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ABSTRACT

Using only information based on current directly-observable market behavior, the paper shows how to make rigorous dynamic welfare comparisons among economies or economic situations having arbitrarily-different endowments and technologies, but sharing a common dynamic preference ordering. The correct answers to seemingly complicated questions, which intrinsically involve comparing wealth-like measures of dynamic well-being, can be translated isomorphically into a simple-minded story told in the familiar language of old-fashioned static consumer-welfare theory.

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Introduction

Recent times have witnessed a greatly heightened awareness of the interactions between economic, social and environmental issues. Terms like “green accounting” and “sustainability” have found their way into the lexicon of popular jargon. There has appeared a widespread interest in the idea of extending the concepts and measurement of national income to include important near-market activities in related areas that bear on welfare and productivity -- such as the environment and natural resources, but also human capital formation, unpaid home production (possibly including leisure-time activities), research and development, and the like.

Many theoretical questions have been raised about augmented national income, ranging from broad concerns, posed at a high level of abstraction, about its welfare foundations, through basic issues touching upon the design of green national income accounts, down to narrow advice on which particular activities to include and how to include them. In response, as if wanting to be able to answer such questions, has arisen a branch of economic analysis that might be called the pure theory of comprehensive national income accounting. Through the core of this theory runs a common strand attempting to connect a currently-observable index of comprehensive net national income or product with some appropriate but not-currently-observable welfare measure of future power to consume, which typically has a “sustainability-like” flavor or undertone.

We seem presently to have created at least a partially-successful body of theory.[1] However, some big pieces of the conceptual puzzle are not yet fitting snugly into a fully-coherent overall picture. One piece is obvious because the existing theory is, almost without exception, built around a fictitious entity of “aggregate consumption,” while what we really want is a general theory that includes heterogeneous consumption as seamlessly as it incorporates heterogeneous capital. Another piece of the puzzle is a seeming disconnect between the idea that, to be observable, net national product or income must be a price-weighted commodity-based index, whereas the concepts that show up naturally in optimal growth theory, such as the Hamiltonian,

[1] For a good overview, with an extensive bibliography, see Aronsson, Johansson and Löfgren [1997].

are mostly utility-based measures. Finally, perhaps the biggest and most critical piece of the puzzle not yet fitting neatly into the existing body of theory concerns the answer to the following question. Exactly how, at least in principle, are we supposed to use national income statistics and other currently-observable market information to make rigorous inferences about welfare differences among economies or across economic situations? Taken seriously, such inferences would appear to require the calculation and comparison of inherently-dynamic wealth-like measures. Is there a way to circumvent these difficult calculations, or at least to relate the dynamic wealth-like measures to some form of a simpler static income-like version?

The purpose of this paper is to take one more step moving in the direction towards a more-fully-coherent theory. The paper is aimed primarily at showing how to make rigorous dynamic welfare comparisons based only on current directly-observable market information. In the course of developing a general methodology to deal with this problem, the paper will treat fully-disaggregated consumption as a natural formulation, and will also show implicitly how to reconcile commodity-based national product with utility-based welfare. And, by embedding short-run consumer behavior within an optimal growth framework, the paper will cast new light on some old controversies in consumer-surplus and index-number theory. While the motivation has been framed here in terms of the theory of national income accounting, the essential contribution of the paper is to provide a proper dynamic generalization of the standard static formula for the welfare evaluation of economic changes.

The Setting of the Model

Previous work on the welfare significance of national income has effectively postulated a single homogeneous "aggregate consumption" good, while allowing multiple capital goods. Here we are covering in full generality the case of heterogeneous investment *and* consumption. Let the vector $\boldsymbol{C}$ represent an m-dimensional fully-disaggregated bundle of consumption flows. More specifically, $C_i(t)$ measures the *flow of consumption services* from consuming $C_i(t)$ units of commodity i at time instant t, for $i=1,2,...,m$.

The consumption vector $\boldsymbol{C}$ is conceptualized as being a complete list containing everything that influences current well being, including environmental amenities and other externalities. Consumption here would ideally include all components that influence the true "standard of living" -- not just the goods we buy in stores and the government services "purchased" with our taxes, but also non-market commodities, such as those produced at home, and environmental services, such as those rendered by natural capital like forests and clean air. For the sake of developing the core theory, initial consumption $C(0)$ is presumed to be fully observable and it is assumed we know at the present time the associated *m*-vector of competitive or efficiency consumption prices. We will also presume to be able to observe the relevant short-run market demand function in the domain over which static comparisons will be made.

