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ABSTRACT

This paper extends a conventional cash-in-advance model to incorporate a real balance effect of the kind described by de Scitovszky, Haberler, Pigou, and Patinkin. When operative, this real balance effect eliminates the liquidity trap, allowing the central bank to control the price level even when the nominal interest rate hits its lower bound of zero. Curiously, the same mechanism that gives rise to the real balance effect also implies that monetary policies have distributional consequences that make some agents much worse off under a zero nominal interest rate than they are when the nominal interest rate is positive.

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1. Old Ideas, New Models

Inflation has come full circle. Low before 1960, it rose during the 1960s and peaked during the 1970s. From this peak, it fell during the 1980s, finally stabilizing during the 1990s at levels similar to those prevailing before 1960. The same circular pattern appears in data from virtually all of the major industrialized countries—in North America, in Europe, and in Asia—as shown, for example, by Mussa (2000, Table 1, p.1103).

Monetary economists and central bankers have also come full circle. Concerned mainly with halting and reversing inflation's upward trend during the 1970s and 1980s, analysts and policymakers have more recently rediscovered some of the special problems that can arise under conditions of price stability. These problems received much attention long ago but were ignored for more than a generation. Now, they have taken center stage once again.

Chief among these problems are those associated with the liquidity trap, which according to Hicks (1937) lies at the core of Keynes' (1936) economics. Krugman (1998) and Svensson (1999) reconsider the idea of the liquidity trap using state-of-the-art monetary models in which optimizing agents have rational expectations. In both Krugman's cash-in-advance model and Svensson's money-in-the-utility

function model, households view money and bonds as perfect substitutes when the nominal interest rate reaches its lower bound of zero. Households then become willing to hoard any additional money that the government chooses to supply. The central bank loses control of the price level and perhaps other key variables as well.

Notably absent from these new models of the liquidity trap, however, is another old idea: that of the real balance effect. First discussed by de Scitovszky (1941), Haberler (1946), and Pigou (1943) and developed most extensively by Patinkin (1965), the real balance effect describes a channel through which a change in real money balances, caused either by a change in the nominal money supply or a change in the nominal price level, impacts on household wealth and thereby affects consumption and output. The real balance effect allows the central bank to influence the economy even after the nominal interest rate hits its lower bound. Yet this effect appears nowhere in Krugman and Svensson's analyses. Why?

It has been widely appreciated, since the publication of Barro's (1974) famous paper on Ricardian equivalence, that government bonds will not be perceived as a source of private-sector wealth if the households owning those bonds are the same households that must pay all of the taxes that will eventually be used to retire the government's debt. Less widely appreciated, however, is a closely related finding, presented most explicitly by Weil (1991) but also implicit in earlier work by Sachs

(1983) and Cohen (1985). These authors show that government-issued fiat money will not be perceived as a source of private-sector wealth if the households owning that money are the same households that, first, receive all of the transfers or pay all of the taxes associated with changing the money stock over time and that, second, incur all of the opportunity costs associated with carrying the money stock between all future periods. In fact, the representative-agent models of Krugman and Svensson describe environments in which money is not net wealth. In these models, therefore, the real balance effect is inoperative.

This paper extends Krugman's cash-in-advance framework by introducing growth in the number of infinitely-lived households as modeled by Weil. The paper shows that with a growing population, households alive in the present pay only a fraction of the taxes levied in the future when the government wants to contract the money supply. Money becomes net wealth, and an operative real balance effect gives the central bank control over the price level even when the nominal interest rate is zero. Only in the special case without population growth—the special case in which the more general model developed here collapses to Krugman's original specification—does the liquidity trap survive.

Introducing population growth in the manner suggested by Weil also serves to resolve a second puzzle that emerges out of Krugman and Svensson's earlier

analyses. By associating the case of zero nominal interest rates with the Keynesian liquidity trap, Krugman and Svensson conjure up images of terrible economic outcomes: the Great Depression in the United States or the ongoing lengthy and severe recession in Japan. As emphasized by Cole and Kocherlakota (1998), however, zero nominal interest rates in models such as Krugman and Svensson's are actually associated with highly desirable resource allocations. In fact, zero nominal interest rates in these models are more closely linked to Friedman's (1969) rule for the "Optimum Quantity of Money" than to what Hicks (1937, p.155) calls the "Economics of Depression." But are zero nominal interest rates always good for the economy?

Once population growth is introduced into the cash-in-advance framework, monetary policies can have important distributional effects across households of different ages; these distributional effects, like the real balance effect itself, are absent in models with a single representative agent. This paper also shows that as a result of these distributional effects, some households are much worse off under zero nominal interest rates than they are under positive nominal interest rates. Curiously, therefore, the same mechanism that reintroduces the real balance effect and eliminates the liquidity trap also works here to make a zero nominal interest rate something to be avoided.

2. An Extended Cash-in-Advance Model

2.1. Overview

Here, Weil's (1991) continuous-time, money-in-the-utility function model with a growing number of infinitely-lived households is recast as a discrete-time, cash-in-advance model. Weil's original specification assigns to each household a utility function that is strictly increasing in two arguments: consumption and real money balances. Since households cannot be satiated by any finite stock of real balances, equilibria in Weil's original model exist only under strictly positive nominal interest rates, ruling out an analysis of the case that Krugman (1998) associates with the liquidity trap. Of course, one could also modify Weil's model in a manner consistent with Svensson (1999) by introducing a satiation point beyond which the marginal utility of real balances equals zero. The cash-in-advance framework used here, however, incorporates the satiation point for real balances in a way that is more naturally linked to the volume of each household's nominal expenditures.

Whitesell (1988) presents a model that is quite similar to Weil's and uses that model to study the effects of money growth on the capital stock and welfare. In fact, both Weil's model and Whitesell's can be viewed as extensions of Blanchard's (1985) model of finite horizons. In Blanchard's model, each agent faces a constant

probability of death; meanwhile, newly-born agents arrive at a rate that keeps the total population constant. Buiter (1988) generalizes Blanchard's model so as to break the tight link between birth and mortality rates. Buiter's analysis reveals that it is the arrival of newly-born agents, rather than the finite horizons of existing agents, that is essential in overturning Barro's (1974) Ricardian equivalence result—a result that, as noted above, relates closely to the presence or absence of monetary wealth effects. Thus, the model used here, like the models used by Weil and Whitesell, retains the essential feature of population growth in an environment where all agents are infinitely lived. This more general model nests, as the special case in which the population growth rate equals zero, the conventional specification that features a single infinitely-lived representative agent.

Weil's model, in which goods are received by each household in the form of a constant endowment, is also extended here by allowing each household to produce output with labor. Here, as in Wilson (1979), Cooley and Hansen (1989), Cole and Kocherlakota (1998), and Ireland (2000), positive nominal interest rates distort households' labor supply decisions. Thus, the structure of production and trade gives rise to a mechanism that might make the central bank want to follow the Friedman (1969) rule, which provides for zero nominal interest rates. And, indeed, the Friedman rule is optimal in the special case where the population growth rate

equals zero. When the population grows at a positive rate, however, the taxes that the government must levy to implement the Friedman rule generate distributional effects that make zero nominal interest rates quite costly for some agents.

