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THE EFFECTS OF DISCOUNTED COST ON THE UNCERTAINTY THRESHOLD PRINCIPLE*

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The optimal stochastic control of a linear system with purely random parameters and with respect to a discounted quadratic index of performance is considered. It is shown that if a function involving the parameter variances and the discount factor exceeds a certain threshold, then the infinite horizon optimization problem has no solution. On the other hand, it is also shown that the existence of optimal infinite horizon rules may not guarantee the stochastic stability of the underlying feedback system.

1. INTRODUCTION

In this paper we consider the problem of stochastic control of a linear system with purely random parameters (i.e. uncorrelated in time) with known statistics. Such a mathematical model for uncertain systems has been advocated by Chow [1], [2], [3] for economic applications; the randomness of the parameters of the econometric model represent the unpredictable future changes of key multipliers.

Apart from certain technical considerations, the optimal stochastic control problem is well defined for such systems for *finite* horizon planning problems; see Chow [1] and Aoki [4]. However, if one considers the infinite horizon problem the results of the Uncertainty Threshold Principle (UTP) (see Athans, Ku, and Gershwin [5], [6]) indicate that an optimal solution will not exist if the standard deviations of the random model parameters is large. In fact even for finite planning horizon problems the optimal cost-to-go undergoes exponential growth with increasing planning horizon (N). In this paper we consider the effects of including *discount factors* in the objective function. Traditionally, discount factors have been used in economic problems to accentuate the near term worth of the utility function as compared to the long-term one. One may then suspect that the inclusion of discount factors in the objective function may increase the threshold at which optimal decision rules for the infinite horizon problem exist; this indeed is the case as it will be shown in the main body of this paper.

However, the analysis of optimization problems involving systems with random parameters and discounted quadratic performance indices brings into the surface another curious phenomenon. One can determine a quantifiable region, involving the statistics of the random parameters

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and the value of the discount factor, in which *optimal long range decision rules exist but the underlying optimal closed loop system is unstable in a mean-square sense.*

The implication of these results is that proper care must be exercised in the interpretation of results of optimization of uncertain systems when discount factors are present. The existence of optimal decision rules does not guarantee the stochastic stability of the system. A separate stability analysis must be carried out.

These issues are demonstrated by the simplest possible scalar example in the main body of this paper.

2. Problem Statement

In this section we summarize the problem statement. The notation is consistent to the degree possible to that used in ref. [5].

Consider a first order stochastic dynamical system with state $x(t)$, control $u(t)$, and process noise $\xi(t)$ described by the difference equation

(1) $$x(t+1) = a(t)x(t) + b(t)u(t) + \xi(t);\ t = 0, 1, 2, \ldots$$

We suppose that the purely random parameters $a(t)$ and $b(t)$ are Gaussian and white (uncorrelated in time) with known constant means $\bar{a}, \bar{b}$ and variances $\sum_{aa}, \sum_{bb}$ respectively. They may be also correlated with (cross) covariance $\sum_{ab}$. More precisely, assume that

(2) $$E\{a(t)\} = \bar{a},\ E\{b(t)\} = \bar{b} \text{ for all } t$$

and that their variances are given by

(3) $$E\{(a(t)\} - \bar{a})(a(\tau) - \bar{a})\} = \Sigma_{aa}\delta(t, \tau)$$
$$E\{(b(t) - \bar{b})(b(\tau) - \bar{b})\} = \Sigma_{bb}\delta(t, \tau)$$
$$E\{(a(t) - \bar{a})(b(\tau) - \bar{b})\} = \Sigma_{ab}\delta(t, \tau)$$

where $\delta(t, \tau)$ is the Kronecker delta ($\delta(t, \tau) = 1$ if $t = \tau$, $\delta(t, \tau) = 0$ if $t \neq \tau$); and

(4) $$\Sigma_{aa}\Sigma_{bb} - \Sigma_{ab}^2 \geq 0$$

It is assumed that the means $\bar{a}, \bar{b}$ and variances Σ_{aa}, Σ_{bb}, Σ_{ab} are constand and known *a priori*.

We assume that the process noise $\xi(t)$ is zero mean, white (i.e., uncorrelated in time), with variance

(5) $$E\{\xi(t)\xi(\tau)\} = \Xi\delta(t, \tau)$$

We further assume that the process noise $\xi(t)$ is independent of the random parameters $a(t)$ and $b(t)$.

