Demographic Change and the Equity Premium[*]

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Abstract

This papers presents an analysis of the financial market effects of demographic change. We first develop a stylized overlapping generations model to derive qualitative general equilibrium predictions on the effects of demographic change on the equity premium, the return differential between a risky and a risk-free investment. As our key insight, we show that the ex-ante equity premium increases when a smaller cohort enters the economy. We then develop a large scale overlapping generations model to provide a realistic quantitative assessment of the effects of demographic change on the equity premium for the U.S. economy. Our simulation model predicts that the expected rate of return to risky physical capital decreases by roughly 1.2 percentage points until 2030 and that the equity premium increases by about 0.28 percentage points.

JEL classification: E17, C68, G12
Keywords: overlapping generations; demographic change; asset pricing; equity premium

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1 Introduction

As in all major industrialized countries the U.S. population is aging reducing the fraction of the population in working age. This process is driven by falling mortality rates and declining birth rates, which substantially reduces population growth rates. Based on United Nations (2002), figure 1 compresses the stylized facts on demographic change by displaying the predicted time paths of two key demographic indicators for the U.S.. The solid line in the figure (left scale) is the predicted working age to population ratio – here defined as the number of the working age population of age 20-64 to the total adult population of age 20-110 – and the dashed line (right scale) is the corresponding time path of the old age dependency ratio – here defined as the number of the population of age 65 and older as a fraction of the working age population. According to this data and our definitions, while the working age to population ratio is projected to decrease by roughly 10 percentage points between 2005 and 2030, which we take as the base years of comparison throughout the paper, the old age dependency ratio increases by about 16 percentage points.[1] These projected developments will make raw labor a scarce factor relative to physical capital with ensuing decreases of the rate of return to capital.

Figure 1: Facts on Future Demographic Change
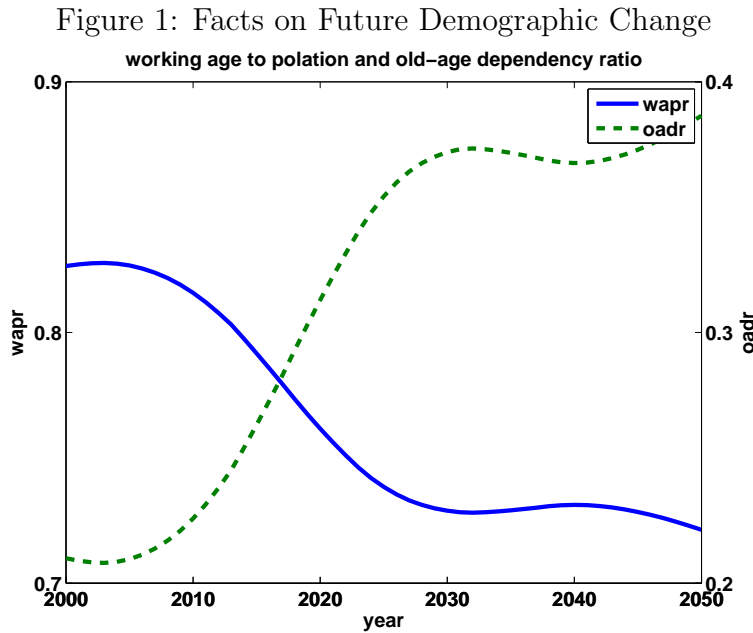
Notes: Blue solid line: working age to population ratio (WAPR). Green dashed line: old age dependency ratio (OADR).

Source: Own calculations based on United Nations (2002).

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[1]The choice of year 2030 as a base year of comparison is motivated by the insight that demographic developments somewhat flatten out after 2030, cf. figure 1, and because demographic projections are inherently more uncertain after a horizon of about 30 years.

What will be the financial market consequences of these demographic developments? No consensus has been reached in the academic literature on this prominent question posed by Abel (2001, 2003), Poterba (2001) and several others. Despite significant effects of demographic change on the rate of return to capital, it has recently been argued that the size of these effects seems too small such that the catchphrase "asset market meltdown" is not justified in the context of population aging, cf., e.g., Börsch-Supan, Ludwig, and Winter (2006) and Krüger and Ludwig (2007). Quite in contrast, there is little agreement on the qualitative as well as the quantitative effects of demographic change on the differential returns between risky and risk-free assets (Bakshi and Chen 1994; Brooks 2002; Börsch-Supan, Ludwig, and Sommer 2003; Geanakoplos, Magill, and Quinzii 2004). While Brooks (2002) reports substantial increases in the equity premium, the approximate calculations in Börsch-Supan, Ludwig, and Sommer (2003) rather suggest a small increase. Geanakoplos, Magill, and Quinzii (2004) conclude that "the equity premium is smaller when the population of savers is older" which the authors interpret as a contradiction to the findings of Bakshi and Chen (1994) and Brooks (2002).

Against this background, the contribution of the present paper is twofold. In a first step we develop a modified two-generations-overlapping Diamond (1965) economy. The two most important features we add to the Diamond model is risky production and a risk-free government bond in positive supply. These extensions provide us with an analytically tractable framework to consistently analyze the causal links between demographic change (aging) and the equity premium in general equilibrium. Such a consistent theoretical treatment is missing in the existing literature. As the central result of our theoretical analysis we show that the equity premium increases when a small cohort enters the labor market.

In a second step we extend our simplifying two generations model to a multi-generation OLG model in the tradition of Auerbach and Kotlikoff (1987). Any serious attempt to quantify the effects of demographic change on asset prices should be based on simulation models with a realistic periodicity of one to at most five years. Models that run at a lower frequency implicitly impose restrictions on household's ability to adjust their portfolio which may severely bias the predictions. The periodicity of our model is therefore annual and we calibrate the model to the projected trends of U.S. demography in the coming decades. Our framework thereby enables us to provide a realistic quantitative assessment of the effects of aging on the equity premium in general equilibrium. We show that the expected decrease of the risky rate of return to capital until 2030 is in the order of magnitude of about 1.2 percentage points. The decrease of the risk-free interest rate on government bonds is slightly higher such that the equity premium increases by about 0.28 percentage points.