For any consumption time series $\{\boldsymbol{C}(t)\}$, it is supposed that it is meaningful to measure overall intertemporal well-being by the expression:

$$W(\{\boldsymbol{C}(t)\}) \equiv \int_0^\infty e^{-\rho t} U(\boldsymbol{C}(t)) \, dt \quad , \tag{1}$$

where $U(\boldsymbol{C})$ is some given concave non-decreasing instantaneous utility function with continuous second derivatives defined over all non-negative consumption flows $\boldsymbol{C}$, while ρ is some given rate of pure time preference. As practically every economist will attest, for better or for worse (1) is the standard workhorse version of the objective function used widely in economics as a maximand in intertemporal optimization problems. Also for what it is worth, a linear functional taking the form of (1) can be given an axiomatic justification as representing the appropriate dynamic preference ordering whenever independence, stationarity, continuity and a few other seemingly-standard conditions are postulated.[2]

The notion of "capital" used in the model is intended to be quite a bit more general than the traditional produced means of production like equipment and structures. Most immediately, pools of natural resources are unquestionably considered to be forms of capital. Forms of human capital, such as education, should in principle be included, and also the knowledge capital

[2] See, e.g., Koopmans [1960].

accumulated from R&D-like activities. Generally speaking, every possible type of capital should be included -- to the extent that we know how to measure and evaluate the associated net investment flows. Under a very broad interpretation, environmental assets generally might be treated as a form of capital. From this perspective, environmental quality would be viewed as a stock of capital that is depreciated by pollution and invested-in by abatement.[3] The underlying ideal is to have the list of capital goods be as comprehensive as possible, subject to the practical limitation that meaningful competitive-market-like efficiency prices are available for evaluating the corresponding net investments.

Suppose that altogether there are n capital goods, including stocks of natural resources and other non-orthodox forms. The stock of capital of type $(1 \leq i \leq n)$ in existence at time t is denoted $K_i(t)$, and its corresponding net investment flow is $I_i(t) = \dot{K}_i(t)$. The n-vector $\mathrm{K} = \{K_i\}$ denotes all capital stocks, while $\mathrm{I} = \{I_i\}$ stands for the corresponding n-vector of net investments. Note that the net investment flow of a natural capital asset like a timber reserve would be negative if the overall extraction rate exceeds the replacement rate. Generally speaking, net investment in environmental capital should be regarded as negative whenever the underlying asset is being depleted or run down more rapidly than it is being replaced or built up.

Again in the spirit of focusing sharply for the sake of developing the core theory, we assume the production system is time autonomous.[4] For theoretical purposes, we are thus imagining an idealized world where the coverage of capital goods is so comprehensive, and the national accounting system is so complete, that there remain no unaccounted-for residual "atmospheric" growth factors. *All* sources of future growth have been identified as proper investments able to be evaluated at their proper efficiency prices.

Unfortunately, we do not now live in a world where national income accounting is

[3] Mäler [1991] includes a good discussion of some of the relevant issues here.

[4] For some treatments of the time-dependent case, see Nordhaus [1995], Weitzman [1997], or Weitzman and Löfgren [1997], and the further references cited therein. Time dependence introduces a host of messy complications, but a modified (and unpretty) version of the result presented here can usually be found, contingent on some simplifying assumptions about the particular form of time dependency.

complete. Completeness is perhaps best envisioned as a limiting case, which some accounting systems approach in coverage but few attain. In our actual world we cannot measure accurately all investments, many externalities are not internalized, it is often difficult to find market-like prices for non-market goods, and there are various "atmospheric" sources of positive or negative growth, which we cannot or do not include in net national product. (The omitted "atmospheric" contributions are identified primarily as a residual, which is obtained by subtracting off from actual growth the effects of all known, properly identified, sources of growth.)

Even if it were to be admitted that we live in a world whose accounting is incomplete, it would still be indispensable for us to understand fully the pure theory of perfectly complete accounting -- if for no other reason than as a base case, or reference, or starting point for a more complete analysis. In actuality, the most important practical reason for studying the pure theory of complete accounting is that it can suggest what things to include, and how best to include them, to "green up" national income -- meaning to make it a more complete aggregate reflecting more accurately what the future portends relative to the present. For this reason, one might say that the pure theory is useful in a world of incomplete accounting *precisely because* it suggests the best way to make the accounting more complete.

In the mathematical language of the model, the accounting system is said to be *perfectly complete*, or, equivalently, national income is said to be *fully comprehensive*, if the attainable-possibilities set at any time t can be described in reduced form as a function only of the capital stocks $\mathrm{K}(t)$ existing at that time. Therefore, by making this assumption we are allowed to denote the $(m+n)$-dimensional attainable possibilities set here as $S(\boldsymbol{K})$. Then the consumption-investment pair $(\boldsymbol{C}(t),\boldsymbol{I}(t))$ is attainable at time t if and only if

$$(\boldsymbol{C}(t),\boldsymbol{I}(t)) \ \varepsilon \ S(\boldsymbol{K}(t)) \quad . \tag{2}$$

As usual, the set $S(\boldsymbol{K})$ is presumed to be convex. Here, purely for ease of exposition, it will be assumed that $S(\boldsymbol{K})$ is *strictly* convex.

In the context of the duality theory of a multi-sector optimal growth problem, specification (2) signifies that attainable possibilities at any time are completely summarized by the state of then-existing capital stocks, whose corresponding net investment flows, which are also completely "accounted for" at known efficiency prices, are included in net national product. A complete accounting system of comprehensive national income, which is embedded in a dynamic economy at competitive equilibrium, corresponds to an infinite horizon optimal control problem whose distinguishing mathematical characteristic is that the only place where time enters explicitly is through the constant exponential discounting term of the objective functional.

