Test, whether the i-th order restriction in $\bar{z}_1^*$ is violated by the current value of the respective variable x_i. If not, go to 1).
3) Solve the following problem:

$$\min_u \; x_i$$

s.t.

(5)
$$z_1^i \leq \bar{z}_1^{*i}$$
$$f(.) = 0$$
$$z_2 \leq z_2^*$$

where $\bar{z}_1^{*i}$ is the truncated vector $\bar{z}_1^*$ containing only the first $i - 1$ elements,[14] i.e. those restrictions which have been dealt with in earlier steps 2) or 3).[15]

4) Substitute

(6)
$$\bar{z}_{1i}^* := \max(\bar{z}_{1i}^*, x_i^{min})$$

where $\bar{z}_{1i}^*$ is the maximal permitted value (of the i-th order restriction) for x_i, and x_i^{min} the minimum achieved in (5). Go to step 1).

Note that step 4) above is essential: we have either constructed a $\bar{z}_1^{*i}$-feasible point or – in the case where x_i^{min} is greater that the original value $\bar{z}_{1i}^*$ – shown that this i-th order restriction is incompatible with the preceding ones of higher importance (including the vector z_2^*). For the solution of problem (5) we make use of a Lagrangean approach (cf. the next chapter for more details), especially to compute the multipliers for binding restrictions.

Going through steps 1) to 4) finally leads to the information output of the ultimate feasible solution as well as the respective results from intermediate steps. If an optimization step 3) was performed, this would include the values of the multipliers[16] to indicate the relative importance of

[14] Or equivalently: a vector $\bar{z}_1^*$, whose elements $i + 1, \ldots, i_{max}$ are fixed at the highest economically still meaningful values. Note that the minimum of x_i is independent of the ordering of $\bar{z}_1^{*i}$.

[15] Note that proceeding this way the restrictions are in any case fulfilled whereas in the weighted criterion function approach it is generally not guaranteed that the solution is "close" to the desired path: cf. e.g. Livesey (1973 a,b;1974).

[16] Evaluation of the multipliers in the present paper means that we make use of the well known property of the Lagrangean multipliers, namely to indicate the derivative of the criterion function w.r.t. changes of the right hand side of the respective inequality constraints: cf. e.g. Luenberger (1969), pp. 221–223, Peterson (1973). This allows for any easy calculation of local elasticities.

the effective $\bar{z}_1^*$- and z_2^*-restrictions, which indicate the sensitivity of the results w.r.t. variations of the z^*-values; this is the information needed for policy analysis. From the well-known manifold evaluation possibilities we only note the following: a high dual for an instrument at the bound states on the one hand its effectiveness and on the other hand the necessity to control it precisely; a high multiplier for an endogenous variable at the bound states the crucial importance of a precise knowledge of the respective $\bar{z}_{1i}^*$-value.[17]

With these results at hand the policymaker has the following option: either to accept the result as definite or, if he considers the informational content of the multipliers to be insufficient, to respecify the vectors $\bar{z}_1^*$ and, possibly, z_2^* concerning values and ordering.[18] In the latter case we would start the computational procedure anew. Hence the essence of this method is to perform an alternating sequence of computation and evaluation steps.[19,20]

3. The Algorithm

For the solution of (5) and the evaluation of alternative policies as outlined above an algorithm is needed which is not restricted to the handling of quadratic criterion functions, which allows for inequality restrictions on endogenous variables as well as on instruments and, finally, computes the values of multipliers at least for the inequality constraints.

The literature on solution methods for this general NLP-problem (5) is not very extensive, especially for the numerical treatment of medium-sized or large-scale problems. In the following we outline our algorithm.[21]

A) Transform all inequality restrictions to equalities by introduction of quadratic slack variables:

[17]Note that in a stochastic system, where the restrictions get a stochastic character too, the multipliers can be evaluated under this aspect. Heuristically, the restriction then could be read as

$$x_i + \varepsilon \leq \bar{z}_{1i}^*, \quad \text{or} \quad x_i \leq \bar{z}_{1i}^* - \varepsilon$$

where ε is a random term.

