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AVERAGE INTEREST

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ABSTRACT

We develop analytic pricing models for options on averages by means of a state-space expansion method. These models augment the class of Asian options to markets where the underlying traded variable follows a mean-reverting process. The approach builds from the digital Asian option on the average and enables pricing of standard Asian calls and puts, caps and floors, as well as other exotica. The models may be used (i) to hedge long period interest rate risk cheaply, (ii) to hedge event risk (regime based risk), (iii) to manage long term foreign exchange risk by hedging through the average interest differential, (iv) managing credit risk exposures, and (v) for pricing specialized options like range-Asians. The techniques in the paper provide several advantages over existing numerical approaches.

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1 Introduction

Options on the average, or Asian options, have become ubiquitious in the financial world. An Asian option is a path-dependent contingent claim whose payoff depends on the arithmetic or geometric average value of the underlying's price over some time period. This paper develops an analytical approach based on expansion of the state-space for modelling these derivative securities.

Asian options have several uses. For example: (i) Banks and corporations may use them to hedge their financing costs over an extended period of time, rather than rely on more traditional contracts such as caps, floors and collars. (ii) Corporations that have cash flows over a period of time may use an Asian option instead of a series of conventional options to hedge the risks associated with these cash flows. Asian options are often cheaper than regular options, and this makes hedging more cost-effective. (iii) The writers of caps and floors may use the Asian option to hedge their risk on these contracts over several maturities. (iv) Interest differentials are known to follow mean-reverting processes, and Asian options written on the average interest differential of two currencies may be used to hedge risk in a portfolio of long term foreign currency options over a range of maturities. (v) Binary Asian options may be used to cover 'event risk'; such contracts pay off a fixed amount only if an event occurs. An example of such contracts is one where two parties contract on whether the EMS (European Monetary System) accord will occur or not. In this setting, the rationale for the binary Asian option lies in the fact that interest rates will be in one of two regimes (high or low) depending on the outcome of EMS. Since regimes are often difficult to detect empirically, writing options on the average of a financial variable over a period of time is more likely to ensure that a financial variable actually resides within a regime, than when a variable is examined only at one point in time. (vi) Likewise, Asian options also offer contracts that are less susceptible to market manipulation by the option's parties, since it is harder to manipulate a market over an extended period of time.

This paper develops a general approach to pricing average rate options based on expansion of the state space. The idea, very simply, is to expand the state space from that of a traditional Black-Scholes/Merton setup with just one state variable, the underlying, to two state variables where the second is the average (i.e. arithmetic integral) of the underlying. Using Fourier inversion methods such as those in Heston [16] and Eydeland and Geman [13], we show that the resulting valuation equations have closed-form solutions for a variety of underlying stochastic processes using arithmetic averaging. The advantage of the closed-form solution over numerical methods is that its speed is not impaired as we increase the maturity of the options, while lattice or simulation approaches would need larger grids and result in slower computational speeds to maintain required accuracy levels. The models in the paper have been implemented and offer extremely rapid and accurate option valuation for any maturity.

To demonstrate the technique, we develop closed form solutions for average-rate options under three different models for the underlying price process. Two of these models deal with mean-reverting processes, on which little work has thus far been done with regards to Asian options. Mean-reverting processes are commonly used to model economic variables such as interest rates, rate differentials, volatility, credit spreads, convenience yields, inflation, equity premia, amongst several others. In addition, we use the techniques to develop a pricing model for Asian options where the underlying is modelled using a CEV process, which would be more appropriate for equities, commodities, etc. Thus, the models presented as examples in this paper should have immediate applicability with a variety of underlying variables. In addition, closed-form solutions for more complex instruments such as average range forwards or instruments utilizing more complicated stochastic processes such as stochastic volatility, stochastic mean, or Poisson processes are also readily obtained using the techniques in the paper.

The work in this paper is related to the methods of Geman and Yor [15]. They implement via Bessel processes an approach to Asian options involving the integral of an exponential of Brownian motion. In contrast, the techniques in this paper may be used to price Asian options involving the integral of both unexponentiated Brownian motion or the exponential of Brownian motion. Therefore, it provides a more general framework for modelling the Asian option problem.

In the existing literature on Asian options, very few closed-form solutions of the type presented in this paper have been found. Analytical solutions exist for pricing Asian options on a geometric Brownian motion process but only when geometric averaging, rather than arithmetic averaging, is used.[1] The notable exception here is the model of Geman and Yor [15], who provide a solution for the arithmetic average option when the underlying follows a Bessel process. Most of the work done on techniques for pricing Asian options focuses on numerical techniques such as Monte Carlo simulation or lattice-based methods. Examples of interesting numerical techniques for the Asian option problem with geometric Brownian

[1] Analytic solutions also exist for arithmetic averaging when the underlying's price process is modelled as arithmetic Brownian motion, but this stochastic process allows the underlying's price to become negative.

motions include Dewynne and Wilmott [11], Yor [28], De-Schepper, Teunen, and Goovaerts [10] and Barraquand and Pudet [1] In addition, the overwhelming majority of work has focused on Asian options written on a stock price or a foreign exchange rate, where the use of geometric Brownian motion may be deemed appropriate.[2] Examples include Kemna and Vorst [17], Turnbull and Wakeman [24], Levy [18], Geman and Yor [15], Carverhill and Clewlow [6], Ruttiens [21], Vorst [27], and Bouaziz, Briys and Crouhy [4]. In contrast, the focus of this paper is on an analytical method for pricing Asian options that can be used potentially with any underlying process, and therefore, applicable for a wider range of underlying securities.

In order to fix ideas, we undertake the discussion of the solution technique primarily as it would apply to Asian options on interest rates. The plan of the paper is as follows. First, we demonstrate the technique in deriving an analytical formula for the binary option on the average of a square-root diffusion, i.e. a binary Asian cap. We also develop a similar equation for the binary option on the terminal interest rate, i.e. a simple binary cap. Using a put-call parity relationship for binary options we present the formulae for pricing binary floors. For completeness we also provide the solutions for the Ornstein-Uhlenbeck (O-U) process in a paradigm similar to the one for the square-root diffusion (this complements the work of Longstaff [19] who has developed a similar model using different methods). It is also important to recognize here the models in Geman and Yor [15], where Bessel process methods are used to value perpetuities in both the O-U and square-root process models[3]. On the other hand, in our paper, alternative methods for *finite time* integrals of mean-reverting Brownian motions are developed by means of state-space expansion. Numerical examples illustrate option pricing under both the square-root and O-U processes. As an example of a complex security, we also price range Asian options. We then show that the pricing technique can be employed with equal success for stochastic processes without mean-reversion. Specifically we develop closed-form solutions for an Asian option written on a CEV process. The final section offers concluding comments.

[2]Actually, in the case of foreign exchange rates, geometric Brownian motion is probably not appropriate since exchange rates tend to display mean-reversion, especially over longer periods, a property not captured by a geometric Brownian motion process.

[3]Perpetuities are also integrals of exponentials of a Brownian motion and hence are logically subsumed within the framework of Geman and Yor [15]. This issue also connects with the work on perpetuities by Dufresne [12].

2 Stochastic Processes

The paper begins with an analysis of the Asian interest rate option for square-root processes. We use this process to demonstrate the use of state-space expansion techniques in pricing path-dependent contingent claims. In Section 6 of the paper, we develop analogous results for the O-U process.

We assume that interest rates $r(t), t \in [0, \infty]$ follow a square-root diffusion as in Cox-Ingersoll-Ross [9]:

$$dr(t) = k\left[\theta - r(t)\right]dt + \eta\sqrt{r(t)}dz(t), \quad r(0) = r_0. \tag{1}$$

The long run mean of the interest rate is θ, and the interest rate reverts to this mean at rate k. The diffusion has square-root volatility with coefficient η, and $z(t)$ is the standard Wiener shock. We now expand the state-space by introducing the state variable $X(T)$ which represents the average of the underlying. To do so, we define the average interest rate over the interval $[0, T]$ as

$$X(T) = \frac{1}{T}\int_0^T r(u)du$$

By expanding the state-space, the valuation equation which governs the price of an average-rate option will depend only on the current values of state variables rather than both current and past values.

It is easier to work with the sum of the interest rate over $[0, T]$ which we denote

$$Y(T) = \int_0^T r(u)du$$

which in differential form may be written as

$$dY(t) = r(t)dt, \quad Y(0) = Y_0. \tag{2}$$

We assume the existence of a risk-neutral equivalent martingale measure Q under which the discounted value of interest rate dependent securities follow martingales. The interest rate process under this measure has a risk-adjusted drift and is re-expressed as

$$dr(t) = \left(k\left[\theta - r(t)\right] - \phi r(t)\right)dt + \eta\sqrt{r(t)}dz(t), \quad r(0) = r_0. \tag{3}$$

The risk adjustment (as derived in Cox-Ingersoll-Ross [9]) is proportional to the level of the interest rate $r(t)$, and depends on coefficient ϕ. Further analysis will be undertaken employing the risk neutral interest rate process in equation (3) and the cumulative interest rate process in equation (2).

3 Binary Asian Options

3.1 Definition

We first derive the equation for pricing binary (i.e. digital) Asian options. The pricing of pure (non-Asian) digital options is undertaken in Turnbull [25].[4] The technique for this requires the derivation of the density function of $X(T)$ which is simply the density function for $\frac{1}{T}Y(T)$. The derivation of this density function is original. We shall then employ the approach developed in Heston [16] to value this digital option. Employing this as a building block, we then derive the pricing model for regular Asian options on interest rates.

