Volume Title: Annals of Economic and Social Measurement, Volume 4, number 2

Volume Author/Editor: NBER

Volume Publisher: NBER

Volume URL: http://www.nber.org/books/aesm75-2

Publication Date: April 1975

Chapter Title: A Comparison of Three Control Algorithms as Applied to the Monetarist-Fiscalist Debate

Chapter Author: Andrew B. Abel

Chapter URL: http://www.nber.org/chapters/c10396

Chapter pages in book: (p. 239 - 252)

Annals of Economic and Social Measurement, 4/2, 1975

A COMPARISON OF THREE CONTROL ALGORITHMS AS APPLIED TO THE MONETARIST-FISCALIST DEBATE

BY ANDREW B. ABEL*

This paper employs an optimal control framework to analyze the relative effectiveness of monetary and fiscal policies. The economy is modeled by a very simple two-equation linear dynamic model with the money supply and government expenditures as two instruments. Three control algorithms, which differ in their treatments of uncertainty and learning, are applied to this model to calculate optimal policies and minimum expected welfare costs. With minimum expected welfare cost serving as the criterion of effectiveness, it appears that fiscal policy is somewhat more effective than monetary policy with respect to the quadratic welfare function employed in this paper, although using both policies together is much more effective than using either policy alone.

1. INTRODUCTION

In this paper we shall use an optimal control framework to examine the relative effectiveness of monetary and fiscal policies for the purpose of controlling the major macroeconomic aggregates. In Section 2, we shall present a very simple linear macroeconomic model with additive Gaussian disturbances. After briefly describing three control algorithms in Section 3, we shall, in Section 4, apply these three control algorithms to our simple model. Monetary policy will be represented by the money supply and fiscal policy will be represented by government expenditures. In evaluating the effectiveness of a given instrument, we shall designate that instrument as a discretionary instrument and the other instrument as a passive instrument, and then solve an optimal control problem. The values of the discretionary instrument are determined subject to feedback control, but the values of the passive instrument are constrained to change at a constant rate over the planning horizon. In addition, we solve the control problem with both instruments assumed to be discretionary. Comparison of the expected welfare costs in the three situations serves to evaluate the effectiveness of each discretionary instrument.

Three different algorithms will be used to perform our analysis of the relative effectiveness of monetary and fiscal policies in order to determine whether our results are sensitive to the choice of control algorithm employed. The three algorithms presented in this paper differ in their treatments of learning. Method I is a certainty equivalence control algorithm formulated by Chow (1972). The assumptions of certainty equivalence preclude the possibility of learning by assuming that the parameters of the linear model are known with certainty. Method II, which is presented by Chow (1973a) recognizes the uncertainties in the parameters of the model but ignores the possibility of learning. Method III, which is a dual adaptive control algorithm presented by Chow (1973c), anticipates that learning will occur through the re-estimation of the unknown parameters of the linear model as additional observations are obtained with the passage of time. Method III is closest to being optimal and contains method I and method II as special cases.

* I would like to express my most sincere thanks to Professor Gregory C. Chow, my thesis advisor, for generously sharing his time and his ideas with me.

2. A Simple Macroeconomic Model

For the policy analysis of this paper, we shall employ a very simple aggregative model. It is based on real quarterly data covering the period from 1954/I to 1963/IV, which corresponds roughly to the period between the end of the Korean War and the beginning of heavy United States involvement in Vietnam. It consists of only two endogenous target variables, consumption (C_t) and investment (I_t), and two instruments, government expenditures (E_t) and the money supply (M_t). We assume that in the short-run, government authorities can control E_t and M_t in real terms since prices do not change rapidly enough to seriously offset their actions. Over the time period covered by our data, the rate of inflation was low enough to make this assumption plausible.

Our model is based on a closed economy. Desired consumption is a linear function of GNP, and the realized period-to-period adjustment in consumption is subject to a partial adjustment factor:

$$C_t = aC_{t-1} + bI_t + bE_t + d. \tag{2.1}$$

The structural equation for investment is based upon a modification of Samuelson's private consumption accelerator. We posit that the desired level of the capital stock is a linear function of consumption and that the realized adjustment of the capital stock is subject to a partial adjustment factor. Since gross investment, I_t, is defined as $K_t - (1 - D)K_{t-1}$, where D is the depreciation rate of the capital stock, we have

$$I_t = eC_t - (1 - D)eC_{t-1} + fI_{t-1} + g. \tag{2.2}$$

In addition, we assume that the level of gross investment is linearly related to the money supply in order to capture some of the effects of interest rates upon investment:

$$I_t = e'C_t - (1 - D)e'C_{t-1} + f'I_t + hM_t + g'. \tag{2.3}$$

The estimated reduced form equations corresponding to the structural equations are

$$\begin{aligned} C_t = \underset{(0.0534)}{0.9266C_{t-1}} - \underset{(0.0916)}{0.0203I_{t-1}} + \underset{(0.1389)}{0.3190E_t} + \underset{(0.1863)}{0.4206M_t} - \underset{(25.7719)}{63.2386}; \end{aligned} \tag{2.4}$$

$$R^2 = 0.9958$$

$$D\text{-}W = 1.7084$$

$$\begin{aligned} I_t = \underset{(0.0781)}{0.1527C_{t-1}} + \underset{(0.1339)}{0.3806I_{t-1}} - \underset{(0.2031)}{0.0735E_t} + \underset{(0.2724)}{1.5389M_t} - \underset{(37.6899)}{210.8994}; \end{aligned} \tag{2.5}$$

$$R^2 = 0.8749$$

$$D\text{-}W = 1.7582.$$

Note that each of these estimated equations has a high value of R^2. In addition, the Durbin–Watson statistic, although biased toward 2.0 because of the lagged endogenous variable, does not suggest significant serial correlation in either equation.

