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UNCERTAINTY AND THE STABILITY OF THE ARMAMENTS RACE*

BY D. L. BRITO AND M. D. INTRILIGATOR

The paper investigates the dynamics of an armaments race in a model of resource allocation. The preordering of weapons space in the allocative model is constructed from a strategic dynamic model of a nuclear war. The strategic model incorporates uncertainty in the effectiveness of weapons and lags in their impact. It was shown that an equilibrium level of armaments exists and the Cournot solution would be stable at this equilibrium.

1. INTRODUCTION

Models about armaments fall into two classes: simple analytical models that derive from the Richardson tradition and very complex computer simulation models. The first class of models cannot address many of the problems that are of interest to policymakers; in particular they cannot address problems that involve interaction of technology, strategy and crisis stability within the context of an armaments race. The second class of models is very useful for simulating particular situations such as an encounter between a submarine and an ASW taskforce or the problem of finding an optimal targeting pattern of the Minutemen force. These models are very useful at the micro-level but are often very complex and *ad hoc*.

This paper explores the interaction between technological and strategic considerations in a potential nuclear war and the implied dynamics of an armaments race. It extends the previous work of both authors in the general area of strategy and arms races [1], [2], [3], [4], [5], [6]. It presents an *analytical* model of the dynamics of a nuclear war which incorporates both time lags and uncertainty. In particular, it investigates the effect of the dynamics of a potential nuclear war that incorporates uncertainty on the stability of an armaments race.

2. INTERACTION BETWEEN WEAPONS CHOICES AND STRATEGIC CHOICES

The interaction between weapons choices and strategic choices is a fundamental element of the arms race. The strategic options open to a nation depend, to a great extent, on its weapons systems, while, conversely, the weapons systems a nation procures depend largely on its strategic goals. The strategic concepts adopted by the various armed services in the recent period, for example, have been choices based to a large extent on the weapons available to the individual services. Thus the Air Force during much of this period advocated a counterforce strategy, according to which prime targets should be enemy military installations, perhaps because its bombers and land based missiles were well suited to such targets. At the same time the Navy advocated a countervalue strategy, according to which prime targets should be enemy cities, perhaps because its submarine based missiles were well suited to such targets [4].

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Both political and military authorities participate in choices of weapons and choices of strategies. Typically the political authorities decide on the allocation of resources to procure weapons while the military authorities decide on the specific types of weapons procured. As to strategic choices, the political authorities typically bear responsibility for decisions regarding war initiation and war termination while the military authorities decide on targets and rates of fire in wartime. The two sets of choices—weapons and strategy—and the two decision makers—political and military—are represented in the basic model [4]. The model is based on previous work by Brito addressed principally to the allocation of resources to procure weapons by the political authorities in the context of an armament race and on previous work by Intriligator addressed principally to the choice of targets and rates of fire in wartime by the military authorities.

3. Review of Previous Work

Brito [2] studied the arms race in the context of a model of resource allocation. He considered a model with two countries denoted by 1 and 2 and indexed by i or j. Each of these countries was assumed to maximize

$$\int_0^\infty e^{-rt} U_i[C_i, D_i(M_i, M_j)]\, dt \qquad i,j = 1,2; \qquad i \neq j \tag{1}$$

subject to the constraints

$$\dot{M}_i = Z_i - \beta_i M_i \tag{2}$$

$$Y_i = Z_i + C_i \tag{3}$$

and the j-th country's reactions. The function $U[C_i, D_i(M_i, M_j)]$ is a twice differentiable strictly concave utility function, where C_i is the consumption of the i-th country, M_i is the weapons stock of the i-th country and $D_i(M_i, M_j)$ is the index of defense, a technological and strategic preordering of weapon space. Z_i is the net investment in weapons at time t, $\beta_i M_i$ is the amount of resources consumed to maintain a weapons stock of size M_i, and Y_i is the net national product of country i. Using (1), (2) and (3), Brito derived a dynamic model of an armament race and showed that finite equilibrium levels of armaments consistent with both maximizations exist given appropriate assumptions about weapons technology. It was assumed that both indices of defense increase (decrease) at a decreasing (increasing) rate with one's own (enemy) weapons. These assumptions are sufficient to ensure that both objective functions (1) will have a maximum, and hence that an equilibrium exists.

The stability of an equilibrium level of armaments depended on more questionable assumptions. If it was assumed that the two countries behave in a myopic manner, there was no question that the equilibrium is stable, as in the Richardson equation model. If, however, it is assumed that the two countries behave in a more sophisticated manner, for example using information about current armament levels and the rate of change of armament levels to predict future levels of armaments, which levels are used to plan current investment in arms, then the stability of the equilibrium is in question.