[18]Reordering of course can only be expected to be effective in the case of a sufficient nonlinearity of the multipliers.

[19]Just to see whether the preferences are feasible, obviously, other (and simpler) procedures would do as well; e.g. the changing of the target variable in (5) in each step would be unnecessary. To choose, however, the lowest preference variable as a fixed target would not give as much information concerning the feasible region as does the changing procedure adopted here.

[20]This sequential procedure is in the spirit of Kornai (1967) and Livesey (1976); cf. ftn. 9 too.

[21]For a more detailed exposition cf. Bock v. Wülfingen/Pauly (1977), pp. 1–10. One of the referees noted that "the superiority of the nonlinear programming algorithm to any other existing algorithm is not at all clear." As a general remark this is correct; but one of the a priori reasons for the development of this algorithm was its high rate of convergence, which in our modification (employing an idea of Laasonen (1969)) is overquadratic. Moreover, the Jacobian in (10) is very easy to compute for econometric models.

(7)
$$z_1 + s_1^2 = \bar{z}_1^*$$
$$z_2 + s_2^2 = z_2^*$$

B) Transform the restricted optimization problem (5) to a non-restricted extremal problem by introduction of the Lagrangean:

(8)
$$L = x_i + M_1'(z_1 + s_1^2 - \bar{z}_1^*) + M_2'(z_2 + s_2^2 - z_2^*) + \Lambda' f(\cdot) \stackrel{!}{=} \min$$

C) Set up the whole set of necessary conditions for an extremum of L

(9)
$$\left(\frac{\partial L}{\partial x}, \frac{\partial L}{\partial u}, \frac{\partial L}{\partial s_1}, \frac{\partial L}{\partial s_2}, \frac{\partial L}{\partial M_1}, \frac{\partial L}{\partial M_2}, \frac{\partial L}{\partial \Lambda}\right)' = 0$$

Note that in any case of nonlinearity in the system (4) the respective derivative equation in (9) contains a nonlinear mixture of system variables and multipliers.

The respective derivative for a variable entering the system only in linear form is, of course, a linear combination of multipliers. The derivatives w.r.t. the slack variables simply state the well known condition that the slack or the multiplier must equal zero.

D) Solve this system (9) simultaneously to get the optimal values of instruments, endogenous variables and multipliers.

The implementation[22] is characterized by the following properties:

- The model f is coded in data form, not in a program or procedure: $\bar{z}_1^*$ and z_2^* are represented simply by indices and critical values of the respective variables. This makes model modifications or changes a very easy task, and in either case we do not need to translate any program anew.
- The system (9) uses analytical derivatives. They are created internally by the program using the input for the model f and the vectors $\bar{z}_1^*$ and z_2^*. Thus we circumvent the rather time consuming approach via a formula interpreter and code generator, whose output normally has to be translated before further processing.
- (9) is solved with the Newton-method. The iteration prescription is[23]

(10)
$$y_{j+1} = y_j - J^{-1}(y_j)F(y_j)$$

where y is the vector of unknowns, which in our application contains the whole set of variables x and u, the slack variables s, and the multipliers Λ and M. Note that the derivatives of (9) in the

[22] Cf. ibid. for a more detailed discussion of the implementational advantages.
[23] Cf. any standard reference, e.g. Ortega/Rheinboldt (1970).

Jacobian J contain the first and second derivatives of the original model (1).

The derivatives needed in the Jacobian J are also computed analytically and generated internally; in each iteration j the whole system (9) is evaluated simultaneously for the computation of the new Jacobian of the last iterated value y_j. F denotes the vector of residuals of all equations for y_j.