A criticism that may be raised against the above model is that it includes only the current values of M_t and E_t among the explanatory variables. However, concerning the lagged or delayed effects of M_t and E_t, our model implicitly assumes a lag structure with geometrically declining weights for M_t and E_t because the lagged endogenous variable appears as an explanatory variable in each equation. In this paper, we do not explore more complicated lag structures.

3. THE ALGORITHMS[1]

The three algorithms presented in this paper are applicable to linear stochastic discrete-time econometric models with unknown parameters and additive Gaussian errors. We shall write the model as a first-order linear difference equation

$$y_t^* = Ay_{t-1}^* + Cx_t + b_t + e_t, \tag{3.1}$$

where A, C, and b_t are random parameters whose values will be estimated using the Bayesian techniques presented by Chow (1973a). The vector y_t^* is a stacked vector containing values of the endogenous variables and the instruments, x_t is a vector of instruments, b_t is a vector which models the effects of the noncontrollable exogenous variables, and e_t is a vector of random variables such that $e_t \sim N(0, \Sigma)$. The e_t are assumed to be serially uncorrelated and uncorrelated with the random parameters A, C, and b_t.

The objective of each of our three control algorithms is to minimize the expected value of the following quadratic welfare cost function

$$W = \frac{1}{2}\sum_{t=1}^{T} (y_t^* - a_t)'K_t(y_t^* - a_t), \tag{3.2}$$

where T is the length of the planning horizon, a_t is the target value of y_t^*, and K_t is a weighting matrix. Observe that (3.2) may be rewritten as

$$W = \frac{1}{2}\sum_{t=1}^{T} y_t^{*\prime}K_t y_t^* + \sum_{t=1}^{T} y_t^{*\prime}k_t + \text{constant}, \tag{3.3}$$

where $k_t = -K_t a_t$. We solve this problem using the method of dynamic programming. Let $E_{t-1}w_t$ denote the expected welfare cost from period t up to and including period T, with the subscript $t - 1$ indicating that the expectation is conditional on information available at the end of period $t - 1$. We first minimize $E_{T-1}w_T$ with respect to x_T. Letting $H_T = K_T$ and $h_T = k_T$, we obtain

$$E_{T-1}w_T = E_{T-1}(\tfrac{1}{2}y_T^{*\prime}H_T y_T^* + y_T^{*\prime}h_T) + \text{constant}'. \tag{3.4}$$

Substituting (3.1) into (3.4) and partially differentiating with respect to x_T, we obtain the following feedback control equation, which yields the optimal value of x_T,

$$\hat{x}_T = G_T y_{T-1}^* + g_T, \tag{3.5}$$

[1] A more complete discussion of these algorithms is presented in Andrew B. Abel, "*A Comparison of Three Optimal Control Algorithms as Applied to the Monetarist-Fiscalist Debate*," Senior Thesis, Princeton University, Department of Economics, 1974.

where

$$G_T = -(E_{T-1}C'H_TC)^{-1}(E_{T-1}C'H_TA)$$

and

$$g_T = -(E_{T-1}C'H_TC)^{-1}[(E_{T-1}C'H_Tb_T) + (E_{T-1}C')h_T].$$

Note that the feedback control equation is not linear in y^*_{T-1} since the parameters G_T and g_T are functions of the posterior density of A, C, and b_T at the end of period $T-1$, which is a function of $y^*_{T-1}, y^*_{T-2}, \ldots$. After substituting $\hat{x}_T$ into (3.4) to obtain the optimal expected welfare cost $\hat{w}_T$ for the last period, we then approximate the function $\hat{w}_T$ by a modified second-order Taylor series expansion to obtain

(3.6) $$w_T \approx \tfrac{1}{2}\sum_{t=1}^{T} y_t^{*\prime}Q_t^T y_t^* + \sum_{t=1}^{T} y_t^{*\prime}q_t^T + c_0.$$ [2]

Using Bellman's principle of optimality, we minimize $E_{T-2}w_{T-1}$ with respect to x_{T-1} under the assumption that the optimal value of x_T, i.e., $\hat{x}_T$, will be selected in period T. Hence, we seek to minimize

(3.7) $$E_{T-2}w_{T-1} = E_{T-2}(\tfrac{1}{2}y^{*\prime}_{T-1}K_{T-1}y^*_{T-1} + y^{*\prime}_{T-1}k_{T-1} + \hat{w}_T) + \text{constant}''.$$

