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MULTISTAGE PRICING UNDER UNCERTAIN DEMAND

BY CHEE-YEE CHONG AND DAVID C. CHENG

The optimal pricing policy of a monopolistic firm facing random demand and maximizing its expected profit over a period of several stages is considered. The demand function is assumed to be time invariant but unknown. A special case when the cost is certain and the demand is a linear function of the price is investigated. This is formulated as a stochastic control problem. It is found that when both the intercept and slope of the demand function are unknown, the optimal pricing policy does not correspond to optimal prices for each individual stage. Approximate methods are used to find the optimal policies. Simulation results are given.

I. INTRODUCTION

There have been a growing number of studies of the behavior of the firm under uncertainty. Most of the existing work is concerned with the single-period analysis of the impact of uncertainty, e.g. Mills (1959), Hymans (1966), Smith (1969), Horowitz (1969), Zabel (1970), Baron (1971) and Leland (1972). The dynamics and the learning behavior of the firm facing uncertainty have largely been ignored. In the work of Clower (1959), Day (1966) and Hadar and Hillinger (1969), some dynamic adjustment processes are introduced to characterize the adaptive behavior of firms. Under certain conditions, they are shown to give rise to convergent time paths of output and price, which are related to optimal price and output decisions under certainty. However, these processes are ad-hoc measures, and the adjustment coefficients have not been derived from optimization procedures. Nevins (1966) conducts simulation studies of dynamic price-setting and quantity-setting policies of a monopoly model. As in the other studies (except for the certainty-equivalence assumption), no consideration has been taken into account to allow for learning about uncertainty.

Dreze (1972) suggests that "full optimization would call for taking into consideration the expected value of the information generated by the decisions, in addition to the expected value of the direct consequences of the decisions. To be concrete, a monopolist may wish to depart from the price which maximizes expected profit, simply to learn more about his demand function." This dual aspect of decision-making was first investigated by Feldbaum (1960) under the title of dual control. Interesting approximations to the optimal solution of dual control problems are suggested recently by MacRae (1974), Tse, Bar-Shalom and Meier (1973) and Tse (1974).

Since learning plays a crucial part in dynamic economic behavior under uncertainty, dual control is a powerful tool of analysis. Chow (1973) and Aoki (1973) and (1974) are among the first to apply dual control to economics. They have been able to shed more light into the complexities of optimizing behavior in dynamic and stochastic economic models.

An attempt has been made to study from the viewpoint of dual control the effect of uncertainty upon the behavior of the firm over time. To highlight the role of learning in intertemporal decision-making, we study a special model of a monopolist which carries no inventory, sets the price and produces instantaneously according to demand. Even though the monopolist tries to maximize his expected profit over a finite horizon, this model is essentially static if no learning is taken into account. The intertemporal optimal pricing policy simply consists of optimal prices for the individual stages. This will no longer be the case when the monopolist utilizes the extra data generated by his pricing strategy to learn more about the uncertain demand curve.

A simple model of the firm is introduced in Section II. In Section III, the equations governing the solution are presented. A special case which can be solved exactly is discussed in Section IV. Three methods to approximate the solution for the general case are given in Section V. Section VI contains some simulation results using two different methods.

II. The Model

We assume that during any period k a monopolist faces the following demand curve.

(2.1) $$q(k) = \alpha p(k) + \beta + \theta(k)$$

where

$p(k)$ is the price charged by the monopolist in period k

$q(k)$ is the quantity demanded

α, β are parameters characterizing the demand curve

$$\alpha < 0, \qquad \beta > 0$$

$\theta(k)$ is the error term (noise) in the demand equation.

The monopolist knows that the demand is linear with constant but unknown parameters α and β. A Bayesian assumption is used, i.e., the monopolist has available to him prior statistics of the parameters α and β as well as of the noise $\theta(k)$ affecting his demand. For instance, these can be obtained from standard econometric models using past data. We assume α is normal with mean $\bar{\alpha}$ and covariance σ_α^2. β is normal with mean $\bar{\beta}$ and variance σ_β^2, and $\theta(k)$ is normal with zero mean and covariance Θ. The random variables are all independent.