The simplest form of binary option is a cash or nothing option. A cash or nothing call option (F) with strike rate K and maturity T on the average interest rate $X(T)$ pays off a fixed amount M if $X(T) \geq K$, else it expires worthless. A cash or nothing put option (F') with strike rate K and maturity T on the average interest rate $X(T)$ pays off a fixed amount M if $X(T) \leq K$, else it expires worthless. We use the terms 'calls' and 'caps' interchangeably here. Likewise, 'puts' and 'floors' are synonymous. We restrict our exposition to calls, and develop the pricing of puts in Section 5 by exploiting binary put-call parity. Define the value of the binary option as $t = 0$ as:

$$\begin{aligned} F &= M \int_K^\infty e^{-\int_0^T r(s)ds} f\left[X(T)\right] dX(T) \\ &= M \int_K^\infty e^{-Y(T)} f\left[X(T)\right] dX(T) \\ &= M \int_{K'}^\infty e^{-Y(T)} f\left[Y(T)\right] dY(T), \end{aligned} \tag{4}$$

where $K' = KT$, and $f[Y(T)]$ is the density function for $Y(T)$ given $Y(0)$. Hence, if we derive the density function for $Y(T)$, i.e. $f\left[Y(T)\right]$, then we would be in a position to compute the value of this option via simple integration. However, the integral itself may be expressed as the solution to a partial differential equation, which we shall derive, and then solve. With this set up, we proceed to derive an analytical expression for the value of the entire integral.

[4] A digital option pays off a constant amount if the option ends up in the money, else pays off nothing.

3.2 Derivation

Here, we now develop the pricing algorithm to exploit the equation (4) above. The price of the Asian binary call may be re-written as

$$
\begin{aligned}
F &= M \int_{K'}^{\infty} e^{-Y(T)} f\left[Y(T)\right] dY(T) \\
&= M E_{Q'}\left[e^{-\int_0^T r(u)du} dY(T)\right] \\
&= M\, E_Q\left[e^{-\int_0^T r(u)du}\right] E_{Q'}\left[\frac{e^{-\int_0^T r(u)du}}{E_Q\left(e^{-\int_0^T r(u)du}\right)}\right] \\
&= M\, P(r_0, T)\, E_{Q'}\left[\frac{e^{-\int_0^T r(u)du}}{E_Q\left(e^{-\int_0^T r(u)du}\right)}\right] \\
&= M\, P(r_0, T)\ \Pi(r_0, Y_0, T)
\end{aligned}
$$

where $E_Q[.]$ and $E_{Q'}[.]$ stand for the expectations operator over the support $[0, \infty]$ and $[K', \infty]$ respectively, and line 3 of the equation above obtains from the law of iterated expectations. $P(r_0, T)$ is simply the CIR bond pricing equation because $E_Q[e^{-\int_0^T r(u)du}]$ is simply the expectation of the line integral over the interest rate sample path, leading to the solution below:

$$
\begin{aligned}
P(r_0, T) &= \exp\left[C(T) + rD(T)\right] \\
C(T) &= \ln\left[C'(T)\right] \\
C'(T) &= \left[\frac{2\gamma e^{(k+\phi+\gamma)T/2}}{(k+\phi+\gamma)\left[e^{\gamma T} - 1\right] + 2\gamma}\right]^{\frac{2k\theta}{\eta^2}} \\
D(T) &= \frac{-2\left[e^{\gamma T} - 1\right]}{(k+\phi+\gamma)\left[e^{\gamma T} - 1\right] + 2\gamma} \\
\gamma &= \sqrt{(k+\phi)^2 + 2\eta^2}
\end{aligned} \tag{5}
$$

Recognize that $\Pi(r_0, Y_0, T)$ is a conditional cumulative probability. It may be interpreted as the probability under the martingale measure that the option expires in the money conditional on (r_0, Y_0) which forms the expanded state spaces. We shall solve for the characteristic function of $\Pi(.)$ and then employ Fourier inversion to obtain the required probability.

For simplicity, normalize $M = 1$, i.e. the binary option has a unitary payoff! The call option then takes value $F_0 = P(r_0, T)\ \Pi(r_0, Y_0, T)$. By the Fundamental theorem of Asset

Pricing (i.e. an exploitation of no-arbitrage), it is easy to show that the call price must satisfy the following partial differential equation:

$$\frac{1}{2}\eta^2 r\frac{\partial^2 F}{\partial r^2}+[k(\theta-r)-\phi r]\frac{\partial F}{\partial r}+r\frac{\partial F}{\partial Y}-\frac{\partial F}{\partial T}-rF=0. \tag{6}$$

subject to the boundary condition $F_T = 1_{Y(T)\geq K'}$. Since $F = P\Pi$, we rewrite equation (6) as

$$\begin{aligned}0 &= \frac{1}{2}\eta^2 r\Pi\frac{\partial^2 P}{\partial r^2}+\eta^2 r\frac{\partial P}{\partial r}\frac{\partial \Pi}{\partial r}+\frac{1}{2}\eta^2 rP\frac{\partial^2 \Pi}{\partial r^2}\\ &\quad+[k(\theta-r)-\phi r]\,\Pi\frac{\partial P}{\partial r}+[k(\theta-r)-\phi r]\,P\frac{\partial \Pi}{\partial r}\\ &\quad+rP\frac{\partial \Pi}{\partial Y}-\Pi\frac{\partial P}{\partial T}-P\frac{\partial \Pi}{\partial T}-rP\Pi.\end{aligned}$$

This equation can then be re-arranged to

$$\begin{aligned}0 &= \left[\frac{1}{2}\eta^2 r\frac{\partial^2 P}{\partial r^2}+[k(\theta-r)-\phi r]\frac{\partial P}{\partial r}-\frac{\partial P}{\partial T}-rP\right]\Pi\\ &\quad+\eta^2 r\frac{\partial P}{\partial r}\frac{\partial \Pi}{\partial r}+\frac{1}{2}\eta^2 rP\frac{\partial^2 \Pi}{\partial r^2}+[k(\theta-r)-\phi r]\,P\frac{\partial \Pi}{\partial r}+rP\frac{\partial \Pi}{\partial Y}-P\frac{\partial \Pi}{\partial T}.\end{aligned}$$

The expression in parenthesis in the first line of the equation above is identical to the PDE obtained in deriving the Cox-Ingersoll-Ross bond pricing model (see page 393 of [9], equation 22). Hence, it is zero, and this reduces our PDE to

$$\begin{aligned}0 &= \frac{1}{2}\eta^2 rP\frac{\partial^2 \Pi}{\partial r^2}+\left[k(\theta-r)-\phi r+\eta^2 r\frac{\partial P}{\partial r}\frac{1}{P}\right]P\frac{\partial \Pi}{\partial r}+rP\frac{\partial \Pi}{\partial Y}-P\frac{\partial \Pi}{\partial T}\\ &= \frac{1}{2}\eta^2 r\frac{\partial^2 \Pi}{\partial r^2}+\left[k(\theta-r)-\phi r+\eta^2 r\frac{\partial P}{\partial r}\frac{1}{P}\right]\frac{\partial \Pi}{\partial r}+r\frac{\partial \Pi}{\partial Y}-\frac{\partial \Pi}{\partial T}.\end{aligned} \tag{7}$$

Note that equation (7) is similar to equation (6) with the additional time-varying term $\left(\eta^2 r\frac{\partial P}{\partial r}\frac{1}{P}\right)$ which make the PDE in (7) difficult to solve. As explained above, Π is a conditional probability. Therefore, the boundary condition for equation (7) is given by $\Pi(r_T, Y_T, 0) = 1_{Y_T\geq K'}$. We do know that the solution for P is given by $P = \exp[C + rD]$ with the form expressed in equation (5). However, we do need to obtain the solution for Π. We guess a solution of the form

$$\Pi=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left[\frac{1}{i\xi}e^{-i\xi K'}H(Y)\right]d\xi$$

which represents $\Pi(.)$ as the Fourier inversion of the function $H(.)$:

$$H(r,Y,T)=\exp\left[(\overline{C}-C)+(\overline{D}-D)r+sY\right],$$

with boundary condition

$$H(r,Y,T=0)=e^{sY}, \quad s=i\xi, \quad i=\sqrt{-1}.$$

where $H(r,Y,T)$ is the characteristic function for the probability function $\Pi(r,Y,T)$. $H(.)$ must also satisfy equation (7). This transforms equation (7) to

$$\begin{aligned}0 &= \frac{1}{2}\eta^2 r(\overline{D}-D)^2+\left[k(\theta-r)-\phi r+\eta^2 r\frac{\partial P}{\partial r}\frac{1}{P}\right](\overline{D}-D)+rs \\ &-\frac{\partial}{\partial T}(\overline{D}-D)r-\frac{\partial}{\partial T}(\overline{C}-C).\end{aligned}$$

Further simplification leads to:

$$\begin{aligned}&\frac{1}{2}\eta^2 r\overline{D}^2+[k(\theta-r)-\phi r]\,\overline{D}-r\overline{D}_T-\overline{C}_T+rs \\ =\ &\frac{1}{2}\eta^2 rD^2+[k(\theta-r)-\phi r]\,D-rD_T-C_T\end{aligned}$$

and the RHS of the above equation simply equates to r (as in CIR) so that

$$\frac{1}{2}\eta^2 r\overline{D}^2+[k(\theta-r)-\phi r]\,\overline{D}-r\overline{D}_T-\overline{C}_T=(1-s)r.$$