A Tale of Two Economies

Suppose we are interested in comparing the dynamic welfare delivered by two different economies or two different economic situations. Let the economy "type" be indexed by the superscript indicator variable j. The index value $j=1$ indicates the given *base* economy. The index value $j=2$ indicates some particular *comparison* economy. Both economies share the *same preferences,* but they may have *arbitrarily different endowments* and/or *arbitrarily different technologies*. The basic contribution of this paper is to compare (1) across the two economies *relying only on currently-observable market information*.

Both economies $j=1$ and $j=2$ behave over time as if they are solutions of a symmetrical pair of optimal growth problems of the form:

maximize

$$\int_0^\infty U(\boldsymbol{C}^j(t))\ e^{-\rho t}\ dt \quad , \tag{3}$$

subject to the constraints

$$(\boldsymbol{C}^j(t),\boldsymbol{I}^j(t))\ \varepsilon\ S^j(\boldsymbol{K}^j(t)) \quad , \tag{4}$$

and the differential equations

$$\dot{\boldsymbol{K}}^j(t) = \boldsymbol{I}^j(t) \quad , \tag{5}$$

and obeying the initial conditions

$$\boldsymbol{K}^j(0) \;=\; \boldsymbol{K}_0^j \quad , \tag{6}$$

where K_0^j *is the initially given capital stocks -- all of the above holding for* $j=1$ *and* $j=2$.

Concerning the above formulation (3)-(6), note that the "technology" $S^j(\boldsymbol{K})$ in (4) and the "endowments" $\boldsymbol{K}_0^j$ in (6) are allowed to differ arbitrarily between the base economy (j=1) and the comparison economy (j=2), while "preferences" are identical, as indicated by the shared objective function (3).

In what follows, it is assumed, purely for ease of exposition, that the two optimal solutions of (3)-(6) corresponding to $j=1$ and $j=2$ not only exist, but are unique. Let $\{\boldsymbol{C}^{*j}(t),\ \boldsymbol{I}^{*j}(t),\ \boldsymbol{K}^{*j}(t)\}$ represent the optimal trajectory for economy j. As is well known from duality theory, the solutions of (3)-(6) for both economies will generate corresponding dynamic competitive prices, denoted here by the m-vector time series $\{\boldsymbol{p}^{*j}(t)\}$ for consumption-goods (money) prices, and by the n-vector time series $\{\boldsymbol{q}^{*j}(t)\}$ for investment-goods (money) prices. Then (money) national income or product for economy j at time t is

$$Y^{*j}(t) \quad \equiv \quad \boldsymbol{p}^{*j}(t)\cdot\boldsymbol{C}^{*j}(t) \;+\; \boldsymbol{q}^{*j}(t)\cdot\boldsymbol{I}^{*j}(t) \quad . \tag{7}$$

Let $\{\lambda^j(t)\}$ represent the non-observable marginal utility of income along an optimal trajectory in economy j (=1,2) at time t. The investment-goods price n-vector, expressed in real current-value utility terms for economy j (=1,2) at time t is then

$$\lambda^j(t)\boldsymbol{q}^{*j}(t) \quad , \tag{8}$$

while the corresponding consumption-goods price m-vector, expressed in real current-value utility terms for economy j (=1,2) at time t is

$$\lambda^j(t)\boldsymbol{p}^{*j}(t) \quad . \tag{9}$$

In the model, $\{\lambda^j(t)\}$ may be chosen arbitrarily because it represents an extra degree of freedom that merely parameterizes the marginal utility of money income, which can be given a life of its own, related behind the scenes of the real economy to the money supply and other background, purely-monetary, factors that determine the price level. What matters for the allocation of resources in the *real* economy -- through the classical-dichotomy veil of arbitrary $\{\lambda^j(t)\}$, so to speak -- is *real* prices (8), (9), which are denominated in terms of the contemporaneous value of utility serving as numeraire.

Next, define the maximized current-value Hamiltonian expression

$$\hat{H}^j(\boldsymbol{K};\boldsymbol{q},\lambda) \equiv \underset{(\boldsymbol{C},\boldsymbol{I})\varepsilon S^j(\boldsymbol{K})}{maximum} \; \{U(\boldsymbol{C}) + \lambda \boldsymbol{q}\cdot\boldsymbol{I}\} \quad . \tag{10}$$

Note in (10) that the dependence of $\hat{H}^j$ upon j comes solely through the constraint-set term $S^j(\boldsymbol{K})$.

As is well known, the duality conditions corresponding to (3)-(6) can be given an interpretation as if describing a decentralized perfectly-competitive economy in dynamic equilibrium with a single representative agent, whose preference ordering is described by (1). We will emphasize this decentralized market interpretation throughout the paper, concentrating especially on how the observable short-run market demand function of the representative consumer-agent can be used to reveal certain relevant aspects of the agent's underlying preferences.