The Intriligator model is a dynamic model of a missile war which explicitly treats strategic choices made by both countries [3]. Assuming again that there are two countries, 1 and 2 indexed by i or j; that there is only one weapon, referred to as a "missile," where $M_i(t)$ is the number of missiles available to country i at time t; and there is an index of casualties in each country, where $N_i(t)$ is the number of casualties in country i at time t, the dynamic model is summarized by the four differential equations

$$\dot{M}_i = -\alpha_i M_i - \alpha_j M_j \beta_j f_j \qquad i, j = 1, 2; \qquad i \neq j \tag{4}$$

$$\dot{N}_i = \alpha_j M_j (1 - \beta_j) v_j \qquad i, j = 1, 2; \qquad i \neq j. \tag{5}$$

Equation (4) shows the time rate of change (decrease) in the missile stocks of country i as stemming from two considerations. First, country i decides to fire its missiles at a certain rate, designated α_i, which can range between zero and some maximum rate, $\bar{\alpha}_i$, assumed given and finite. Second, country i is losing missiles as they are being destroyed by counterforce missiles of country j. Of the missiles fired by j, represented by $\alpha_j M_j$, the fraction β_j are targeted at enemy missiles (counterforce) where β_j can range between zero (pure countervalue) and one (pure counterforce). The missiles fired counterforce by j destroy $\alpha_j M_j \beta_j f_j$ of the i missiles, where f_j is the counterforce effectiveness ratio, the number of i missiles destroyed by one counterforce missile. Equation (5) shows the time rate of change (increase) in casualties in country i stemming from countervalue missiles launched by j. Thus, of the $\alpha_j M_j$ missiles fired, the fraction $(1 - \beta_j)$ fired countervalue inflict $\alpha_j M_j (1 - \beta_j) v_j$ casualties, where v_j is the countervalue effectiveness ratio, the number of i casualties inflicted by one j countervalue missile.

Equations (4) and (5), in which all variables are time dependent, together with the boundary conditions stating that each country starts with a given stock of missiles and no casualties at the outset of the war at time 0, determine the evolution of the war over time. The payoff country i is assumed to depend on values of missiles and casualties at the end of the war, T, and optimal strategies are obtained for i by maximizing the payoff function. It is shown in [3] that the optimal strategies are switching strategies, with targets switching from pure counterforce to pure countervalue and rates switching from the maximum rate to the zero rate.

Since the maximized value of the payoff function depends on the initial conditions, an index of defense similar to the $D_i(\cdot\cdot)$ function above can be constructed. This function depends on initial technological and strategic considerations. A more complete description of this synthesis can be found in [4].

4. The Enlarged Model

The enlarged model of a nuclear war introduces two additional considerations to the analysis. First, it incorporates the significant lag between the time a missile is launched and the time it hits its target. As a consequence of this lag, if the other side detects the launch it has the option of launching its missiles before they are destroyed. Second, it incorporates uncertainty. As the war progresses with missiles being launched by both sides, unless the parties have perfect reconnaissance capabilities, each side does not know for certain whether a given missile site is empty. Clearly the probability that a given warhead will destroy a missile declines as the probability that a given missile site is empty increases. These elements can be incorporated in an enlarged version of the basic model of a missile war.

Using a notation slightly different from (4) and (5), let $\alpha_i(t)$ be the number of missiles the i-th country fires at the j-th country's missiles at time t, and let $\beta_i(t)$ be the number of missiles the i-th country fires at the j-th country's cities, each measured as missiles launched per unit time.[1] Let f_j be the expected value of the number of the i-th country's warheads that will be destroyed by one of the j-th country's warheads given that the missile site attacked is not empty. f_j is then the product of the probability that a given attacking warhead destroys a missile site and the number of warheads in the missile site. Let p_i be the probability that a given missile has been fired and let L be the lag between the time a missile is fired and the time it strikes the target. The equation that describes the change in $M_i(t)$ in the missile war is then

(6) $$\dot{M}_i = -[\alpha_i(t) + \beta_i(t)] - f_j[1 - p_i(t)]\alpha_j(t - L).$$

If we define v_j to be the expected number of casualties inflicted on the i-th country by an attacking warhead then the equation that describes the change in the i-th country's casualties $N_i(t)$ is given by

(7) $$\dot{N}_i(t) = v_j\beta_j(t - L).$$

The only equations necessary to complete the system are those for $p_i(t)$.

At this point it is useful to introduce three more assumptions. First, assume that at any point in time, t, the missiles fired are from sites that had not been attacked as of $t = L$ and the missiles to be fired are selected from a uniform distribution; second, assume neither side has perfect reconnaissance[2] capabilities; and third, assume that targeting is one missile site per targeted warhead. These assumptions will be relaxed in future work.