Collecting terms in r this resolves into 2 ordinary equations using standard separability arguments:

$$\begin{aligned}\frac{d\overline{D}}{dT} &= \frac{1}{2}\eta^2\overline{D}^2-(k+\phi)\overline{D}-(1-s), \quad \overline{D}(0)=0, \\ \frac{d\overline{C}}{dT} &= k\theta\overline{D}, \quad \overline{C}(0)=0.\end{aligned}$$

As shown in the appendix, the solution to this set of ODEs is

$$\begin{aligned}H(r,Y,T) &= \exp\left[(\overline{C}-C)+(\overline{D}-D)r+sY\right], \\ C(T) &= \ln\left[C'(T)\right], \\ C'(T) &= \left[\frac{2\gamma e^{(k+\phi+\gamma)T/2}}{(k+\phi+\gamma)\left[e^{\gamma T}-1\right]+2\gamma}\right]^{\frac{2k\theta}{\eta^2}}, \\ D(T) &= \frac{-2\left[e^{\gamma T}-1\right]}{(k+\phi+\gamma)\left[e^{\gamma T}-1\right]+2\gamma}, \\ \gamma &= \sqrt{(k+\phi)^2+2\eta^2}, \\ \overline{C}(T) &= \frac{2k\theta}{\eta^2}\ln\left[\frac{r_1-r_2}{r_1e^{r_2T}-r_2e^{r_1T}}\right],\end{aligned}$$

$$\begin{aligned}\overline{D}(T) &= \frac{2}{\eta^2}\left[\frac{r_1 r_2\left(e^{r_1 T}-e^{r_2 T}\right)}{r_1 e^{r_2 T}-r_2 e^{r_1 T}}\right],\\ r_1 &= -\frac{1}{2}(k+\phi)+\sqrt{(k+\phi)^2+2(1-s)\eta^2}\\ r_2 &= -\frac{1}{2}(k+\phi)-\sqrt{(k+\phi)^2+2(1-s)\eta^2}\end{aligned}$$

Using Fourier inversion to transform from $H(.)$ to $\Pi(.)$, the Asian binary call (ABC) option is then written as

$$\begin{aligned}F &= M\,P(r_0,T)\;\Pi(r_0,Y_0,T;K)\\ &= M\,P(r_0,T)\;\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left[\frac{1}{i\xi}e^{-i\xi KT}H(r_0,Y_0,T)\right]d\xi\right)\\ &= M\,P(r_0,T)\;\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left[\frac{1}{i\xi}e^{-i\xi K'}H(r_0,Y_0,T)\right]d\xi\right)\end{aligned}$$

where K is the strike price of the option relative to X, and enters the expression above as $K' = KT$ since we are working with $Y(T)$, and not $X(T)$ and we know that $Y(T) = TX(T)$.

If the option is being priced at inception, the value for Y will be $Y_0 = 0$ in the equation above. If the option is being priced midway through its life (say, t years after inception) the current average of the interest rate since inception (denoted X') is used and we set $Y_0 = tX'$. The expression above is extremely easy to evaluate using standard mathematical software, and pricing takes a few seconds.

4 Pricing Binary Interest Rate Caps

So far, we have priced binary average rate options. In this section we develop a model for simple binary interest rate caps (i.e. non-average rate options, with payoffs determined only by the terminal interest rate). For consistency, we employ the same methodology as we did with the Asian option. Once again, we assume the interest rate process in equation (1). The binary cap with a nominal value of \$1 at a strike rate K,and maturity T is as follows

$$J = \int_K^\infty \exp\left[-\int_0^T r_T(u)du\right] f\left(r_T\right) dr_T$$

where r_T is the uncertain terminal interest rate, $f(r_T)$ is the probability density of r_T given an initial value of the interest rate r_0 at time 0. It is required that J satisfy the fundamental

PDE

$$\frac{1}{2}\eta^2 r\frac{\partial^2 J}{\partial r^2} + [k(\theta - r) - \phi r]\frac{\partial J}{\partial r} - \frac{\partial J}{\partial T} - rJ = 0. \tag{8}$$

subject to the boundary condition $J_T = 1_{r_T \geq K}$.

We can write the expression for J as

$$\begin{aligned} J &= E_{Q'}\left[e^{-\int_0^T r(u)du} dr(T)\right] \\ &= E_Q\left[e^{-\int_0^T r(u)du}\right] E_{Q'}\left[\frac{e^{-\int_0^T r(u)du}}{E_Q\left(e^{-\int_0^T r(u)du}\right)}\right] \\ &= P(r_0, T)\, E_{Q'}\left[\frac{e^{-\int_0^T r(u)du}}{E_Q\left(e^{-\int_0^T r(u)du}\right)}\right] \\ &= P(r_0, T)\ \Omega(r_0, T) \end{aligned}$$

and $E_Q[.]$ and $E_{Q'}[.]$ stand for the expectations operator over the support $[0, \infty]$ and $[K, \infty]$ respectively, and $P(r_0, T)$ is the CIR bond pricing equation given in (5). It is clear that $\Omega(.)$ is a probability, and we can solve for its characteristic function, which we shall denote $G(.)$. Then, by Fourier inversion we can evaluate the probability. We guess a solution to Ω as the Fourier inversion of its characteristic function $G = \exp[A(T) + rB(T)]$. We already know the solution to $P = \exp[C(T) + rD(T)]$ [see equation (5)]. Rewriting the PDE in equation (8), (using $J = PG$, since G must also satisfy the PDE) we obtain

$$\begin{aligned} 0 &= [k(\theta - r) - \phi r]\left[P\frac{\partial G}{\partial r} + G\frac{\partial P}{\partial r}\right] + \frac{1}{2}\eta^2 r\left[P\frac{\partial^2 G}{\partial r^2} + 2\frac{\partial P}{\partial r}\frac{\partial G}{\partial r} + G\frac{\partial P}{\partial r}\right] \\ & -P\frac{\partial G}{\partial T} - G\frac{\partial P}{\partial T} - rPG, \end{aligned}$$

which we rewrite as

$$\begin{aligned} 0 &= G\left[[k(\theta - r) - \phi r]\frac{\partial P}{\partial r} + \frac{1}{2}\eta^2 r\frac{\partial P}{\partial r} - \frac{\partial P}{\partial T} - rP\right] \\ & [k(\theta - r) - \phi r]P\frac{\partial G}{\partial r} + \frac{1}{2}\eta^2 r\left[P\frac{\partial^2 G}{\partial r^2} + 2\frac{\partial P}{\partial r}\frac{\partial G}{\partial r}\right] - P\frac{\partial G}{\partial T}, \end{aligned}$$

and since the first line in the equation above contains the CIR partial differential equation and must equal zero, this expression simplifies to

$$\begin{aligned} 0 &= [k(\theta - r) - \phi r]P\frac{\partial G}{\partial r} + \frac{1}{2}\eta^2 r\left[P\frac{\partial^2 G}{\partial r^2} + 2\frac{\partial P}{\partial r}\frac{\partial G}{\partial r}\right] - P\frac{\partial G}{\partial T} \\ &= [k(\theta - r) - \phi r]\frac{\partial G}{\partial r} + \frac{1}{2}\eta^2 r\left[\frac{\partial^2 G}{\partial r^2} + 2\frac{1}{P}\frac{\partial P}{\partial r}\frac{\partial G}{\partial r}\right] - \frac{\partial G}{\partial T}. \end{aligned}$$

Using the posited solutions for P, G we further obtain

$$0=[k(\theta-r)-\phi r]B+\frac{1}{2}\eta^2 rB^2+\eta^2 rDB-A_T-rB_T.$$

By separation of variables we arrive at

$$0=r\left[\frac{1}{2}\eta^2B^2-(k+\phi-\eta^2D)B-B_T\right]+\left[-A_T+k\theta B\right],$$

which results in two ordinary differential equations. The solution to the two ODEs is derived in the Appendix and is presented below:

$$\begin{aligned}
G(r,T) &= \exp\left[A(T)+rB(T)\right]\\
A(T) &= \frac{-2k\theta}{\eta^2}\ln\left[\frac{1}{2}\eta^2 sD+1\right]\\
B(T) &= \frac{-2sD'}{2+\eta^2 sD}\\
D(T) &= \frac{-2\left[e^{\gamma T}-1\right]}{(k+\phi+\gamma)\left[e^{\gamma T}-1\right]+2\gamma},\\
D' &= \frac{\partial D}{\partial T}=-4\gamma^2\left[\frac{e^{\gamma T/2}}{(k+\gamma)(e^{\gamma T}-1)+2\gamma}\right]^2\\
\gamma &= \sqrt{(k+\gamma+\phi)^2+2\eta^2}.
\end{aligned}$$

Using Fourier inversion to transform $G(.)$ to $\Omega(.)$, the binary call option with face value M is then written as

$$\begin{aligned}
J &= M\,P(r_0,T)\;\Omega(r_0,T;K)\\
&= M\,P(r_0,T)\;\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left[\frac{1}{i\xi}e^{-i\xi K}G(r_0,T)\right]d\xi\right)
\end{aligned}$$

where K is the strike price of the option relative to r. Once again, this expression is evaluated in a few seconds on a personal computer.