The monopolist produces a homogenous product in each period and his objective is to maximize the expected profit over N period. Thus his utility function is linear in risk. We also assume that once he selects a price $p(k)$, he can produce, and supply the quantity demanded according to equation (2.1).

The profit for each period is

(2.2) $$p(k)q(k) - cq(k)$$

where c is a known and constant marginal cost.

The expected profit over N periods is thus

(2.3) $$J = E\left\{\sum_{k=1}^{N} p(k)q(k) - cq(k)\right\}$$

and the monopolist has to choose optimal prices $p(k)$, $k = 1, \ldots, N$ in order to maximize his expected profit.

The model we study is one that is frequently used. Although assumed time-invariant here, α and β can be relaxed to be time-varying. The normal distributions are assumed for convenience but are not overly restrictive. Most of the existing work dealing with uncertain demand treats static cases with the monopolist optimizing his expected utility. Differences between price setting and quantity setting were illustrated (e.g. Leland, etc.). In this paper we try to reflect the fact that most monopolists do not choose their price only once but can in effect vary it from time to time. If his objective is to maximize the expected profit over several periods, then the pricing strategy which maximizes the expected profit of each period may not be the one to use. The intertemporal optimal strategy has to take into account the fact that learning is possible, and thus should be adaptive in nature. Some related results along this line have been obtained by Aoki (1973) and (1974).

III. Optimal Pricing Strategies

In principle, the solution to our problem can be found using dynamic programming (Aoki, 1967). The following equation has to be solved recursively to obtain the optimal $p(k)$.

$$J(I(k-1), k) = \max_{p(k)} E\{p(k)q(k) - cq(k) + J(I(k), k+1) | I(k-1)\} \tag{3.1}$$

with

$$E\{J(I(N-1), N) | I(N-1)\} = E\{p(N)q(N) - cq(N) | I(N-1)\}. \tag{3.2}$$

$I(k-1)$ is the information available to the monopolist up to the beginning of period k and consists of all the past prices and the quantities demanded, i.e.,

$$I(k-1) = \{p(0), \ldots, p(k-1), q(0), \ldots, q(k-1)\}. \tag{3.3}$$

The past stream of profits is also information available to the monopolist. When the cost of production c is assumed to be known, this information is redundant since profit is given by $p(k)q(k) - cq(k)$. When c is not known exactly, then the past profit will be useful in determining the optimal pricing policy.

It is well known that all the information in $I(k-1)$ can be replaced by the following estimates (Athans, 1974) of α and β which can be generated recursively. Let

$$\hat{\lambda}(k) = \begin{bmatrix} \hat{\alpha}(k) \\ \hat{\beta}(k) \end{bmatrix} \triangleq E\left\{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \middle| I(k) \right\} \tag{3.4}$$

$$\Sigma(k) \triangleq \operatorname{cov}\left\{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \middle| I(k) \right\}. \tag{3.5}$$

Then

$$\begin{bmatrix}\hat{\alpha}(k+1)\\ \hat{\beta}(k+1)\end{bmatrix} = \begin{bmatrix}\hat{\alpha}(k)\\ \hat{\beta}(k)\end{bmatrix} + \Sigma(k+1)\begin{bmatrix}p(k+1)\\ 1\end{bmatrix} \times \Theta^{-1}(q(k+1) - p(k+1)\hat{\alpha}(k) - \hat{\beta}(k)) \tag{3.6}$$

$$\Sigma(k+1) = \left(\Sigma^{-1}(k) + \begin{bmatrix}p(k+1)\\ 1\end{bmatrix}\Theta^{-1}[p(k+1), 1]\right)^{-1} \tag{3.7}$$

$$\begin{bmatrix}\hat{\alpha}(0)\\ \hat{\beta}(0)\end{bmatrix} = \begin{bmatrix}\bar{\alpha}\\ \bar{\beta}\end{bmatrix}, \quad \Sigma(0) = \begin{bmatrix}\sigma_\alpha^2 & 0\\ 0 & \sigma_\beta^2\end{bmatrix}. \tag{3.8}$$