Define $y_i(t)$ to be the number of warheads in the i-th country's missiles sites that have not been attacked at time t; define $z_i(t)$ to be the number of warheads that have been fired from unattacked sites in the interval $(0, t)$. $z_i(t)$ divided by the number of warheads in a missile site is then the number of empty unattacked missile sites. The probability that an unattacked site is empty is

(8) $$p_i(t) = \frac{z_i(t)}{y_i(t)}.$$

Since it is assumed that missiles to be fired are selected from a uniform distribution, the probability that at time $r < t$ a missile from $y(t)$ will be selected is $y(t)/y(r)$. The number of warheads that have been fired from unattacked missile sites can thus be approximated by

(9a) $$z_i(t) = \int_0^t \frac{y_i(t)}{y_i(r)}[\alpha_i(r) + \beta_i(r)]\, dr, \qquad 0 \leq t \leq T_i^0,$$

where T_i^0 is defined as

(9b) $$\int_0^{T_i^0} \frac{\alpha_i(r) + \beta_i(r)}{y_i(r)}\, dr = 1,$$

[1] Note that α_i and β_i are not the same as the α_i and β_i that appear in equations (4) and (5). It will be assumed that $\alpha_i(t) + \beta_i(t) \leq k_i$ where k_i is a constraint on the rate of fire.

[2] We are assuming that each side can monitor the other side's rate of fire but cannot determine the exact source of attacking missiles.

that is, the time when all unattacked missile sites are empty. The probability that a given unattacked missile site is empty is then

$$(10) \qquad p_i(t) = \int_0^t \frac{\alpha_i(r) + \beta_i(r)}{y_j(r)}\,dr, \qquad 0 \le t \le T_i^0.$$

Differentiating (10) yields

$$(11) \qquad \dot{p}_i(t) = \frac{\alpha_i(t) + \beta_i(t)}{y_i(t)}, \qquad 0 \le t \le T_i^0.$$

The boundary conditions for (11) are

$$(11a) \qquad p_i(0) = 0$$

$$p_i(t) = 1 \qquad \text{for } t \ge T_i^0.$$

Equation (11) gives the change in the probability that a missile site in country i will be empty as a function of the current rate of fire of country i and its remaining unattacked missile sites. The boundary conditions state that at the outset of war none of its missile sites are empty and at and after the time when all unattacked missiles have been fired the probability is one.

The number of remaining unattacked missile sites, $y_i(t)$, it itself determined by the rate of fire of the other country.

$$(12) \qquad \dot{y}_i(t) = -\alpha_j(t - L),$$

i.e., the decrease in unattacked missile sites is exactly the rate of fire against missiles, with a lag of length L. The boundary conditions for (12) are clearly

$$(12a) \qquad y_i(0) = M_i(0),$$

i.e., at the outset of war all missile sites are unattacked.

Equations (6), (7), (11) and (12) summarize the dynamics of a missile war. Following the scenario described in detail in (4), the military authorities in each country use these equations to choose optimal *contingency* strategies. The values of these optimal contingency strategies are then communicated to the political authorities.

The optimal contingency strategy will be worked out for one important case. Consider the case in which country 1 initiates the war; country 2 reacts with a lag L_2 and country 1 assumes that country 2 is choosing its strategy in a manner that will minimize the value of the country 1 objective function.[3] The last assumption, which is a zero-sum assumption, is not as objectionable as it might seem at first glance since the discussion involves *contingency* strategies, which can be considered a "worse case" analysis.

The military authority is given an objective function by the political authorities of the form:

$$(13) \qquad V_1 = m_1^* M_1(T) + m_2^* M_2(T) + n_1^* N_1(T) + n_2^* N_2(T)$$

where m_i^* and n_i^* are the values assigned to $M_i(t)$ and $N_i(T)$ by the political authorities. To compute the contingency strategy the military authorities in the first

[3] Note that lag L_2 is a reaction lag, not the technical time of flight lag introduced above.

country maximize (13) subject to each country's missile, casualty and probability equations (6), (7), (11) and (12); initial conditions, and the constraint:

$$\text{(14)} \qquad \alpha_2(t) + \beta_2(t) = 0 \qquad \text{for } 0 \le t \le L_2.$$

The solution to this probem is $V_1^*(M_1(0), M_2(0))$. To solve this problem, let ρ_i be the costate variable associated with y_i; γ_i with p_i; λ_i with M_i; and μ_i with N_i. The Hamiltonian is given by:

$$\begin{aligned}\text{(15)}\ H(N, M, \mathrm{v}, P, \gamma, \lambda, \alpha, \beta) = {} & -\rho_1\alpha_2(t-L) + \gamma_1\left[\frac{\alpha_1(t)+\beta_1(t)}{y_1(t)}\right] \\ & +\lambda_1\{-[\alpha_1(t)+\beta_1(t)] - f_2[1-p_1(t)]\alpha_2(t-L)\} \\ & +\mu_1 v_2\beta_2(t-L) + \rho_2\alpha_2(t-L) \\ & +\gamma_2\left[\frac{\alpha_2(t)+\beta_2(t)}{y_2(t)}\right] + \lambda_2\{-[\alpha_2(t)+\beta_2(t)] \\ & -f_1[1-p_2(t)]\alpha_1(t-L)\} + \mu_2 v_1\beta_1(t-L).\end{aligned}$$