2.2. Demographic Structure

A new cohort of infinitely-lived households is born at the beginning of each period $t = 0, 1, 2, ...$ Those households born in a particular period $t = s$ belong to cohort s. The arrival of new cohorts causes the total number of households to grow at the constant rate $n \geq 0$. Let N_t denote the number of households alive during period t. Then given $N_0 > 0$,

$$N_{t+1} = (1 + n)N_t$$

for all $t = 0, 1, 2, ...$

Households of a given cohort are identical, so that it is possible to consider a representative household for each cohort. The representative household of cohort s has preferences described by the utility function

$$\sum_{t=s}^{\infty} \beta^{t-s} \ln[c_t^s - (1/\gamma)(h_t^s)^{\gamma}], \tag{1}$$

where $1 > \beta > 0$, $\gamma > 1$, c_t^s denotes the household's consumption, and h_t^s denotes the household's hours worked during period t. This specification for utility, borrowed from Greenwood, Hercowitz, and Huffman (1988), implies that the marginal rate of substitution between consumption and hours worked depends only on hours worked; here, this special assumption facilitates the aggregation of quantities chosen by households of different cohorts.

Thus, during any given period, the economy consists of many infinitely-lived agents of varying ages. As suggested by Weil (1991) and Whitesell (1988), therefore, the population growth rate n serves as a measure of financial disconnectedness and heterogeneity in the economy as a whole. In the special case with $n = 0$, however, the model collapses to the more familiar one in which there is a single infinitely-lived representative agent.

2.3. Timing of Events

The representative household of cohort s enters each period $t = s, s+1, s+2, ...$ with money M_t^s and bonds B_t^s. Only the initial cohort is endowed with money at birth, and no cohort is endowed with bonds at birth, so that $M_0^0 > 0$ but $M_s^s = 0$ for all $s = 1, 2, 3, ...$ and $B_s^s = 0$ for all $s = 0, 1, 2, ...$ As emphasized by Weil (1991) and Whitesell (1988), these initial conditions formalize the idea that

newly-born households are not linked financially to older dynasties.

Each household that is alive during period t receives the same lump-sum monetary transfer T_t from the central bank at the beginning of the period. Also at the beginning of the period, existing bonds mature, providing the representative household of cohort s with B_t^s additional units of money. The household uses some of its money to purchase B_{t+1}^s new bonds at the price of $1/(1+r_t)$ units of money per bond, where r_t denotes the net nominal interest rate between t and $t+1$; the household carries the rest of its money into the goods market.

The description of goods production and trade builds on Lucas' (1980) interpretation of the cash-in-advance model. Each household consists of two members: a shopper and a worker. The shopper from the representative household of cohort s purchases c_t^s units of output from workers from other households, subject to the cash-in-advance constraint

$$M_t^s + T_t + B_t^s - B_{t+1}^s/(1+r_t) \geq P_t c_t^s, \tag{2}$$

where P_t denotes the nominal price of goods during period t. Meanwhile, the worker from the representative household of cohort s uses h_t^s units of labor to produce y_t^s units of output according to the constant-returns-to-scale technology

that yields one unit of output for every unit of labor input:

$$y_t^s = h_t^s.$$

The worker sells this output to shoppers from other households for $P_t h_t^s$ units of money. The representative household's two members then reunite to consume the shopper's purchases. The household carries M_{t+1}^s units of money into period $t+1$; its choices must satisfy the budget constraint

$$M_t^s + T_t + B_t^s + P_t h_t^s \geq P_t c_t^s + B_{t+1}^s/(1+r_t) + M_{t+1}^s. \tag{3}$$

In addition to the cash-in-advance and budget constraints (2) and (3), which must hold for all $t = s, s+1, s+2, ...$, the representative household's choices must satisfy a set of nonnegativity constraints:

$$h_t^s \geq 0,\ M_{t+1}^s \geq 0,\ c_t^s - (1/\gamma)(h_t^s)^\gamma > 0 \tag{4}$$

for all $t = s, s+1, s+2, ...$ The first two constraints in (4) are standard; the third must be imposed given the special form of the utility function (1).

The representative household of cohort s can borrow by choosing negative

values for B^s_{t+1} but is not allowed to engage in Ponzi schemes through which it borrows more than it can ever repay. To formalize the constraints that rule out such Ponzi schemes, let $Q_{\mathbf{0}} = 1$ and

$$Q_t = \prod_{u=\mathbf{0}}^{t-1} \left(\frac{1}{1+r_u} \right)$$

for all $t = 1, 2, 3, ...$ Then for any $T \geq t \geq 0$, Q_T/Q_t measures the present discounted value at the beginning of period t of one unit of money received at the beginning period T. The no-Ponzi-scheme constraints are

$$W^s_{t+1} = M^s_{t+1} + B^s_{t+1} + \sum_{u=t+1}^{\infty} (Q_u/Q_{t+1})(T_u + P_u h^s_u) \geq 0 \tag{5}$$

for all $t = s, s+1, s+2, ...$ Part 1 of the appendix shows that these no-Ponzi-scheme constraints imply that the infinite-horizon budget constraint

$$Q_t(M^s_t + B^s_t) + \sum_{u=t}^{\infty} Q_u(T_u + P_u h^s_u) \geq \sum_{u=t}^{\infty} Q_u \left[P_u c^s_u + \left(\frac{r_u}{1+r_u} \right) M^s_{u+1} \right] \tag{6}$$

applies to the household's choices from period t forward. This infinite-horizon budget constraint includes, as sources of funds, the household's beginning-of-period nominal balances M^s_t as well as the present discounted value of the monetary

transfers that the household receives from period t forward. It also includes, as uses of funds, the present discounted value of the opportunity costs that the household incurs when it carries money instead of bonds between all future periods. Ultimately, a comparison between the values of these three items will determine whether or not the real balance effect is operative in general equilibrium.

2.4. Household Optimization

Taking the initial conditions M_s^s and B_s^s as given, the representative household of cohort s chooses sequences $\{c_t^s, h_t^s, M_{t+1}^s, B_{t+1}^s\}_{t=s}^{\infty}$ to maximize the utility function (1) subject to the constraints (2)-(5), each of which must hold for all $t = s, s+1, s+2, ...$ Equivalently, (3) and (5) can be replaced by (6) in this statement of the household's problem.