Our quantitative analysis contributes to and borrows model elements from several strands of the literature. We extend the analysis by Brooks (2002) – who uses an OLG model with only 4 generations – to a more realistic annual periodic-

ity. While such a large-scale model is, in our opinion, key for providing a realistic quantitative assessment, it also potentially implies tremendous computational costs. To overcome these we adopt the risky human capital framework developed in Krebs (2003) and Krebs and Wilson (2004) in an overlapping generations setup. Consequently, there are three assets in the economy: risky human capital, risky physical capital and risk-free government bonds. This setup considerably simplifies the numerical solution of the model's household sector for given expectations on aggregate prices, also see Merton (1969) and Samuelson (1969). On the aggregate side, while the current version of the paper applies a "semi-deterministic" solution method, we will follow the literature (e.g., Gomes and Michaelides (2006) and Storesletten, Telmer, and Yaron (2007)) in future versions and compute an approximate rational expectations equilibrium in our model by applying a variant of the Krusell-Smith methodology (Krusell and Smith 1997; Krusell and Smith 1998) that we suitable modify to account for the fact that demographic change enters the model through a time-varying exogenous process.

The remainder of this paper is structured as follows. In section 2 we develop our stylized two generations OLG model in order to illustrate the key mechanisms at work in our quantitative model and to derive the qualitative conclusions on the relationship between demographic change and the equity premium. Section 3 extends the simplified setup of our two period model to a large scale overlapping generations model and section 4 discusses calibration and the numerical solution. Section 5 presents the simulation results of our quantitative model. Finally, section 6 concludes.

2 A Stylized OLG Economy

Our stylized two generations OLG model is a simplified version of the model developed in Kuhle (2007). We extend the Diamond (1965) model by introducing aggregate risk to the production function. As a consequence of aggregate risk, future output and interest rates are unknown to the representative agent. This adds an additional choice to the household problem: the individual has to decide how to allocate savings between risky capital and risk free government bonds.

2.1 Population

The working age population is assumed to grow at an exogenously given rate n_t. Hence we have

$$N_{t+1} = (1 + n_t)N_t. \tag{1}$$

Below, we focus on the following scenario: an exogenously given baby boom/bust, i.e. the growth rate n_t will be in-(de-)creased for one period.

2.2 Firms

The production technology is given by a continuous constant returns to scale aggregate production function,

$$Y = z_t F_t(K_t, N_t); \quad F_K > 0, \quad F_{KK} < 0, \quad F_N > 0, \quad F_{NN} < 0, \tag{2}$$

where the signs of the respective partial derivatives with respect to the inputs capital K and labor N ensure that $F_t(\cdot)$ is concave. Production is subject to an aggregate technology shock z_t. This technology shock is assumed to be log-normally distributed and hence output and factor prices are log-normal too.

Once the respective realization of the shock is known, each firm will rent capital and hire labor up to the point where the marginal products are equal to the market prices such that

$$r_t = z_t \frac{\partial F}{\partial K_t} \tag{3}$$

$$w_t = z_t \frac{\partial F}{\partial N_t} \tag{4}$$

2.3 Government Debt

In order to appreciate the fact that the government is the only entity supplying bonds that are (in absence of inflation risks) risk free we introduce a government that pursues a certain debt policy. Different debt policies of the government are perceivable and Kuhle (2007) discuss two such policies. Here, we focus the analysis on a policy that holds the debt to GDP ratio constant as suggested by the Maastricht criteria. The budget constraint of the government is given by

$$B_{t+1} + N_t \tau = (1 + r_t^f) B_t + G_t. \tag{5}$$

where B_t is the amount of outstanding and B_{t+1} the amount of newly issued debt in period t and $r_t^f = \frac{1}{q_{t-1}} - 1$ is the rate of interest on government debt which was issued at time $t - 1$ at price q_{t-1}. Government consumption is given by G_t.

In the following we will assume that the rate of interest earned on government debt is deterministic, i.e., at time t the government issues debt with a guaranteed rate of return r_{t+1}^f. Hence, in an economy that is inhabited by risk averse agents with concave utility, the rate of return on government debt will always be below the expected return on risky capital, i.e., the expected equity premium must be positive.

For our Maastricht policy of constant per GDP debt, where

$$\frac{B_{t+1}}{Y_t} = b \quad \forall t \tag{6}$$

we can now use (6) to solve (5) for per capita government consumption as

$$g_t = \tau + \left(y_t - \frac{(1 + r_t^f)}{(1 + n_{t-1})}y_{t-1}\right)b. \tag{7}$$

In the following we keep the tax rate τ constant such that equation (7) determines per capita government consumption g_t.

2.4 The Household

The representative household lives for two periods and supplies labor inelastically in the first period only. Towards the end of the first period the household faces a consumption/saving and a portfolio decision. As in Abel (1999) and Bohn (2001) preferences over current and future consumption, $c_{t,0}$ and $c_{t+1,1}$, respectively, are described by a simplified Epstein and Zin (1989) utility function:

$$u_t = \ln(c_{t,0}) + \beta\frac{1}{1-\theta}\ln\mathbb{E}_t[(c_{t+1,1})^{1-\theta}]; \quad 0 < \theta \neq 1, \quad 0 < \beta < 1. \tag{8}$$

Equation (8) indicates that we use a utility function where the elasticity of inter-temporal substitution is set to unity, which implies that the individual savings/consumption decision is independent of the interest rate. This assumption is reasonable as long as the influence of changes in the rate of interest on savings is not too large. The assumption is also necessary to keep the general equilibrium analysis tractable. The parameter θ is the coefficient of relative risk aversion with respect to second period consumption and, as Epstein and Zin (1989) discuss, allows to disentangle the distinct concepts of intertemporal substitution and the preferences with respect to temporal risks.