Since there are no lags in the *state* variables the costate equations are

$$\begin{aligned}\text{(16)} \qquad \dot{\rho}_i &= \frac{\alpha_i(t)+\beta_i(t)}{[y_i(t)]^2} \\ \dot{\gamma}_i &= f_j\alpha_j(t-L) \\ \dot{\lambda}_i &= 0 \\ \dot{\mu}_i &= 0.\end{aligned}$$

Assume in this example that the solution is in the interior. (If the solution were to be on a boundary the dual variable associated with the nonnegativity constraint would appear in the transversality conditions). The transversality conditions are

$$\begin{aligned}\text{(17)} \qquad \rho_1(T) &= 0 \qquad \lambda_1(T) = m_1^* \\ \gamma_1(T) &= 0 \qquad \lambda_2(T) = m_2^* \\ \rho_2(T) &= 0 \qquad \mu_1(T) = n_1^* \\ \gamma_2(T) &= 0 \qquad \mu_2(T) = n_2^*\end{aligned}$$

Integrating the differential equations of the costate variables for the appropriate boundary conditions results in

(18)

$$\lambda_1 = m_1^*$$

$$\lambda_2 = m_2^*$$

$$\mu_1 = n_1^*$$

$$\mu_2 = n_2^*$$

$$\rho_1(t) = \int_t^{T_0} \frac{\alpha_1(r) + \beta_1(r)}{[y_1(r)]^2}\, dr$$

$$\rho_2(t) = \int_t^{T_0} \frac{\alpha_2(*) + \beta_2(r)}{[y_2(r)]^2}\, dr$$

$$\gamma_1(t) = \int_t^{T} f_2\alpha_2(r - L)\, dr$$

$$\gamma_2(t) = \int_t^{T} f_1\alpha_1(r - L)\, dr.$$

Let $t + L = \theta$, the military authority in country 1 will pick at each point in time $\alpha_i(t)$ and $\beta_i(t)$ in a manner that will maximize (see [7])

$$H(\theta) + H(t). \tag{19}$$

Assume the military authorities in country 1 are operating under the assumption that the military authorities in country 2 are choosing $\alpha_2(t)$ and $\beta_2(t)$ in a manner that minimizes (19). Let $\delta_i(t)$ be the Lagrange multiplier constraint associated with the constraint $\alpha_1(t) + \beta_1(t) = k_1$ and $\delta_2(t)$ be the multiplier associated with the constraint $\alpha_2(t) + \beta_2(t) = k_2$. The Kuhn–Tucker conditions for the i-th country's control variables are

$$-m_2^* f_1[1 - p_2(\theta)] - m_1^* - \rho_2^*(\theta) + \frac{\gamma_1(t)}{y_1^*(t)} - \delta_1^*(t) \le 0 \tag{20}$$

$$\alpha_1^*(t)\left\{-m_2^* f_1[1 - p_2(\theta)] - m_1^* - \rho_2^*(\theta) + \frac{\gamma_1^*(t)}{y_1^*(t)} - \delta_1^*(t)\right\} = 0$$

$$n_2^* v_1 - m_1^* + \frac{\gamma_1^*(t)}{y_1^*(t)} - \delta_1^*(t) \le 0$$

$$\beta_1^*(t)\left[n_2^* v_1 - m_1^* + \frac{\gamma_1^*(t)}{y_1^*(t)} - \delta_1^*(t)\right] = 0$$

and for the second country

$$-m_1^* f_2[1 - p_1(\theta)] - m_2^* - \rho_1^*(\theta) + \frac{\gamma_2^*(t)}{y_2^*(t)} - \delta_2^*(t) \geq 0 \tag{21}$$

$$\alpha_2^*(t)\left\{-m_1^* f_2[1 - p_1(\theta)] - m_2^* - \rho_1^*(\theta) + \frac{\gamma_2^*(t)}{y_2^*(t)} - \delta_2^*(t)\right\} = 0$$

$$n_1^* v_2 - m_2^* + \frac{\gamma_2^*(t)}{y_2^*(t)} - \delta_2^*(t) \geq 0$$

$$\beta_2^*(t)\left[n_1^* v_2 - m_2^* + \frac{\gamma_2^*(t)}{y_2^*(t)} - \delta_2^*(t)\right] = 0.$$

The star (*) on the control variable designates an optimal solution.