Substituting the Taylor series approximation for $\hat{w}_T$ and combining like terms in y^*_{T-1} within the expectation operator, we obtain

(3.8) $$E_{T-2}w_{T-1} = E_{T-2}(\tfrac{1}{2}y^{*\prime}_{T-1}H_{T-1}y^*_{T-1} + y^{*\prime}_{T-1}h_{T-1}) + \text{constant}''',$$

where $H_{T-1} = K_{T-1} + Q^T_{T-1}$, $h_{T-1} + q^T_{T-1}$, and constant''' absorbs those terms in (3.6) which are not dependent upon x_{T-1} and y^*_{T-1}. It should be observed that (3.8) is identical in form to (3.4). Hence, we may solve for $\hat{x}_{T-1}$ in the same manner that we solved for $\hat{x}_T$. This backward induction procedure is repeated until we obtain values for $\hat{x}_1$ and $\hat{w}_1$. This algorithm, which we shall call method III, is a dual adaptive control algorithm. It gives only approximate solutions since it involves a quadratic approximation about a somewhat arbitrary tentative path.[3]

In the certainty equivalence algorithm (method I), it is assumed that the values of the random parameters, A, C, and b_t are equal, with certainty, to their respective conditional expectations at time 0, i.e., $\bar{A}$, $\bar{C}$, and $\bar{b}_t$. Hence, the parameters of the feedback control equations are $G_T = -(\bar{C}'H_T\bar{C})^{-1}(\bar{C}'H_T\bar{A})$ and $g_T = -(\bar{C}'H_T\bar{C})^{-1}[(\bar{C}'H_T\bar{b}_T) + \bar{C}'h_T]$. The feedback control equation is strictly linear in y^*_{T-1} and the function $\hat{w}_T$ is truly quadratic in y^*_{T-1}. Therefore, there is no need to approximate $\hat{w}_T$ by a second-order Taylor series expansion and thus the certainty equivalence solution to the control problem is exact.

Method II, which is another special case of method III, takes account of uncertainty in the parameters but does not anticipate future learning. In method II, all conditional expectations are evaluated at time 0 so that the coefficients of the

[2] In the calculations summarized later we employ a standard Taylor series with cross-partials, and (3.2) includes terms of the form $(y_t^* - a_t)'K_{t,s}(y_s^* - a_s)$. However, we let $K_{t,s} = 0$ for all $t \neq s$.

[3] In our calculations, we obtain the tentative path by using the certainty equivalence algorithm to determine $\hat{x}_t$, and then apply the estimated model, without random disturbance, to $\hat{x}_t$, to generate y_t^*, for $t = 1, \ldots, T-1$.

feedback control equation are $G_T = -(E_0C'H_TC)^{-1}(E_0C'H_TA)$ and $g_T = -(E_0C'H_TC)^{-1}[(E_0C'H_Tb_T) + (E_0C')h_T]$. As in method I, the parameters of the feedback control equation are independent of y^*_{T-1}, and hence $\hat{x}_T$ is a linear function of y^*_{T-1}. Thus $\hat{w}_T$ is truly quadratic in y^*_{T-1} and method II yields an exact solution to the modified control problem.

4. Policy Analysis Using the Simple Model

Studies of the relative effectiveness of monetary and fiscal policy often focus on the size of long-run and short-run multipliers of the monetary and fiscal instruments, e.g., Kmenta and Smith (1973). However, Brainard (1967) argues that an examination of the minimum expected welfare cost attainable with a given set of instruments is a more meaningful approach to the question of the effectiveness of the instruments than is an examination of the multipliers of these instruments. Indeed, from the point of view of maximizing social welfare, any relevant features of the multipliers will be reflected in the minimum expected value of the welfare cost function and hence multiplier analysis is unnecessary, if not misleading. In this paper we shall compare the minimum expected welfare cost attainable using only a discretionary monetary instrument with the minimum expected welfare cost attainable using only a discretionary fiscal instrument. However, before solving the control problem using only one discretionary instrument at a time, we solve the control problem using both M_t and E_t as discretionary instruments subject to feedback control. We rewrite the reduced form equations (2.4) and (2.5) as $y^*_t = Ay^*_{t-1} + Cx_t + b_t + e_t$, where

$$y^{*\prime}_t = (C_t, I_t, E_t, M_t), \tag{4.1}$$

$$A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & 0 \end{array}\right)$$

is a 4 × 4 matrix containing the 2 × 2 matrix A_1,

$$C = \left(\begin{array}{c} C_0 \\ \hline I \end{array}\right)$$

is a 4 × 2 matrix containing the 2 × 2 matrix C_0,

$$x'_t = (E_t, M_t)$$

is the vector of instruments,

$$b_t = \left(\begin{array}{c} \bar{b}_t \\ \hline 0 \end{array}\right)$$

is a 4-vector containing the 2-vector $\bar{b}_t$, and

$$e_t = \left(\begin{array}{c} \varepsilon_t \\ \hline 0 \end{array}\right)$$

is a 4-vector containing the 2-vector ε_t.