This result is obtained by applying the standard Kalman filtering algorithm to the filtering problem given by the trivial difference equation

$$\begin{bmatrix}\alpha(k+1)\\ \beta(k+1)\end{bmatrix} = \begin{bmatrix}\alpha(k)\\ \beta(k)\end{bmatrix} \tag{3.9}$$

with measurement

$$q(k) = [p(k) \quad 1]\begin{bmatrix}\alpha(k)\\ \beta(k)\end{bmatrix} + \theta(k). \tag{3.10}$$

Equation (3.9) is a statement that the parameters α and β are constant. The details of this can be found in (Athans, 1974).

Note that in general the estimation error $\Sigma(k)$ of the parameters will depend on the past policies $p(k)$. However, when α is known, then the estimation error of β no longer depends on the decisions of the monopolist. As we shall see in the next section, the pricing strategy then becomes very simple.

IV. A Special Case: α is Known

When α is known, only the intercept of the demand curve is uncertain. Under such circumstances the error covariance of the parameter β becomes

$$\Sigma_\beta(k+1) = (\Sigma_\beta^{-1}(k) + \Theta^{-1})^{-1}. \tag{4.1}$$

This is independent of $p(k)$.

At $k = N$

$$J(\hat{\beta}(N-1), N) = \max_{p(N)} E\{p(N)q(N) - cq(N)|\hat{\beta}(N-1)\}. \tag{4.2}$$

Maximization of this gives

$$p^*(N) = -\frac{1}{2\alpha}(\hat{\beta}(N-1) - \alpha c) \tag{4.3}$$

$$J(\hat{\beta}(N-1), N) = -\frac{1}{4\alpha}(\hat{\beta}(N-1) + \alpha c)^2. \tag{4.4}$$

At $k = N - 1$

$$J(\hat{\beta}(N-2), N-1) = \max_{p(N-1)} E\{p(N-1)q(N-1) - cq(n-1) + J(\hat{\beta}(N-1), N)|\hat{\beta}(N-2)\}. \tag{4.5}$$

But

$$E\{J(\hat{\beta}(N-1), N)|\hat{\beta}(N-2)\} = -\frac{1}{4\alpha}(\hat{\beta}(N-2) + \alpha c)^2 + \frac{1}{4\alpha}[\Sigma_\beta(N-1) - \Sigma_\beta(N-2)] \tag{4.6}$$

is independent of $p(N-1)$. For details of this result, see Appendix. Thus

$$p^*(N-1) = -\frac{1}{2\alpha}(\hat{\beta}(N-2) - \alpha c). \tag{4.7}$$

In general

$$p^*(k) = -\frac{1}{2\alpha}(\hat{\beta}(k-1) - \alpha c). \tag{4.8}$$

This is the optimal pricing policy for the single period k if all the past prices and quantities are used to generate a better estimate of the parameter β. Except for the use of estimates instead of actual parameters, this policy is the same as in the static and deterministic case. Thus, although the monopolist is optimizing over N periods, the statistical assumption is such that all he has to consider is the immediate future. The optimal pricing policy for each period has the same form and is independent of the total number of periods. We may expect, however, that his estimate should improve with more measurements, and this may affect the actual prices used.