From the Kuhn–Tucker conditions it follows that if $\alpha_1(t) + \beta_1(t) > 0$ then the control chosen and switch points depend on the sign of the switching function

$$\{-m_2^* f_1[1 - p_2(\theta)] - \rho_2(\theta)\} - n_2^* v_1 \gtreqless 0. \tag{22}$$

As long as $\alpha_i(t) + \beta_i(t) \neq 0, i = 1, 2$, (22) will not be an equality almost everywhere; strategies will then be corner solutions. We can thus define $\{t_{1s}^k\} k = 1, m, t_{1s}^k$ in $(0, T)$ as strategy switch points for the i-th country. That is to say the points where the i-th country changes from counterforce to countervalue or vice-versa.

Proposition 1

If country 1 at some time employs a counterforce strategy and at another time employs a countervalue strategy, then country 1 will initially employ a counterforce strategy, making a single switch to countervalue strategy, i.e., for country 1, $\alpha_1^*(t) > 0$ for some t and $\beta_1^*(t) > 0$ for some t imply that $\alpha_1^*(t) = k_1$ for t in $[0, t_{1s}]$ and $\beta_1^*(t) = k_1$ for t in $[t_{1s}, T - L]$.

Proof. It is clear from the definition of $p_2^*(t)$ that if $p_2^*(t_1) > 0$ for some t_1 in $(0, T)$ then $p_2^*(t) > 0$ for all t in (t_1, T). Second if $\alpha_1^*(t) = 0$ for t in (t_2, t_3) then $p_2^*(t)$ is non-decreasing in (t_2, t_3). Finally $\dot{\rho}_2 \geq 0$ so ρ_2^* is non-decreasing. The control chosen and switch points thus depend on the sign of (22). If at $t = \bar{t}$ the term in braces is larger than $n_2^* v_1$, then $\alpha_1(t) = k_1$, if it is smaller then $\beta_1(t) = k_1$, and if it is equal, then $\bar{t}$ is a switch point. At $t = 0$ there are two interesting possibilities: first, if the expression given by (22) is negative, then the first country will initially choose a countervalue strategy and since the term in braces is non-increasing then $\beta_1(t) = k$ for all t in $(0, T - L)$, and second if (22) is positive then the first country will initially choose a counterforce strategy. There are then two possibilities: first, (22) will be positive for all t in $(0, T - L)$ and $\alpha_1(t) = k$ for all t in $(0, T - L)$; second, (22) is zero for some t'_{1s} in $(0, T - L)$ then $\beta_1(t) = k$ for t in (t'_{1s}, t). But since the term in braces is non-increasing in $(t'_{1s}, \hat{t}), \hat{t} = T - L)$. This proves the proposition.

Define t_{ic} as the time the i-th country ceases fire, that is $\alpha_i(t) + \beta_i(t) = 0$ for all $t > t_{ic}$.

Proposition 2

If country 2 ceases firing before country 1 has switched to countervalue then country 1 will not switch to countervalue until all country 2 missiles have been targeted, i.e., $t_{2c} < t_{1s}$ implies $\alpha_1(t) = k_1$ for all t in $[t_{2c}, M_2(0)/k_1]$ where $M_2(0)/k_1$ is the time at which all of the second country's missile sites have been attacked.

Proof. The change in the sign of inequality (22) depends only on $p_2(\theta)$ and $\rho_2(\theta)$ where $\dot{\rho}_2(\theta) = 0$ and $\dot{p}_2 = 0$ if $\alpha_2 + \beta_2 = 0$. Thus if the inequality (22) is positive at $t = t_{2c}$ then the inequality will remain positive for all t in $[t_{2c}, M_2(0)/k_1]$.

5. Existence and Stability of an Equilibrium

This section gives sufficient conditions for the existence and stability of an equilibrium level of armaments in the enlarged model. With these conditions, if the dynamic equations for the arms race are given by

$$\dot{M}_1 = F_1(M_1, M_2) \qquad (23)$$
$$\dot{M}_2 = F_2(M_1, M_2)$$

then there exists equilibrium levels of missile stocks, $\overline{M}_1$ and $\overline{M}_2$, such that

$$F_1(\overline{M}_1, \overline{M}_2) = 0 \qquad (24)$$
$$F_2(\overline{M}_1, \overline{M}_2) = 0$$

Let us approach the proof by first solving a simpler problem. Let us suppose that M_j is fixed at $\overline{M}_j$.