The first type of optimality condition requires that the Hamiltonian expression (10) should actually attain its maximum everywhere along an optimal trajectory. In the representative-agent interpretation, maximizing the Hamiltonian is equivalent to a combination of the condition describing the representative *consumer*'s decentralized static-equilibrium behavior:

$$U(\boldsymbol{C}^{*j}(t)) + \lambda^j(t)\boldsymbol{q}^{*j}(t)\cdot\boldsymbol{I}^{*j}(t) = \underset{\boldsymbol{p}^{*j}(t)\cdot\boldsymbol{C}+Z=Y^{*j}(t)}{maximum} \; \{U(\boldsymbol{C}) + \lambda^j(t)Z\} \quad , \tag{11}$$

along with the condition describing the representative *producer*'s decentralized static-equilibrium behavior:

$$\boldsymbol{p}^{*j}(t)\cdot\boldsymbol{C}^{*j}(t) + \boldsymbol{q}^{*j}(t)\cdot\boldsymbol{I}^{*j}(t) = \underset{(\boldsymbol{C},\boldsymbol{I})\varepsilon S^{j}(\boldsymbol{K}^{*j}(t))}{maximum} \{\boldsymbol{p}^{*j}(t)\cdot\boldsymbol{C} + \boldsymbol{q}^{*j}(t)\cdot\boldsymbol{I}\} \quad . \tag{12}$$

A second set of optimality conditions can be translated as describing a perfect capital/stock market in dynamic competitive equilibrium:

$$\frac{d}{dt}[\lambda^{j}(t)\boldsymbol{q}^{*j}(t)] - \rho\lambda^{j}(t)\boldsymbol{q}^{*j}(t) = -\left.\frac{\partial\hat{H}^{j}}{\partial\boldsymbol{K}}\right|_{*j(t)} \quad , \tag{13}$$

where the notation ‘ $|_{*j(t)}$’ means evaluation along the optimal trajectory of economy *j* at time *t*.

Finally, the third optimality condition is the transversality requirement

$$\lim_{t\to\infty} e^{-\rho t}\ \lambda^{j}(t)\ \boldsymbol{q}^{*j}(t)\cdot\boldsymbol{K}^{*j}(t) = 0 \quad . \tag{14}$$

If conditions (13) or (14) did *not* hold, then pure positive profits could be made by intertemporal arbitrage operations, which would induce a change in (13), (14) -- meaning these equations could *not* have been describing a dynamic competitive equilibrium in the first place.

Because of the underlying convexity of problem (3)-(6), the duality conditions (11)-(14) are both necessary and sufficient for an optimal solution.[5]

Current Directly-Observable Market Information

From this point on, the paper deals with market-behavior observations made *only* at the present time $t=0$. More precisely, we take on faith that the dynamic optimality-equilibrium conditions describing the coupled system (3)-(6), (10)-(14) will hold over all future time, but, aside from this general knowledge, all that we are permitted to know or infer at the present time

[5] This aspect, along with the representative-agent dynamic-competitive-equilibrium interpretation of duality, is discussed in several advanced theory textbooks. For an exposition whose notation is very close to this paper, see Weitzman [1970] and/or Weitzman [1973].

$t=0$ must be based solely on what is, at least in principle, the *current directly-observable market behavior* of the representative consumer. In keeping with this restriction on knowable information, the symbol $X^{*j}(0)$ -- for all pertinent X -- is henceforth in the paper replaced simply by the symbol X^j. Thus, the representative consumer in economy *j* (=1,2) currently (at the present time $t=0$) faces prices $\boldsymbol{p}^j$ and income Y^j, where, from (7),

$$Y^j \quad = \quad \boldsymbol{p}^j \cdot \boldsymbol{C}^j \; + \; \boldsymbol{q}^j \cdot \boldsymbol{I}^j \quad . \tag{15}$$

Note that consumption $\boldsymbol{C}^j$ in (15) is conceptualized entirely as a *flow* concept, as is investment $\boldsymbol{I}^j$ and income Y^j. Strict adherence here to this economic, service-flow view of consumption will imply some surprisingly useful consequences.

The presently-observable short-run market demand function in economy *j* is the representative consumer-agent's response to the following counterfactual question. "All other things being equal, how much would you choose to buy and consume now if the short-run market prices of consumption were $\boldsymbol{p}$?" The answer is given by envisioning the consumer's short-run demand behavior as a reaction-function of $\boldsymbol{p}$, where $\boldsymbol{p}$ is viewed in this context as a counterfactual parametrically-given version of the variable $\boldsymbol{p}^{j*}(t)$ appearing in the constraint set of equation (11) for $t=0$.

To the representative consumer-agent in situation *j* (=1,2), then, the act of "maximizing the Hamiltonian" translates behaviorally from (11) into having this consumer solve a decentralized problem of the reduced form:

Maximize

$$U(\boldsymbol{C}) \quad + \quad \lambda^j \, Z \quad , \tag{16}$$

subject to the budget constraint

$$\boldsymbol{p} \cdot \boldsymbol{C} \; + \; Z \quad = \quad Y^j \quad , \tag{17}$$

where $\boldsymbol{p}$ stands for the counterfactual parametrically-fixed short-run consumption prices, Y^j represents the given as-if-fixed national-income budget, λ^j is the (not observable to an outsider) given as-if-fixed marginal utility of income, and Z symbolizes aggregate investment, to be chosen along with $\boldsymbol{C}$ by the representative consumer in j.

A critical, if simple, observation from (16), (17) is that *the Hamiltonian itself is in the form of a quasilinear utility function* -- which means it is an objective function having several very important properties and implications.

