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THE TERM STRUCTURE OF INTEREST RATE DIFFERENTIALS IN A TARGET ZONE:  
THEORY AND SWEDISH DATA

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ABSTRACT

The term structure of interest rate differentials is derived in a model of a small open economy with a target zone exchange rate regime. The target zone is modeled as a regulated Brownian motion. The interest rate differentials are computed as the solution to a parabolic partial differential equation with derivative boundary conditions, both via a Fourier-series analytical solution and via a direct numerical solution. Several specific properties of the term structure of interest rate differentials are derived. For instance, for given time to maturity the interest rate differential is decreasing in the exchange rate, and for given exchange rate the interest rate differential's absolute value and its instantaneous variability are both decreasing in the time to maturity. Devaluation/realignment risks are incorporated and imply upward shifts of the interest rate differentials. Some implications of the theory are found to be broadly consistent with data on Swedish exchange rates and interest differentials for the period 1986-1989.

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## 1. Introduction

After the breakdown of the Bretton Woods system, several countries have for many years maintained target zone exchange rate regimes, rather than completely fixed exchange rate regimes. In these target zones the exchange rate is allowed to float more or less freely within a specified band, but foreign exchange interventions prevent the exchange rate from moving outside the band. The European Exchange Rate Mechanism within the European Monetary System has exchange rate bands for the member countries of  $\pm 2.25$  percent, except Spain which has a  $\pm 6$  percent band (Italy's band was recently reduced from  $\pm 6\%$  to  $\pm 2.25\%$ ). The Nordic countries outside EMS have unilateral target zones; Sweden, for instance, has a band of  $\pm 1.5$  percent. Actually, most historical fixed exchange rate regimes were in practice target zones, although with narrower bands: even the Gold Standard had gold points, with a narrow band in between (see Yeager (1976, p. 19-21, 322)).

The existence of target zone exchange rate regimes gives rise to many questions. For instance, do target zone regimes with relatively narrow bands differ in essential ways from completely fixed exchange rate regimes? Does the width of the exchange rate band matter, and if so for what? Is there an optimal band? Do target zones give central banks some degree of monetary independence, even if there is relatively free capital mobility? With regard to the last question, it has indeed been argued that the Swedish target zone gives Sveriges Riksbank some control over domestic interest rates. For instance, if the Bank moves the krona close to the strong edge of its band, the krona can only depreciate and not appreciate, and expectations of a depreciation within the band will increase domestic interest rates above foreign interest rates (see for instance Järnhäll and Lindenius (1989)).

The traditional theoretical literature on exchange rate regimes does not distinguish narrow target zones from completely fixed exchange rates. Fixed exchange rate regimes have traditionally been modeled as having completely fixed exchange rates, with free

capital mobility resulting in zero differentials between home and foreign interest rates (absent devaluation risks) and a complete loss of monetary autonomy for small economies. This framework is clearly inadequate for a discussion of the questions mentioned above. Only very recently have researchers started to rigorously model and understand the details of exchange rate determination within an explicit target zone.

Krugman (1988) presented the first model of an explicit rational expectations target zone for the nominal exchange rate, using the framework of regulated Brownian motion.<sup>1</sup> He assumed infinitesimal foreign exchange interventions at the edges of a band for a 'fundamental,' the (log of the) ratio of money supply to money demand, preventing the fundamental from moving outside the band. He showed that this implies an exchange rate band which is narrower than the fundamental band (the 'honeymoon effect'), hence there is an inherent stabilizing effect of a target zone. The reason for this inherent stabilizing is that a weak currency implies expectations of future interventions to reduce the money supply and strengthen the currency, and expectations of a future appreciation leads to an immediate current strengthening of the currency. Krugman (1988) also extended the analysis to an imperfectly credible target zone regime, where the regime collapses once and for all to a free float with a given probability when the exchange rate reaches the edge of its band.

A rapidly growing literature has since extended the analysis of target zones in various directions.<sup>2</sup> As a step in the understanding and evaluation of target zones it is important

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<sup>1</sup> Krugman (1987) was a discrete-time forerunner.

Before the development of the nominal exchange rate target zone model Dumas (1989a) specified a two-country general equilibrium model with physical capital movements where the *real* exchange rate moves in a band exactly like a target zone.