The current value Hamiltonian for maximizing (1) given (2) and (3) is given by

$$U_i[C_i, D_i(M_i, M_j)] + q_i(Z_i - \beta_i M_i) + \lambda_i(Y_i - Z_i - C_i). \qquad (25)$$

The i-th country wants to pick Z_i and C_i in a manner that will maximize the Hamiltonian for the q_i on the trajectory of a solution to the differential equation

$$\dot{q}_i = -\frac{\partial U_i}{\partial D_i}\frac{\partial D_i}{\partial M_i} + [r + \beta_i]q_i \qquad (26)$$

and the transversality conditions

$$\lim_{t \to \infty} e^{-rt} q_i(t) M_i(t) = 0. \qquad (27)$$

Let $q_i^*(M_i, \overline{M}_j)$ be the set of prices that solve the optimization problem in the sense given $M_j = \overline{M}_j$, maximizing the Hamiltonian with respect to $q_i = q_i(M_i, \overline{M}_j)$ will result in an optimal solution assuming this optimal solution exists.

If we use the Kuhn–Tucker Theorem to maximize the Hamiltonian, we have as the saddlepoint conditions,

$$\frac{\partial U_i}{\partial C_i} - \lambda_i \le 0; \qquad C_i\left[\frac{\partial U_i}{\partial C_i} - \lambda_i\right] = 0; \qquad (28)$$
$$q_i - \lambda_i \le 0; \qquad Z_i[q_i - \lambda_i] = 0.$$

C_i cannot be zero since the marginal utility would then be infinite. Therefore

$$\frac{\partial U_i}{\partial C_i} = \lambda_i. \qquad (29)$$

The strictly concavity of $U_i(\cdot, \cdot)$ implies, using the Implicit Function Theorem, that

$$C_i = C_i(\lambda_i, M_i, \overline{M}_j), \tag{30}$$

and we can show $\partial C_i/\partial\lambda_i < 0$, $\partial C_i/\partial M_i > 0$, $\partial C_i/\partial\overline{M}_j \leq 0$. Also either $Z_i = 0$ or $q_i = \lambda_i$. Letting

$$\bar{q}_i(M_1, M_2) = \left.\frac{\partial U_i}{\partial C_i}\right|_{C_i = Y_i}; \tag{31}$$

the differential equation for $\dot{M}_i$ is given by

$$\dot{M}_i = Y_i - C_i[q_i(M_1, \overline{M}_2)] - \beta_i M_i \qquad \text{if } q_i \geq \bar{q}_i \tag{32}$$

and

$$\dot{M}_i = -\beta_i M_i \qquad \text{if } q_i < \bar{q}_i. \tag{33}$$

These equations can also be written as

$$\dot{M}_i = F_i(M_i, \overline{M}_j) \tag{34}$$

Let $\overline{M}_i$ be the solution to $\dot{F}_i(M_i, \overline{M}_j) = 0$. It is clear that if $\overline{M}_i > 0$ then $\dot{F}_i(M_i, \overline{M}_j) = 0$ implies that $\dot{q}_i(M_i, \overline{M}_j) = 0$, and that $Z_i = \beta_i M_i$; so $q_i = \lambda_i$ and

$$\frac{\partial U}{\partial C_i} - q_i = 0, \tag{35}$$

$$\frac{\partial U_i}{\partial D_i}\frac{\partial D_i}{\partial M_i} - (r + \beta_i)q_i = 0.$$

Combining these two equations to eliminate the multiplier we have

$$\frac{\partial U_i}{\partial C_i}(r + \beta_i) = \frac{\partial U}{\partial D_i}\frac{\partial D_i}{\partial M_i} \tag{36}$$

and

$$C_i = Y_i - \beta_i M_i. \tag{37}$$

Lemma 1

If

$$\frac{\partial D_i(M_i, M_j)}{\partial M_i} \geq 0, \tag{38}$$

$$\frac{\partial D_i(M_i, M_j)}{\partial M_j} \leq 0$$

$$\lim_{M_i \to \infty} \frac{\partial^2 D(M_i, M_j)}{\partial M_i^2} \leq 0$$

$$\lim_{M_i \to \infty} \frac{\partial^2 D(M_i, M_j)}{\partial M_i \partial M_j} \geq 0.$$

Then there exists $M_i \leq \hat{M}_i$, such that a solution exists for the optimization and for all $M_i \geq \hat{M}_i$, equations (36) and (37) can be solved such that $M_i = g_i(M_j)$ and $g_i'(M_j) \geq 0$.