V. General Case: Both α and β Uncertain

The relevant equation to be solved is given by (3.1). At $k = N$, the solution of this equation gives

$$p^*(N) = -\frac{1}{2\hat{\alpha}(N-1)}(\hat{\beta}(N-1) - \hat{\alpha}(N-1)c) \tag{5.1}$$

where $\hat{\alpha}(N-1)$ and $\hat{\beta}(N-1)$ are as given in Section III. The pricing policy is similar to the static and deterministic case except for the use of estimates. This is expected since at that stage, the monopolist essentially faces a static problem. The expected profit is given by

$$J(I(N-1), N) = \frac{-1}{4\hat{\alpha}(N-1)}(\hat{\beta}(N-1) + \hat{\alpha}(N-1)c)^2. \tag{5.2}$$

At $k = N - 1$, we have to solve

$$(5.3) \quad J(I(N-2), N-1) = \max_{p(N-1)} E\{p(N-1)q(N-1) - cq(N-1) - \frac{1}{4\hat{\alpha}(N-1)}(\hat{\beta}(N-1) + \hat{\alpha}(N-1)c)^2 | I(N-2)\}.$$

From Section III, we can see that the last term in this maximization depends on $p(N-1)$ and $q(N-1)$. Moreover, the dependence is nonlinear and not quadratic. An analytic solution is thus not possible. Various approximations have been suggested.

(a) *Open-Loop Feedback Optimal*

At any time k, an open-loop problem is solved which assumes that no additional information will be available in the future. Thus the problem is to choose a deterministic sequence of prices

$$(5.4) \quad \{p(k), p(k+1), \ldots, p(N)\}$$

to maximize the expected future profit

$$(5.5) \quad E\left\{\sum_{i=k}^{N} p(i)q(i) - cq(i) | I(k-1)\right\}.$$

Only $p(k)$ is used. Once $q(k)$ is observed, the estimates on α and β are updated and the problem is solved again. For the problem under consideration the solution is extremely simple. The optimal price is given by a certainty equivalence policy which is the same as that for a static one-period case.

$$(5.6) \quad p(k) = \frac{-1}{2\hat{\alpha}(k-1)}(\hat{\beta}(k-1) - \hat{\alpha}(k-1)c).$$

This price policy does not take an active role in reducing the uncertainty. Since our model is uncoupled temporally except for the flow of information, the optimal pricing policy thus reduces to that of a one-period problem.

(b) *Wide-Sense Adaptive Dual Control*

In this approach (Tse, Bar-Shalom, Meier,- 1973), emphasis is placed on finding an approximate representation for the function $J(I(k), k+1)$ in equation (3.1) given an arbitrary price $p(k)$. The expression in (3.1) is then solved numerically to obtain $p(k)$. Specifically, we use the following method to approximate the expected profit.

1. Assume a price $p(k)$ has been chosen at time k.
2. $\Sigma(k)$, the error covariance of the parameters can then be found using equation (3.7). The predicted values of the parameters remain $\hat{\alpha}(k-1)$ and $\hat{\beta}(k-1)$ because of the special nature of our problem.
3. A nominal pricing policy $\{p_0(k+1), p_0(k+2), \ldots, p_0(N)\}$ is then chosen which depends only on the predicted values of $\hat{\alpha}(k-1)$ and $\hat{\beta}(k-1)$.

One possible candidate is the certainty equivalence type strategy described in Section V(a). Thus

$$(5.7)\quad p_0(j) = -\frac{1}{2\hat{\alpha}(k-1)}(\hat{\beta}(k-1) - \hat{\alpha}(k-1)c) \qquad j = k+1,\ldots,N$$

$$(5.8)\quad q_0(j) = \hat{\alpha}(k-1)p_0(j) + \hat{\beta}(k-1) \qquad j = k+1,\ldots,N.$$

The nominal profit function is

$$(5.9)\qquad J_0(k+1) = -\frac{N-k}{4\hat{\alpha}(k-1)}(\hat{\beta}(k-1) + \hat{\alpha}(k-1)c)^2.$$