The present value budget constraint to the household problem can be written as:

$$W_t = w_t - \tau = a_{t+1} + c_{t,0}; \quad a_{t+1} = a_{t+1}^b + a_{t+1}^s, \tag{9}$$

$$c_{t+1,1} = \left((1 + r_{t+1}^f)a_{t+1}^b + (1 + r_{t+1})a_{t+1}^s\right), \tag{10}$$

where W_t is the total wealth of a young agent, a_{t+1} are the asset holdings (savings) after first period consumption is realized whereby a_{t+1}^b and a_{t+1}^s are the respective amounts invested into the risk free and risky asset. Denoting the portfolio share of risky assets by $\omega_{t+1} = \frac{a_{t+1}^s}{a_{t+1}}$ and the share of risk-free assets by $1 - \omega_{t+1} = \frac{a_{t+1}^b}{a_{t+1}}$ yields, according to (8), the following household problem:

$$\max_{c_t,\omega_{t+1},a_{t+1}} u_t = \ln(W_t - a_{t+1}) +$$

$$\beta\ln(a_{t+1}) + \frac{\beta}{1-\theta}\ln\mathbb{E}_t\left[\left(1 + r_{t+1}^f + \omega_{t+1}\left(r_{t+1} - r_{t+1}^f\right)\right)^{1-\theta}\right].$$

Optimal savings are then given by

$$a_{t+1} = \frac{\beta}{1+\beta} W_t, \tag{11}$$

where the propensity to save out of wealth is $\frac{\beta}{1+\beta}$. The implicit condition for ω_{t+1}, the optimal portfolio share in the risky asset, is the first-order condition

$$\mathbb{E}_t \left(\left[1 + r_{t+1}^f + \omega_{t+1} \left(r_{t+1} - r_{t+1}^f \right) \right]^{-\theta} \left(r_{t+1} - r_{t+1}^f \right) \right) = 0. \tag{12}$$

To abbreviate subsequent expressions we define R as a shorthand for principal and interest earned on one dollar invested in the portfolio such that

$$R_{t+1} \equiv 1 + r_{t+1}^f + \omega_t (r_{t+1} + r_{t+1}^f). \tag{13}$$

Condition (12) allows to derive the following proposition.

Proposition 1. *The signs of the partial derivatives of the portfolio share with respect to the risk-free (risky) rate of return and the coefficient of relative risk aversion, respectively, are as follows:*

$$\frac{d\omega}{dr^f} < 0; \quad \frac{d\omega}{dr} > 0; \quad \theta \leq 1 \tag{14}$$

$$\frac{d\omega}{d\theta} < 0 \quad \forall \theta. \tag{15}$$

Proof. See Kuhle (2007). ◻

As proposition 1 indicates we cannot determine the sign of ω_r and ω_{rf} unambiguously for $\theta > 1$. In the following we will assume that an increase of the expected risky (risk-free) return will, ceteris paribus, increase (decrease) the share ω invested in the risky asset, irrespective of the value of θ.

2.5 Equilibrium

After having completed the partial analysis of the firm, the government and the household we can now turn towards the conditions for the bond, equity and asset markets. Accordingly, the capital market equilibrium condition reads as

$$N_{t+1} k_{t+1} = N_t \omega_{t+1} \frac{\beta}{1+\beta} W_t. \tag{16}$$

The bond market equilibrium condition with a Maastricht policy reads as

$$N_t y_t b^g = N_t (1 - \omega_{t+1}) \frac{\beta}{1+\beta} W_t. \tag{17}$$

Finally, the asset market equilibrium condition is given by

$$N_{t+1}(k_{t+1} + b) = N_t \frac{\beta}{1+\beta} W_t. \tag{18}$$

It is important to note that the equilibrium conditions (16) and (17), for the capital and bond market form a system of two linearly independent equations that imply the equilibrium path of the two interest rates, r and r^f. Adding (16) and (17) yields the linearly dependent asset market equilibrium condition (18). The resulting ex-ante equity premium is given by

$$\mu_{t+1} = r_{t+1} - r_{t+1}^f. \tag{19}$$

2.6 The Impact of a Baby-Boom on the Equity-Premium

In this subsection we discuss the short run implications of a baby boom/bust i.e. a high/low realization of n_t in the instance of a constant debt to output ratio (Maastricht policy). Total differentiation of equations (16) and (17) with respect to dn_t, dk_{t+1}, dr_{t+1}^f and dr_{t+1} yields, after using (3) and rearranging:

$$\frac{dr_{t+1}}{dn_t} = -\frac{k_{t+1}}{(1+n_t)} f''(k_{t+1}) > 0, \tag{20}$$

and

$$\frac{d\omega_t}{dn_t} = \omega_{rf} \frac{dr_{t+1}^f}{dn_t} + \omega_r \frac{dr_{t+1}}{dn_t} = 0. \tag{21}$$

Interpretation of these two equations is straightforward: since a change in the growth rate of population does not change government taxes, the present value of lifetime income W_t out of which individuals save a constant fraction remains unchanged. Thus an increase in the relative size of the next cohort lowers the capital intensity and increases the expected future return on risky investments, cf. equation (20). Equation (21) follows from the bond market equilibrium condition and indicates that, for $\frac{dr_{t+1}}{dn_t} > 0$, the government has to offer a higher risk-free rate to sell a given amount of debt. With respect to the ex-ante equity premium we can now use the individual portfolio choice behavior described in Kuhle (2007) to show that

$$\frac{d\mu_{t+1}}{dn_t} < 0. \tag{22}$$

We can therefore state the following proposition:

Proposition 2. *A baby bust (boom) will increase (decrease) the equity premium.*

Proof. See Kuhle (2007). □

With respect to our stylized economy we have now come full circle. We have sketched a tractable model, which allows to study the relation between the age distribution in the economy and the equity premium. Contrary to the previous literature, i.e. Brooks (2004) and Geanakoplos, Magill, and Quinzii (2004), who discuss economies where government bonds are in zero net supply, we have appreciated the fact that the government is the only entity that can supply safe debt.

In such a framework, we find that the entrance of a small cohort into the labor market leads to an increase in the capital intensity, and thus to a lower risky rate. At the same time the government issues a given amount of debt. To allow for market clearing, the share invested in the risk-free asset has to remain constant. This implies that the risk-free rate must decrease alongside with the risky rate. To this point the change in the equity premium is ambiguous.

The resulting change in the equity premium is then basically independent of the age distribution, since it purely follows from the portfolio adjustment of the current working population. This adjustment process indicates that the equity premium has to increase since the risky rate has to fall by less than the risk-free rate to keep the portfolio shares constant.