Define the real variables

$$m_t^s = M_t^s/P_t,\ b_t^s = B_t^s/P_t,\ \tau_t = T_t/P_t,$$

and let π_t denote the net inflation rate between $t-1$ and t:

$$1 + \pi_t = P_t/P_{t-1}.$$

In addition, let

$$1 + x_t = (1 + r_t)/(1 + \pi_{t+1}) \tag{7}$$

define the net real interest rate x_t during period t, and let

$$a_t^s = m_t^s + b_t^s \tag{8}$$

summarize the representative household's real asset position at the beginning of period t. Part 2 of the appendix demonstrates that in terms of these newly-defined variables, the conditions

$$h_t^s = \left(\frac{1}{1 + r_t}\right)^{1/(\gamma-1)}, \tag{9}$$

$$(1 + \pi_{t+1})m_{t+1}^s \geq h_t^s,\ r_t \geq 0,\ r_t[(1 + \pi_{t+1})m_{t+1}^s - h_t^s] = 0, \tag{10}$$

$$\begin{aligned} c_t^s \;=\; & \frac{1}{\gamma}\left(\frac{1}{1 + r_t}\right)^{\gamma/(\gamma-1)} \\ & +(1 - \beta)\left\{a_t^s + \sum_{u=t}^{\infty}\left[\prod_{v=t}^{u-1}\left(\frac{1}{1 + x_v}\right)\right]\left[\tau_u + \left(\frac{\gamma - 1}{\gamma}\right)\left(\frac{1}{1 + r_u}\right)^{\gamma/(\gamma-1)}\right]\right\}, \end{aligned} \tag{11}$$

and

$$a_{t+1}^s = (1 + x_t)\left[a_t^s + \tau_t + \left(\frac{1}{1 + r_t}\right)^{\gamma/(\gamma-1)} - c_t^s\right] \tag{12}$$

for all $t = s, s+1, s+2, ...$ and

$$\lim_{t\to\infty}\left[\prod_{v=s}^{t}\left(\frac{1}{1+x_v}\right)\right]a_{t+1}^{s}=0 \tag{13}$$

are both necessary and sufficient for a solution to the household's problem.

Equation (9) confirms that positive nominal interest rates distort the household's labor supply decisions, as discussed by Wilson (1979), Cooley and Hansen (1989), Cole and Kocherlakota (1998), and Ireland (2000). Equation (10) restates the cash-in-advance constraint. It reveals that when the nominal interest rate hits its lower bound of zero, the household views money and bonds as perfect substitutes, so that the cash-in-advance constraint no longer binds; this is the case that Krugman (1998) associates with the liquidity trap.

Equation (11) defines the household's consumption function, which according to the permanent income hypothesis links consumption to total wealth. Embedded into the right-hand side of (11) are the same three components of monetary wealth identified in (6): the household's current money balances, the present discounted value of the future monetary transfers, and the present discounted value of the opportunity costs associated with carrying money instead of bonds between future periods. Once again, a comparison between the values of these three items will

determine whether or not the real balance effect appears in equilibrium.

Equation (12) governs the evolution of the household's financial wealth. It shows that the household accumulates wealth as it earns interest on its existing assets and as it receives monetary transfers from the government; the household also accumulates wealth by working more and consuming less. Finally, (13) is the household's transversality condition. If the limit on the left-hand side of (13) was negative, then the household would be violating the no-Ponzi-scheme constraints in (5); if, on the other hand, the limit was positive, then the household could achieve a preferred consumption profile, without violating any of its constraints, by drawing down its stock of financial assets.

2.5. Aggregation

Define aggregate per-household financial wealth during period t as

$$a_t = \frac{N_0 a_t^0 + \sum_{s=1}^{t}(N_t - N_{t-1})a_t^s}{N_t},$$

and define aggregate per-household real money balances m_t, real bond holdings b_t, hours worked h_t, and consumption c_t similarly. In terms of these aggregates,

(8)-(12) become

$$a_t = m_t + b_t, \tag{14}$$

$$h_t = \left(\frac{1}{1+r_t}\right)^{1/(\gamma-1)}, \tag{15}$$

$$(1+n)(1+\pi_{t+1})m_{t+1} \geq h_t,\ r_t \geq 0,\ r_t[(1+n)(1+\pi_{t+1})m_{t+1} - h_t] = 0, \tag{16}$$

$$\begin{aligned} c_t &= \frac{1}{\gamma}\left(\frac{1}{1+r_t}\right)^{\gamma/(\gamma-1)} \\ &\quad +(1-\beta)\left\{a_t + \sum_{u=t}^{\infty}\left[\prod_{v=t}^{u-1}\left(\frac{1}{1+x_v}\right)\right]\left[\tau_u + \left(\frac{\gamma-1}{\gamma}\right)\left(\frac{1}{1+r_u}\right)^{\gamma/(\gamma-1)}\right]\right\}, \end{aligned} \tag{17}$$

and

$$a_{t+1} = \left(\frac{1+x_t}{1+n}\right)\left[a_t + \tau_t + \left(\frac{1}{1+r_t}\right)^{\gamma/(\gamma-1)} - c_t\right] \tag{18}$$

for all $t = 0, 1, 2, ...$ While (14)-(17) are straightforward analogs to (8)-(11), a comparison of (18) to (12) reveals that aggregate per-capita financial wealth grows at a slower rate than each individual household's financial wealth, since newly-born households start their lives without money and bonds.

2.6. Steady States under Constant Money Growth

Equations (7) and (14)-(18) form a system of six equations in the nine aggregate variables x_t, r_t, π_{t+1}, a_t, m_t, b_t, h_t, c_t, and τ_t. This system can be closed by imposing the market-clearing condition for goods and labor,

$$h_t = c_t, \tag{19}$$

and by making assumptions about the government's supply of money and bonds.

Accordingly, suppose now that the government issues no bonds and expands the total money supply at the constant rate σ. Then, in equilibrium,

$$b_t = 0 \tag{20}$$

and

$$\tau_t = \sigma m_t \tag{21}$$

must also hold for all $t = 0, 1, 2, ...$ Note that (20) only requires that aggregate per-household bonds equal zero. Indeed, at the level of the individual household, b_t^s will typically be nonzero in equilibrium. Older and wealthier households will lend to—that is, buy bonds from—younger households; younger households will

then use the borrowed money to finance their purchases of consumption.

Under government policies described by (20) and (21), steady-state equilibria exist in which each of the nine aggregate variables is constant over time. In particular, part 3 of the appendix boils the nine equations (7) and (14)-(21) down to a smaller system of five equations that determine the steady-state values of $x_t = x$, $r_t = r$, $\pi_{t+1} = \pi$, $m_t = m$, and $c_t = c$:

$$1 + x = (1 + r)/(1 + \pi), \tag{22}$$

$$1 + \pi = (1 + \sigma)/(1 + n), \tag{23}$$

$$c = \left(\frac{1}{1+r}\right)^{1/(\gamma-1)}, \tag{24}$$

$$(1 + \sigma)m \geq c,\ r \geq 0,\ r[(1 + \sigma)m - c] = 0, \tag{25}$$

and

$$\begin{aligned} c \;=\; & \frac{1}{\gamma}\left(\frac{1}{1+r}\right)^{\gamma/(\gamma-1)} \\ & +(1-\beta)\left\{\left[1+\left(\frac{1+x}{x}\right)\sigma\right]m+\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{1+x}{x}\right)\left(\frac{1}{1+r}\right)^{\gamma/(\gamma-1)}\right\}. \end{aligned} \tag{26}$$

Given these five steady-state values, (14) and (19)-(21) determine $a_t = m$, $h_t = c$,

$b_t = 0$, and $\tau_t = \sigma m$.