Before proceeding with the application of the control algorithms, we must specify the following parameters of the welfare function: (1) T, the number of periods in the planning horizon; (2) a_t, the target values of y_t^*; and (3) K_t, the weighting matrices. We shall solve the control for a 6-period planning horizon, i.e., $T = 6$. In order to select appropriate target growth rates for C_t and I_t, we examine the historical percentage growth rates shown in Table 1.[4] It should be noted that the growth rate for I_t for the 11 quarters ending with 1963/IV is much higher than the growth rate for the 40 quarters ending with 1963/IV because investment was near a cyclical low in 1961/II. With these historical growth rates in mind, we somewhat arbitrarily choose target growth rates of 1.25 percent per quarter for C_t and I_t.

TABLE 1
QUARTERLY GROWTH RATES (%)

Period	C_t	I_t
1954/I to 1963/IV	0.91	1.14
1961/II to 1963/IV	1.10	2.61

For the weighting matrices in the welfare function, we set

$$K_t = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right), \qquad t = 1, \ldots, T,$$

where I is the 2×2 identity matrix. Note that since the instruments are assigned zero weight in each K_t, they do not explicitly appear as arguments of the welfare function. Since the ultimate objective of our analysis is to compare the relative effectiveness of monetary and fiscal policy in reaching given targets over time, we shall examine the welfare cost net of the costs directly associated with the instruments. We will, however, examine the stability of the instruments in each solution to make sure that they do not fluctuate excessively.

Method I is applied to the two-instrument, two-target control problem, with

$$\bar{A}_1 = \begin{pmatrix} 0.9266 & -0.0203 \\ 0.1527 & 0.3806 \end{pmatrix}, \quad \bar{b}_t = \begin{pmatrix} -63.2386 \\ -210.8994 \end{pmatrix}$$

and

$$\bar{C}_0 = \begin{pmatrix} 0.3190 & 0.4206 \\ -0.0735 & 1.5389 \end{pmatrix}.$$

Then method II and method III are applied to the same problem. For the quadratic approximation of method III, a deterministically generated tentative path derived from the solution of method I is used. The solutions to this problem by the three algorithms are presented in Table 4.

In order to investigate the effectiveness of one instrument alone, we shall assume that the instrument under consideration is a discretionary instrument

[4] Since E_t and M_t have zero weight in the welfare function, it is not necessary to specify targets for these variables.

the values of which are chosen by the policy maker subject to feedback control. It is assumed that the values of the other instrument are determined by a passive policy of a constant percentage change per quarter. In the notation of (3.1), the discretionary instrument is represented by the scalar x_t, and the passive instrument is modeled as a noncontrollable exogenous variable which is absorbed in the value of b_t. The solution to the control problem will be sensitive to the values of b_t, $t = 1, \ldots, T$, and hence the values of the passive instrument must be chosen judiciously. To determine the values of the passive policy variable, we solve the six-period, two-instrument control problem in which each of the instruments is constrained to change at a constant rate throughout the planning horizon. The solution to this problem, calculated under the assumption of certainty equivalence, is shown in Table 2. There is no *a priori* reason to believe that the growth rate obtained in this manner for each instrument will be optimal when the values of the other instrument are chosen subject to feedback control. A better approach to selecting an optimal growth rate for the passive instrument would be to solve the control problem repeatedly with one discretionary instrument and one passive instrument, allowing the growth rate for the passive instrument to vary in successive computations of the solution. The optimal growth rate for the passive instrument is the growth rate for which the optimal expected welfare cost is minimized.

TABLE 2
CERTAINTY EQUIVALENCE SOLUTION TO CONTROL PROBLEM WHEN BOTH INSTRUMENTS ARE PASSIVE

Instrument \ Period	1	2	3	4	5	6	Rate of Change per Period
E_t (billions)	110.9	112.4	113.9	115.4	116.9	118.4	+1.313%
M_t (billions)	143.6	143.4	143.3	143.1	142.9	142.7	−0.122%

Note: This solution was obtained using the OPTCDIAG option of the certainty equivalence program described in Douglas R. Chapman and Gregory C. Chow, "*Optimal Control Programs: User's Guide*," Econometric Research Program, Princeton University, Research Memorandum No. 141, May, 1972. Slight inconsistencies may appear above as a result of rounding since the program used a percentage growth rate with 6 decimal places. Also note that $\hat{w}_1 = 68.3074$.

In lieu of performing an extensive search to determine the optimal growth rate for each instrument when the other instrument is discretionary, we merely examine two other growth rates for each instrument to check whether the growth rates shown in Table 2 appear to be approximately optimal. The optimal welfare costs shown in Table 3 were obtained from the solution, by method I, to the control problem in which the passive instrument grows at the given rate and the discretionary instrument is determined by feedback control. Note that for each instrument, the value of $\hat{w}_1$ obtained using the growth rate from Table 2, is smaller than the values of $\hat{w}_1$ obtained using growth rates 0.1 percent larger and 0.1 percent smaller than the growth rate in Table 2. This result lends some credence to the assertion that, for each instrument, the growth rate in Table 2 reasonably approximates the optimal rate when the other instrument is determined by feedback control.