3   Quantitative Model

Our quantitative model is based on Ludwig (2007) and extends the simple model from the previous section to a multi-period setup as in Auerbach and Kotlikoff (1987) and also adds additional idiosyncratic risks. On the household side, the novelty in this paper is to assume that human capital of households is a choice variable rather than being exogenously given. We implement this feature by adopting the risky human capital framework developed in Krebs (2003) and Krebs and Wilson (2004) in an overlapping generations setup. In each period, a household of a given age chooses to invest a fraction of her overall wealth in human capital, respectively financial assets. As for the fraction of wealth invested in financial assets, the household solves a standard portfolio allocation problem as in our simple model by choosing how much to invest into risky physical capital and risk-free government bonds.[2] Consequently, there are three assets in the economy: risky human capital, risky physical capital and risk-free government bonds. In this setup, once portfolio allocation decisions are made and for given expectations on aggregate prices, household consumption and savings policies are linear functions of total household wealth, cf. Merton (1969) and Samuelson (1969). This feature of our model is particularly useful because it enables us to solve a large-scale OLG model with rather complex economic and population dynamics

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[2]In contrast to the simple model, our multi-period setup in this section implies that the bond is risk-free only for one period, which, in our context, corresponds to one calendar year.

without incurring tremendous computational costs. On the firm side, our model is standard.

3.1 Risk and Time

Time is discrete and runs from $t = 0, \ldots, \infty$ whereby one period corresponds to one calendar year. Aggregate risk is represented by an event tree. The economy starts with some fixed event s_0, and each node of the tree is a history of exogenous shocks $s^t = (s_0, s_1, \ldots, s_t)$. The shocks are assumed to follow a Markov chain with finite support S and strictly positive transition matrix π. Let $\pi(s^t \mid s_0)$ denote the probability that the node s^t occurs. For notational convenience, unless needed, we will suppress the dependency of variables on s^t but it is understood that all choice variables are history dependent. We allow for two aggregate shocks, a productivity shock as in the simple model and an additional shock to the depreciation rate of physical capital, see subsection 3.3. In addition, households are subject to idiosyncratic depreciations shocks of their human capital depreciation, see subsection 3.6.

3.2 Demographics

The economy is populated with $J+1$ overlapping generations and the underlying population dynamics are the exogenous driving force our model. Households enter the model at the age of 20 $(j = 0)$ and live at most until 110 $(j = J = 90)$. Population of age j in time period t is given recursively as

$$N_{t,j} = \begin{cases} N_{t-1,j-1}\varsigma_{t-1,j-1} & \text{for } j = 1, \ldots, J \\ \sum_{j=0}^{j_f} f_{t-1,j-1} N_{t-1,j-1} & \text{for } j = 0 \end{cases}$$

where $\varsigma_{t,j}$ denotes time and age-specific survival rates and $f_{t,j}$ are age-specific fertility rates whereby j_f is the age of menopause. Defining the time specific Leslie matrices Π_t^l we can compress the population dynamics as

$$N_{t,j} = \Pi_t N_{t-1,j-1}, \quad \Pi_0 \text{ given}, \quad \text{whereby} \tag{23}$$

$$\Pi_t^l = \begin{bmatrix} f_{t,0} & \ldots & f_{t,j_f} & 0 & \ldots & 0 \\ \varsigma_{t,0} & 0 & \ldots & \ldots & \ldots & \ldots \\ 0 & \varsigma_{t,1} & \ldots & \ldots & \ldots & 0 \\ \ldots & & & & & \\ 0 & 0 & \ldots & \ldots & \ldots & \varsigma_{t,J-1} \end{bmatrix}$$

Processes governing mortality and fertility are assumed to be non-stochastic.

3.3 Production

In contrast to our simple model of section 2 we here specify an explicit technology by assuming that firms employ a standard Cobb-Douglas production function. As

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the number of firms is indeterminate, we assume one representative firm in the economy that produces total output at time t, Y_t, by

$$Y_t = z_t K_t^\alpha H_t^{1-\alpha}. \tag{24}$$

K_t denotes the aggregate stock of physical capital, H_t is the aggregate stock of human capital and z is a productivity shock whereby $z_t = z(s_t)$.

Profit maximization gives

$$r_t + \delta_t = z_t \alpha k_t^{\alpha-1} \tag{25}$$

and

$$r_t^h = z(s_t)(1-\alpha)k_t^\alpha \tag{26}$$

where $k_t = \frac{K_t}{H_t}$ is capital intensity, r_t is the rate of return to physical capital and r_t^h is the rate of return to aggregate human capital. Following Krüger and Kübler (2006), Storesletten, Telmer, and Yaron (2007), Gomes and Michaelides (2006) and others we here assume that the depreciation rate of physical capital, $\delta_t = \delta(s_t)$ is stochastic.

3.4 Government

Government policy is as in our simple model of section 2 whereby we replace the lump-sum taxes τ by taxes on human capital income at the time constant rate τ^h. Furthermore, we assume that the government taxes bequested wealth of households at a confiscatory rate of 100% and denote this income from bequest taxation by T_t. Accordingly, the government budget constraint is given by

$$B_{t+1} + \tau^h r_t^h H_t + T_t = (1 + r_t^f)B_t + G_t. \tag{27}$$

3.5 Preferences

The life-time utility function of a household born in period t is given by

$$\mathbb{E}_0 \sum_{j=0}^{J} \beta^j \varphi_{t,j} u(c_{t+j,j}) \tag{28}$$

where $\mathbb{E}$ is the expectations operator, β is the raw time discount factor and $c_{t,j}$ is consumption at time t, age j. $\varphi_{t,j}$ is the probability of an agent born in period t to survive until age j, hence

$$\varphi_{t,j} = \prod_{i=0}^{j-1} \varsigma_{t+i,i}$$

where $\varsigma_{t,i}$ are the age-specific probabilities to survive in period t from age i to $i+1$.

In contrast to the simple model of section 2, we work with standard CRRA preferences. The per period utility function is accordingly given by

$$u(c_{t,j}) = \begin{cases} \frac{1}{1-\theta}c_{t,j}^{1-\theta} & \text{if } \theta \neq 1 \\ \ln(c_{t,j}) & \text{if } \theta = 1 \end{cases} \tag{29}$$

where θ is the coefficient of relative risk aversion.