Equation (22) defines the steady-state real interest rate as the difference between the nominal interest rate and the inflation rate; similarly, (23) determines the steady-state inflation rate as the difference between the money growth rate and the population growth rate. Equation (24) goes beyond (9) and (15) by showing that in equilibrium, higher nominal interest rates reduce consumption and output as well as employment. Equation (25), derived from the cash-in-advance constraint, describes the aggregate demand for money, while (26) is the aggregate consumption function with the steady-state conditions imposed.

3. The Liquidity Trap and the Real Balance Effect

What do the steady-state conditions (22)-(26) imply about the behavior of the economy and the efficacy of monetary policy under zero nominal interest rates? To answer this question, it is helpful to consider two cases. The first case is the one in which $n = 0$, so that there is no population growth. This first case is therefore the special case in which the more general model developed above reduces to the familiar specification, used by Krugman (1998) and many others, in which there is a single representative agent. And, indeed, Krugman's liquidity trap appears in this special case: the central bank loses control over the price level when the

nominal interest rate hits its lower bound of zero. In the second case with $n > 0$, however, a real balance effect emerges, enabling the central bank to control the price level even under a zero nominal interest rate.

3.1. The Liquidity Trap

When $r = 0$ and $n = 0$, so that both the nominal interest rate and the population growth rate equal zero, (22)-(24) and (26) imply that

$$1 + \sigma = 1 + \pi = \beta, \tag{27}$$

$$1 + x = 1/\beta, \tag{28}$$

and

$$c = 1. \tag{29}$$

In this steady state, the central bank follows the Friedman (1969) rule, contracting the money stock at the rate of time preference and generating a rate of deflation that is consistent with the zero nominal interest rate. As in Sidrauski's (1967) famous model, the steady-state real interest rate is pinned down by the rate of time preference; and as discussed below, consumption, output, and employment

are at their Pareto optimal levels.

But while (27)-(29) provide unique solutions for π, x, and c, the cash-in-advance constraint (25) requires only that

$$m \geq 1/\beta. \tag{30}$$

Since $r = 0$, the opportunity cost of holding money instead of bonds is zero. Households are therefore willing to hoard arbitrarily large stocks of real money balances. A continuum of steady-state equilibria exist, each corresponding to a value of m that satisfies (30).

Thus, in this case without population growth, the model exhibits what McCallum (1986, p.137) refers to as solution "multiplicity," as opposed to the less severe problem of price-level "indeterminacy." Multiple values of the real balance variable m satisfy (30). Hence, even if the central bank chooses an initial value M_0^0 for the level of the nominal money supply in addition to the constant money growth rate σ, there are still many distinct time paths for the price level that are consistent with all of the steady-state conditions.

One can, therefore, follow Krugman (1998) by associating this case with the Keynesian liquidity trap. Here, variations in the government's choice of M_0^0,

holding the money growth rate σ fixed, need not be associated with movements in the price level. With nominal interest rates frozen at their lower bound of zero, the central bank loses the ability to influence the behavior of prices.

3.2. The Real Balance Effect

When $r = 0$ but $n > 0$, the nominal interest rate continues to equal zero but the population grows at a positive rate. Equations (22)-(24) and (26) imply that

$$1 + \pi = (1 + \sigma)/(1 + n), \tag{31}$$

$$1 + x = (1 + n)/(1 + \sigma), \tag{32}$$

$$c = 1, \tag{33}$$

and

$$m = \frac{(\gamma - 1)[\beta(1 + n) - (1 + \sigma)]}{\gamma(1 - \beta)(1 + \sigma)n}, \tag{34}$$

while (25) requires that the money growth rate satisfy

$$\beta(1 + n) - n(1 - \beta)\left(\frac{\gamma}{\gamma - 1}\right) \geq 1 + \sigma. \tag{35}$$

There is, in addition to (35), a second condition that places restrictions on the money growth rate when $n > 0$: the condition $c_t^s - (1/\gamma)(h_t^s)^\gamma > 0$ from the set of nonnegativity constraints in (4). Part 4 of the appendix shows that in a steady state, this additional condition holds if and only if

$$1 + \sigma > \beta. \tag{36}$$

Intuitively, (35) requires the money growth rate to be low enough to be consistent with a zero nominal interest rate, while (36) guarantees that the lump-sum taxes required to implement a policy of zero nominal interest rates do not become so large that newly-born households cannot afford to pay them and still consume. So long as β is sufficiently close to one or, more precisely, so long as $\beta > \gamma/(2\gamma - 1)$, the upper bound in (35) exceeds the lower bound in (36), and there is a range of values for σ that satisfy both constraints.

Equation (32) reveals that in this case with population growth, the steady-state real interest rate is no longer tied to the rate of time preference; instead, a Tobin (1965) effect arises through which the real interest rate falls when the money growth rate rises. This Tobin effect also appears under positive nominal interest rates, as discussed by Weil (1991) and, more extensively, Whitesell (1988).

Equation (34), meanwhile, serves to uniquely determine the level of steady-state real balances. Thus, by selecting the initial value M_0^0 for the level of the nominal money supply as well as the money growth rate σ, the central bank can, through its choice of policy, determine a unique path for the nominal price level. This result—that when $n > 0$, m is uniquely determined, even when $r = 0$—cannot be found in Weil (1991) or Whitesell (1988), since their money-in-the-utility function specifications require the nominal interest rate to be positive. But why does Krugman's (1998) liquidity trap vanish when n becomes positive?

Sachs (1983), Cohen (1985), and Weil (1991) identify the three components of the private sector's monetary wealth that appear explicitly in the infinite-horizon budget constraint (6) and implicitly in the consumption functions (11), (17), and (26). First, there is the value of the current period's money supply. Second, there is the present discounted value of all future transfers or taxes that households will receive or pay as the government expands or contracts the money supply over time. Third, there is the present discounted value of the opportunity costs that households incur as they carry money instead of bonds between all future periods. When the nominal interest rate equals zero, only the first two of these three components remain, so that aggregate per-household real monetary wealth

during period t is measured by

$$\Omega_t = m_t + \sum_{u=t}^{\infty} \left[\prod_{v=t}^{u-1} \left(\frac{1}{1+x_v} \right) \right] \tau_u.$$

In a steady state with constant money growth, (21) implies that Ω_t is constant and equal to

$$\Omega = \left[1 + \left(\frac{1+x}{x} \right) \sigma \right] m. \tag{37}$$

In general, this measure of monetary wealth enters into the aggregate consumption function (26). In the special case with $n = 0$ and $r = 0$, however, (27), (28), and (37) imply that

$$\Omega = \left[1 + \left(\frac{1}{1-\beta} \right) (\beta - 1) \right] m = 0. \tag{38}$$

Without population growth, the households owning the current period's money stock are exactly the same households that pay all of the taxes required to implement a policy of zero nominal interest rates. Thus, as noted by Weil, an argument analogous to the one underlying Barro's (1974) Ricardian equivalence theorem implies that government-issued money, like government-issued bonds, will not be a source of private net wealth.