Proof. A sufficient condition for a maximum to exist is that $U[C_i, D_i(M_i, M_j)]$ is strictly concave: Given the assumptions about $U(\cdot, \cdot)$ the only difficulty can be caused by $D(M_1, M_2)$. The Hessian is

$$\begin{vmatrix} \dfrac{\partial^2 U_i}{\partial C_i^2} & \dfrac{\partial^2 U_i}{\partial C_i \partial D_i}\dfrac{\partial D_i}{\partial M_i} \\[2ex] \dfrac{\partial^2 U_i}{\partial C_i \partial D_i}\dfrac{\partial D_i}{\partial M_i} & \dfrac{\partial^2 U_i}{\partial D_i^2}\left(\dfrac{\partial D_i}{\partial M_i}\right)^2 + \dfrac{\partial U_i}{\partial D_i}\dfrac{\partial^2 D_i}{\partial M_i^2} \end{vmatrix}$$

and the only ambiguity in the sign of the second minor is introduced by the term

$$\frac{\partial^2 U_i}{\partial D_i}\frac{\partial^2 D_i}{\partial M_i^2}.$$

By assumption there exists M_i^* such that the ambiguity disappears since $\partial^2 D_i/\partial M_i^2$ can be made arbitrarily small or negative. The second part of the lemma can be shown by differentiating equations (36) and (37) with respect to M_j and solving for dM_i/dM_j where $F_i(M_i, M_j) = 0$.

$$\frac{dM_i}{dM_j} = \frac{-\left(\dfrac{\partial^2 U_i}{\partial D_i^2}\dfrac{\partial D_i}{\partial M_i}\dfrac{\partial D_i}{\partial M_j} + \dfrac{\partial U_i}{\partial D_i}\dfrac{\partial^2 D_i}{\partial M_i \partial M_j} - r + \beta_i \dfrac{\partial U_i}{\partial D_i \partial C_i}\dfrac{\partial D_i}{\partial M_j}\right)}{\dfrac{\partial^2 U_i}{\partial D_i^2}\dfrac{\partial D_i}{\partial M_i}2 + \dfrac{\partial U_i}{\partial D_i}\dfrac{\partial^2 D_i}{\partial M_i^2} - (r+\beta_i)\dfrac{\partial^2 U_i}{\partial C_i \partial D_i}\dfrac{\partial D_i}{\partial M_i} - \beta_i\left[\dfrac{\partial^2 U_i}{\partial C_i \partial D_i}\dfrac{\partial D_i}{\partial M_i} - (r+\beta_i)\dfrac{\partial U_i}{\partial C_i}\right]} \tag{39}$$

Given the assumptions about $U_i(C_i, D_i)$ and $D_i(M_i, M_j)$ the ambiguities in the sign of dM_i/dM_j are caused by $\partial U_i/\partial D_i\, \partial^2 D_i/\partial M_i^2$ and $\partial U_i/\partial D_i\, \partial^2 D_i/\partial M_i \partial M_j$. Again, by assumption there exists an M_j^{**} such that the ambiguity disappears. Let $\hat{M}_i = \max\,[M_i^*, M_i^{**}]$.

Lemma 2

$M_i = g_i(M_j) < Y_i/\beta_i$ for all M_j.

Proof. The lemma follows from the assumption that the marginal utility of zero consumption is infinite. But if consumption is positive then $Z_i < Y_i$, implying $M_i < Y_i/\beta_i$.

Proposition 3

If the preordering of weapon space for the political authorities $D_i[M_i(0), M_j(0)]$ is a positive linear combination of $V_i(\cdot, \cdot)$, defined above as the solution to maximizing (13), treated as a function of the initial missile stocks, then the preordering will satisfy the following conditions

$$\frac{\partial D_i(M_i(0), M_j(0))}{\partial M_i(0)} \geq 0 \tag{40}$$

(41) $$\frac{\partial D_i[M_i(0), M_j(0)]}{\partial M_j(0)} \leq 0$$

(42) $$\lim_{M_i(0)\to\infty} \frac{\partial^2 D_i[M_i(0), M_j(0)]}{\partial^2 M_i(0)} = 0$$

(43) $$\lim_{M_i(0)\to\infty} \frac{\partial^2 D_i[M_i(0), M_j(0)]}{\partial M_i(0)\, \partial M_j(0)} = 0.$$

Proof. $V_i(\cdot, \cdot)$ is linear in $M_i(\hat{t})$, and $M_j(\hat{t})$, $\hat{t} = \max[t_{is}, t_{js}]$, so the proof involves determining the sign of $\partial M_i(t_{js})/\partial M_i(0)$, $\partial^2 M_i(t_{js})/\partial M_i(0)^2$, $\partial^2 M_i(t_{js})/\partial M_i(0)\, \partial M_j(0)$ for all $t \leq \min[t_{js}, t_{ic}]$.