3.6   Endowments

When entering the economy at age $j = 0$, households are endowed with an initial level of human capital, $h_{t,0} = h_0$ for all $t = 0, 1, \ldots$. Each period, households choose to invest a fraction of their total wealth - which, as we shall demonstrate below, is the sum of financial assets and human capital - in financial assets, respectively in human capital. Let $i_{t,j}^h$ denote the investment in human capital.

As in Huggett, Ventrua, and Yaron (2007), human capital earns a gross rate of return of $r_t^h l_j$ whereby the two components are the marginal product of human capital, r_t^h, cf. equation (26) and a labor supply component l_j that varies across age. The labor supply component is calibrated to match the actual hump-shaped average age specific labor supply patterns in the data and is therefore constant across time. Details on the calibration of $\{l_j\}$ are provided below. As the return from the labor component, l_j, is hump-shaped and goes to zero for ages above 80 $(j = 60)^3$, the household at some age chooses to drive down human capital investments such that next periods human capital stock hits the lower bound of zero, that is, the household chooses to retire. Consequently, retirement in our model is endogenous at some age j_t^r.

In addition, returns to human capital are subject to human capital taxes, τ^h, such that the net return on human capital is given by $r_{t+1}^h l_{j+1}(1-\tau^h)$. Notice that τ^h just replaces the lump-sum taxes we used in our simple model in section 2. Furthermore, human capital adjustments are assumed to be costly. Adjustment costs enable us to calibrate the average return on human capital relative to the return on physical capital, see below. In order to preserve analytical tractability, adjustment costs are linear and governed by the adjustment cost parameter γ^h.

Before the investment decision is made, the household is hit by an idiosyncratic shock to the depreciation rate of human capital, δ^h. The shock is additive and denoted by $\eta = \eta(s_t)$. Although the shock is idiosyncratic, it depends on the current state of the economy, s_t, because, as further discussed below, the variance of idiosyncratic human capital shocks depend on the current state of the economy. Collecting all these elements, the human capital accumulation equation

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[3]This is the maximum age with data on age-specific labor supply.

in period t, age j, is given by

$$h_{t+1,j+1} = h_{t,j}(1 - \delta^h + \eta) + i_{t,j}^h, \qquad h \geq 0. \tag{30}$$

As for the investment in financial assets, the household chooses to invest in period t, age j, a fraction in risky physical capital and a fraction in government bonds. Let $\omega_{t,j}^s$ be the fraction of holdings of risky physical capital in period t, age j. Accordingly, let $\omega_{t,j}^b = 1 - \omega_{t,j}^s$ be bond holdings. Consequently, the dynamic asset accumulation equation in period t, age j, is given by

$$a_{t+1,j+1} = a_{t,j}(1 + r_t^f + \omega_{t,j}^s(r_t - r_t^f)) + r_t^h l_j(1 - \tau^h)h_{t,j} - (1 + \gamma^h)i_{t,j}^h - c_{t,j} \tag{31}$$

In the following we work on equations (30) and (31) in order to derive a recursive law of motion of total wealth of households. Total wealth will be defined below. Combining (30) and (31) we have

$$a_{t+1,j+1} + h_{t+1,j+1} = a_{t,j}(1 + r_t^f + \omega_{t,j}^s(r_t - r_t^f)) +$$
$$h_{t,j}(1 + r_t^h l_j(1 - \tau^h) - \delta^h + \eta) - \gamma^h i_{t,j}^h - c_{t,j}$$

Again using (30) in the above we get

$$a_{t+1,j+1} + h_{t+1,j+1} = a_{t,j}(1 + r_t^f + \omega_{t,j}^s(r_t - r_t^f)) + h_{t,j}(1 + r_t^h l_j(1 - \tau^h) - \delta^h + \eta) -$$
$$\gamma^h \left(h_{t+1,j+1} - h_{t,j}(1 - \delta^h + \eta) \right) - c_{t,j}$$

and therefore

$$a_{t+1,j+1} + h_{t+1,j+1} \left(1 + \gamma^h \right) = a_{t,j}(1 + r_t^f + \omega_{t,j}^s(r_t - r_t^f)) +$$
$$h_{t,j} \left(r_t^h l_j(1 - \tau^h) + \left(1 + \gamma^h \right) (1 - \delta^h + \eta) \right) - c_{t,j}$$

Next, let $\tilde{h}_{t,j} = h_{t,j} \left(1 + \gamma^h \right)$ and $\tilde{r}_{t,j}^h = \frac{r_t^h}{1+\gamma^h} l_j(1 - \tau^h) - \delta^h + \eta$, then

$$a_{t+1,j+1} + \tilde{h}_{t+1,j+1} = a_{t,j}(1 + r_t^f + \omega_{t,j}^s(r_t - r_t^f)) + \tilde{h}_{t,j}(1 + \tilde{r}_{t,j}^h) - c_{t,j}$$

Now define by $\hat{\omega}_{t,j}^s = \frac{\omega_{t,j}^s a_{t,j}}{w_{t,j}}$ and by $\hat{\omega}_{t,j}^b = \frac{(1-\omega_{t,j}^s)a_{t,j}}{w_{t,j}}$ the share of total "wealth", $w_{t,j} = a_{t,j} + \tilde{h}_{t,j}$, invested in physical capital and bonds, respectively, and let $\hat{\omega}_{t,j}^h = \frac{\tilde{h}_{t,j}}{w_{t,j}}$ be the share invested in human capital including the adjustment costs. Observe that $\hat{\omega}_{t,j}^b = 1 - \hat{\omega}_{t,j}^s - \hat{\omega}_{t,j}^h$. We then finally have a dynamic budget constraint in terms of total wealth which is given by

$$w_{t+1,j+1} = w_{t,j}(1 + r_t^f + \hat{\omega}_{t,j}^s(r_t - r_t^f) + \hat{\omega}_{t,j}^h(\tilde{r}_{t,j}^h - r_t^f)) - c_{t,j} \tag{32}$$
$$= w_{t,j}R_{t,j} - c_{t,j},$$

where $R_{t,j}$ is the return on the total portfolio in period t, age j.