5 Pricing Floors using Binary Parity

Pricing binary puts is extremely easy employing a simple parity relationship that exists between binary calls and puts. Since the binary Asian call was denoted F, we shall denote the binary Asian put as F'. The parity relationship that holds is as follows:

$$F(r_0,Y_0,T,K)+F'(r_0,Y_0,T,K)=P(r_0,T)$$

Similarly for the simple (i.e. non-Asian) binary options a parity relationship is applicable:

$$J(r_0, T, K) + J'(r_0, T, K) = P(r_0, T)$$

6 Equations for the Ornstein-Uhlenbeck Process

In this section, we provide solutions to the prices of binary Asian options and caps on the interest rate when the interest rate process follows a mean-reverting Ornstein-Uhlenbeck process as in [26]. For yields, Longstaff [19] also provides a model using the Vasicek process. The interest rate is assumed to be governed by the following stochastic differential equation:

$$dr(t) = k\,[\theta - r(t)]\,dt + \eta dz(t), \quad r(0) = r_0.$$

where the coefficients are as defined before. Under the martingale measure, the interest rate dynamics are given by:

$$dr(t) = k\left[\theta - r(t) - \frac{\eta\phi}{k}\right] dt + \eta dz(t), \quad r(0) = r_0.$$

where ϕ represents the coefficient of the market price of risk. The price $P(r_0, T)$ at time 0 of a zero-coupon bond maturing at time T is then given by

$$\begin{aligned} P(r_0, T) &= \exp\left[C(T) + D(T)r\right] \\ C(T) &= \frac{\eta^2}{4k^3}\left(1 - e^{-2kT}\right) + \frac{\eta^2 - kc}{k^3}\left(e^{-kT} - 1\right) + \frac{\eta^2 - 2kc}{2k^2}T \\ D(T) &= \frac{1}{k}\left(e^{-kT} - 1\right) \\ c &= k\theta - \eta\phi \end{aligned} \tag{9}$$

6.1 Binary Asian Options

As before define $Y(T) = \int_0^T r(u)\,du$. Then, the price F at time 0 of a binary Asian option with strike K satisfies the following partial differential equation:

$$\frac{1}{2}\eta^2\frac{\partial^2 F}{\partial r^2} + [k(\theta - r) - \phi\eta]\frac{\partial F}{\partial r} + r\frac{\partial F}{\partial Y} - \frac{\partial F}{\partial T} - rF = 0. \tag{10}$$

with the boundary condition $F_T = M \times 1_{Y(T) \geq K}$. As before, we assume that the option has a unitary payoff, i.e., $M = 1$. The price of the option then takes the form

$F = P(r_0,T)\Pi(r_0,Y_0,T)$ where $\Pi(r_0,Y_0,T)$ can be interpreted as the probability under the martingale measure that the option expires in the money. Following a process similar to that used earlier, this solution form may be substituted into equation (10) and simplified to yield the following partial differential equation for $\Pi(r_0,Y_0,T)$:

$$0 = \frac{1}{2}\eta^2\frac{\partial^2\Pi}{\partial r^2} + \left[k(\theta - r) - \phi\eta + \eta^2\frac{\partial P}{\partial r}\frac{1}{P}\right]\frac{\partial\Pi}{\partial r} + r\frac{\partial\Pi}{\partial Y} - \frac{\partial\Pi}{\partial T}$$

Since $\Pi(r_0,Y_0,T)$ is a risk-neutral probability, the boundary condition is given by $\Pi(r_T,Y_T,0) = 1_{Y(T)\geq K'}$. Alternatively, we can solve for the Fourier transform, $\hat{H}(r,Y,T)$, of Π. The Fourier transform satisfies the same partial differential equation as Π, but the boundary condition is given by

$$\hat{H}(r_T,Y_T,0) = e^{sY},\ \ s = i\xi,\ \ i = \sqrt{-1}.$$

The solution for the Fourier transform of Π is[5] given by

$$\begin{aligned}
\hat{H}(r,Y,T) &= \exp\left[\bar{C}(T) - C(T) + \left(\bar{D}(T) - D(T)\right)r + sY\right] \\
\bar{C}(T) &= \frac{a^2\eta^2}{4k}\left(1 - e^{-2kT}\right) + \frac{a^2\eta^2 - ac}{k}\left(e^{-kT} - 1\right) + \left(\frac{1}{2}a^2\eta^2 - ac\right)T \\
\bar{D}(T) &= a\left(e^{-kT} - 1\right) \\
a &= \frac{1-s}{k} \\
c &= k\theta - \eta\phi
\end{aligned}$$

and $C(T)$ and $D(T)$ are given in the Vasicek bond pricing formula in equation (9). Consequently, the formula for $\Pi(r_0,Y_0,T)$ can be calculated as

$$\Pi(r_0,Y_0,T) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{1}{i\xi}e^{-i\xi K'}H(r_0,Y_0,T)\right]d\xi$$

where $K' = KT$. Thus, the price of a binary Asian option with a unitary payoff is given by

$$F_0 = P(r_0,T)\ \left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{1}{i\xi}e^{-i\xi K'}H(r_0,Y_0,T)\right]d\xi\right)$$

6.2 Binary Interest Rate Caps

In this section, we develop the pricing formulae for binary interest rate caps for the O-U process. As with the binary Asian options in the previous section, we follow a procedure

[5]The proof is omitted and is available on request.

similar to that used above for the square-root process. Let J represent the price at time 0 of a binary interest rate cap with a unitary payoff if the interest rate is above the strike price. The cap satisfies the fundamental partial differential equation

$$\frac{1}{2}\eta^2\frac{\partial^2 J}{\partial r^2} + [k(\theta - r) - \phi\eta]\frac{\partial J}{\partial r} - \frac{\partial J}{\partial T} - rJ = 0. \tag{11}$$

with the boundary condition $J_T = 1_{r_T \geq K}$. We again assume that the price of the option takes the form $J_0 = P(r_0, T)\Omega(r_0, T)$ where $\Omega(r_0, T)$ has the interpretation of being the probability under the martingale measure that the option expires in the money. Substituting this solution form into equation (11) and simplifying results in a partial differential equation for $\Omega(r_0, T)$.

$$0 = \frac{1}{2}\eta^2\frac{\partial^2\Omega}{\partial r^2} + [k(\theta - r) - \phi\eta + \eta^2\frac{1}{P}\frac{\partial P}{\partial r}]\frac{\partial\Omega}{\partial r} - \frac{\partial\Omega}{\partial T}$$

Since Ω may be thought of as a conditional probability, the boundary condition for this equation is given by $\Omega(r_T, 0) = 1_{r_T \geq K}$. Rather than solving for Ω directly, we can solve for its Fourier transform, $\hat{G}(r_0, T)$. The Fourier transform satisfies the same partial differential equation as Ω but with the boundary condition $\hat{G}(r_T, 0) = e^{sr}$. Therefore, the solution for $\hat{G}(r_0, T)$ is given by[6]

$$\begin{aligned}
\hat{G}(r_0, T) &= \exp\left[\bar{C}(T) - C(T) + \left(\bar{D}(T) - D(T)\right) r\right] \\
\bar{C}(T) &= \frac{a^2\eta^2}{4k}\left(1 - e^{-2kT}\right) + \frac{a\eta^2 - ack}{k^2}\left(e^{-kT} - 1\right) + \left(\frac{\eta^2}{2k^2} - \frac{c}{k}\right)T \\
\bar{D}(T) &= ae^{-kT} - \frac{1}{k} \\
a &= s + \frac{1}{k} \\
c &= k\theta - \eta\phi
\end{aligned}$$

Using Fourier inversion of $\hat{G}(r_0, T)$, we obtain the price of the cap as

$$J = P(r_0, T)\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{1}{i\xi}e^{-i\xi K}G(r_0, T)\right]d\xi\right)$$

7 Probability density functions

Fourier inversion also provides the probability density functions (h and $\hat{h}$) for the two average interest rate processes, i.e. when the interest rate follows (i) a square-root process and (ii)

[6] Once again, the proof is omitted.

an Ornstein-Uhlenbeck process. The equation for obtaining the density function for the square-root process is

$$h(r_0, Y_0, T; Y) = \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}[\exp(-i\xi Y)H(r_0, Y_0, T)]\, d\xi$$

The equation for obtaining the density function for the O-U process is

$$\hat{h}(r_0, Y_0, T; Y) = \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}[\exp(-i\xi Y)\hat{H}(r_0, Y_0, T)]\, d\xi$$

The plots for the square-root density and O-U density are provided in Figures 1 and 2 respectively.

8 Numerical Examples

While the procedure provided above is accurate and fast, it is also of interest to examine how Asian interest rate options behave for different parameter values. So far, the pricing models in the paper have discussed only binary options: the Asian binary call (F), and simple binary call (J). We now introduce formulae for non-binary (regular) options, which are derived by employing the binary options as raw material, and integrating over a continuum of strike prices. We price four different types of options: (i) Asian binary call (or cap), (ii) Non-Asian binary call, i.e. a simple digital option on the interest rate, (iii) Asian non-binary (regular) call, i.e. a non-digital call on the average interest rate, and (iv) Regular (non-Asian) call on the interest rate. Thus, we consider Asian vs non-Asian options and their combinations with binary vs non-binary forms.

8.1 *Regular (non-binary) Options*

The formula for the Asian call (F_1) at a strike price X_1 is simply a sum of Asian binary calls (F) at ascending strikes ΔX apart:

$$F_1 = \sum_{n=0}^{\infty} F(., X_1 + n\Delta X)\, \Delta X$$

which in the limit is

$$F_1 = \int_{X_1}^{\infty} F(., X)\, dX$$

Likewise, the formula for the regular call (J_1) at a strike price X_1 is a sum of Asian binary calls (J) at ascending strikes ΔX apart:

$$J_1 = \sum_{n=0}^{\infty} J(., X_1 + n\Delta X)\,\Delta X$$

which in the limit is

$$J_1 = \int_{X_1}^{\infty} J(., X)\,dX$$

This simple exploitation of the building block property of digital options extends the use of the derived results in the paper to standard options.