When $n > 0$, on the other hand, (32) and (37) imply that

$$\Omega = n\left(\frac{1+\sigma}{n-\sigma}\right) m > 0.$$

In this case, households alive during any period t pay only a fraction of the future taxes required to keep the nominal interest rate at zero; households born in later periods share the total tax burden. Hence, money is a component of private net wealth. Since real balances enter nontrivially into the aggregate consumption function (26), m is uniquely determined, even when the cash-in-advance constraint (25) does not bind. The central bank retains control over the price level, even when the nominal interest rate is zero.

de Scitovszky (1941), Haberler (1946), Pigou (1943), and Patinkin (1965) describe the real balance effect. According to these authors, real money balances form a component of private-sector wealth and therefore enter into the aggregate consumption function. As a result, a change in the level of real balances, brought about either by a change in the nominal money supply or a change in the nominal price level, gives rise to changes in consumption and output. Thus, the real balance effect allows the central bank to influence the economy even after the nominal interest rate reaches its lower bound. Here, the real balance effect oper-

ates in exactly this way, so long as the population grows at a positive rate. Only in the special case without population growth, where money is not net wealth, does the liquidity trap survive.

4. The Welfare Cost of Deflation

The results from above resolve one of the puzzles that emerges from Krugman (1998) and Svensson's (1999) recent analyses of the liquidity trap. These results show that a real balance effect of the kind described by de Scitovszky (1941), Haberler (1946), Pigou (1943), and Patinkin (1965) fails to appear in Krugman and Svensson's models because these models, which feature a single infinitely-lived representative agent, describe economic environments in which government-issued money is not a component of aggregate private-sector wealth. When population growth is introduced into one of these models, in the manner suggested by Weil (1991) and Whitesell (1988), money becomes net wealth. The real balance effect appears, and the central bank retains control over the price level even when the nominal interest rate equals zero. The real balance effect appears because monetary policies have distributional consequences: the households owning the current period's money supply pay only some of the taxes or receive only some of the transfers associated with future changes in the money supply.

The same distributional consequences help resolve a second puzzle emerging from Krugman and Svensson's analyses. By associating the case of zero nominal interest rates with the Keynesian liquidity trap, Krugman and Svensson conjure up images of economic depression. But in fact, Wilson (1979), Cole and Kocherlakota (1998), and Ireland (2000) derive results associating zero nominal interest rates with Pareto optimal resource allocations in representative-agent models such as Krugman and Svensson's. These optimality results can be rederived for the cash-in-advance model developed here in the special case without population growth.

When $n = 0$ in the model from above, there is a single representative household that lives from the beginning of period $t = 0$ forward. In equilibrium, this household's consumption and hours worked coincide with the per-household aggregates, so that according to (15) and (19),

$$c_t^0 = h_t^0 = \left(\frac{1}{1 + r_t}\right)^{1/(\gamma - 1)} \tag{39}$$

for all $t = 0, 1, 2, ...$ Now consider a social planner, who chooses $\{c_t^0, h_t^0\}_{t=0}^{\infty}$ to maximize the representative household's utility

$$\sum_{t=0}^{\infty} \beta^t \ln[c_t^0 - (1/\gamma)(h_t^0)^\gamma],$$

subject only to the aggregate resource constraints

$$h_t^0 \geq c_t^0$$

for all $t = 0, 1, 2, ...$ The solution to this planning problem, which describes the unique symmetric Pareto optimal allocation, sets

$$c_t^0 = h_t^0 = 1 \tag{40}$$

for all $t = 0, 1, 2, ...$

Comparing (39) and (40) reveals that equilibrium and optimal allocations coincide when monetary policy provides for zero nominal interest rates. Since positive nominal interest rates serve only to distort the representative household's labor supply decisions, zero nominal interest rates are good, not bad. They are more appropriately associated with Friedman's (1969) rule for the optimum quantity of money than with Keynes' (1936) theories of economic depression.

When $n = 0$, the representative household can always use its initial stock of real balances to finance the lump-sum taxes required to contract the money supply; this result follows directly from (38), which shows that in the case without

population growth, the value of the stock of real balances exactly offsets the present discounted value of the future taxes needed to implement a policy of zero nominal interest rates. When $n > 0$, however, some of the taxes associated with monetary contraction must be paid by households that are born without financial assets. And as the money growth rate approaches its lower bound from (36), the tax burden on newly-born households becomes heavier and heavier, to the point where these households can scarcely afford to consume.

Thus, when the population grows, monetary policies have distributional consequences that potentially make deflation quite costly for younger agents. On the other hand, even when $n > 0$, (15) and (19) associate lower nominal interest rates—brought about through deflation—with higher levels of aggregate consumption and output. Thus, monetary contraction has both costs and benefits. The key question becomes: how large are the costs, compared to the benefits?

To answer this question, consider adopting as a welfare criterion for monetary policy the lifetime utility achieved by a representative household that is born into the model's steady state. Woodford (1990) vigorously defends this measure of welfare in models, like the one used here, in which heterogeneous agents are distinguished by their dates of birth. Whitesell (1988) finds that steady-state utility is maximized under positive money growth rates and, indeed, Whitesell's

result carries over to the variant of his model developed here.

As an example, suppose that $\beta = 0.99$, so that each period in the model can be identified as one quarter year. Let $\gamma = 1.6$, the value used by Greenwood, Hercowitz, and Huffman (1988) to match estimates of the labor supply elasticity $1/(\gamma - 1)$, and let $n = 0.0025$, corresponding to an annualized rate of population growth of about one percent. With these parameter settings, numerical analysis reveals that steady-state utility is maximized when $\sigma = 0.0046$, so that the nominal money stock grows at the annualized rate of 1.87 percent. This optimal policy gives rise to an annualized inflation rate of 0.85 percent and an annualized nominal interest rate of 5.02 percent. The annualized real interest rate of 4.13 percent exceeds the annualized discount factor of 4.10 percent, so that each individual household chooses a growing path for consumption. Aggregate consumption in the optimal steady state is constant at 0.9798, more than 2 percent below the level that, according to (33), is achieved in a steady state with a zero nominal interest rate. But despite this reduction in aggregate consumption, the representative household prefers the steady state with positive money growth.