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3.7 The household problem and equilibrium

We now define recursively the household problem for a given law of motion of the aggregate state of the economy. Rather than using $w_{t,j}$ as a state variable, it is convenient to solve the household problem in terms of total resources available. Let $x_{t,j} = w_{t,j}R_{t,j}$ be total resources, or, alternatively, "cash-on-hand" (Deaton 1991). Observe that

$$x_{t+1,j+1} = (x_{t,j} - c_{t,j})R_{t+1,j+1}. \tag{33}$$

Furthermore, it is convenient to express next period's values with symbol $'$, irrespective of whether next period values are only time dependent or age and time dependent. The states of the household problem are the exogenous states t, j and s, the endogenous cash-on-hand of the household, x, as well as the endogenous aggregate state of the economy, G, with associated law of motion $G' = \Phi(G, s, s')$. The household problem in period t, age j is then given by

$$V(x; s, t, j; G) = \max_{c, \hat{\omega}^{s'}, \hat{\omega}^{h'}, x'} \{u(c) + \beta\varsigma\mathbb{E}V(x'; s', t+1, j+1; G')\} \tag{34}$$

subject to

$$x' = (x - c)R'$$
$$R' = 1 + r^{f'} + \hat{\omega}^{s'}(r' - r^{f'}) + \hat{\omega}^{h'}(\tilde{r}^{h'} - r^{f'})$$
$$G' = \Phi(G, s, s').$$

The expectation $\mathbb{E}$ above is taken with respect to the realization of tomorrow's aggregate state s' conditional on state s today and thereby with respect to the technology shocks, z', the aggregate physical capital depreciation shock, δ', and the idiosyncratic depreciation shock η' which all are functions of tomorrow's state s'.

Using results derived in Samuelson (1969) we can now state the following property of the optimal consumption policy functions.

Proposition 3. *Denote by $\hat{\omega}^{s\star'}$ and $\hat{\omega}^{h\star'}$ the optimal portfolio decisions that are the solutions to*

$$\mathbb{E}\left[(R'm')^{-\theta}(r' - r^{f'})\right] = 0$$
$$\mathbb{E}\left[(R'm')^{-\theta}(\tilde{r}^{h'} - r^{f'})\right] = 0$$

where m' denotes the marginal propensity to consume out of cash-on-hand in the next period. Then the optimal consumption function is linear in cash-on-hand,

$$c = m \cdot x$$

whereby the marginal propensity m to consume out of cash-on-hand x is given by

$$m = \frac{(\varsigma \beta R^{\star\prime})^{-\frac{1}{\theta}}}{1 + (\varsigma \beta R^{\star\prime})^{-\frac{1}{\theta}}} \quad where$$

$$R^{\star\prime} = \mathbb{E}\left[m'^{-\theta}(1 + r^{f\prime} + \hat{\omega}^{s\star}(r' - r^{f\prime}) + \hat{\omega}^{h\star}(\tilde{r^h}' - r^{f\prime}))^{1-\theta} \right].$$

Proof. See Ludwig (2007). □

As in our simple model of section 2 the portfolio decisions do not depend on current cash-on-hand and the policy functions of consumption are linear. These features are due to the homotheticity of preferences and are particularly useful in the numerical solution of our simulation model.

Equilibrium in the economy is defined recursively and requires market clearing in all periods, while optimal decisions and aggregation conditions have to hold. Details on the formal definition of equilibrium are provided in Ludwig (2007).

4 Calibration and numerical solution

4.1 Calibration

Calibration of our model is in part by reference to other studies and in part by informal matching of moments procedures. Many of these choices are somewhat *ad hoc* and future versions of the paper will have a much more careful calibration. Table 1 summarizes some of our structural model parameters. Parameters governing stochastic processes are described in the text.

Table 1: Calibration parameters

Firm sector	
Capital share, α	0.33
Mean depreciation rate of capital, δ_0	0.05
Household sector	
Discount factor, β	0.96
Coefficient of relative risk aversion, θ	4.0
Initial human capital, h_0	1.0
Mean depreciation rate of human capital, δ^h	0.01
Adjustment costs to human capital, γ^h	2.0
Government sector	
Debt to GDP ratio, b	0.38
Taxes on human capital income, τ^h	0.28

Aggregate states and shocks. We assume that aggregate risk is driven by a four state Markov chain with support $S = \{s_1, \ldots, s_4\}$ and transition matrix $\pi = (\pi_{ij})$. Each aggregate state maps into a combination of low or high technology shocks and low or high physical capital depreciation. Precisely, we assume that

$$z_t = z(s_t) = \begin{cases} 1 + z^l \text{ for } s \in s_1, s_2 \\ 1 - z^h \text{ for } s \in s_3, s_4 \end{cases} \quad \text{and } \delta_t = \delta(s_t) = \begin{cases} \delta_0 + \psi \text{ for } s \in s_1, s_3 \\ \delta_0 - \psi \text{ for } s \in s_2, s_4. \end{cases} \tag{35}$$

The 4×4 transition matrix of the aggregate state is given by

$$\Pi = \begin{bmatrix} (0.879)_{2\times2} & (0.121)_{2\times2} \\ (0.121)_{2\times2} & (0.879)_{2\times2} \end{bmatrix}. \tag{36}$$

The transition probabilities are based on an estimation of deviations of Solow residuals from linear trends for the U.S. economy, cf. Silos (2004). The support of the technology shocks z is set to $\{0.9795, 1.0205\}$ such that $z^l = z^h = 0.0205$ which is based on the same estimation. The values of shocks to the aggregate depreciation rate of physical capital are based on Gomes and Michaelides (2006) and set to $\psi = 0.16$.

Population data. Our annual and age-specific demographic data for the population dynamics in (23) are based on the United Nations population projections United Nations (2002). More details on the construction of these data are provided in Krüger and Ludwig (2007).

Production sector. The value of the capital share parameter $\alpha = 0.33$ is based on an estimation of the aggregate production function for the U.S., cf. Krüger and Ludwig (2007) and lies in the usual range considered in the literature. The value of the mean depreciation rate of physical capital, $\delta_0 = 0.05$ is also standard and corresponds with empirical estimates.