Since, from (6) and Proposition 1

(44) $$M_i(t) = M_i(0) - k_i t - \int_L^t [1 - p_i(s)] f_j k_j \, ds$$

(45) $$\frac{\partial M_i(t)}{\partial M_i(0)} = 1 + \int_L^t f_j k_j \frac{\partial p_i(s)}{\partial M_i(0)} ds.$$

The assumption that the contingency strategies are optimal requires that $\partial M_i(t)/\partial M_i(0) > 0$. $p_i(s)$ is defined for all $t < \min[t_{js}, t_{ic}]$ from (10) and (12) as

(46) $$p_i(s) = \int_L^S \frac{k_i\, dS}{M_i(0) - k_j(S - L)} + \frac{k_i L}{M_i(0)}.$$

(47) $$\frac{\partial p_i(s)}{\partial M_i(0)} = \int_L^S \frac{-k_i\, dS}{[M_i(0) - k_j(S - L)]^2} - \frac{k_i L}{M_i(0)^2} < 0$$

(48) $$\frac{\partial^2 p_i(s)}{\partial M_i(0)^2} = \int_L^S \frac{2k_i(M_i(0) - k_j(S - L)}{[M_i(0) - k_j(S - L)]^3} dS + \frac{2k_i L}{M_i(0)^3} > 0$$

From (22) it follows that $\partial t_{js}/\partial M_i(0) \geq 0$ so $0 < \partial M_i(t_{js})/\partial M_i(0) < 1$, and, since $\partial^2 M_i(t)/\partial M_i(0)^2$ is strictly positive, this implies that

(49) $$\lim_{M_i(0)\to\infty} \frac{\partial^2 M_i(t)}{\partial M_i(0)^2} = 0.$$

To show that $\partial M_i(t_{js})/\partial M_j(0) \leq 0$, note from (22) that if $t_{js} < t_{jc}$ then $\partial t_{js}/\partial M_j(0) = 0$ and thus $\partial^2 M_i(t_{js})/\partial M_j^2(0) = 0$; if $t_{js} = t_{ic}$ then $\partial t_{js}/\partial M_j(0) \geq 0$ and $\partial M_i(t_{jc})/\partial M_j(0) \leq 0$.

Finally for all

$$M_j > \min[M_j | t_{js} \geq t_{jc}], \qquad \frac{\partial^2 M_i(0)}{\partial M_i(0)\, \partial M_j(0)} = 0.$$

Proposition 4

There exists an equilibrium level of armaments.

Proof

$$\text{Let } \Lambda = \left\{ M_1, M_2 | \hat{M}_1 \leq M_1 \leq \frac{Y_1}{\beta_1}, \hat{M}_2 \leq M_2 \leq \frac{Y_2}{\beta_2} \right\}$$

where $\hat{M}_1$ and $\hat{M}_2$ are chosen such that $g_i(M_j)$ exists and $g_i'(M_j) \geq 0$. Define

$$g(M_i, M_j) = \begin{bmatrix} g_i(M_j) \\ g_j(M_i) \end{bmatrix}. \tag{50}$$

Because of Lemmas 1 and 2 $g: \Lambda \rightarrow \Lambda$ and therefore using the Brouwer Theorem a fixed point $\overline{M}_1, \overline{M}_2$ exists.

Proposition 5

The equilibrium point is stable if each country attempts to behave in a myopically optimal manner.

Proof. Suppose $M_i > g_i(M_j)$, $q_i^*(M_i, M_j)$ a solution to the optimization problem implies that $q_i^*(M_i, M_j) < q_i^*[g_i(M_j), M_j]$. This in turn implies

$$C_i[p_i^*(M_i, M_j), M_i, M_j] > C_i\{p_i^*[g_i(M_j), M_j], g_i(M_j), M_j\}$$

and thus $F_i(M_i, M_j) > 0$ if $M_i < g_i(M_j)$. If we examine the resulting phase diagram for M_1 and M_2 we see that the equilibrium point is stable. See Figure 1.

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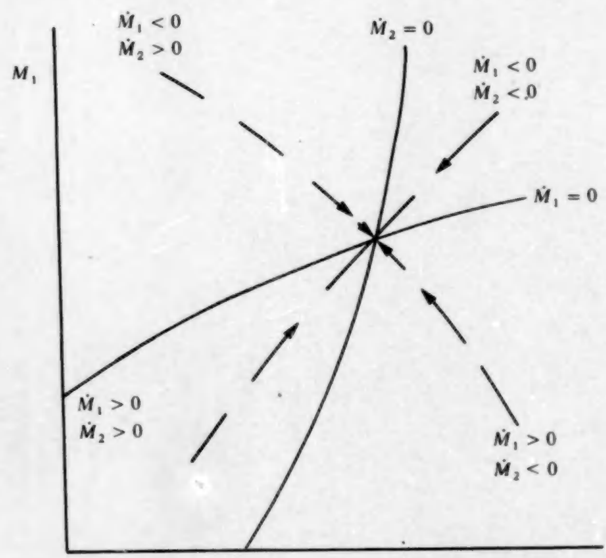

Figure 1

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