TABLE 3
OPTIMAL EXPECTED WELFARE COSTS (by Method I) WHEN ONE INSTRUMENT IS PASSIVE AND ONE INSTRUMENT IS DISCRETIONARY

E_t is passive		M_t is passive	
Growth rate of E_t	$\hat{w}_1$	Growth rate of M_t	$\hat{w}_1$
+1.213%	48.7176	−0.022%	48.8780
+1.313%	48.0508	−0.122%	44.1556
+1.413%	48.1178	−0.222%	48.6696

The control problem is now solved using methods I, II, and III under the assumption that E_t is a discretionary instrument and M_t is a passive instrument exogenously set equal to the values given in Table 4.2. This procedure is then repeated with M_t as the discretionary instrument and E_t as the passive instrument. The results of the control computations for period 1 are presented in Table 4. Let $w_i(P)$ be the optimal expected welfare cost function from period 1 to period T where $i \in \{\text{I, II, III}\}$ refers to the algorithm employed and $P \subseteq P^* = \{E_t, M_t\}$ refers to the set of discretionary policy variables used in the application of the algorithm. Note that for each $i \in \{\text{I, II, III}\}$, $\min_{P \subseteq P^*} w_i(P) = w_i(\{E_t, M_t\})$, which is an illustration of the well-known fact that in a control problem with two targets, a lower optimal expected welfare cost is attainable using two instruments subject to feedback control than by using only one of these instruments subject to feedback control. More significant for our economic analysis, however, is the result that for each i, $w_i(\{E_t\}) < w_i(\{M_t\})$. Therefore, assuming that the economy of the United States is appropriately modeled by (2.4) and (2.5), fiscal policy as represented by E_t is somewhat more effective with respect to the given welfare function than is monetary policy represented by M_t. We note, however, that this difference is small, especially when we allow for uncertainty in method II.

To study our solution more closely, the value of $w_I(P)$ can be decomposed into a deterministic welfare cost and a stochastic welfare cost. To compute the deterministic welfare cost, we first generate a deterministic time path for each of the endogenous variables by assuming that each parameter of the linear model (3.1) is equal to its point estimate at time 0, and that $e_t = 0$, for $t = 1, \ldots, T$. The deterministic welfare cost is weighted sum of squared deviations of the deterministic time paths of the endogenous variables from their respective targets. The stochastic welfare cost is due to the randomness in y_t^* and results from the additive stochastic disturbance e_t. Substituting the optimal value of x_t from (3.3) into (3.1), we obtain

$$y_t^* = (A + CG_t)y_{t-1}^* + Cg_t + b_t + e_t. \tag{4.2}$$

Assuming that the covariance matrix of e_t is Σ_e for $t > 0$, it can be shown that the covariance matrix of y_t^* is given by the recursive formula

$$\Sigma_{y_t^*} = (A + CG_t)\Sigma_{y_{t-1}^*}(A + CG_t)' + \Sigma_e. \tag{4.3}$$

TABLE 4

SOLUTIONS USING THE THREE CONTROL ALGORITHMS

Instrument Set and Results	Method	I	II	III
$\{E_t, M_t\}$	$H^1_{1,1}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.4121 & -0.1265 & 0 & 0 \\ -0.1265 & 1.0789 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.6397 & -0.1974 & -0.0184 & -0.0125 \\ -0.1974 & 1.1434 & 0.0040 & -0.0331 \\ -0.0184 & 0.0040 & 0.0062 & 0.0016 \\ 0.0125 & -0.0331 & 0.0016 & 0.0563 \end{bmatrix}$
	G_1	$\begin{bmatrix} -2.6095 & 0.3666 & 0 & 0 \\ -0.2239 & -0.2298 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1.6856 & 0.0647 & 0 & 0 \\ -0.2497 & -0.1997 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1.7174 & 0.0765 & 0 & 0 \\ -0.2497 & -0.1991 & 0 & 0 \end{bmatrix}$
	g'_1	[1013.0050 243.1293]	[709.0372 249.7009]	[719.3068 249.0192]
	$\hat{x}'_1$	[111.7262 142.8996]	[111.7831 142.8544]	[111.6835 142.8729]
	$\hat{w}_1$	35.5479	51.3291	48.3926
$\{E_t\}$	$H^1_{1,1}$	$\begin{bmatrix} 1.1545 & 0.1586 & 0 \\ 0.1586 & 1.1628 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.7558 & 0.0296 & 0 \\ 0.0296 & 1.2169 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 2.3791 & 0.2670 & -0.0309 \\ 0.2670 & 1.9123 & 0.0219 \\ -0.0309 & 0.0219 & 0.0116 \end{bmatrix}$
	G_1	[−2.7949 0.1763 0]	[−1.7012 0.0018 0]	[−1.6065 −0.0991 0]
	g_1	1094.6816	719.4268	694.4397
	$\hat{x}_1$	110.3943	111.0514	111.1035
	$\hat{w}_1$	44.1556	63.4141	74.2185
$\{M_t\}$	$H^1_{1,1}$	$\begin{bmatrix} 3.0666 & -0.2904 & 0 \\ -0.2904 & 1.0408 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 3.1216 & -0.2876 & 0 \\ -0.2876 & 1.1196 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 7.0334 & -1.1361 & -0.0257 \\ -1.1361 & 1.3879 & -0.0329 \\ -0.0257 & -0.0329 & 0.0616 \end{bmatrix}$
	G_1	[−0.3827 −0.2075 0]	[−0.3580 −0.1732 0]	[−0.4210 −0.1451 0]
	g_1	298.0639	286.1260	306.2156
	$\hat{x}_1$	142.9770	142.8952	142.8845
	$\hat{w}_1$	48.0508	64.7409	77.2147