8.2 *Range-Asian Options*

Apart from the four types of options introduced above, a second set of results looks at the valuation of "range-Asian" options (denoted by the value R). These are options that have daily pay offs which are based on whether the average interest rate up to time t lies within a prespecified range $[a(t), b(t)], a(t) < b(t), \forall t \in [0, T]$. In the paper we assume that $a(t) = a$ and $b(t) = b$, without loss of generality. The value of these options is simply

$$\begin{aligned} R[a(t), b(t), T] &= \frac{1}{d}\sum_{j=1}^{d}[Q(a(t), t) - Q(b(t), t)] \\ d &= \text{Flr}(T * 365) \\ t &= \frac{j}{365} \end{aligned}$$

where $\text{Flr}(x)$ is a function that returns the greatest integer less than or equal to x. Our analyses utilize both, the square-root and the O-U process models.

8.3 *Analysis*

Tables 1 thru 4 present option prices for the square-root model. Tables 5 thru 6 present option prices for the O-U model. Table 7 presents values for the range-Asian option. The notional values underlying all the options is a dollar. Several results emanate from the numerical analysis. Binary options are worth more than regular options. This is because the binary options always pay off a dollar. The regular options would most likely never pay off more than 30 cents on the dollar, and would require terminal interest rates of over 100% to pay off a dollar.

It is useful to obtain a visual feel for the probability distribution of the average interest rate. This provides a means to directly compare Asian and non-Asian option values. In Figures 1 and 2, we present the probability density functions for the average interest rate and terminal interest rate. Figure 1 is for the square-root model and Figure 2 for the O-U model. The density of the terminal interest rate is fatter-tailed than that of the average interest rate. These density representations are quite intuitive since it is reasonable that the average of the interest rate will display a tighter density function, as it has less variability. For the square-root model the terminal interest rate density tends to be negatively skewed relative to that of the average rate density, which may explain why at-the-money simple digital options are priced cheaper than the comparable Asian digital options. This effect appears to be reversed for the O-U model where the terminal interest rate density does not display this negative tendency.

How does the moneyness of the option affect the comparison of Asian and non-Asian options? Here we may refer to Tables 1 thru 6. When the exercise price X is below the interest rate r, the Asian binary option is worth more than the non-Asian binary option. This is because an option which is in-the-money at the outset is more likely to sustain that value if it is an Asian option than a non-Asian option, since even if interest rates fall, the average of the interest rate will not fall as fast. Likewise, when the exercise price X is above the interest rate r, the Asian binary option is worth less than the non-Asian binary option. This is because an option which is out of the money at the outset is more likely to remain out of the money if it is an Asian option than a simple option, since even if interest rates rise, the average of the interest rate will not rise as fast.

Is the effect of volatility any different from that with equity options? When volatility increases (as in Table 2 versus Table 1), the options that are out-of-the-money increase in value, since the likelihood of the options finishing in-the-money increases when volatility increases. Conversely, the in-the-money options decline in value when volatility increases. This effect holds for both the digital Asian option and the simple digital option. This matches the features we observe when pricing equity options.

Since mean reversion is a feature that distinguishes interest rate dependent securities from equity dependent securities, it is instructive to look at the parameter θ, the long-run mean of the interest rate. In the presence of mean reversion, the location of the long run mean is critical for the pricing of options. A comparison of Tables 1,3 and 4 reveals that the location of the mean rate (θ) impacts options values quite severely. When the mean is low ($\theta = 0.05$) option values drop substantially. This is especially true for the non-Asian options and the effect is less marked for the Asian options. This is because when pricing calls, if the mean rate is low, then mean-reversion drags down the value of the terminal interest rate and the average interest rate. Likewise, when the mean is high ($\theta = 0.15$), option values are substantially higher. Hence, unlike with equity options, even if the option is deep in-the-money, the interest rate option may still not offer much value, if the interest rate quickly reverts to a mean level which is quite low.

One question of importance when pricing Asian options on interest rates, is that of model choice. Are the values of Asian options very sensitive to the specific choice of stochastic process for interest rates? To answer this question, we compare prices from the O-U model with those of the square-root diffusion, taking care to ensure that the average volatility in both models is held constant. Tables 5 and 6 provide results for the O-U process. They are analogous to Tables 1 and 2 for the square-root process. The average volatility of the O-U process has been set approximately equal to that of the square-root process by means of the following equation: $\eta_{OU} = \eta_{SQR}\sqrt{r_0}$, where η_{OU} is the O-U process volatility coefficient, η_{SQR} is the coefficient for the square-root process, and $r_0 = \theta$. Therefore, in Table 5 we use $\eta_{OU} = 0.20\sqrt{0.1} = 0.063246$. In Table 6, the value is $\eta_{OU} = 0.30\sqrt{0.1} = 0.094868$. In general, the prices from the O-U model are fairly close to, though higher than that of the square-root model. We can thus conclude that the choice of interest rate model does not substantially impact Asian option prices. One reason for this is that mean-reversion makes the level-dependent volatility of the square-root diffusion likely to be quite stable, and hence, especially for the Asian option, this average volatility will be quite close to the constant level of volatility in the O-U model.

From the analysis so far, it is clear that one of the most interesting differences between equity options and interest rate options is the feature of mean-reversion. Mean-reversion tends to reduce average volatility over time, reducing option values in most cases. However, it also affects the direction (skewness) of the interest rate depending on where the current rate of interest is relative to its long-run mean level. Depending on how strong the rate of mean reversion is, it may cause away from the money options to demonstrate interesting behavior. For example, if the rate of mean reversion (k) is high, out-of-the-money options become more likely to swing into-the-money, and vice versa for in-the-money options. Mean reversion (k) has an interesting impact on option prices as time to maturity (T) varies. As T increases, options prices increase at first and then decline as the effect of mean reversion begins to negate the effect of volatility. To understand mean reversion, we decided to explore a more exotic option, the range-Asian. The range-Asian is an interesting option to analyze because it offers a good setting in which the joint effects of the mean rate θ, and the rate of mean-reversion k, may be examined. In general, an Asian option is one which pays off a certain amount each day if the value of the underlying variable lies within a pre-specified range. The range-Asian pays off each day that the current average up to that date remains within pre-specified limits. In Table 7 we present prices of range-Asian options. These prices increase when the range widens. When the mean rate θ lies inside the range, increases in mean reversion (k) drive the price upwards. This is because, as k rises, the likelihood of the interest rate remaining within the range increases, thereby raising value. When the mean lies outside the range, option prices decrease when k increases because the interest rate is less likely to remain in the desired range. This is true of both cases, when the mean is above and below the range, i.e. $\theta = 0.15$ and $\theta = 0.05$ respectively.

9 Asian Options on Equities

While the pricing formulas thus far developed have been for mean-reverting processes, the techniques are equally effective in developing pricing formulas for stochastic processes that do not display mean-reversion. Such processes are important for modelling equity and commodity prices. In this section, we demonstrate the use of the pricing techniques developed above for the case of binary Asian options when the underlying is a CEV process.

Let S_t represent the price of a stock, for example. We assume that the price process for the stock is given by the CEV process

$$dS_t = \mu S_t \, dt + \sigma \sqrt{S_t} \, dW_s$$

Empirical evidence has shown that the CEV process may be a better descriptor of stock and commodity price behavior than the more commonly used lognormal model[7] because the CEV process allows for a non-zero elasticity of return variance with respect to price. The market price of risk in this model is given by $\frac{\mu - r}{\sigma}\sqrt{S_t}$, which unlike the lognormal model, depends on the price level of the stock. The interest rate, r_t, is assumed to be constant.

As before, we expand the state-space by introducing the state variable Y

$$Y_t = \int_0^t S_v \, dv$$

[7]See, for example, Campbell and Hentschel [5], Christie [8], French, Schwert, and Stambaugh [14] and Schwert [23]. See also Schroder [22], Beckers [2], and Choi and Longstaff [7] for pricing regular options with the CEV process.

This can be written in differential form as

$$dY_t = S_t\,dt$$

Therefore, the partial differential equation governing the price of an Asian call option, F_t, whose terminal payoff is M if $\frac{1}{T}Y_T > K$ is given by

$$\frac{1}{2}\sigma^2 S \frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} + S\frac{\partial F}{\partial Y} - \frac{\partial F}{\partial T} = rF \tag{12}$$

where the boundary condition is simply $F_T = M \times 1_{Y_T \geq K'}$ where $K' = KT$. We can rewrite F using the Feynman-Kac functional representation

$$\begin{aligned} F_t &= \int_K^\infty e^{-\int_t^T r_v\,dv} M f(Y_T)\,dY_T \\ &= Me^{-rT}\Pi(S_0, Y_0, T) \end{aligned}$$

where Π represents the risk-neutral probability that the option ends up in the money. By substituting this expression back into (12), it is easy to show that Π_t satisfies the following PDE

$$\frac{1}{2}\sigma^2 S \frac{\partial^2 \Pi}{\partial S^2} + rS\frac{\partial \Pi}{\partial S} + S\frac{\partial \Pi}{\partial Y} - \frac{\partial \Pi}{\partial T} = 0$$

subject to the boundary condition $\Pi(Y_T, T) = 1_{Y_T \geq K'}$. The solution to this equation is given by

$$\Pi(S_0, Y_0, T) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{1}{i\xi}e^{-i\xi K'}H(S_0, Y_0, T)\right]d\xi$$

where $i = \sqrt{-1}$ and

$$\begin{aligned} H(S_0, Y_0, T) &= \exp[A(T)S_0 + i\omega Y_0] \\ A(T) &= \frac{2uv(e^{uT} - e^{vT})}{\sigma^2 u e^{vT} - \sigma^2 v e^{uT}} \\ u &= \frac{1}{2}r + \frac{1}{2}\sqrt{r^2 - 2i\omega\sigma^2} \end{aligned}$$

Thus, the price of a binary Asian option is given by

$$F_0 = Me^{-rT}\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{1}{i\xi}e^{-i\xi K'}H(S_0, Y_0, T)\right]d\xi\right)$$

10 Concluding Comments

This paper extends the considerable literature on average rate option pricing by introducing a state-space expansion approach to developing Asian option models, in particular for mean-reverting processes. Models are built for the two most popular processes employed in the financial engineer's arsenal: the square-root diffusion and the O-U process. We also develop a model for the constant-elasticity of volatility (CEV) diffusion for stock options. By exploiting the particle-like nature of Asian digital options, we are able to price more standard options, such as regular calls and range notes. The existence of mean reversion results in interesting pricing characteristics for Asian interest rate options, and we provide numerical analyses to explore this feature.