Virtually all economists agree that consumption should be conceptualized as a *flow* of services. At least in principle, the appropriate market prices of consumer durables, like owner occupied houses or cars, should be interpreted as *imputations* -- namely, the *imputed rental prices*, which *would be* observed in a competitive market economy *if* the flow of consumption services were fixed at a corresponding level.[6] Pushing this service-flow view of consumption to its logical limit, there is a simple but important implication, which has largely gone unnoticed in the literature. Under the standard economic assumptions, as we have shown, the short-run consumer objective function is quasilinear. If consumption is conceptualized strictly as a *flow*, then the corresponding short-run consumer demand function is independent of income. Intuitively, this kind of envelope-type result occurs because the consumer can fully offset, via changes in savings behavior, any and all possible income effects of short-run price changes on instantaneous consumption flows -- merely by shifting investment income across time.

We write the directly-observable *short-run consumer-demand function* in economy j (=1,2) as $\boldsymbol{D}^j(\boldsymbol{p})$. The vector function $\boldsymbol{D}^j(\boldsymbol{p})$ is the implicit solution of the above problem (16), (17), which therefore satisfies, for all parametrically-given hypothetical values of $\boldsymbol{p} \geq \boldsymbol{0}$, the standard duality conditions

$$\boldsymbol{U}\ (\boldsymbol{D}^j(\boldsymbol{p})) \ \leq\ \lambda^j \boldsymbol{p} \quad , \tag{18}$$

and

$$[\lambda^j \boldsymbol{p} - \boldsymbol{U}\ (\boldsymbol{D}^j(\boldsymbol{p}))] \cdot \boldsymbol{D}^j(\boldsymbol{p}) \ =\ 0 \ . \tag{19}$$

Because there are no income effects as $\boldsymbol{p}$ is varied in the short run, λ^j is seen by the representative

[6] See, e.g., Boskin et al [1998] for a good discussion aimed at practical applications.

consumer as if being fixed in (18), (19), even though it is not directly observable to an outsider.

Next, for all consumption flows $\boldsymbol{C} \geq \mathbf{0}$, define the directly-observable *short-run inverse-demand function* in economy *j* (=1,2), denoted $\boldsymbol{P}^j(\boldsymbol{C})$, to be the solution of the equation:

$$\boldsymbol{P}^j(\boldsymbol{C}) \quad \equiv \quad \min \ \{\boldsymbol{p} \mid \ \boldsymbol{D}^j(\boldsymbol{p}) = \boldsymbol{C}\} \quad . \tag{20}$$

The corresponding short-run *consumer-expenditure function* in economy *j* (=1,2) is

$$E^{\,j}(\boldsymbol{C}) \ \equiv \ \boldsymbol{P}^j(\boldsymbol{C}) \cdot \boldsymbol{C} \ . \tag{21}$$

The expenditure formula (21) describes *the expense to consumers in economy j of purchasing the fixed market basket of consumption goods **C**.* In other words, expression (21) is just exactly the familiar revenue function from elementary economics, which a hypothetical monopolist would face in economy *j*.

We now define what might be called an ideal "CPI-type" (consumer-price-index-type) price deflator, or, equivalently here, an ideal "PPP-type" (purchasing-power-parity-type) price deflator -- as a function of the parametrically-fixed market basket of consumption goods ***C***.

Definition. An *ideal CPI/PPP-type price deflator* for converting from the current prices of comparison economy 2 into the current prices of base economy 1, evaluated at the fixed-benchmark market basket of consumption goods ***C***, is defined as the Laspeyres-type ratio:

$$\theta(\boldsymbol{C}) \ \equiv \ \frac{E^{\,1}(\boldsymbol{C})}{E^{\,2}(\boldsymbol{C})} \quad . \tag{22}$$

That there may be some kind of an imputation issue involved in calculating (22) should come as no more of a surprise here than the idea that the appropriate "price" of owner-occupied housing needs to be imputed as what would be the observed rental price in the economy at some given level of housing consumption-flow services. The appropriate prices to use in (21) and (22) are the imputed, counterfactual, other-things-being equal *prices that would actually be observed in the marketplace* of each economy (*j* =1 and *j*=2), *if the consumption-flow basket being purchased were actually observed to be the benchmark **C**.* In practice, this is typically not a difficult imputation to make for economies that are structurally very similar, like the U.S. and Canada, or like the U.S. from one year to the next, because the index number comparison then typically reduces to using the existing market prices for a given well-specified representative market basket of consumer goods. Even so, in any actual real-world comparison-pricing exercise, surprisingly many imputations are required to deal with "comparison-resistant" items of the same quantity and quality as the particular consumption market basket chosen to be "representative" in the comparisons.[7] And there is absolutely no way of escaping the central necessity to make some genuine imputations in CPI/PPP-type price deflators if the two comparison economies differ substantially in structure -- so that, for example, one economy may have commodities in its marketplace that are not purchased at all in the marketplace of the other economy. The following concept may help to shed some analytic light on this important set of issues.

Definition: An ideal CPI/PPP-type price deflator is called "*benchmark-invariant*" if $\theta(\boldsymbol{C})$ defined by (22) is independent of the benchmark market basket of consumption goods $\boldsymbol{C}$, for all current consumption flows $\boldsymbol{C} \geq \mathbf{0}$.

The following result is of theoretical importance for the paper, but also, I believe, has some real-world implications for how best to actually perform the imputations that are required to construct CPI/PPP-type price deflators in practice.