The stochastic welfare cost is equal to

(4.4) $$w_s = \frac{1}{2} \sum_{t=1}^{T} \text{tr}\,(K_t V_t),$$

where V_t is the estimate of Σ_{yt} based on the estimate of Σ_e at the current time.

In Figure 1 we present the target time path for C_t and the deterministic time paths for C_t using the instrument sets $\{E_t\}$ and $\{M_t\}$. In addition to the deterministic time path for each instrument set, we present the values of C_t one standard deviation above and below the deterministic time path of C_t. For each time period,

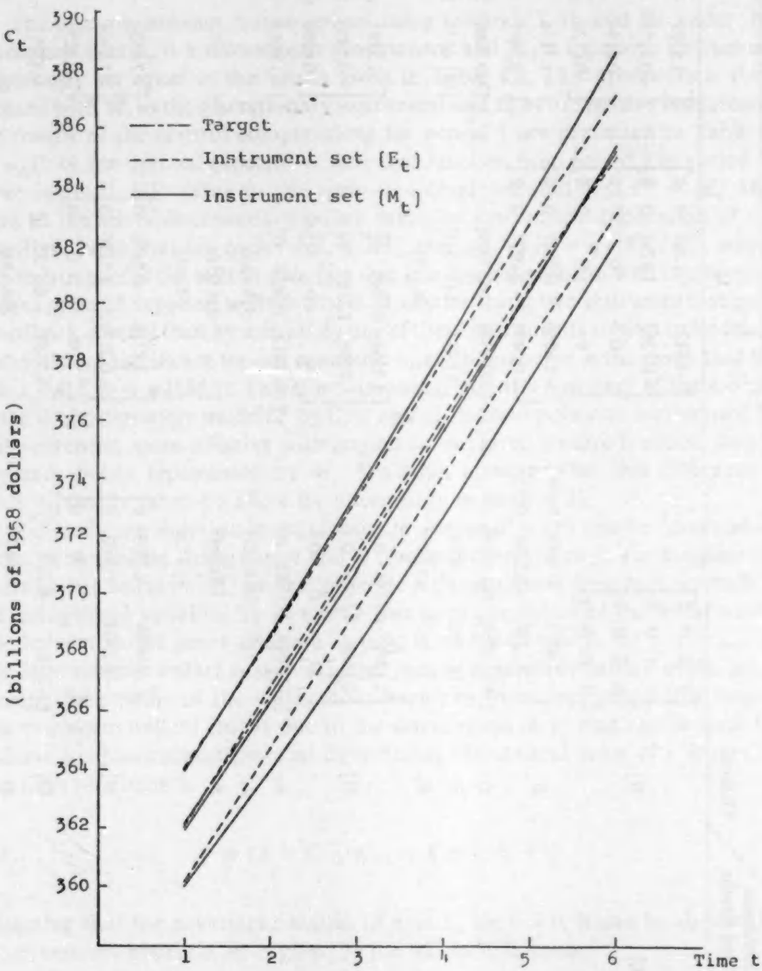

Figure 1 Expected Time Paths of C_t with Standard Deviation Bands (Obtained from Certainty Equivalence Solutions)

the vertical distance between the deterministic time path of C_t for a given instrument set and the target time path of C_t essentially measures the square root of the deterministic cost attributable to C_t.[5] Similarly, for each period, the standard deviation of C_t around its deterministic time path reflects the stochastic cost attributable to C_t in that period. Note that the deterministic time path of C_t for the instrument set $\{E_t\}$ is generally above the target time path whereas for the instrument set $\{M_t\}$ it is generally below the target time path. If the deterministic cost comprised a major portion of the expected welfare cost, we might have to consider the given quadratic welfare cost function to be inappropriate because it assigns costs to the overachievement of the targets for C_t through the use of the instrument set $\{E_t\}$ as well as to the underachievement of the targets for C_t through the use of $\{M_t\}$. However, we note that for each instrument set, the standard deviation of the stochastic variation around the deterministic time path of C_t far outweighs the deterministic "standard deviation" of the deterministic time path around the target time path. Hence, the adverse effects of assigning deterministic costs to expected positive deviations from the target values may be neglected since they appear to be unimportant. We also note that the standard deviation band around the deterministic time path obtained using $\{E_t\}$ lies within the standard deviation band around the deterministic time path obtained using $\{M_t\}$, except for period 1. Hence, the stochastic cost attributable to C_t is smaller for $\{E_t\}$ than for $\{M_t\}$. Figure 2 is analogous to 1 except that the endogenous target variable is I_t. As in Figure 1, we observe that the stochastic welfare cost is much larger than the deterministic welfare cost.