The results in this paper would be useful for a wide range of applications, such as hedging interest rate options books containing a range of maturities, hedging long dated foreign currency positions using options on average interest differentials, and for event risk contracts keying off varying economic regimes. The methods here also do not suffer from accuracy problems which is a concern when employing numerical procedures such as simulation.

Appendix

A Proof of the equation for $H(r, Y, T)$:

The solution for the Fourier transform, $H(r, Y, T)$, of Π is given by $\exp\left[(\overline{C} - C) + (\overline{D} - D)r + sY\right]$, where $\bar{C}$ and $\bar{D}$ solve the following ordinary differential equations:

$$\begin{aligned} \frac{d\overline{D}}{dT} &= \frac{1}{2}\eta^2\overline{D}^2 - (k+\phi)\overline{D} - (1-s), \quad \overline{D}(0) = 0, \\ \frac{d\overline{C}}{dT} &= k\theta\overline{D}, \quad \overline{C}(0) = 0. \end{aligned}$$

To solve the equation for $\bar{D}$, we first employ the transformation

$$\bar{D} = -\frac{2}{\eta^2 W(T)}\frac{dW}{dT}$$

The resulting differential equation is a homogeneous, second-order differential equation with constant coefficients.

$$\frac{d^2W}{dT^2} + (k+\phi)\frac{dW}{dT} - \frac{1-s}{2}\eta^2 = 0$$

This equation can be solved using well-known methods, i.e., by substituting in the guess $C_1 e^{r_1 T} + C_2 e^{r_2 T}$, where C_1 and C_2 are constants to be solved using the boundary condition, and r_1 and r_2 are simply the roots to the characteristic equation of the differential equation. This yields the solution

$$\begin{aligned} \overline{D}(T) &= \frac{2}{\eta^2}\left[\frac{r_1 r_2\left(e^{r_1 T} - e^{r_2 T}\right)}{r_1 e^{r_2 T} - r_2 e^{r_1 T}}\right], \\ r_1 &= -\frac{1}{2}(k+\phi) + \sqrt{(k+\phi)^2 + 2(1-s)\eta^2} \\ r_2 &= -\frac{1}{2}(k+\phi) - \sqrt{(k+\phi)^2 + 2(1-s)\eta^2} \end{aligned}$$

With $\bar{D}$ known, it is a simple matter of integration to solve for $\bar{C}$.

$$\begin{aligned} \bar{C} &= \int k\theta\bar{D}(T)\,dT \\ &= \frac{2k\theta}{\eta^2}\ln\left[\frac{r_1 - r_2}{r_1 e^{r_2 T} - r_2 e^{r_1 T}}\right] \end{aligned}$$

B Proof of the equation for $G(r, T)$:

First, the ODE for $B(T)$.

$$\frac{dB}{dT} = \frac{1}{2}\eta^2 B^2 - (k - \eta^2 D)B,$$

which is solved subject to $G(0) = e^{sr}$ which implies $B(0) = s$. Let

$$B = -\frac{2}{\eta^2 X}\frac{dX}{dT}. \tag{13}$$

Then, using

$$\frac{dB}{dT} = -\frac{2}{\eta^2 X}\frac{d^2X}{dT^2} + \frac{2}{\eta^2 X^2}\left(\frac{dX}{dT}\right)^2$$

we can transform the equation to

$$\frac{d^2X}{dT^2} + (k + \phi - \eta^2 D)\frac{dX}{dT} = 0.$$

Let

$$Y = \frac{dX}{dT}, \tag{14}$$

then

$$\frac{dY}{dT} + (k - \eta^2 B)Y = 0, \quad Y(0) = Y_0.$$

The solution is

$$Y(T) = Y_0\left[\frac{e^{\gamma T/2}}{(k + \phi + \gamma)(e^{\gamma T} - 1) + 2\gamma}\right]^2.$$

Notice that

$$\frac{dD}{dT} = -4\gamma^2\left[\frac{e^{\gamma T/2}}{(k + \phi + \gamma)(e^{\gamma T} - 1) + 2\gamma}\right]^2,$$

therefore

$$Y(T) = Y_0\left(\frac{-1}{4\gamma^2}\right)\frac{dD}{dT} = \frac{dX}{dT}. \tag{15}$$

Integrating to get $X(T)$ from equation (15), we get

$$X(T) = \frac{-Y_0}{4\gamma^2}D(T) + c_2, \tag{16}$$

where c_2 is simply the integration constant. Substituting equations (15, 16) into equation (13) results in

$$B = \frac{-2}{\eta^2 X}\frac{dX}{dT}$$

$$\begin{aligned}
&= \frac{-2}{\eta^2}\left[\frac{1}{c_2 - \frac{Y_0 D}{4\gamma^2}}\right]\left(\frac{-Y_0}{4\gamma^2}\frac{dD}{dT}\right) \\
&= \frac{Y_0}{2\eta^2\gamma^2}\frac{dD}{dT}\left[\frac{1}{c_2 - \frac{Y_0 D}{4\gamma^2}}\right] \\
&= \frac{2}{\eta^2}\frac{dD}{dT}\left[\frac{1}{4\frac{c_2}{Y_0}\gamma^2 - D}\right]. \qquad (17)
\end{aligned}$$

Apply the boundary condition ($D(0) = 0, D'(0) = -1$)

$$\begin{aligned}
B(0) &= s = \frac{2}{\eta^2}\frac{dD}{dT}\left[\frac{1}{4\frac{c_2}{Y_0}\gamma^2}\right] \\
s &= \frac{-2}{\eta^2}\left[\frac{1}{4\frac{1}{c_1}\gamma^2}\right], \quad c_1 = \frac{Y_0}{c_2}.
\end{aligned}$$

This gives

$$c_1 = -2s\eta^2\gamma^2.$$

Substituting this back into equation (17),

$$\begin{aligned}
B(T) &= \frac{2}{\eta^2}\frac{dD}{dT}\left[\frac{1}{4\frac{1}{c_1}\gamma^2 - D}\right] \\
&= \frac{2}{\eta^2}\frac{dD}{dT}\left[\frac{1}{4\gamma^2\left(\frac{-1}{2s\eta^2\gamma^2}\right) - D}\right] \\
&= \frac{-2s}{2+\eta^2 sD}\frac{dD}{dT} \\
&= \frac{-2sD'}{2+\eta^2 sD}.
\end{aligned}$$

Now we solve the second ODE for $A(T)$:

$$\frac{dA}{dT} = k\theta B,$$

which implies

$$A = k\theta \int B dT + c_3,$$

where c_3 is the integration constant. Now using equation (13)

$$\begin{aligned}
B &= -\frac{2}{\eta^2 X}\frac{dX}{dT} \\
\int B dt &= \frac{-2}{\eta^2}\int \frac{1}{X} dX.
\end{aligned}$$

Therefore,

$$\begin{aligned} A &= k\theta \int B dT + c_3, \\ &= \frac{-2k\theta}{\eta^2} \int \frac{1}{X} dX + c_3 \\ &= \frac{-2k\theta}{\eta^2} \ln(X) + c_3. \end{aligned} \tag{18}$$

Applying the boundary condition, $A(0) = 0$:

$$\begin{aligned} A(0) &= 0 = \frac{-2k\theta}{\eta^2} \ln(X(0)) + c_3 \\ c_3 &= \frac{2k\theta}{\eta^2} \ln(X(0)) \end{aligned} \tag{19}$$

Then using equation (16) we get

$$\begin{aligned} X(T) &= \frac{-Y_0}{4\gamma^2} D(T) + c_2, \\ X(0) &= \frac{-Y_0}{4\gamma^2} D(0) + c_2 \\ &= c_2. \end{aligned}$$

Which eventually gives, employing equations (18) and (19), and noting that $c_1 = Y_0/c_2 = -2s\eta^2\gamma^2$,

$$\begin{aligned} A(T) &= \frac{-2k\theta}{\eta^2} \ln\left[\frac{X(T)}{X(0)}\right] \\ &= \frac{-2k\theta}{\eta^2} \ln\left[\frac{\frac{-Y_0}{4\gamma^2} D(0) + c_2}{c_2}\right] \\ &= \frac{-2k\theta}{\eta^2} \ln\left[\frac{-Y_0}{c_2} \frac{D}{4\gamma^2} + 1\right] \\ &= \frac{-2k\theta}{\eta^2} \ln\left[-c_1 \frac{D}{4\gamma^2} + 1\right] \\ &= \frac{-2k\theta}{\eta^2} \ln\left[2s\eta^2\gamma^2 \frac{D}{4\gamma^2} + 1\right] \\ &= \frac{-2k\theta}{\eta^2} \ln\left[\frac{1}{2}\eta^2 s D + 1\right] \end{aligned}$$

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Table 1: **Options Prices for the square-root model (at inception,** $\eta = 0.20$)
This table presents the values of four options: (i) Asian binary call, (ii) Regular binary call, (iii) Asian call and (iv) Regular call. The parameters that are varied are: (a) exercise price (X), (b) time to maturity (T). The base parameters used are: volatility ($\eta = 0.2$), initial interest rate ($r_0 = 0.1$), mean reversion ($k = 1.5$), mean rate ($\theta = 0.1$), market price of risk ($\phi = 0$).