Lemma: *When consumption is measured as a strict flow of services, so that (18) and (19) describe the short-run demand function, then the ideal CPI/PPP-type price deflator (22) is benchmark-invariant.*

Proof: see Appendix.

As the lemma permits it, we will henceforth *replace the symbol* $\theta(\boldsymbol{C})$ *by the symbol* θ, which stands here for '*the*' ideal CPI/PPP-type price deflator.

Dynamic Welfare Comparisons

We come now to the basic result of the paper. With complete accounting, all relevant information for making dynamic welfare comparisons is contained in market behavior that is

[7] For a practical overview with further references, see Summers and Heston [1991].

currently observable within the domain of the relevant current static comparison. Equation (23) shows that the theoretically-correct but non-observable dynamic welfare index on the left-hand side of the equality sign is exactly the familiar, even famous, currently-observable static welfare expression on the right-hand side.

Theorem: Under the assumptions of the paper,

$$\frac{\rho}{\lambda^1}\left[\int_0^\infty U(\boldsymbol{C}^{*2}(t))e^{-\rho t}dt - \int_0^\infty U(\boldsymbol{C}^{*1}(t))e^{-\rho t}dt\right] = \theta Y^2 - Y^1 + \int_{\theta \boldsymbol{p}^2}^{\boldsymbol{p}^1} \boldsymbol{D}^1(\boldsymbol{p})\cdot d\boldsymbol{p} \quad . \qquad \textbf{(23)}$$

Proof: see Appendix.

Expression (23) can be conceptualized as "compressing" or "reducing" the wealth-like "true" dynamic welfare ordering on the left-hand side of the equation into the isomorphic income-like static welfare ordering on the right-hand side. A way to think about the theoretical equivalence of these two welfare orderings is to envision economic situations j=1 and j=2 as varying over all possible technologies and initial endowments. Then situation j=2 will be "better" than situation j=1 by welfare criterion (1) if and only if the right-hand side of equation (23) is positive. It follows that, for purposes of comparison, the dynamic welfare ordering induced by (1) is equivalent to the static welfare ordering induced by the expression

$$\theta Y^2 - Y^1 + \int_{\theta \boldsymbol{p}^2}^{\boldsymbol{p}^1} \boldsymbol{D}^1(\boldsymbol{p})\cdot d\boldsymbol{p} \quad . \qquad \textbf{(24)}$$

The basic result (23) can thus be interpreted as proving that expressions (1) and (24) are here just *different representations of the same underlying dynamic welfare ordering*. The currently-observable static expression (24) might even be called a *sufficient statistic* for comparisons based upon the standard but not-currently-observable dynamic welfare criterion (1) -- because expression (24) exhausts all of the welfare-comparison information contained in (1).

The unobservable "normalization constant"

$$\frac{\rho}{\lambda^1} \quad , \qquad \textbf{(25)}$$

which appears on the left-hand side of (23), involves a compounding of two "conversion coefficients." The coefficient $1/\lambda^1$ represents an arbitrary and inessential scaling constant for converting from units of utility into units of current income in the base economy. The pure-time-preference coefficient ρ converts the utility *wealth* expression within the square brackets of (23) into an annuitized *flow* of stationary-equivalent or sustainable-equivalent utility. Net national income itself is here interpretable as measuring sustainable-equivalent money-metricized utility.[8]

Note the very simple form of the isomorphism parable being told by (23). The difference in money-metricized sustainable-equivalent utility between any two comparison economies comes exactly in the form of an answer to the following standard question of classical static welfare analysis. "How much extra money must a consumer facing observable prices ^{1} with observable income Y^1 (and corresponding non-observable-but-constant marginal utility of income λ^1) be paid to be equally as well off as when facing observable prices p^2 with observable income Y^2 (and corresponding non-observable-but-constant marginal utility of income λ^2)?" The answer to this standard question in economic statics is given by the famous expression (24), where the term

$$\int_{\theta \boldsymbol{p}^2}^{\boldsymbol{p}^1} \boldsymbol{D}^1(\boldsymbol{p})\cdot d\boldsymbol{p} \qquad \textbf{(26)}$$

stands for the appropriate change in old-fashioned (Marshall-Dupuit) consumer surplus. It is because the Hamiltonian itself is in the form of a quasilinear utility function that the answer to the static question above (as well as to the larger dynamic-welfare question answered by (23)) is such a simple direct function of observable short-run market demands, entirely free of messy and extraneous income-effect corrections.

[8] This interpretation, which is further elaborated in Weitzman [1999], can be proved here directly from (23).

It follows that there no need to apologize *at all* for using consumer surplus routinely in welfare comparisons -- whenever consumption is conceptualized as being a short-run flow of services embedded within a larger dynamic-optimization problem of the generic form (3)-(6), which is, at least implicitly, the background setting for most economic applications. Actually, the burden of proof in this context might well appear to rest upon any economist trying here to use one of the well-known “variation” measures *instead of* consumer surplus.[9] Such a person might be asked to begin by explaining how the artificial constructs of “compensating variation” or “equivalent variation” can possibly represent a useful extension of the more intuitive and more practical concept of consumer surplus -- in a generic context where both variation measures are operationally indistinguishable from a change in consumer surplus in the first place.