Table 5 summarizes the results presented in Figures 1 and 2. In this table, the deterministic welfare cost is expressed as an average over the six-period planning horizon. The stochastic welfare cost for each target variable is the average variance of that variable around its deterministic time path. The last two columns of Table 5 present the square roots of the corresponding values in the first two columns of the table and represent deviations in terms of 1958 dollars. It is clear from Table 5 that the total cost attributable to I_t is greater than the total cost attributable to C_t, and the stochastic cost is much greater than the deterministic cost for each instrument set.

Since the instruments receive zero weights in the welfare function, we shall briefly examine the dynamic characteristics of the time paths of the instruments (derived from method I) to determine whether they are highly volatile. We note that for the instrument set $\{E_t, M_t\}$, the period-to-period fluctuations of E_t along its deterministic time path are all less than 1.3 billion 1958 dollars, and for $\{E_t\}$ all of the deterministic changes are less than 2.5 billion 1958 dollars. Furthermore for each instrument set, the standard deviation of E_t around its deterministic time path remains fairly stable and is less than 5.3 billion 1958 dollars in each period. For each of the instrument sets $\{E_t, M_t\}$ and $\{M_t\}$, the deterministic period-to-period fluctuations of M_t are all less than 0.1 billion 1958 dollars. The standard deviation of M_t remains fairly stable at about 0.9 billion 1958 dollars for $\{E_t, M_t\}$ and about 1.1 billion 1958 dollars for $\{M_t\}$. Hence, it appears that for each instrument, neither the deterministic period-to-period fluctuations nor

[5] This expected welfare cost ignores the factor of $\frac{1}{2}$ in (3.2).

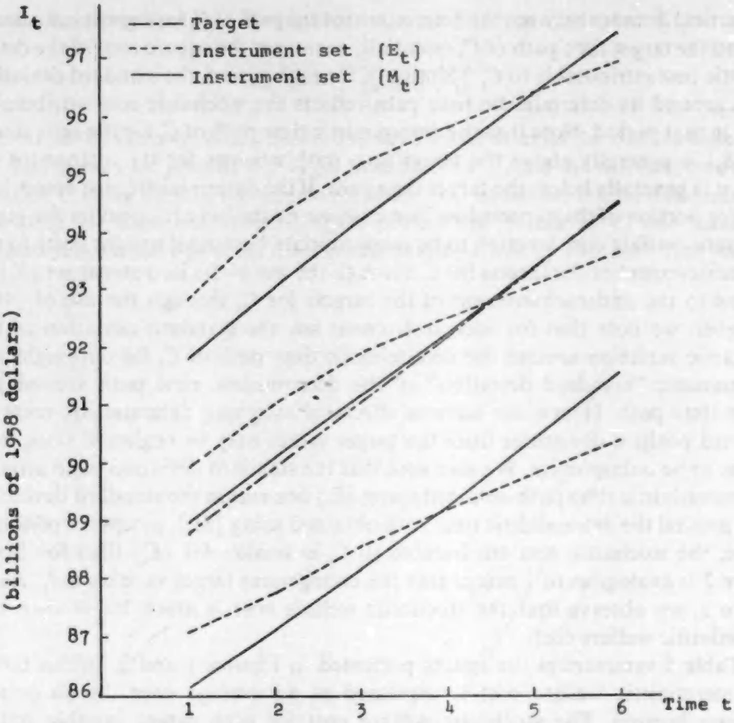

Figure 2 Expected time paths of I_t with standard deviation bands (obtained from certainty equivalence solutions)

TABLE 5

AVERAGE WELFARE COSTS PER PERIOD

Instrument Set	Target	Variance (Welfare Cost) (billions of 1958 dollars)^2			Standard Deviations (billions of 1958 dollars)	
		Deterministic	Stochastic	Total	Deterministic	Stochastic
$\{E_t\}$	C_t	0.028	3.816	3.844	0.167	1.953
	I_t	0.696	10.178	10.874	0.834	3.190
	Total	0.724	13.994	14.718	—	—
$\{M_t\}$	C_t	0.072	7.273	7.345	0.268	2.697
	I_t	0.016	8.656	8.672	0.126	2.942
	Total	0.088	15.929	16.017	—	—
$\{E_t, M_t\}$	C_t	0	3.775	3.775	0	1.943
	I_t	0	8.074	8.074	0	2.841
	Total	0	11.849	11.849	—	—

Note: Costs exclude factor of $\frac{1}{2}$ which appears in (2.4) and (4.6).

the stochastic variation around the deterministic time path is large enough to present serious problems of implementation.

We observe in Table 4 that for a given instrument set, the coefficients of the feedback control equation are subject to considerable variation across algorithms with the introduction of uncertainty and the anticipation of learning. However, it should be noted that the optimal values of the instrument do not appear to be very sensitive to the presence of uncertainty or to the anticipation of learning. In Table 6, we present the percentage variation across the three algorithms of the optimal first-period settings of the instruments for each of the three control problems. Note that when the policy maker treats both E_t and M_t as discretionary instruments, there is an extremely small percentage variation in $\hat{x}_1$ across the three algorithms. This result suggests that for the purpose of determining the optimal values of E_1 and M_1, it makes little difference whether the effects of uncertainty and learning are considered.