4. A perturbation analysis is done about the nominal of equation (5.8) to obtain the model

$$(5.10)\qquad \delta q(j) = \hat{\alpha}(k-1)\delta p(j) + \delta\alpha p_0(j) + \delta\beta + \theta(j).$$

The incremental expected profit about this nominal is

$$(5.11)\quad \begin{aligned} \delta J(k+1) &= J(I(k), k+1) - J_0(k+1) \\ &= E\left\{\sum_{j=k+1}^{N} \delta p(j)q_0(j) + p_0(j)\delta q(j) - c\delta q(j) \right. \\ &\qquad \left. + \delta p(j)\delta q(j) | I(k)\right\}. \end{aligned}$$

Equations (5.10) and (5.11) define an optimization problem similar to the special case discussed in Section IV. In fact the optimal incremental price is given by

$$(5.12)\quad \delta p(j) = -\frac{1}{2\hat{\alpha}(k-1)}[\delta\hat{\alpha}(j-1)p_0(j) + \delta\hat{\beta}(j-1)] \qquad j = k+1,\ldots,N$$

where the incremental parameter estimates are given by equations similar to those in Section III. The error covariances of the incremental estimates are, however, independent of the incremental price.

Let

$$(5.13)\qquad \Sigma_0(j) \triangleq \operatorname{cov}\left\{\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \middle| I(j-1)\right\}.$$

Then

$$(5.14)\qquad \Sigma_0^{-1}(j) = \Sigma_0^{-1}(j-1) + \mathbf{K}_0(j) \qquad j = k+1,\ldots,N$$

where

$$(5.15)\qquad \mathbf{K}_0(k) = \begin{bmatrix}p_0(j)\\ 1\end{bmatrix}\Theta^{-1}[p_0(j), 1]$$

is a constant matrix since $p_0(j)$ given by equation (5.7) is constant for $j > k$

$$(5.16)\qquad \Sigma_0^{-1}(k) = \Sigma^{-1}(k-1) + \begin{bmatrix}p(k)\\ 1\end{bmatrix}\Theta^{-1}[p(k), 1].$$

The optimal expected incremental profit from period $k + 1$ to N is

$$\text{(5.17)}\quad \delta J(k+1) = -\frac{N-k}{4\hat{a}(k-1)}(\delta\hat{a}(k)p_0(k) + \delta\hat{\beta}(k))^2 + (N-k)[p_0(k)\,\delta\hat{\beta}(k) + \delta\hat{a}(k)p_0^2(k) - c\,\delta\hat{a}(k)p_0(k) - c\,\delta\hat{\beta}(k)] + \frac{1}{4\hat{a}(k-1)}\sum_{j=k}^{N-2}(N-j-1)(\Sigma_0(j+1) - \Sigma_0(j))\mathbf{K}_0(k).$$

The quantity on the right hand side of equation (3.1) to be maximized can then be approximated by

$$\text{(5.18)}\quad E\{p(k)q(k) - cq(k) + J_0(k+1) + \delta J(k+1)|I(k-1)\}.$$

Since $J_0(k + 1)$ is independent of $p(k)$, the quantity to be maximized is then

$$\text{(5.19)}\quad J_d(p(k)) = E\{p(k)q(k) - cq(k)|I(k-1)\} + \frac{1}{4\hat{a}(k-1)}\operatorname{tr}[\Sigma_0(N-1) + \Sigma_0(N-2) + \dots + \Sigma_0(k)]\mathbf{K}_0(k) \qquad k = 1, \dots, N-1$$

$$\text{(5.20)}\quad J_d(p(k)) = E\{p(k)q(k) - cq(k)|I(k-1)\}, \qquad k = N.$$

5. After $p^*(k)$ is found, a new estimate of the parameters $\hat{a}(k)$, $\hat{\beta}(k)$ and their covariances $\Sigma(k)$ are then updated. The whole process is then repeated to find $p^*(k + 1)$.