As is well known, with a quasilinear utility function the utility difference between any two static economic situations differing in income and prices can be measured by the famous static welfare formula of type (24) -- consisting of the change in real income plus consumer surplus.[10] An exact statement in the notation of this paper is:

$$\frac{U(\boldsymbol{C}^2) - U(\boldsymbol{C}^1)}{\lambda^1} = \theta Y^2 - Y^1 + \int_{\theta \boldsymbol{p}^2}^{\boldsymbol{p}^1} \boldsymbol{D}^1(\boldsymbol{p}) \cdot d\boldsymbol{p} \quad . \qquad \textbf{(27)}$$

It is the welfare relation (27) that is cited behind the scenes to justify using (24) as a veritable workhorse of applied partial-equilibrium analysis. Equation (23) represents a true generalization of (27) from a static to a dynamic context. The static equation (27) is just a very special case of the far more general equation (23), where the relevant utility function $U(\boldsymbol{C},x)$ selected to appear on the left-hand side of (23) is of the familiar Hamiltonian quasilinear form

$$U(\boldsymbol{C},x) \equiv U(\boldsymbol{C}) + x \quad , \qquad \textbf{(28)}$$

[9] I realize that such statements may sound heretical, but at the same time believe it is important to say plainly that the standard optimal-growth framework opens the door to rehabilitating old-fashioned consumer surplus as a useful apparatus of some general applicability.

[10] See, e.g., Varian [1992], Section 10.4.

while the (m+1)-dimensional attainable-possibilities set relevant for this special case is simply

$$S^{j}(\boldsymbol{K}) \equiv \{(\boldsymbol{C},x)\colon \boldsymbol{p}^{j}\cdot\boldsymbol{C} + \frac{x}{\lambda^{j}} \leq Y^{j}\} \quad . \tag{29}$$

As (27) is a special static case of (23), and as (27) has proved itself to be of great practical importance in many fields of applied economic analysis, it might be hoped that its dynamic generalization (23) may also find useful applications. The basic result (23) shows that (1) and (24) are operationally-equivalent representation forms of the same underlying dynamic preference ordering. An economist is therefore free to choose whichever representation is more convenient to work with. For most economic applications, the income-like form (24) is vastly simpler, more intuitive, more observable, and more operational than the equivalent wealth-like form (1).

Thus, a relatively-straightforward, simple-minded shorthand application of static consumer-welfare theory -- which involves only comparing presently-observable prices and quantities along the relevant part of the short-run consumer-demand function -- gives the “correct answers” to some seemingly very complicated questions, the longhand versions of which must intrinsically involve comparing wealth-like “true indicators” of dynamic welfare. Put slightly differently, every time we perform a familiar, static, consumer-surplus-like economic analysis of the welfare difference between two situations, we are implicitly answering a dynamic question posed in terms of an underlying dynamic welfare comparison.

Conclusion

This paper has derived a kind of “dynamic welfare-comparison principle,” which lets us compare dynamic welfare situations rigorously, yet relies only on currently observable prices and quantities evaluated along the current short-run consumer-demand function within the current consumption-comparison domain. The underlying isomorphism assures us that it is ‘OK’ to translate dynamic welfare comparisons into a simple as-if-static story told in terms of conventional, old-fashioned consumer-welfare theory. The simple-minded story gives the correct answers to complicated questions that intrinsically involve comparing wealth-like dynamic welfare measures across any two economic situations differing arbitrarily in technologies or endowments.

It is anticipated that there may be useful applications of a welfare-comparison principle having this kind of simplicity and generality.

Appendix

Proof of Lemma:

From (18), (19) and the definition (20), it follows that the equation

$$\boldsymbol{U}\ (\boldsymbol{C}) \ = \ \lambda^j \ \boldsymbol{P}^j(\boldsymbol{C}) \tag{30}$$

holds for j=1, j=2, all $\boldsymbol{C} \geq \mathbf{0}$.

An immediate consequence of comparing (30) with the definitions (21), (22) is that

$$\theta(\boldsymbol{C}) \ = \ \theta \ \equiv \ \frac{\lambda^2}{\lambda^1} \tag{31}$$

for all $\boldsymbol{C} \geq \mathbf{0}$. ■

Equation (31) represents a stronger-than-required form of the conclusion to be proved in the lemma, but a form which will nevertheless be needed in the following proof of the theorem.

Proof of Theorem:

A basic result from Weitzman [1970; page 15, equation (16)], transposed to the notation of this paper, is

$$\rho \int_0^\infty U(\boldsymbol{C}^{*j}(t)) \ e^{-\rho t} \ dt \ = \ U(\boldsymbol{C}^j) \ + \ \lambda^j \boldsymbol{q}^j \cdot \boldsymbol{I}^j \ . \tag{32}$$

Taking the difference of (32) between comparison and base economies gives

$$\rho \ [\int_0^\infty U(\boldsymbol{C}^{*2}(t)) e^{-\rho t} dt \ - \ \int_0^\infty U(\boldsymbol{C}^{*1}(t)) e^{-\rho t} dt] \ = \ U(\boldsymbol{C}^2) \ + \ \lambda^2 \boldsymbol{q}^2 \cdot \boldsymbol{I}^2 \ - \ U(\boldsymbol{C}^1) \ - \ \lambda^1 \boldsymbol{q}^1 \cdot \boldsymbol{I}^1 \ . \tag{33}$$