- The main computational burden in the approach is associated with the solution of J^{-1} or equivalently the solution of a large scale linear equation system. In our implementation we first extract from the original system (9) the upper and lower triangular part, then test the remaining interdependent system for blockdiagonal structure[24] and if it has one repeat the triangulization and decomposition procedure for each block and so on. Eventually (10) is applied to each indecomposable block separately, employing if necessary sparse matrix techniques as familiar from LP-implementations.

4. An Example

The function of the following chapter is to give a rough impression of the working of the procedure. For expository purposes we have applied our method to a medium-sized, nonlinear, and interdependent theoretical model. It can be characterized as a modified Keynes-Wicksell monetary growth model[25] of an open economy[26] with price flexibility and labor market disequilibrium; special emphasis has been put on the introduction of stock and flow constraints as well as on the fulfilment of the government budget constraint.[27] It is a condensed version of a larger two-sector model, which has been described in detail elsewhere.[28] In order to focus primarily on the application of the proposed algorithm and the specific form of the criterion function, in the present paper a more comprehensive discussion of the model has been skipped.[29]

Within the context of this model we try to solve the following problem: assume there is an ordered vector z_1 of targets of economic policy

[24] The basic ideas have been outlined e.g. in Hellerman/Rarick (1972).

[25] Cf. e.g. Fischer (1972).

[26] For a survey of models of this kind cf. Prachowny (1975).

[27] The fundamental importance of these aspects in macroeconomic policy analysis has recently been pointed out e.g. by Branson (1976).

[28] Cf. Bock v. Wülfingen/Pauly (1977), pp. 10–28. Even this condensed version sums up to about 100 equations, approximately half of them are essentially interdependent. With regard to the degree of disaggregation as well as to the basic conceptual framework the model is very much in the spirit of recently developed more comprehensive models, cf. e.g. Mussa (1976).

[29] A detailed version of the equation system is, of course, available from the authors on request.

		Bounds		z_2 – Feasible Starting Vector	STEP					
		Lower	Upper		I	II	III	IV	V	VI
TARGETS	$\hat{P}$		2	.74	Fulfilled	1.62	Fulfilled	1.38	2.00 Dual: .31	Fulfilled
	U		3	3.1		$\boxed{2.8}$		3.0 Dual: .13	2.7	
	DEF	–1	2	–3.6		–4.1		$\boxed{-.90}$	1.6	
	BPA	–15	10	–16.7		–23.6		–17.2	$\boxed{-3.1}$	
INSTRUMENTS	G	14	25	18		22.4		14 Dual: .42	14 Dual: .16	
	w	.95	1.05	1.03		1.03*		1.03*	1.05 Dual: .78	
	OM			0		0*		12.1	20.5	

Notes: * = fixed at that value for optimization

$\boxed{}$ = target variable in the minimization step

Figure 1

$$z_1 = \begin{bmatrix} \hat{P} \\ U \\ \mathrm{DEF} \\ -\mathrm{DEF} \\ -\mathrm{BPA} \\ \mathrm{BPA} \end{bmatrix} \quad \text{with} \quad z_1^* = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \\ 15 \\ 10 \end{bmatrix}$$

where $\hat{P}$ = rate of inflation
U = unemployment rate
DEF = budget deficit
BPA = balance of payments account

At the policymakers' disposal in this (fixed exchange rate) system there are three instruments: government expenditures (G), the amount of the monetary authority's autonomous open market transactions (OM), and the exchange rate (w); two of these three instruments, however, are constrained as well, namely

$$z_2 = \begin{bmatrix} G \\ -G \\ w \\ -w \end{bmatrix} \quad \text{with} \quad z_2^* = \begin{bmatrix} 25 \\ -14 \\ 1.05 \\ -\ .95 \end{bmatrix}$$

The optimization results can be drawn from Fig. 1. Since in our application there are upper and lower bounds on the same variable a slight modification of the general procedure is applied: if the violation of the upper (lower) restriction in step 2) makes an optimization (5) necessary we account for the respective lower (upper) constraint at the same time; this is just to reduce the computational expense. Note further that we adopted the special case referred to in the exposition of successive activation of policy instruments.