In equation (5.19) except for $k = N$ the quantity to be maximized consists of two parts. The term inside the conditional expectation is the expected profit influenced by the price for that stage. The second part consists of the error covariances of the parameter estimates about the nominal. They depend on $p(k)$ since $\Sigma_0(k)$ depends on $p(k)$. Since $\hat{a}(k - 1)$ is negative, the maximization of this second part is equivalent to the minimization of the future error covariances of the estimates. The optimal $p^*(k)$ is neither the price which maximizes the profit for period k nor the one which minimizes the error covariances of the future. The dual nature of $p^*(k)$ is thus very clearly displayed. At the last stage, however, $p^*(k)$ will simply be the maximization of profit.

(c) *Approximate solution of equation* (3.1)

The method presented in Section V(b) is one which has to be done in real time. At each period k, the effect of the price $p(k)$ on the expected total profit from that period to the final period is investigated by doing a perturbation analysis. This includes computation of all the incremental covariances $\Sigma_0(k), \dots, \Sigma_0(N - 1)$, and doing an optimization of a nonlinear and nonquadratic function. In the problem under consideration, this is a realistic way since there is usually sufficient time between the times when the price is changed. Moreover, the quantity to be optimized is simple enough so that an optimization can be done easily using numerical techniques. Other methods are also available which approximate the solution to equation (3.1) (e.g. Chow, 1974). The optimal price $p^*(k)$ at each stage is then a precomputable function of the parameter estimates. This approach is more suitable for situations when $p(k)$ has to be changed rapidly so that insufficient time is available for carrying out the optimization problem.

Specializing to our problem, the following steps are necessary.

1. A nominal price sequence is chosen to be $\{p_0(1), \ldots, p_0(N)\}$.
2. This gives rise to a nominal sequence of quantities demanded

$$\{q_0(1), \ldots, q_0(N)\}.$$

   and nominal estimates

$$\{\hat{\alpha}_0(1), \ldots, \hat{\alpha}_0(N-1)\}, \{\hat{\beta}_0(1), \ldots, \hat{\beta}_0(N-1)\}.$$

3. Starting at the final period N, equation (5.2) is approximated by a Taylor series expansion about the nominal, retaining only the linear and quadratic terms of $p(N-1)$, $\hat{\alpha}(N-2)$ and $\hat{\beta}(N-2)$.
4. Equation (5.3) can then be solved and will have a form dependent only on $\hat{\alpha}(N-2)$ and $\hat{\beta}(N-2)$.
5. This process is repeated until $k = 1$. A sequence of $p(k)$ in terms of the estimates will have been obtained.
6. $p(k)$ can then be used to generate a new nominal and the steps 1 to 5 are then repeated. This can be done as often as possible.

The main feature of this approach is that the resulting pricing policies are simple in structure. They will be of the form given by equation (5.1). However, the optimality of the method depends on how far away the actual values are from the nominal last used.

VI. Simulation Results

To illustrate the ideas of this paper, a numerical example is used. The demand function is assumed to be

$$q(k) = -2p(k) + 24 + \theta(k). \tag{6.1}$$

Thus

$$\alpha = -2 \qquad \beta = 24. \tag{6.2}$$

The constant cost of production is 2. The number of periods N is assumed to be 10 and 5.

The monopolist has the following prior statistical information on α and β

$$\begin{aligned} \bar{\alpha} &= -2.5 \qquad \sigma_\alpha^2 = 1 \\ \bar{\beta} &= 20 \qquad \sigma_\beta^2 = 4 \end{aligned} \tag{6.3}$$

$\theta(k)$ has a covariance of 1 which is assumed known to the monopolist.

If the monopolist knows the exact values of α and β, his optimal price for each period will be

$$p^*(k) = 7 \qquad k = 1, \ldots, N. \tag{6.4}$$

His profit over ten and five periods will then have means of 500 and 250 and standard deviations of 50 and 25 respectively.