Now just using basic mathematical considerations arising from smooth differentiability of the function $U(\boldsymbol{C})$, we have

$$\int_{\boldsymbol{C}^1}^{\boldsymbol{C}^2} \boldsymbol{U}(\boldsymbol{C})\cdot d\boldsymbol{C} = U(\boldsymbol{C}^2) - U(\boldsymbol{C}^1) \quad , \tag{34}$$

where the left-hand-side integral of (34) is path-independent because the second mixed partial derivatives of $U(\boldsymbol{C})$ are equal by the assumption of continuous second derivatives.[11]

Now (30) implies directly that

$$\int_{\boldsymbol{C}^1}^{\boldsymbol{C}^2} \boldsymbol{U}(\boldsymbol{C})\cdot d\boldsymbol{C} = \lambda^1 \int_{\boldsymbol{C}^1}^{\boldsymbol{C}^2} \boldsymbol{P}^1(\boldsymbol{C})\cdot d\boldsymbol{C} \quad . \tag{35}$$

Because $\{\boldsymbol{P}^1(\bullet)\}$ and $\{\boldsymbol{D}^1(\bullet)\}$ from (18), (19), (20) are inverse functions to each other, integration by parts along any continuous connecting path yields the equation

$$\int_{\boldsymbol{C}^1}^{\boldsymbol{C}^2} \boldsymbol{P}^1(\boldsymbol{C})\cdot d\boldsymbol{C} = \boldsymbol{P}^1(\boldsymbol{C}^2)\cdot\boldsymbol{C}^2 - \boldsymbol{P}^1(\boldsymbol{C}^1)\cdot\boldsymbol{C}^1 - \int_{\boldsymbol{P}^1(\boldsymbol{C}^1)}^{\boldsymbol{P}^1(\boldsymbol{C}^2)} \boldsymbol{D}^1(\boldsymbol{p})\cdot d\boldsymbol{p} \quad . \tag{36}$$

Picking $\boldsymbol{C}=\boldsymbol{C}^2$ in (30) for j=1 and for j=2, and then comparing the resulting expression with (31) implies

$$\boldsymbol{P}^1(\boldsymbol{C}^2) = \theta\ \boldsymbol{P}^2(\boldsymbol{C}^2) \quad . \tag{37}$$

Now, by the definition (20),

[11] Actually, because the function $U(\boldsymbol{C})$ is concave, the assumption of differentiability is not even required here, since the singular points where the second derivatives do not exist or are not continuous have measure zero in the relevant domain. However, the slight gain in generality of recasting the paper without any differentiability assumptions is not worth the messy and excessively mathematical notation that is thereby required. But it could be done!

$$\boldsymbol{P}^j(\boldsymbol{C}^j) \quad = \quad \boldsymbol{p}^j \quad . \tag{38}$$

Making use of (37) and (38), expression (36) can be transformed into the equivalent form

$$\int_{\boldsymbol{C}^1}^{\boldsymbol{C}^2} \boldsymbol{P}^1(\boldsymbol{C})\cdot d\boldsymbol{C} \quad = \quad \theta\boldsymbol{p}^2\cdot\boldsymbol{C}^2 \; - \; \boldsymbol{p}^1\cdot\boldsymbol{C}^1 \; - \int_{\boldsymbol{p}^1}^{\theta\boldsymbol{p}^2} \boldsymbol{D}^1(\boldsymbol{p})\cdot d\boldsymbol{p} \; . \tag{39}$$

Next, substitute (39) into (35) into (34) to yield the equation

$$U(\boldsymbol{C}^2) \; - \; U(\boldsymbol{C}^1) \quad = \quad \lambda^1 \; [\theta\boldsymbol{p}^2\cdot\boldsymbol{C}^2 \; - \; \boldsymbol{p}^1\cdot\boldsymbol{C}^1 \; - \int_{\boldsymbol{p}^1}^{\theta\boldsymbol{p}^2} \boldsymbol{D}^1(\boldsymbol{p})\cdot d\boldsymbol{p}] \; . \tag{40}$$

Finally, substitute (40) into the right-hand side of equation (33) and use (31) to obtain the expression

$$\rho[\int_0^\infty U(\boldsymbol{C}^{*2}(t))e^{-\rho t}dt \; - \int_0^\infty U(\boldsymbol{C}^{*1}(t))e^{-\rho t}dt] \; = \; \lambda^1[\theta\boldsymbol{p}^2\cdot\boldsymbol{C}^2 + \theta\boldsymbol{q}^2\cdot\boldsymbol{I}^2 - \boldsymbol{p}^1\cdot\boldsymbol{C}^1 - \boldsymbol{q}^1\cdot\boldsymbol{I}^1 - \int_{\boldsymbol{p}^1}^{\theta\boldsymbol{p}^2} \boldsymbol{D}^1(\boldsymbol{p})\cdot d\boldsymbol{p}]. \tag{41}$$

Using (15) to abbreviate (41) and rearranging terms, we have, at last, equation (23), which is the result desired to be proved. ■

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