We consider the minimization of the discounted quadratic cost functional

$$J = E\frac{1}{N}\left\{\sum_{t=0}^{N} \alpha^t(Qx^2(t) + Ru^2(t))\right\} \tag{6}$$

where N is the planning horizon time and $Q > 0$, $R > 0$. The scalar α is the *discount factor*. We assume that

$$0 < \alpha < 1 \tag{7}$$

When $\alpha = 1$, then we have the no-discount case discussed in ref [5].

We assume that the state $x(t)$ can be measured exactly.

3. Problem Solution for Finite Planning Horizon

The solution can be obtained by standard stochastic dynamic programming methods; the derivations represent a trivial exercise and hence will not be given. We summarize the results below.

The optimal feedback control at each instant of time is given by a linear transformation of the state, i.e.

$$u(t) = -G(t)x(t) \tag{8}$$

The scalar linear gain $G(t)$ is given by

$$G(t) = \frac{\alpha K(t+1)[\sum_{ab} + \bar{a}\bar{b}]}{R + \alpha K(t+1)[\sum_{bb} + \bar{b}^2]} \tag{9}$$

The scalars $K(t)$ are related by a Riccati-like recursive equation (The UTP equation [6]) by

$$K(t) = Q + \alpha K(t+1)(\textstyle\sum_{aa} + \bar{a}^2) - \frac{\alpha^2 K^2(t+1)(\sum_{ab} + \bar{a}\bar{b})^2}{R + \alpha K(t+1)(\sum_{bb} + \bar{b}^2)}; \; K(N) = 0 \tag{10}$$

The optimal cost is given by

$$J^* = \frac{1}{N}\left\{K(0)x^2(0) + \sum_{t=0}^{N-1} \alpha^{t+1}K(t+1)\Xi\right\} \tag{11}$$

4. The Infinite Horizon Case

The optimal solution stated above exists for all finite values of the planning horizon time N. However, the solution to the optimization problem may fail to exist (in the sense that the optimum cost is infinite) for the infinite horizon case. The precise result is stated as follows.

Theorem 1

Let $N \to \infty$. Define the *undiscounted threshold parameter* m [5], [6] by

$$m \triangleq (\Sigma_{aa} + \bar{a}^2) - \frac{(\Sigma_{ab} + \bar{a}\bar{b})^2}{\Sigma_{bb} + \bar{b}^2} \tag{12}$$

Then the solution to the optimal infinite horizon problem exists *if and only* if

$$m \leq \frac{1}{\alpha} \tag{13}$$

Proof

Let $\tilde{a}(t) \triangleq \sqrt{\alpha}\, a(t)$ and $\tilde{R} = R/\alpha$. Then after some algebra, equation (10) reduces to

$$K(t) = Q + K(t+1)(\tilde{\Sigma}_{aa} + \tilde{\bar{a}}^2) - \frac{K^2(t+1)(\tilde{\Sigma}_{ab} + \tilde{\bar{a}}\bar{b})^2}{\tilde{R} + K(t+1)(\Sigma_{bb} + \bar{b}^2)} \tag{14}$$

where the $\sim$ quantities refer to the statistics of $\tilde{a}(t)$. The structure of eq. (14) is identical to that given in ref [5] and hence the result follows. QED

The above result implies that if (13) holds then the limiting solution of eq. (10) exists, is bounded and approaches a constant K i.e.,

$$\lim_{N \to \infty} K(t) = K < \infty \tag{15}$$

and it is the positive solution to the algebraic equation

$$K = Q + \alpha K(\Sigma_{aa} + \bar{a}^2) - \frac{\alpha^2 K^2(\Sigma_{ab} + \bar{a}\bar{b})^2}{R + \alpha K(\Sigma_{bb} + \bar{b}^2)} \tag{16}$$

and, consequently, the linear gain $G(t)$ of eq. (9) also approaches a constant value

$$G = \lim_{N \to \infty} G(t) = \frac{\alpha K(\Sigma_{ab} + \bar{a}\bar{b})}{R + \alpha K[\Sigma_{bb} + \bar{b}^2]} \tag{17}$$