More generally, the welfare effects of different money growth rates can be summarized as follows. Let U^0 denote the lifetime utility achieved by a representative household that is born into the model's steady state when the money supply is

held constant or, equivalently, when the money growth rate equals zero. Next, let $\{c_t^s(\sigma)\}_{t=s}^{\infty}$ and $\{h_t^s(\sigma)\}_{t=s}^{\infty}$ denote the sequences of consumption and hours worked chosen by this representative household in the alternative steady state in which the money growth rate equals σ. Finally, let $\omega(\sigma)$ be defined implicitly by

$$U^0 = \sum_{t=s}^{\infty} \beta^{t-s} \ln\{[1+\omega(\sigma)/100]c_t^s(\sigma) - (1/\gamma)[h_t^s(\sigma)]^{\gamma}\}.$$

Then $\omega(\sigma)$ measures the permanent percentage increase in consumption that makes the representative household as well off under the money growth rate σ as it is under the benchmark of zero money growth; Cooley and Hansen (1989) and Lucas (2000) use similar measures of the welfare cost of inflation.

Table 1 summarizes the effects of changes in the steady-state money growth rate σ and reports the value of $\omega(\sigma)$ for various choices of σ when, as in the example from above, $\beta = 0.99$, $\gamma = 1.6$, and $n = 0.0025$. The function ω takes on negative values for annualized money growth rates as high as 3.65 percent, indicating that the representative household prefers small but positive values of σ to the benchmark setting of $\sigma = 0$. The function ω reaches its minimum at the optimal setting of $\sigma = 0.0046$.

As σ rises above 0.01, the negative effects of money growth on aggregate

output begin to overwhelm the positive distributional effects, so that ω turns positive. The largest values of ω, however, occur for negative values of σ that make the nominal interest rate equal to zero. A representative household born into the steady state with $\sigma = -0.008$ needs a permanent 5.25 percent increase in consumption to be as well off as under a constant money supply. And as the money growth rate approaches -0.010, the lower bound from (36), the tax burden associated with the zero nominal interest rate becomes so heavy that the household needs almost 60 percent more consumption to be as well off as under a constant money supply.

Thus, in one way, the introduction of the real balance effect into an otherwise conventional cash-in-advance model works exactly as promised by Pigou, Patinkin, and others: it eliminates the liquidity trap, giving the central bank control over the price level even when the nominal interest rate hits its lower bound of zero. Yet here, the same distributional effects that allow the real balance effect to operate also make zero nominal interest rates quite costly for some agents. Paradoxically, a zero nominal interest rate is something to be achieved in the conventional model, where the liquidity trap survives. With the introduction of the real balance effect, a zero nominal interest rate becomes something to be avoided.

5. Appendix

5.1. Deriving the Infinite-Horizon Budget Constraint

To derive the infinite-horizon budget constraint (6), multiply the single-period budget constraint (3) by Q_t and rearrange to obtain

$$Q_t(M_t^s + B_t^s) + Q_t(T_t + P_t h_t^s) \geq Q_t P_t c_t^s + (Q_t - Q_{t+1})M_{t+1}^s + Q_{t+1}(M_{t+1}^s + B_{t+1}^s).$$

Sum from t through $T \geq t$ to obtain

$$\begin{aligned} & Q_t(M_t^s + B_t^s) + \sum_{u=t}^{T} Q_u(T_u + P_u h_u^s) \\ \geq & \sum_{u=t}^{T} Q_u \left[P_u c_u^s + \left(\frac{r_u}{1+r_u} \right) M_{u+1}^s \right] + Q_{T+1}(M_{T+1}^s + B_{T+1}^s). \end{aligned}$$

Now use the no-Ponzi-scheme constraint (5) at $t = T$ to obtain

$$Q_t(M_t^s + B_t^s) + \sum_{u=t}^{\infty} Q_u(T_u + P_u h_u^s) \geq \sum_{u=t}^{T} Q_u \left[P_u c_u^s + \left(\frac{r_u}{1+r_u} \right) M_{u+1}^s \right].$$

Finally, take the limit as $T \to \infty$ to arrive at (6).

5.2. Solving the Household's Problem

Let λ_t^s and μ_t^s denote the nonnegative Lagrange multipliers on the household's budget and cash-in-advance constraints for period t. Since the household's utility function is increasing and concave, necessary conditions for optimality include the usual first-order and complementary slackness conditions, which are given by

$$\frac{1}{c_t^s - (1/\gamma)(h_t^s)^\gamma} = \lambda_t^s + \mu_t^s, \tag{A.1}$$

$$\frac{(h_t^s)^{\gamma-1}}{c_t^s - (1/\gamma)(h_t^s)^\gamma} = \lambda_t^s, \tag{A.2}$$

$$\frac{\lambda_t^s}{P_t} = \frac{\beta(\lambda_{t+1}^s + \mu_{t+1}^s)}{P_{t+1}}, \tag{A.3}$$

$$\frac{\lambda_t^s + \mu_t^s}{(1+r_t)P_t} = \frac{\beta(\lambda_{t+1}^s + \mu_{t+1}^s)}{P_{t+1}}, \tag{A.4}$$

$$\frac{M_t^s + T_t + B_t^s}{P_t} + h_t^s = c_t^s + \frac{B_{t+1}^s}{(1+r_t)P_t} + \frac{M_{t+1}^s}{P_t}, \tag{A.5}$$

$$\frac{M_t^s + T_t + B_t^s}{P_t} - \frac{B_{t+1}^s}{(1+r_t)P_t} \geq c_t^s, \tag{A.6a}$$

$$\mu_t^s \geq 0, \tag{A.6b}$$

and

$$\mu_t^s \left[\frac{M_t^s + T_t + B_t^s}{P_t} - \frac{B_{t+1}^s}{(1+r_t)P_t} - c_t^s \right] = 0 \tag{A.6c}$$

for all $t = s, s+1, s+2, ...$

Necessary conditions also include the transversality condition

$$\lim_{t\to\infty} Q_{t+1}W^s_{t+1} = \lim_{t\to\infty} Q_{t+1}(M^s_{t+1} + B^s_{t+1}) = 0. \tag{A.7}$$

To derive (A.7), note first that since the net nominal interest rate must always be nonnegative, the sequence $\{Q_t\}_{t=0}^{\infty}$ is nonincreasing, with

$$Q_t = (1 + r_t)Q_{t+1} \geq Q_{t+1}$$

for all $t = 0, 1, 2, ...$ Note also that $\{Q_{t+1}W^s_{t+1}\}_{t=s}^{\infty}$ is nonincreasing, since for any $t = s+1, s+2, s+3, ...,$ the definition of W^s_{t+1}, the period t budget constraint, the fact that $Q_t \geq Q_{t+1}$, and the nonnegativity constraints from (4) imply

$$\begin{aligned}
Q_{t+1}W^s_{t+1} - Q_tW^s_t &= Q_{t+1}(M^s_{t+1} + B^s_{t+1}) - Q_t(M^s_t + B^s_t) - Q_t(T_t + P_th^s_t) \\
&\leq Q_{t+1}(M^s_{t+1} + B^s_{t+1}) - Q_t[P_tc^s_t + B^s_{t+1}/(1 + r_t) + M^s_{t+1}] \\
&= (Q_{t+1} - Q_t)M^s_{t+1} - Q_tP_tc^s_t \\
&\leq 0.
\end{aligned}$$