The certainty equivalence (CE) policy in Section V(a) is compared with the wide sense adaptive dual control (WSADC) policy in Section V(b). The method in Section V(c) will be compared elsewhere. A linear search with quadratic interpolation is used to maximize $J_d(p(k))$ in equation (5.19). Ten simulation runs are obtained for each policy.

The results for $N = 10$, $\bar{\alpha} = -2.5$, $\bar{\beta} = 20$, $\sigma_\alpha^2 = 1$, $\sigma_\beta^2 = 4$ are tabulated below in Tables 1 to 3.

TABLE 1

COMPARISON BETWEEN CE PRICING POLICY AND WSADC POLICY

	Average Profit	Range of Profit	Range of $p(k)$	$P(1)$
CE	476.02	457.12–496.78	5.000–10.750	5.00
WSADC	484.85	462.56–501.96	6.307– 9.605	6.30

TABLE 2

RESULT OF CE POLICY FOR ONE SAMPLE RUN

k	$p(k)$	$q(k)$	$\hat{\alpha}(k)$	$\hat{\beta}(k)$	Cumulative Profit
1	5.000	14.157	−1.390	20.888	42.472
2	8.511	5.935	−1.842	22.213	81.116
3	7.030	9.008	−1.852	22.198	126.421
4	6.993	10.539	−1.816	22.268	179.040
5	7.131	10.591	−1.780	22.273	233.384
6	7.256	10.314	−1.753	22.242	287.599
7	7.346	9.027	−1.762	22.260	335.854
8	7.316	8.851	−1.774	22.280	382.906
9	7.279	10.373	−1.755	22.258	437.663
10	7.341	8.575	−1.771	22.289	483.461

TABLE 3

RESULT OF WSADC POLICY FOR ONE SAMPLE RUN

k	$p(k)$	$q(k)$	$\hat{\alpha}(k)$	$\hat{\beta}(k)$	Cumulative Profit
1	6.307	11.643	−1.470	20.653	49.717
2	8.031	6.896	−1.703	21.268	91.304
3	7.410	8.247	−1.723	21.276	135.920
4	7.384	9.756	−1.580	21.269	188.449
5	7.440	9.973	−1.643	21.238	242.704
6	7.486	9.862	−1.617	21.203	296.765
7	7.556	8.606	−1.628	21.225	344.581
8	7.521	8.442	−1.640	21.246	391.185
9	7.493	9.945	−1.622	21.224	445.811
10	7.543	8.170	−1.637	21.257	491.099

Our results show that the WSADC policy always gives a higher profit than the CE pricing policy. On the other hand, the parameter estimates using CE pricing are better than those of WSADC pricing. The most dramatic difference between these two policies is in period 1. From Table 1, $p(1)$ is 5.000 for CE and is 6.307 for WSADC policy. This is so because in the former case no use is made of the fact that information will be available in the future. Because of the nature of our problem, only the profit for that period is involved in the optimization. Thus $p(1)$ is selected on the basis of a priori means $\bar{\alpha}$ and $\bar{\beta}$ to maximize the profit for period 1. The result is a very poor performance for period 1 which contributes to the total profit. On the other hand, the WSADC approach takes into consideration the fact that more measurements will be available in future. The future and the present are no longer uncoupled. This results in a $p(1)$ which is more optimal for the overall problem. $p(1)$ gives a bigger profit for period 1 and also improves the estimates $\hat{\alpha}(1)$ and $\hat{\beta}(1)$ leading to a better $p(2)$. The reason that the future estimates $\hat{\alpha}(k)$ and $\hat{\beta}(k)$ are not as good as those given by certainty equivalence is probably an accident.

Other priori statistical data have also been used. If σ_β^2 is modified to 16, the following results are obtained.