Starting with the choice of a z_2 – feasible vector the solution proceeds along the following steps:[30]

I) Add: $\hat{P} \leq 2$; no computation
II) Add: $U \leq 3$; minimize U.
III) Add: $\mathrm{DEF} \leq 2$; no computation.
IV) Add: $-\mathrm{DEF} \leq 1$; minimize $-\mathrm{DEF}$.
V) Add: $-\mathrm{BPA} \leq 15$; minimize $-\mathrm{BPA}$.
VI) Add: $\mathrm{BPA} \leq 10$; no computation.

The final solution V has the undesirable property that the most important restriction is at its bound; we would assume that the policy-

[30]Note that, since the instruments have been set free successively, the value of a target variable may, in a subsequent step, still fall below that value achieved previously, when the respective variable was chosen as a target variable.

maker takes this final result as the basis to ask for further analysis of the local properties. Nevertheless it turns out in this example that the policymaker's preferences are at least feasible. If this were not the case we had to ask the policymaker to respecify his preferences, e.g. by weakening the restrictions in $\bar{z}_1^*$. Then the procedure had to be applied anew.

5. Modifications and Extensions

Up to this point we have explicitly considered only one-period problems. In a multiperiod approach there could be slight modifications of the formulation. E.g.: the z^*-value in z_1^* would be treated as restrictions on the average value of the respective variables:

$$(12) \qquad 1/T \cdot \sum_{1}^{T} z_{1t} \leq \bar{z}_1^*$$

where T is the long run planning horizon. If it is the policymaker's preference not to allow some variables to deviate in a single period from the average value by more than a certain amount, we would add the following restrictions:

$$(13) \qquad z_{1t} \leq b \cdot \bar{z}_1^* \quad 1 \leq t \leq T$$

where

$$b = \mathrm{diag}\,(1 + 0.01 \cdot b_i)$$

and $b_i \geq 0$ is the maximal tolerable (percentual) deviation from the respective average value.[31]

If—in the multiperiod case—the politician's preference actually is to maximize a certain variable (say capacity growth or per capita consumption) upon holding of the restrictions formulated above, this would cause no computational problem: the last step of the procedure described in ch. 3 would be the maximization of that particular variable under the restriction of (4). A discounting in the criterion function would be unnecessary because we can reformulate any kind of intertemporal preference into an adequate inequality restriction.[32]

Of course concerning our main aim, the entire elimination of the weights from the approach, the linear criterion function in (5) is not essential. Any other, e.g. a quadratic function containing only one single variable would do as well and be conceptually equivalent. In this case,

[31] The problems arising in the multiperiod case concerning the algorithm have been discussed in Bock v. Wülfingen/Pauly (1977), p.6.

[32] In a recent paper Ku/Athans/Varaiya (1977) have pointed out the crucial importance of the numerical value of the discount rate for the stability properties of dynamic systems. Despite that the introduction of a discounting factor would only be another element of arbitrariness.

however, the numerical evaluation of the multipliers would have to take into account the specific form of the criterion function; this would be the case particularly for piecewise defined functions.

From the exposition of the computational procedure in ch. 5 it should be evident that the assumption of z_2^* containing only instruments and z_1^* containing only endogenous variables is not essential either. With z_1^* containing also instruments the approach would be identical. If there are endogenous variables in z_2^*, the construction of a feasible starting point would have to be slightly modified, because it could involve still some optimization steps as in (5).

One of the fundamental ingredients of the present procedure is a systematic evaluation of the multipliers. Using an algorithm, which does not compute the values of the multipliers, changes of the criterion function value with regard to variations of the restrictions on the instruments might as well be computed simply by simulation runs with modified values of the instruments (as long as no restrictions on endogenous variables are or become active). In contrast to that the computations of multipliers w.r.t. restrictions on endogenous variables would deserve for repeated optimization runs for slightly altered z*-values of the respective elements.