Household sector. The value of household's raw time discount factor $\beta = 0.96$ is at the lower range of values considered in the literature and provides us with a hump-shaped consumption profile with a peak around the age of 70 which is a bit late compared to the data, cf. Fernández-Villaverde and Krüger (2006). Higher values of the discount factor would imply a consumption profile peaking even later in life and we therefore opted for this value. The coefficient of relative risk aversion θ is at the upper bound of the usual interval $[1, 4]$ considered in the literature. With this value, our model generates an equity premium of 3.5 percentage points in 2005. A value of $\theta = 2$ would have resulted in an equity premium of roughly 1.5 percentage points and we therefore chose the higher value. Due to our homothetic preferences, the initial level of human capital h_0 is irrelevant and we normalize human capital by setting $h_0 = 1$.

The mean depreciation rate of human capital δ^h is set to 0.01 which is in the range of values estimated by Ludwig, Schelkle, and Vogel (2007). Idiosyncratic depreciation shocks to human capital, η, are uncorrelated but the variance of η

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depends on the current state of the economy (Constantinides and Duffie 1996). We thereby follow the approach of Storesletten, Telmer, and Yaron (2007) and set the standard deviation $\sigma(\eta)$ to

$$\sigma(\eta) = \begin{cases} 0.2 & \text{for } s \in s_1, s_2 \\ 0.1 & \text{for } s \in s_3, s_4 \end{cases}. \tag{37}$$

which is within the range considered in Krebs and Wilson (2004).

The age-specific hours $\{l_j\}$ are taken as the averages of labor supply shares for the U.S. for the period $1960 - 2003$ based on the OECD and are additionally weighted by a correction factor for hours worked. As far as the adjustment cost parameter to human capital γ^h is concerned we have determined it such that the model generates a reasonable capital to output ratio in our base year 2005. This explains the choice of $\gamma^h = 2.0$. With these values, the year 2005 capital output ratio is 2.75 which gives an expected rate of return to risky physical capital of 0.07 and an average rate of return to human capital of $\bar{\bar{r}}^h = 0.036$.

Government sector. The aggregate supply of government bonds is set to 38% of GDP which is based on U.S. data, cf. Gomes and Michaelides (2006). The tax rate on human capital is set to $\tau^h = 0.28$ which is an estimate of effective average tax rates on labor income for the U.S. based on Ruggeri and Vincent (2000).

4.2   Solution method

Below, we report results from a simulation based on a "semi-deterministic" solution of our model. More precisely, we solve the model by setting all shocks to their expected values and also assume that agents have correct expectations about the dynamics of the relevant aggregate state variables such that they can correctly predict the expected capital intensity k_t and the bond price q_t. This approximate solution of our model can be solved by application of standard procedures for the solution of deterministic OLG models, cf. Ludwig (2006). Precisely, we loop on the capital intensity $\{k_t\}$ and the expected ex-ante equity premium, $\{\mu_t^e\} = \{\mathbb{E}_{t-1} r_t - r_t^f\}$, until convergence of the time paths of these variables. Results in future versions of this paper will be based on a stochastic simulation using a modified Krusell-Smith (Krusell and Smith 1997; Krusell and Smith 1998) method for solution. Details of this extension are discussed in Ludwig (2007).

Although the current solution procedure is approximate, we are confident that a more elaborate model will not change our predictions on the time paths of the average expected risky interest rate, $\{r_t\}$, the average risk-free interest rate $\{r_t^f\}$, respectively the average expected ex-ante equity premium, $\{\mu_t^e\}$. Whether this conjecture is correct depends on the linearity of decision rules and the importance of Jensen's inequality for our predictions. Since the savings decisions are linear functions of current state variables, non-linearities enter into our model only by

the portfolio allocation decisions. When computing averages of portfolio decision rules, our approximation errors are however relatively small.

5 Results

As a starting point, we first look in figure 2 at the projected time paths for the physical capital to output ratio – $\frac{K}{Y}$, the solid line – and the human capital to output ratio – $\frac{H}{Y}$, the dashed line. Notice that the time paths of these figures inherit the properties of the exogenous demographic variation already shown in figure 1. That is, as the working age to population ratio decreases, the human capital to output ratio decreases and the low frequency fluctuations of the population data map into these macroeconomic aggregates. The physical capital to output ratio is predicted to increase from an initial value of about 2.75 in 2005 to 3.05 in 2030, an increase of about 10 percent. At the same time, the ratio of human capital to output decreases by roughly 5 percent.

Figure 2: Ratios of Physical Capital and Human Capital to Output
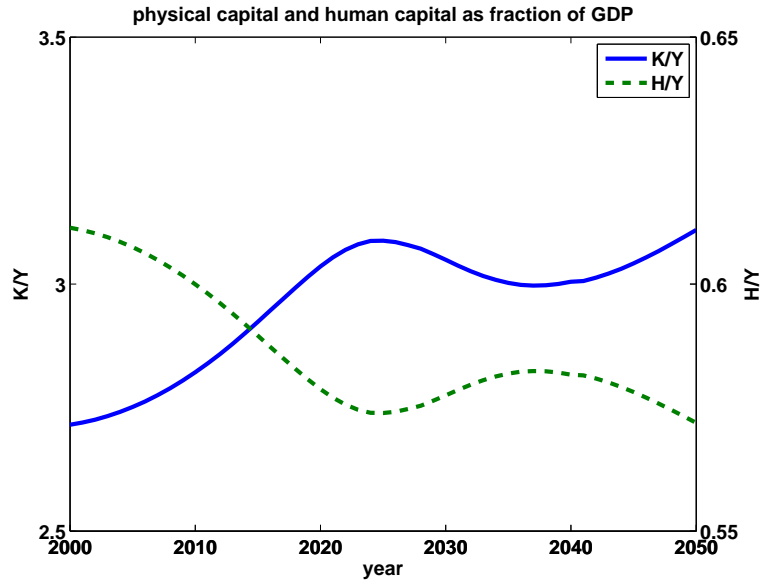
Notes: Blue solid line: physical capital to output ratio ($\frac{K}{Y}$). Green dashed line: human capital to output ratio ($\frac{H}{Y}$).
Source: Own calculations based on United Nations (2002).