Maturity	Option Type	$X = 0.08$	$X = 0.09$	$X = 0.10$	$X = 0.11$	$X = 0.12$
$T = 0.10$	Asian Binary call	0.9303	0.7426	0.4914	0.2420	0.0585
$T = 0.10$	Regular Binary call	0.8528	0.6852	0.4753	0.2804	0.1403
$T = 0.10$	Asian call	0.2093	0.1596	0.1007	0.0471	0.0108
$T = 0.10$	Regular call	0.1918	0.1473	0.0974	0.0546	0.0259
$T = 0.50$	Asian Binary call	0.8018	0.6377	0.4452	0.2718	0.1458
$T = 0.50$	Regular Binary call	0.6759	0.5563	0.4360	0.3261	0.2333
$T = 0.50$	Asian call	0.1804	0.1371	0.0912	0.0530	0.0269
$T = 0.50$	Regular call	0.1520	0.1196	0.0893	0.0635	0.0431
$T = 1.00$	Asian Binary call	0.7300	0.5779	0.4124	0.2661	0.1565
$T = 1.00$	Regular Binary call	0.6162	0.5097	0.4055	0.3112	0.2311
$T = 1.00$	Asian call	0.1642	0.1242	0.0845	0.0519	0.0289
$T = 1.00$	Regular call	0.1386	0.1095	0.0831	0.0607	0.0427
$T = 2.00$	Asian Binary call	0.6643	0.5193	0.3633	0.2291	0.1316
$T = 2.00$	Regular Binary call	0.5507	0.4553	0.3630	0.2797	0.2090
$T = 2.00$	Asian call	0.1494	0.1116	0.0744	0.0446	0.0243
$T = 2.00$	Regular call	0.1239	0.0979	0.0744	0.0545	0.0386

Table 2: **Options Prices for the square-root model (at inception, $\eta = 0.30$)** This table presents the values of four options: (i) Asian binary call, (ii) Regular binary call, (iii) Asian call and (iv) Regular call. The parameters that are varied are: (a) exercise price (X), (b) time to maturity (T). The base parameters used are: volatility ($\eta = 0.3$), initial interest rate ($r_0 = 0.1$), mean reversion ($k = 1.5$), mean rate ($\theta = 0.1$), market price of risk ($\phi = 0$).

Maturity	Option Type	$X = 0.08$	$X = 0.09$	$X = 0.10$	$X = 0.11$	$X = 0.12$
$T = 0.10$	Asian Binary call	0.8662	0.6950	0.4832	0.2769	0.1189
$T = 0.10$	Regular Binary call	0.7444	0.6087	0.4655	0.3328	0.2229
$T = 0.10$	Asian call	0.1948	0.1494	0.0990	0.0540	0.0220
$T = 0.10$	Regular call	0.1675	0.1308	0.0954	0.0649	0.0412
$T = 0.50$	Asian Binary call	0.6907	0.5605	0.4299	0.3127	0.2165
$T = 0.50$	Regular Binary call	0.5800	0.4958	0.4159	0.3431	0.2786
$T = 0.50$	Asian call	0.1554	0.1205	0.0881	0.0609	0.0400
$T = 0.50$	Regular call	0.1305	0.1065	0.0852	0.0669	0.0515
$T = 1.00$	Asian Binary call	0.6204	0.5039	0.3924	0.2942	0.2132
$T = 1.00$	Regular Binary call	0.5251	0.4510	0.3820	0.3197	0.2646
$T = 1.00$	Asian call	0.1395	0.1083	0.0804	0.0573	0.0394
$T = 1.00$	Regular call	0.1181	0.0969	0.0783	0.0623	0.0489
$T = 2.00$	Asian Binary call	0.5594	0.4464	0.3408	0.2503	0.1780
$T = 2.00$	Regular Binary call	0.4677	0.4015	0.3402	0.2851	0.2365
$T = 2.00$	Asian call	0.1258	0.0959	0.0698	0.0488	0.0329
$T = 2.00$	Regular call	0.1052	0.0863	0.0697	0.0555	0.0437

Table 3: **Options Prices for the square-root model (at inception, $\eta = 0.20$, $\theta = 0.05$)** This table presents the values of four options: (i) Asian binary call, (ii) Regular binary call, (iii) Asian call and (iv) Regular call. The parameters that are varied are: (a) exercise price (X), (b) time to maturity (T). The base parameters used are: volatility ($\eta = 0.2$), initial interest rate ($r_0 = 0.1$), mean reversion ($k = 1.5$), mean rate ($\theta = 0.05$), market price of risk ($\phi = 0$).

Maturity	Option Type	$X = 0.08$	$X = 0.09$	$X = 0.10$	$X = 0.11$	$X = 0.12$
$T = 0.10$	Asian Binary call	0.8750	0.6580	0.3983	0.1655	0.0151
$T = 0.10$	Regular Binary call	0.7474	0.5409	0.3321	0.1720	0.0755
$T = 0.10$	Asian call	0.1968	0.1414	0.0816	0.0322	0.0028
$T = 0.10$	Regular call	0.1681	0.1163	0.0680	0.0335	0.0139
$T = 0.50$	Asian Binary call	0.5523	0.3537	0.1966	0.0958	0.0414
$T = 0.50$	Regular Binary call	0.3545	0.2484	0.1661	0.1066	0.0658
$T = 0.50$	Asian call	0.1242	0.0760	0.0403	0.0186	0.0076
$T = 0.50$	Regular call	0.0797	0.0534	0.0340	0.0207	0.0121
$T = 1.00$	Asian Binary call	0.3474	0.2049	0.1097	0.0539	0.0245
$T = 1.00$	Regular Binary call	0.2108	0.1419	0.0927	0.0590	0.0367
$T = 1.00$	Asian call	0.0781	0.0440	0.0224	0.0105	0.0045
$T = 1.00$	Regular call	0.0474	0.0305	0.0190	0.0115	0.0067
$T = 2.00$	Asian Binary call	0.1692	0.0842	0.0386	0.0166	0.0067
$T = 2.00$	Regular Binary call	0.1253	0.0794	0.0491	0.0298	0.0177
$T = 2.00$	Asian call	0.0380	0.0181	0.0079	0.0032	0.0012
$T = 2.00$	Regular call	0.0282	0.0170	0.0100	0.0058	0.0032

Table 4: **Options Prices for the square-root model (at inception, $\eta = 0.20$, $\theta = 0.15$)** This table presents the values of four options: (i) Asian binary call, (ii) Regular binary call, (iii) Asian call and (iv) Regular call. The parameters that are varied are: (a) exercise price (X), (b) time to maturity (T). The base parameters used are: volatility ($\eta = 0.2$), initial interest rate ($r_0 = 0.1$), mean reversion ($k = 1.5$), mean rate ($\theta = 0.15$), market price of risk ($\phi = 0$).

Maturity	Option Type	$X = 0.08$	$X = 0.09$	$X = 0.10$	$X = 0.11$	$X = 0.12$
$T = 0.10$	Asian Binary call	0.9728	0.8183	0.5841	0.3267	0.1145
$T = 0.10$	Regular Binary call	0.9214	0.8041	0.6211	0.4136	0.2350
$T = 0.10$	Asian call	0.2188	0.1759	0.1197	0.0637	0.0211
$T = 0.10$	Regular call	0.2073	0.1728	0.1273	0.0806	0.0434
$T = 0.50$	Asian Binary call	0.9129	0.8410	0.7082	0.5321	0.3531
$T = 0.50$	Regular Binary call	0.8709	0.8066	0.7198	0.6166	0.5064
$T = 0.50$	Asian call	0.2054	0.1808	0.1451	0.1037	0.0653
$T = 0.50$	Regular call	0.1959	0.1734	0.1475	0.1202	0.0936
$T = 1.00$	Asian Binary call	0.8672	0.8255	0.7406	0.6140	0.4660
$T = 1.00$	Regular Binary call	0.8346	0.7911	0.7303	0.6547	0.5692
$T = 1.00$	Asian call	0.1951	0.1774	0.1518	0.1197	0.0862
$T = 1.00$	Regular call	0.1877	0.1700	0.1497	0.1276	0.1053
$T = 2.00$	Asian Binary call	0.7627	0.7496	0.7107	0.6340	0.5224
$T = 2.00$	Regular Binary call	0.7341	0.7045	0.6618	0.6065	0.5413
$T = 2.00$	Asian call	0.1716	0.1611	0.1457	0.1236	0.0966
$T = 2.00$	Regular call	0.1651	0.1514	0.1356	0.1182	0.1001

Table 5: **Options Prices for the O-U model (at inception, $\eta = 0.063246$)** This table presents the values of four options: (i) Asian binary call, (ii) Regular binary call, (iii) Asian call and (iv) Regular call. The parameters that are varied are: (a) exercise price (X), (b) time to maturity (T). The base parameters used are: volatility ($\eta = 0.1$), initial interest rate ($r_0 = 0.1$), mean reversion ($k = 1.5$), mean rate ($\theta = 0.063246$), market price of risk ($\phi = 0$).