TABLE 6
PERCENTAGE VARIATION IN x ACROSS THE THREE ALGORITHMS

Instrument Set	Instrument	% Variation Across Algorithms
$\{E_t, M_t\}$	E_t	0.09
	M_t	0.03
$\{E_t\}$	E_t	0.64
$\{M_t\}$	M_t	0.06

Note: The percentage variation is the ratio of the range of $\hat{x}_1$ to the value of $\hat{x}_1$ obtained for method I.

5. CONCLUDING REMARKS

Using the very simple macro-econometric model presented in Section 2, we found that fiscal policy, represented by E_t, is more effective than monetary policy, represented by M_t, with respect to the given welfare function. Note, however, that this result does not imply that the policy maker should treat M_t as a passive instrument not subject to feedback control. The results presented in Table 4 indicate that the minimum expected welfare cost is significantly lower when the policy maker selects the values of both E_t and M_t subject to feedback control than when E_t is the only discretionary instrument. We also observed that the values of the instruments required to achieve the minimum expected welfare cost appear to be free from wild fluctuations over time and do not thereby present a difficult problem of implementation.

In the evaluation of the relative effectiveness of the monetary and fiscal instruments, we allowed for uncertainty and the possibility of learning in the computation of the optimal control solutions and the associated welfare losses. By examining the effectiveness of policy within the framework of the three different algorithms and their different assumptions regarding uncertainty and learning, our analysis has a broader basis than if we had used only method I with its restrictive assumptions of certainty equivalence. We noted that although the introduction of uncertainty may significantly change the coefficients of the feedback control equation, the optimal first-period policy is rather insensitive to uncertainty in the parameters of the linear model. The implication of this result for policy formulation is that we may fairly accurately determine the optimal values of E_1 and M_1 by any of the three algorithms discussed in this paper.

Massachusetts Institute of Technology

BIBLIOGRAPHY

Abel, Andrew B. "A Comparison of Three Control Algorithms As Applied to the Monetarist-Fiscalist Debate," A.B. Thesis submitted to the Department of Economics, Princeton University, April, 1974.

Athans, Michael. "The Discrete Time Linear-Quadratic-Gaussian Stochastic Control Problem," *Annals of Economic and Social Measurement* 1 (October, 1972): 449–91.

Athans, Michael and Chow, Gregory C. "Introduction to Stochastic Control Theory and Economic Systems," *Annals of Economic and Social Measurement* 1 (October, 1972): 375–82.

Brainard, William. "Uncertainty and the Effectiveness of Policy," *American Economic Review* 57 (May, 1967): 411–25.

Chapman, Douglas R. and Chow, Gregory C. "Optimal Control Programs: User's Guide." Econometric Research Program, Princeton University, Research Memorandum No. 141, May, 1972.

Chow, Gregory C. "Multiplier, Accelerator, and Liquidity Preference in the Determination of National Income in the United States." *The Review of Economics and Statistics* 49 (February, 1967): 1–15.

———. "How Much Could be Gained by Optimal Stochastic Control Policies," *Annals of Economic and Social Measurement* 1 (October, 1972) 391–406.

———. "Effect of Uncertainty on Optimal Control Policies," *International Economic Review* 14 (October, 1973): 632–45.

———. "Problems of Economic Policy from the Viewpoint of Optimal Control," *American Economic Review* 63 (December, 1973): 825–37.

———. "A Solution to Optimal Control of Linear Systems with Unknown Parameters," Econometric Research Program, Princeton University, Research Memorandum No. 157, December, 1973, to appear in the *Review of Economics and Statistics*.

Kmenta, J. and Smith, P. E. "Autonomous Expenditures Versus Money Supply: An Application of Dynamic Multipliers." *The Review of Economics and Statistics* 55 (August, 1973): 299–307.

MacRae, Elizabeth Chase. "Linear Decision with Experimentation," *Annals of Economics and Social Measurement* 1 (October, 1972): 437–47.

Pindyck, Robert S. "Optimal Stabilization Policies Via Deterministic Control." *Annals of Economic and Social Measurement* 1 (October, 1972): 385–90.

Prescott, Edward C. "The Multi-period Control Problem under Uncertainty." *Econometrica* 40 (November, 1972): 1043–58.

Rausser, Gordon C. and Freebairn, John W. "Approximate Adaptive Control Solutions to U.S. Beef Trade Policy." *Annals of Economic and Social Measurement* 3 (January, 1974): 177–203.

Samuelson, Paul A. "Interactions between the Multiplier Analysis and the Principle of Acceleration." *Review of Economic Statistics* 21 (May, 1939): 75–8.

Tse, Edison. "Adaptive Dual Control Methods." *Annals of Economic and Social Measurement* 3 (January, 1974): 65–83.

U. S. Board of Governors of the Federal Reserve System. *Federal Reserve Bulletin* 56 (December, 1970).

U. S. Department of Commerce. *Survey of Current Business* 45 (August, 1965), 46 (July, 1966), 47 (July, 1967).