In correspondence with the relative abundance of physical capital and scarcity of human capital in the economy, the rates of return to the two risky production factors are projected to decrease, respectively to increase. Of key importance for our analysis is the projected decrease of the rate of return to physical capital, the risky asset held by households. According to figure 3 it is projected to decrease by a bit more than 1 percentage point until 2030 which is in the range of results

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reported in many other studies, cf. Börsch-Supan, Ludwig, and Winter (2006), Krüger and Ludwig (2007) and the literature cited therein.

Figure 3: Rates of Return to Physical and Human Capital

Notes: Blue solid line: rate of return to physical capital (r). Green dashed line: rate of return to human capital (r^h).
Source: Own calculations based on United Nations (2002).

Based on the intuition developed in our simple model of section 2 we can expect that the rate of return to risk-free assets is also going to decrease with the aging of population. Furthermore, although the structure of the quantitative model differs in many respects from our simple model, we can expect that the return on government bonds, the risk-free interest rate, decreases by more than the return on risky physical capital. That these conjectures are right is also supported by the life-cycle profiles of holdings of risky assets (physical capital) and risk-free government bonds displayed in figure 4 for cohorts born in year 2005. As the graphs in the figure show, our model predicts positive bond demand of households for ages of 56 and older. Since the mass of these older agents is increasing in an aging society and because overall bond supply is determined by a government policy that is neutral with respect to demographic change, we can expect that the return to risk-free government bonds decreases. Furthermore, life-cycle bond holdings exceed life-cycle holdings of risky capital for ages of 61 and older. We can therefore also expect that the bond return decreases by more than the return on risky capital and that therefore the equity premium increases.

Figure 5 finally shows the projected time paths of the bond return – r_t^f, solid line – and the expected ex-ante equity premium – μ_t^e, dashed line. As is readily observed, the bond return decreases by slightly more than the rate of return to risky capital and the equity premium indeed increases. The effect is, however,

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Figure 4: Life Cycle Holdings of Risky Assets and Risk-Free Bonds
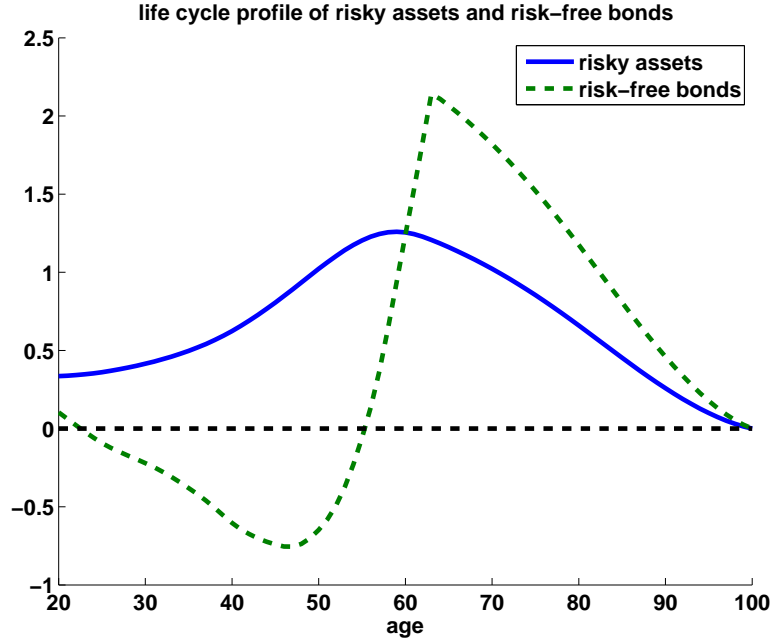
Notes: Life-cycle holdings of risky assets and risk-free governments in year 2005. Blue solid line: risky assets. Green dashed line: risk-free government bonds.
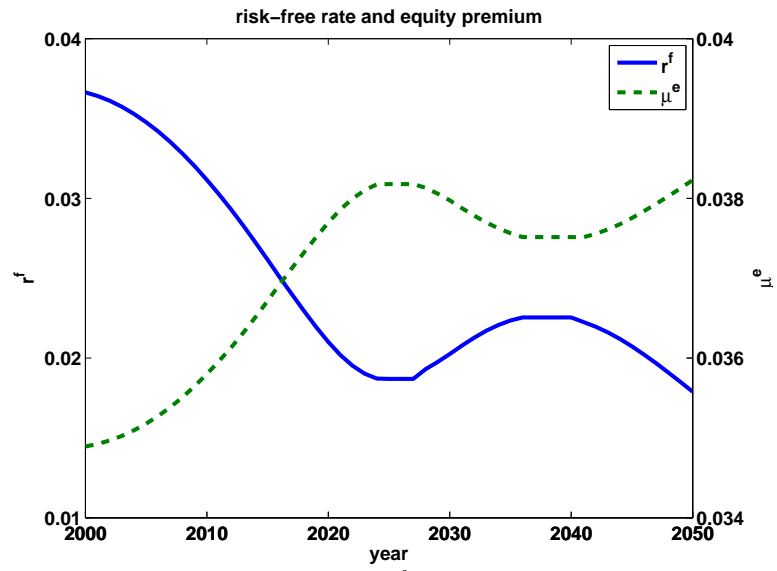*Source:* Own calculations based on United Nations (2002).

not very large: from 2005 to 2030, our model predicts an increase of the equity premium by roughly 0.28 percentage points. Notice that this translates into an overall decrease of the risk-free rate of return by 1.5 percentage points.

6   Conclusion

As the population in all major industrialized countries the U.S. population is aging, bringing with it a potentially large impact on the returns to the risky production factors physical and human capital and risk-free government bonds. Against this background, this paper, first, develops a stylized theoretical two-generations model to illustrate the qualitative effects of demographic change on asset prices with a particular emphasis on the equity premium. We show that the equity premium increases when a small cohort enters the economy. Second, we develop a large-scale simulation model to provide a realistic quantitative answer on the order of magnitude by which rates of returns to different asset categories are affected by demographic change. We show that the expected rate of return to risky physical capital decreases until 2030 by roughly 1.2 percentage points and that the expected equity premium is going to increase by about 0.28 percentage points.

Figure 5: Bond return and Equity Premium

risk–free rate and equity premium

Notes: Blue solid line: risk-free rate of return (r^f). Green dashed line: ex-ante equity premium (μ^e).

Source: Own calculations based on United Nations (2002).

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