On the other hand if

$$m > \frac{1}{\alpha} \tag{18}$$

$\lim_{N \to \infty} K(t)$ is not defined, and in fact $K(t)$ grows exponentially as

$$\lim_{N \to \infty} K(t) \approx e^{\alpha m N} \tag{19}$$

Note that the more the future is discounted ($\alpha \to 0$), the more uncertainty can be tolerated in the random system parameters (reflected in the numerical value of undiscounted threshold parameter m) and still have an optimal decision rule for the infinite horizon case. Thus in the case that the solution exists ($m \leq 1/\alpha$), the use of the optimum decision rule (8) together with the optimal constant value of the gain G given by eq. (17) will result in the following *optimum* evolution of state, according to the stochastic difference equation (obtained by substituting (8) and (17) into (11)

$$
\begin{aligned}
(20) \qquad x(t+1) &= [a(t) - Gb(t)]x(t) \\
&= \left[a(t) - \left(\frac{\alpha K(\Sigma_{ab} + \bar{a}\bar{b})}{R + \alpha K(\Sigma_{bb} + \bar{b}^2)}\right) b(t)\right] x(t)
\end{aligned}
$$

5. Stochastic Stability Considerations

One may suspect that the existence of an optimal decision rule in the case $m \leq 1/\alpha$, will cause the dynamic evolution of the state according to eq. (2) to "behave" and to have certain stability properties. This is *not* the case! *In this section we shall demonstrate that the optimal closed-loop system (20) is unstable in a mean-square sense in the region*

$$(21) \qquad 1 \leq m \leq 1/\alpha$$

in spite of the existence of an optimal decision rule in the range given by (21).

Consider the stochastic system (1) and any linear control law

$$(22) \qquad u(t) = h(t)x(t)$$

Then the closed loop system will propagate according to the stochastic equation

$$(23) \qquad x(t+1) = [a(t) + h(t)b(t)]x(t) = c(t)x(t)$$

Since the $c(t)$ are uncorrelated in time, one can calculate the ratio

$$(24) \qquad \frac{E\{x^2(t+1)\}}{E\{x^2(1)\}} = E\{c^2(1)\}E\{c^2(2)\} \ldots E\{c^2(t)\} \triangleq S(t)$$

The value of $S(t)$ is a measure of how the second moment of the state propagates in time. The larger the value of $S(t)$, the more variable the state is. In particular, if

$$(25) \qquad \lim_{t \to \infty} S(t) \to \infty$$

the system (23) is unstable in a mean square sense.

The value of $S(t)$ will be influenced in part by the value of the feedback gain $h(t)$ in eq. (22). So one can seek the value of $h(t)$ which will *minimize* the ratio $S(t)$ given by eq. (24).

Obviously, $S(t)$ is minimized if each element of the product

$$(26) \qquad E\{c^2(t)\} = E\{[a(t) + h(t)b(t)]^2\}$$

is minimized by $h(t)$. But

$$(27) \qquad E\{c^2(t)\} = E\{a^2(t) + h^2(t)b^2(t) + 2h(t)a(t)b(t)\}$$
$$= E\{a^2(t)\} + h^2(t)E\{b^2(t)\} + 2h(t)E\{a(t)b(t)\}$$

Therefore, the best value of $h(t)$, denoted by $h^*(t)$, is obtained from the solution of

$$(28) \qquad 0 = \frac{\partial E\{c^2(t)\}}{\partial h(t)} = 2h(t)\ E\{b^2(t)\} + 2h(t)\ E\{a(t)b(t)\}$$

which yields

$$(29) \qquad h^* = h^*(t) = -\frac{E\{a(t)b(t)\}}{E\{b^2(t)\}} = -\frac{\sum_{ab} + \bar{a}\bar{b}}{\sum_{bb} + \bar{b}^2} = \text{constant}$$

Hence the minimum value of $E\{c^2(t)\}$ is

$$(30) \qquad E\{c^2(t)\}_{\min} = E\{[a(t) + h^*b(t)]^2\}$$
$$= \sum_{aa} + \bar{a}^2 - \frac{(\sum_{ab} + \bar{a}\bar{b})^2}{\sum_{bb} + \bar{b}^2} = m$$

where m is the *undiscounted threshold parameter* given by eq. (12).