Next, note that if $\{c^s_t, h^s_t, M^s_{t+1}, B^s_{t+1}\}_{t=s}^{\infty}$ are optimal choices for the represen-

tative household of cohort s, the implied sequence $\{Q_{t+1}W^s_{t+1}\}^{\infty}_{t=s}$ must satisfy

$$\inf_{t\geq s} Q_{t+1}W^s_{t+1} = 0.$$

To see this, suppose to the contrary that there exists an $\varepsilon > 0$ such that $Q_{t+1}W^s_{t+1} \geq \varepsilon$ for all $t = s, s+1, s+2, ...$ and construct new sequences $\{\tilde{c}^s_t, \tilde{h}^s_t, \tilde{M}^s_{t+1}, \tilde{B}^s_{t+1}\}^{\infty}_{t=s}$ as

$$\tilde{c}^s_s = c^s_s + \frac{\varepsilon}{Q_s P_s},\ \tilde{c}^s_t = c^s_t \text{ for } t = s+1, s+2, s+3, ...,$$

$$\tilde{h}^s_t = h^s_t \text{ for } t = s, s+1, s+2, ...,$$

$$\tilde{M}^s_{t+1} = M^s_{t+1} \text{ for } t = s, s+1, s+2, ...,$$

and

$$\tilde{B}^s_{t+1} = B^s_{t+1} - \frac{\varepsilon}{Q_{t+1}} \text{ for } t = s, s+1, s+2, ...$$

These new sequences satisfy all of the household's constraints: (2)-(5) for all $t = s, s+1, s+2, ...$ Moreover, they provide the household with a higher level of utility than the original sequences. But this contradicts the assumption that the original sequences are optimal. Hence, $\inf_{t\geq s} Q_{t+1}W^s_{t+1} = 0$ must hold.

Together, $\{Q_{t+1}W^s_{t+1}\}^{\infty}_{t=s}$ nonincreasing and $\inf_{t\geq s} Q_{t+1}W^s_{t+1} = 0$ imply that

(A.7) must hold at the optimum. This establishes that (A.1)-(A.7) are necessary conditions for optimality.

To prove that (A.1)-(A.7) are also sufficient conditions for optimality, suppose that $\{c_t^s, h_t^s, M_{t+1}^s, B_{t+1}^s\}_{t=s}^{\infty}$ satisfy (A.1)-(A.7), but that the alternative sequences $\{\tilde{c}_t^s, \tilde{h}_t^s, \tilde{M}_{t+1}^s, \tilde{B}_{t+1}^s\}_{t=s}^{\infty}$ satisfy (2)-(5) for all $t = s, s+1, s+2, ...$ and provide the household with a higher level of utility. Then

$$
\begin{aligned}
0 &< \lim_{T\to\infty} \sum_{t=s}^{T} \beta^{t-s} \{\ln[\tilde{c}_t^s - (1/\gamma)(\tilde{h}_t^s)^\gamma] - \ln[c_t^s - (1/\gamma)(h_t^s)^\gamma]\} \\
&< \lim_{T\to\infty} \sum_{t=s}^{T} \beta^{t-s} \left\{ \left[\frac{1}{c_t^s - (1/\gamma)(h_t^s)^\gamma} \right] (\tilde{c}_t^s - c_t^s) - \left[\frac{(h_t^s)^{\gamma-1}}{c_t^s - (1/\gamma)(h_t^s)^\gamma} \right] (\tilde{h}_t^s - h_t^s) \right\} \\
&= \lim_{T\to\infty} \sum_{t=s}^{T} \beta^{t-s} [\lambda_t^s (\tilde{c}_t^s - c_t^s) - \lambda_t^s (\tilde{h}_t^s - h_t^s) + \mu_t^s (\tilde{c}_t^s - c_t^s)] \\
&\leq \lim_{T\to\infty} \sum_{t=s}^{T} \beta^{t-s} \lambda_t^s \left[\frac{\tilde{M}_t^s - M_t^s}{P_t} + \frac{\tilde{B}_t^s - B_t^s}{P_t} - \frac{\tilde{M}_{t+1}^s - M_{t+1}^s}{P_t} - \frac{\tilde{B}_{t+1}^s - B_{t+1}^s}{(1+r_t)P_t} \right] \\
&\quad + \sum_{t=s}^{T} \beta^{t-s} \mu_t^s \left[\frac{\tilde{M}_t^s - M_t^s}{P_t} + \frac{\tilde{B}_t^s - B_t^s}{P_t} - \frac{\tilde{B}_{t+1}^s - B_{t+1}^s}{(1+r_t)P_t} \right] \\
&= \lim_{T\to\infty} \frac{\beta^{T-s} \lambda_T^s (M_{T+1}^s - \tilde{M}_{T+1}^s)}{P_T} + \frac{\beta^{T-s} (\lambda_T^s + \mu_T^s)(B_{T+1}^s - \tilde{B}_{T+1}^s)}{(1+r_T)P_T} \\
&= \left(\frac{\lambda_s^s + \mu_s^s}{Q_s P_s} \right) \lim_{T\to\infty} [Q_{T+1}(M_{T+1}^s + B_{T+1}^s) - Q_{T+1}(\tilde{M}_{T+1}^s + \tilde{B}_{T+1}^s)] \\
&= -\left(\frac{\lambda_s^s + \mu_s^s}{Q_s P_s} \right) \lim_{T\to\infty} Q_{T+1}(\tilde{M}_{T+1}^s + \tilde{B}_{T+1}^s) \\
&\leq 0
\end{aligned}
$$

by the concavity of the utility function, by (A.1) and (A.2), by (2), (3), (A.5),

and (A.6c), by (A.3) and (A.4), by (A.3) and (A.4) again, by (A.7), and by (5). But all of this contradicts the assumption that $\{\tilde{c}_t^s, \tilde{h}_t^s, \tilde{M}_{t+1}^s, \tilde{B}_{t+1}^s\}_{t=s}^{\infty}$ provide the household with higher utility than $\{c_t^s, h_t^s, M_{t+1}^s, B_{t+1}^s\}_{t=s}^{\infty}$. Hence, (A.1)-(A.7) are both necessary and sufficient for a solution to the household's problem.

Now let m_t^s, b_t^s, τ_t, π_t, x_t, and a_t^s be as defined in the text. Substitute (A.4) into (A.3) to obtain

$$\mu_t^s = r_t \lambda_t^s, \tag{A.8}$$

and combine this result with (A.1) and (A.2) to arrive at (9) from the text. Use (A.5) and (A.8) to rewrite (A.6a)-(A.6c) as (10) from the text.