In our example we gave the multipliers only for those steps in which an optimization was necessary and trivially the duals are zero for nonbinding restrictions. To gain a deeper insight into the implications of the given preference order, we could perform step 3) after adding any element of z_1^*, irrespective of whether the restriction was effective or not. Furthermore, we could in any case insert the following step:

(3a) Set each element of z_1^*, where the restriction is not binding at a value with a zero slack.[33] Now compute the multipliers of the restrictions,[34] the information contained herein being of special importance in the stochastic case (see ftn. 17).

Finally let us look at the requirement of strong ordering of all restrictions. A natural weakening would be a grouped ordering in the sense that some *-values would be considered of equal importance by the politician. In this case we would have to add not only one but several restrictions in a particular step of the computation procedure.

If none of the group is violated by the solution of the last step, there is no further problem and we can proceed to the next group or single restriction. Otherwise we first try with some auxiliary criterion function to construct a new solution which fulfills those restrictions to be added. If that turns out to be impossible, we raise all restrictions of the group

[33] If e.g. $\hat{p} = 1.62 < 2 = \hat{p}^*$, set $\hat{p}^* = 1.62 - \delta$, with δ small, but nonzero. Obviously $\delta = 0$ would lead to degeneracy.

[34] But the new solution will of course have not all restrictions binding; so to get the multiplier information possibly deserves for more than one additional optimization step

simultaneously by the same rate to achieve feasibility.[35] Note that this states an infeasibility of the original vector z too.

We have tried to show that the present approach is a very flexible tool and allows for a variety of extensions and modifications; the fundamental idea in any case is based on one single unweighted target variable in the course of solution and on the manifold application and evaluation of the multipliers.

6. Summary

In the present paper we set forth an alternative approach to the formulation of the criterion function in multiple target optimization problems based on the use of one- or two-sided inequality constraints on targets and instruments. The computational aspects of its implementation have been discussed in detail. As far as the application to problems of economic policy optimization is concerned, the main advantages of our proposal can be summarized as follows:

- It allows to separate distinctly between targets of economic policy and more or less technically determined restrictions.
- Most of the notorious problems involved in the specification of the weighting matrix in the standard quadratic or otherwise penalized deviations approach can be avoided.
- The problem of sensitivity of the results w.r.t. variations in the weighting matrix is circumvented.
- The formulation of targets and restrictions requires a considerably lower degree of preference revelation. Actually, all we need is an ordering of the targets—at least grouped—and numerical values of upper and lower bounds for targets and instruments.
- As far as policy evaluation for different preferences, i.e. modified sets of *- or +-values is concerned, we suppose our approach to be more suitable than the resort to a single welfare index in the standard approach.
- The local elasticities of the target values w.r.t. the restrictions upon other targets and/or instruments may be evaluated making use of the Lagrangean multipliers.
- If a global analysis is preferred, in this approach a systematic variation of the bounds allows for a straightforward construction of feasible regions of solution.

[35] Assuming implicitly some sort of "local homogeneity" of the politicians' preferences. In spite of the grouping this may be questionable, just as the following procedure: add all restrictions of the group simultaneously and change the bounds successively until all multipliers of the group have the same value (which is practicable of course only in the case of sufficient nonlinearity; it furthermore introduces an additional problem of choice, since the duals are conditional on the respective target variable).

- Although we make no stochastic optimization in our procedure, the stochastic nature of a particular system can be taken into account by an appropriate evaluation of the constraints' dual variables.

As far as our experience indicates these advantages outweigh the computational impediments which may be connected with the application of our method in a particular problem. Thus we are not in a position to share the pessimism sometimes expressed concerning the formulation of economic policy problems in terms and techniques of NLP. A systematic comparison, however, must be left to future research.

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