Maturity	Option Type	$X = 0.08$	$X = 0.09$	$X = 0.10$	$X = 0.11$	$X = 0.12$
$T = 0.10$	Asian Binary call	0.9306	0.7449	0.4947	0.2445	0.0591
$T = 0.10$	Regular Binary call	0.8502	0.6973	0.4946	0.2920	0.1393
$T = 0.10$	Asian call	0.2093	0.1601	0.1014	0.0476	0.0109
$T = 0.10$	Regular call	0.1913	0.1499	0.1014	0.0569	0.0257
$T = 0.50$	Asian Binary call	0.7997	0.6556	0.4718	0.2889	0.1470
$T = 0.50$	Regular Binary call	0.6947	0.5888	0.4727	0.3568	0.2517
$T = 0.50$	Asian call	0.1799	0.1409	0.0967	0.0563	0.0271
$T = 0.50$	Regular call	0.1563	0.1266	0.0969	0.0695	0.0465
$T = 1.00$	Asian Binary call	0.7318	0.6015	0.4444	0.2889	0.1624
$T = 1.00$	Regular Binary call	0.6405	0.5474	0.4470	0.3471	0.2552
$T = 1.00$	Asian call	0.1646	0.1293	0.0911	0.0563	0.0300
$T = 1.00$	Regular call	0.1441	0.1176	0.0916	0.0677	0.0472
$T = 2.00$	Asian Binary call	0.6632	0.5419	0.3955	0.2519	0.1375
$T = 2.00$	Regular Binary call	0.5742	0.4913	0.4025	0.3143	0.2329
$T = 2.00$	Asian call	0.1492	0.1165	0.0810	0.0491	0.0254
$T = 2.00$	Regular call	0.1291	0.1056	0.0825	0.0612	0.0430

Table 6: **Options Prices for the O-U model (at inception, $\eta = 0.094868$)** This table presents the values of four options: (i) Asian binary call, (ii) Regular binary call, (iii) Asian call and (iv) Regular call. The parameters that are varied are: (a) exercise price (X), (b) time to maturity (T). The base parameters used are: volatility ($\eta = 0.1$), initial interest rate ($r_0 = 0.1$), mean reversion ($k = 1.5$), mean rate ($\theta = 0.094868$), market price of risk ($\phi = 0$).

Maturity	Option Type	$X = 0.08$	$X = 0.09$	$X = 0.10$	$X = 0.11$	$X = 0.12$
$T = 0.10$	Asian Binary call	0.8675	0.7030	0.4944	0.2860	0.1217
$T = 0.10$	Regular Binary call	0.7553	0.6331	0.4944	0.3558	0.2338
$T = 0.10$	Asian call	0.1952	0.1511	0.1013	0.0557	0.0225
$T = 0.10$	Regular call	0.1699	0.1361	0.1013	0.0693	0.0432
$T = 0.50$	Asian Binary call	0.7080	0.5953	0.4700	0.3452	0.2342
$T = 0.50$	Regular Binary call	0.6244	0.5494	0.4712	0.3933	0.3188
$T = 0.50$	Asian call	0.1593	0.1280	0.0963	0.0673	0.0433
$T = 0.50$	Regular call	0.1405	0.1181	0.0966	0.0766	0.0589
$T = 1.00$	Asian Binary call	0.6458	0.5472	0.4405	0.3349	0.2392
$T = 1.00$	Regular Binary call	0.5772	0.5118	0.4445	0.3774	0.3128
$T = 1.00$	Asian call	0.1453	0.1176	0.0903	0.0653	0.0442
$T = 1.00$	Regular call	0.1298	0.1100	0.0911	0.0735	0.0578
$T = 2.00$	Asian Binary call	0.5808	0.4883	0.3888	0.2913	0.2042
$T = 2.00$	Regular Binary call	0.5171	0.4590	0.3994	0.3401	0.2831
$T = 2.00$	Asian call	0.1306	0.1050	0.0797	0.0568	0.0377
$T = 2.00$	Regular call	0.1163	0.0987	0.0818	0.0663	0.0523

Table 7: **Range Asian Option Prices**
This table presents the values of the range Asian option. This option is written over a fixed number of days. Every day the option pays off if the average interest rate up to that day lies within a range (a, b). The payoff is a dollar divided by the number of days the option is written for. The values in this table are for a range asian option with maturity $T = 0.2$ years, i.e. 73 days. The parameters that are varied are: (a) mean reversion (k), (b) lower range limit (a) (c) upper range limit (b). The base parameters used are: initial interest rate ($r_0 = 0.1$), time to maturity ($T = 0.2$), mean rate ($\theta = 0.1$), market price of risk ($\phi = 0$), volatility ($\eta = 0.2$).

$\theta = 0.05$

Range	$k = 0.5$	$k = 1.5$	$k = 2.5$
$a = 0.09, b = 0.11$	0.3718	0.3539	0.3075
$a = 0.08, b = 0.12$	0.6631	0.6461	0.5923
$a = 0.07, b = 0.13$	0.8388	0.8353	0.8066

$\theta = 0.10$

Range	$k = 0.5$	$k = 1.5$	$k = 2.5$
$a = 0.09, b = 0.11$	0.3818	0.4120	0.4424
$a = 0.08, b = 0.12$	0.6741	0.7132	0.7488
$a = 0.07, b = 0.13$	0.8435	0.8714	0.8922

$\theta = 0.15$

Range	$k = 0.5$	$k = 1.5$	$k = 2.5$
$a = 0.09, b = 0.11$	0.3823	0.3810	0.3406
$a = 0.08, b = 0.12$	0.6717	0.6646	0.6150
$a = 0.07, b = 0.13$	0.8374	0.8265	0.7938

Figure 1: **Probability Density Functions for the Square-root Process** The plots below depict the probability density functions for the average interest rate and terminal interest rate at time t. The parameters on which these densities are conditioned are: mean reversion ($k = 1.5$), interest rate mean ($\theta = 0.10$), volatility ($\eta = 0.2$), initial interest rate ($r_0 = 0.10$). The plots provide a comparison of the density for pricing Asian interest rate options versus the density for pricing regular (non-Asian) options. The left hand side plot is for $t = 0.5$ years, and the right hand side plot is for $t = 1.0$ years.

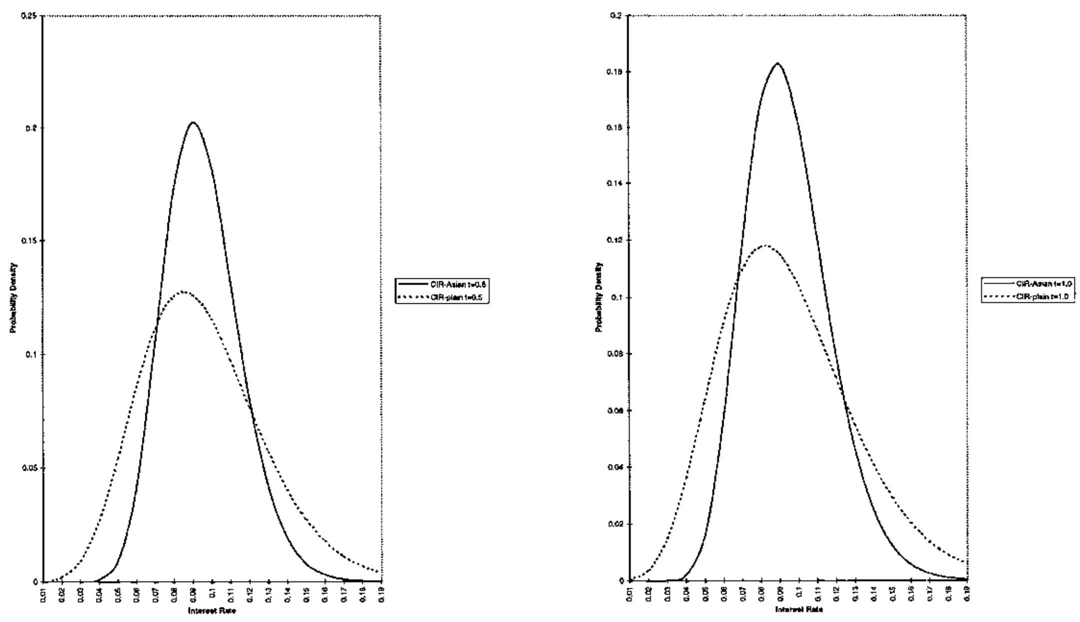

Figure 2: **Probability Density Functions for the O-U Process** The plots below depict the probability density functions for the average interest rate and terminal interest rate at time t. The parameters on which these densities are conditioned are: mean reversion ($k = 1.5$), interest rate mean ($\theta = 0.10$), volatility ($\eta = 0.08$), initial interest rate ($r_0 = 0.10$). The plots provide a comparison of the density for pricing Asian interest rate options versus the density for pricing regular (non-Asian) options. The left hand side plot is for $t = 0.5$ years, and the right hand side plot is for $t = 1.0$ years.