Next, consider (A.4), which can be rewritten using (7) and (A.1) as

$$c_{t+1}^s - (1/\gamma)(h_{t+1}^s)^{\gamma} = \beta(1 + x_t)[c_t^s - (1/\gamma)(h_t^s)^{\gamma}], \tag{A.9}$$

which is the Euler equation linking the household's intertemporal marginal rate of substitution to the real interest rate. Multiply (A.5) by $P_t Q_t$ and, as above, sum from t through $T \geq t$ and take the limit as $T \to \infty$ to obtain

$$Q_t(M_t^s + B_t^s) + \sum_{u=t}^{\infty} Q_u(T_u + P_u h_u^s) = \sum_{u=t}^{\infty} Q_u \left[P_u c_u^s + \left(\frac{r_u}{1 + r_u} \right) M_{u+1}^s \right], \tag{A.10}$$

which is just (6) with equality. Since

$$\frac{Q_u}{Q_t P_t} = \left[\prod_{v=t}^{u-1}\left(\frac{1}{1+x_v}\right)\right]\frac{1}{P_u},$$

(A.10) can be rewritten as

$$\begin{aligned} & a_t^s + \sum_{u=t}^{\infty}\left[\prod_{v=t}^{u-1}\left(\frac{1}{1+x_v}\right)\right](\tau_u + h_u^s) \qquad (A.11) \\ = & \sum_{u=t}^{\infty}\left[\prod_{v=t}^{u-1}\left(\frac{1}{1+x_v}\right)\right]\left[c_u^s + \frac{r_u(1+\pi_{u+1})m_{u+1}^s}{1+r_u}\right]. \end{aligned}$$

Substitute (9), (10), and (A.9) into (A.11) to obtain the consumption function (11) from the text.

Use (7) and (8) to recast (A.5) in real terms:

$$a_t^s + \tau_t + h_t^s = c_t^s + \left(\frac{1}{1+x_t}\right)(a_{t+1}^s + r_t m_{t+1}^s). \qquad (A.12)$$

Then use (9) and (10) to rewrite (A.12) as (12) from the text. Finally, use

$$\frac{Q_{t+1}}{Q_s P_s} = \left[\prod_{v=s}^{t}\left(\frac{1}{1+x_v}\right)\right]\frac{1}{P_{t+1}}$$

and (8) to replace (A.7) with (13) from the text.

5.3. Deriving the Steady-State Conditions

Equations (7) and (14)-(21) form a system of nine equations that describe the behavior of the nine variables x_t, r_t, π_{t+1}, a_t, m_t, b_t, h_t, c_t, and τ_t in equilibria in which the government issues no bonds and expands the money stock at a constant rate. Equations (14) and (19)-(21) can be used to substitute out for a_t, h_t, b_t, and τ_t. After making these substitutions and imposing the steady-state conditions $x_t = x$, $r_t = r$, $\pi_{t+1} = \pi$, $m_t = m$, and $c_t = c$, (7), (15), and (17) can be rewritten as (22), (24), and (26).

In a steady state, (18) becomes

$$m = \left(\frac{1+x}{1+n}\right)\left[m + \sigma m - \left(\frac{1}{1+r}\right)rc\right],$$

or, using (16),

$$m = \left(\frac{1+x}{1+n}\right)\left[m + \sigma m - \left(\frac{1}{1+r}\right)r(1+n)(1+\pi)m\right].$$

Divide both sides of this last equality by m and rearrange using (22) to obtain (23); (23) then allows (16) to be rewritten as (25).

5.4. Deriving the Lower Bound on Money Growth

In light of the Euler equation (A.9), $c_t^s - (1/\gamma)(h_t^s)^\gamma > 0$ for all $t = s, s+1, s+2, ...,$ as required by (4), if and only if $c_s^s - (1/\gamma)(h_s^s)^\gamma > 0$. Combining (9), (11), and (21) with the initial condition $a_s^s = 0$, which applies to any household born into a steady state with $n > 0$, reveals that

$$c_s^s - (1/\gamma)(h_s^s)^\gamma = (1-\beta)\left(\frac{1+x}{x}\right)\left[\sigma m + \left(\frac{\gamma-1}{\gamma}\right)\left(\frac{1}{1+r}\right)^{\gamma/(\gamma-1)}\right]$$

in any steady state with $n > 0$. Equivalently, using (26),

$$c_s^s - (1/\gamma)(h_s^s)^\gamma = c - \frac{1}{\gamma}\left(\frac{1}{1+r}\right)^{\gamma/(\gamma-1)} - (1-\beta)m.$$

When $r = 0$, (33) and (34) imply that the right-hand side of this last equality is strictly positive if and only if the money growth rate satisfies (36).

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Table 1. The Effects of Steady-State Money Growth

σ	π	r	x	c	m	$\omega(\sigma)$
−0.010	−0.0125	0.0000	0.0126	1.0000	37.5000	58.7945
−0.009	−0.0115	0.0000	0.0116	1.0000	22.3259	23.8144
−0.008	−0.0105	0.0000	0.0106	1.0000	7.1825	5.2543
−0.007	−0.0095	0.0006	0.0102	0.9990	1.0060	0.0096
−0.006	−0.0085	0.0016	0.0102	0.9973	1.0034	0.0077
−0.005	−0.0075	0.0026	0.0102	0.9957	1.0007	0.0060
−0.004	−0.0065	0.0036	0.0102	0.9940	0.9980	0.0045
−0.003	−0.0055	0.0046	0.0102	0.9923	0.9953	0.0031
−0.002	−0.0045	0.0056	0.0102	0.9907	0.9927	0.0019
−0.001	−0.0035	0.0066	0.0102	0.9890	0.9900	0.0009
0.000	−0.0025	0.0076	0.0102	0.9874	0.9874	0.0000
0.001	−0.0015	0.0087	0.0102	0.9857	0.9848	−0.0007
0.002	−0.0005	0.0097	0.0102	0.9841	0.9821	−0.0012
0.003	0.0005	0.0107	0.0102	0.9825	0.9795	−0.0016
0.004	0.0015	0.0117	0.0102	0.9808	0.9769	−0.0018
0.005	0.0025	0.0127	0.0102	0.9792	0.9743	−0.0018
0.006	0.0035	0.0137	0.0102	0.9776	0.9718	−0.0017
0.007	0.0045	0.0147	0.0102	0.9760	0.9692	−0.0013
0.008	0.0055	0.0157	0.0102	0.9744	0.9666	−0.0009
0.009	0.0065	0.0167	0.0102	0.9727	0.9641	−0.0002
0.010	0.0075	0.0177	0.0102	0.9711	0.9615	0.0006
0.020	0.0175	0.0278	0.0102	0.9553	0.9366	0.0176
0.030	0.0274	0.0379	0.0102	0.9399	0.9125	0.0506
0.040	0.0374	0.0479	0.0102	0.9249	0.8893	0.0994
0.050	0.0474	0.0580	0.0102	0.9103	0.8669	0.1635
0.100	0.0973	0.1084	0.0102	0.8424	0.7658	0.7031
0.150	0.1471	0.1588	0.0101	0.7822	0.6802	1.5812
0.200	0.1970	0.2092	0.0101	0.7287	0.6072	2.7662

Note: Figures listed for $\sigma = -0.010$ are the limits as σ approaches -0.010 from above.