TABLE 4

$N = 10,\ \alpha = -2.5,\ \beta = 20,\ \sigma_\alpha^2 = 1,\ \sigma_\beta^2 = 16$

COMPARISON BETWEEN CE PRICING POLICY AND WSADC POLICY

	Average Profit	Range of Profit	Range of $p(k)$	$P(1)$
CE	482.76	462.97–499.09	5.00–8.81	5.00
WSADC	485.03	463.43–501.21	5.31–8.92	5.31

The CE policy performs almost as well as the WSADC policy and the prices chosen at each period are almost identical. The parameter estimates are more accurate than those obtained previously.

We also investigate how the time horizon affects the pricing policies. If the time horizon is cut by half, i.e., $N = 5$ the prices used are almost the same as the first five used with a longer time horizon. With certainty equivalence policies this has been expected since the prices are independent of the time horizon. The similar results for the WSADC policy indicate that in this case the learning effect is not very strong. This may have to do with the assumption on the demand curve. Since the demand is assumed to be linear with constant slopes and intercepts, knowledge of two sets of prices and quantities will be sufficient to determine the demand function. It is probable that when the demand function is more complicated, e.g., with time varying parameters, the effect of learning will be more significant.

VII. CONCLUSION

In this paper we have discussed the behavior of a firm under uncertainty. A simple classical model of monopoly is chosen for study. The monopolist facing unknown demand is assumed to have linear risk, constant marginal cost, finite planning horizon and instantaneous production capabilities. Without learning, this model is static in the sense that optimal pricing for each individual stage is also optimal for the whole planning horizon. However, active learning changes this picture. Except for the special case when the slope of the demand curve is known, the multiperiod nature of the problem has important effects on the pricing policy of the monopolist. The additional data collected by the monopolist can always be used to update his information on the demand curve. However, his policy also depends on the availability of future information. If he assumes that the future data are not available, the optimal pricing policy is essentially the same as the optimal one for each period. When future data are assumed to be available, this information will be used in his policy decision. Dual control methods are applied to find approximate optimal solutions. The simulation experiments indicate that by including uncertainty, we improve the performance of the monopolist though in some cases the improvement is quite insignificant. It is, however, difficult to distinguish between the effects due to the inclusion of uncertainty and those due to the learning aspects of the algorithm.

In this paper, we have concentrated on the effect of uncertainty on multiperiod problems which are essentially static in nature except for the propagation of information. Natural extensions of this present research will be the investigation of non-linear risk, the effect of risk aversion upon dual control strategy, the inclusion of the possibility of inventory accumulation, and the comparison between the price setting strategy and the output-setting strategy in the framework of dual control.

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APPENDIX

$$E\{J(\hat{\beta}(N-1), N) | \hat{\beta}(N-2)\} = E\left\{-\frac{1}{4\alpha}(\hat{\beta}(N-1) + \alpha c)^2 | \hat{\beta}(N-2)\right\}$$

$$= -\frac{1}{4\alpha}E\{E\{(\hat{\beta}(N-1) + \alpha c)^2 | \hat{\beta}(N-1)\} | \hat{\beta}(N-2)\}$$

$$= -\frac{1}{4\alpha}E\{(\beta + \alpha c)^2 - \Sigma_\beta(N-1) | \hat{\beta}(N-2)\}.$$

But $\Sigma_\beta(N-1)$ is independent of $\hat{\beta}(N-2)$ by equation (4.1). Thus

$$E\{J(\hat{\beta}(N-1), N) | \hat{\beta}(N-2)\} = -\frac{1}{4\alpha}E\{(\beta + \alpha c)^2 | \hat{\beta}(N-2)\} + \frac{1}{4\alpha}\Sigma_\beta(N-1)$$

$$= -\frac{1}{4\alpha}(\hat{\beta}(N-2) + \alpha c)^2 - \frac{1}{4\alpha}\Sigma_\beta(N-2)$$

$$+ \frac{1}{4\alpha}\Sigma_\beta(N-1)$$

which gives equation (4.6).