<sup>2</sup> The exchange rate target zone model based on regulated Brownian motion has been extended by Miller and Weller (1988, 1989a,b,c), Flood and Garber (1989), Froot and Obstfeld (1989a, b), Klein (1989a, b), Krugman (1989), Bertola and Caballero (1989), and Delgado and Dumas (1990). Pesach and Razin (1990) test the theory on Israeli data.

Harrison (1985), Dixit (1989a, b) and Dumas (1989b) discuss solution techniques for problems with regulated Brownian motion. Dixit (1989a, b) also give references to the rapidly growing literature in which the techniques of regulated Brownian motion are applied to economic problems other than target zones, notably problems of irreversible investment and hysteresis.

to understand the relation between exchange rate and interest rate determination. A small economy's trade-off between the width of the exchange rate band and the variability of interest rate differentials was examined in Svensson (1989). There I found that, for reasonable parameters, the interest rate differential's *asymptotic* (unconditional) variability is not monotonic but increases with the exchange rate band for narrow bands, reaches a maximum, and then slowly decreases for wide exchange rate bands. In contrast, the interest rate differential's *instantaneous* (conditional) variability is monotonic and always decreasing in the exchange rate band. A devaluation/realignment risk was incorporated and shown to imply an upward shift in the interest rate differential.<sup>3</sup>

The negative trade-off between the exchange rate band and the interest rate differential's instantaneous variability implies that narrow target zones have the largest instantaneous variability of the interest rate differential. An exchange rate band of about  $\pm 1$  percent is in this aspect quite different from a completely fixed exchange rate regime, with a zero variability of the interest rate differential. With a  $\pm 1.5$  percent exchange rate band, for reasonable parameters the interest rate differential's instantaneous standard deviation is about 3 percent per year. (The expected time before the exchange rate reaches the edge, starting from the middle of its band, may for such a narrow exchange rate band still be almost a year (see appendix), so the instantaneous variability is certainly relevant.)

The analysis in Svensson (1989) only concerns the instantaneous interest rates, say overnight interest rates. The implications for longer term interest rates are at least as important. For instance, it is often argued that the three-month interest rate has the largest effect on short term capital movements. Most borrowing and lending is certainly not at overnight interest rates. There is also reason to expect that longer interest rates behave differently from shorter ones. For instance, since in the longer run (at least with credible bands) the exchange rate variability is bounded by the band, the exchange rate

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<sup>3</sup> I also showed that an endogenous risk premium is sufficiently small as not to matter, in particular for narrow exchange rate bands, and that therefore uncovered interest arbitrage seems to be a reasonable approximation for narrow bands.

variability per unit of time to maturity decreases with the time to maturity. Therefore, we might expect both the variability and absolute value of interest rate differentials for long terms to be less than for short terms.

In the present paper, the interest rate differentials for arbitrary terms are derived and examined. That is, we compute the complete term structure of interest rate differentials and derive its properties. This task is complicated by the fact that the exchange rate in a target zone is in theory a complicated non-linear and heteroscedastic stochastic process. We will see that the term structure of interest rate differentials can be expressed as the solution to a so-called parabolic partial differential equation with derivative boundary conditions (smooth pasting conditions). This equation is a variant of the partial differential equations arising in option pricing theory (cf. Merton (1973)). The solution to the equation will be computed in two different ways: via the analytical Fourier-series solution (using methods described in Churchill and Brown (1987)) and via a direct numerical method (using methods described in Gerald and Wheatley (1989)). The solution is then used to derive some testable implications about the term structure of interest rate differentials. We also consider the effect of devaluation/realignment risks on the term structure.

Data on the term structure of Swedish interest rate differentials are examined, and some implications of the theory are found to be broadly consistent with the data.

There exists an extensive and elaborate literature on theory and empirical estimates of the term structure of interest rates (see for instance the surveys by Shiller and McCulloch (1987) and Singleton (1987)).<sup>4</sup> The theory of the term structure of interest rate *differentials* developed here should of course be seen as complement and not a substitute to that theory: For a small open economy which faces an exogenous term structure of world interest rates, the domestic term structure of interest rates follows from the term structure of interest rate differentials. The term structure of interest rate differentials is

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<sup>4</sup> Ekdahl and Warne (1990) provide a study of the Swedish term structure of interest rates.

related via interest parity conditions to the term structure of expected currency depreciation, if domestic and world capital markets are sufficiently integrated. The term structure of expected currency depreciation can be explicitly characterized in an exchange rate target zone. This way, the route via the term structure of interest rate differentials offers a shortcut. Simultaneously, the term structure of world interest rate may be determined according to the established theories, and the term structure of the domestic interest rates may also fulfill the equilibrium conditions of the established theories.