It follows that

$$(31) \qquad S(t)_{\min} = m^t$$

and hence that

$$(32) \qquad \lim_{t \to \infty} S(t)_{\min} < \infty \qquad \text{if } m < 1$$

Hence we have proved

Theorem 2

The stochastic system (1) is stabilizable by linear feedback in a mean square sense if and only if the undiscounted threshold parameter m, defined by eq. (12), is less than unity. In particular, the optimal closed loop system of eq. (20) is *not* stable in a mean square sense in the range $1 < m \leq 1/\alpha$, where α is the discount factor.

6. Discussion

The implications of the above results are best understood by reference to Figure 1. The undiscounted threshold parameter m can be

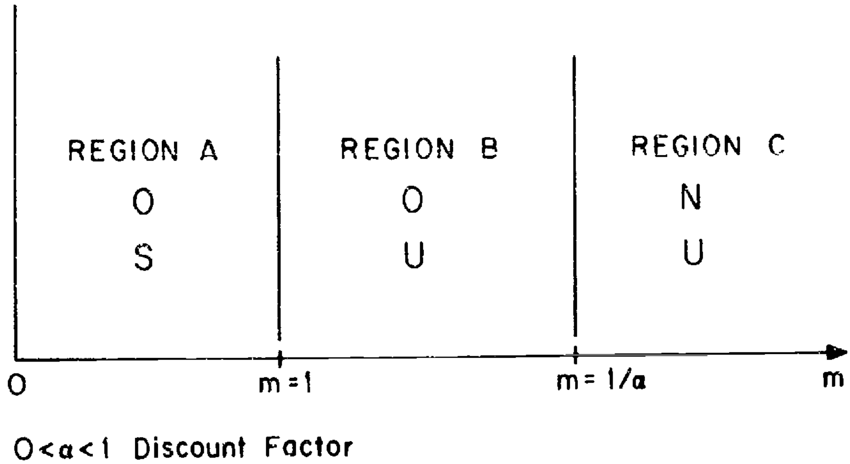

Figure 1 Behaviour of Solution as a Function of the Threshold Parameter m

O: optimal infinite horizon decision rules exist
N: optimal infinite horizon decision rules do not exist
S: closed loop system stochastically stable
U: closed loop system stochastically unstable

thought of as a measure of the system parameter uncertainty since for any given mean values $\bar{a}$, $\bar{b}$ of the random (white) parameters $a(t)$ and $b(t)$, m increases monotonically with both parameter variances $\sum_{aa}$ and $\sum_{bb}$. Note that m is uniquely characterized by the stochastic system itself and is *independent* of the performance criterion employed. For any given discount factor $0 < \alpha < 1$, if the system uncertainty is large enough (Region C in Figure 1) no optimal decision rules exist for the infinite horizon case, and the system is not stabilizable. If the system uncertainty is sufficiently small (Region A in Figure 1) then optimal and stabilizing decision rules exist.

The curious (and surprising?) phenomenon occurs in Region B; note that the size of this region increases as the future is discounted more and more ($\alpha \to 0$). In region B optimal rules exist, but the resultant optimal closed loop system is unstable in a mean square sense. The existence of optimal decision rules in this region is solely due to the inclusion of a discount factor in the performance index.

The implication of the above remarks seem to imply that one has to be careful on interpreting results obtained through the use of discount factors for stochastic optimization problems, and that an independent stochastic stability analysis should be carried out. In most linear-quadratic stochastic optimization problems solved to-date optimality and stability are not in conflict; optimal decision rules result in stable systems. This is clearly not the case for uncertain systems in which the randomness enters in a multiplicative rather than additive way (such as in the standard Linear-Quadratic-Gaussian problem [7]).

By the above results we do not imply that this paper is the only one that points out the interplay between stability and optimality, when discount factors are present. In the economics literature such problems have received attention even in the deterministic case (see for example [8] and [9]). Problems in capital accumulation and business cycles involving state dependent noise have been treated recently by Magill [10] in the continuous time framework. Our remarks are primarily oriented towards linear-quadratic problems. If there is no parameter uncertainty ($m = 0$) there is no conflict between stability and optimality, independent of the magnitude of the additive uncertainty $\xi(t)$.

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