The paper is organized as follows: Section 2 presents Krugman's (1988) target zone model. Section 3 derives the interest rate differential for different terms. Section 4 discusses the interest rate differentials' variability. Section 4 incorporates a devaluation/realignment risk. Section 5 presents the empirical results. Section 6 includes a summary and conclusions. An appendix specifies the expression for the expected time to hit the edge of an exchange rate band, a variant of the Feynman-Kac formula for regulated Brownian motion, the analytical Fourier-series solution to the resulting partial differential equation, and the direct numerical solution to the partial differential equation. The appendix also includes a discussion of the interest rate differential's variability.

## 2. The Exchange Rate

We use the basic loglinear model of the exchange rate. The log of the exchange rate at time  $\tau$ ,  $e(\tau)$ , is equal to a 'fundamental,'  $f(\tau)$ , plus a term proportional to the expected change in the log exchange rate,

$$(2.1) \quad e(\tau) = f(\tau) + \alpha E[de(\tau)]/d\tau, \quad \alpha > 0,$$

where  $E$  is the expectations operator.<sup>5</sup>

This exchange rate equation can be seen as a general asset-pricing relation. It can also be seen as the result of a simple monetary exchange rate theory for a small open

<sup>5</sup>  $E[de(\tau)]/d\tau$  denotes  $\lim_{s \rightarrow 0+} \{E[e(\tau + s)] - e(\tau)\}/s$ .

economy. In this case the fundamental is the sum of two components,

$$(2.2) \quad f(\tau) = m(\tau) + u(\tau),$$

(the log of) the domestic money supply,  $m$ , and (the negative of the log of) a composite money demand shock,  $v$ , called velocity. Velocity is given by

$$(2.3) \quad u(\tau) = -\psi y(\tau) + q(\tau) - p^*(\tau) + \alpha i^*(\tau) + \alpha \rho(\tau) - \epsilon(\tau).$$

Here  $\psi$  is the elasticity of money demand with respect to output,  $y$  is the log of home output,  $q$  is the log of the real exchange rate,  $p^*$  is the log of the foreign price level,  $\alpha$  is (the absolute value of) the semi-elasticity of money demand with respect to the home nominal interest rate,  $i^*$  is the foreign nominal interest rate,  $\rho$  is an exogenous risk premium which equals the interest rate differential less the expected exchange rate depreciation, and  $\epsilon$  is a money demand disturbance.<sup>6</sup>

In what follows, velocity will be an exogenous stochastic process whereas the money supply will be a stochastic process under direct control by a monetary authority. Together these stochastic processes will determine the endogenous stochastic process of the exchange rate via equation (2.1).

The saddle-path solution to (2.1) can be written

$$(2.4) \quad e(\tau) = E_{\tau} \int_{s=\tau}^{\infty} \{\exp[-(s-\tau)/\alpha] f(s)/\alpha\} d\tau,$$

the expected present value of the path of future fundamentals over  $\alpha$ , discounted by  $1/\alpha$ , where  $E_{\tau}$  denote expectations conditional upon information available at time  $\tau$ . The saddle-path solution excludes bubbles. The expected value of a bubble would grow exponentially in the present model. Since we are going to discuss exchange rates which are restricted to a target zone, it makes particular sense to exclude bubbles. Henceforth

<sup>6</sup> The exchange rate is measured in units of home currency per unit of foreign currency. The model consists of the money demand function,  $m - p = \psi y - \alpha i + \epsilon$ , the definition of the real exchange rate,  $q = e + p^* - p$ , and the definition of the risk premium,  $\rho = i - i^* - E[de]/dt$ . Elimination of  $p$  and  $i$ , assumed to be endogenous and flexible, gives (2.1)-(2.3).

See Froot and Obstfeld (1989a) for an interpretation in terms of a two-country model. See also Miller and Weller (1988, 1989b) for an interpretation in terms of Dornbusch's overshooting model.

we shall therefore only discuss saddle-path solutions to (2.1).<sup>7</sup>

In order to model a target zone for the exchange rate, we assume that foreign exchange market interventions, which directly affect the money supply, are undertaken to prevent the fundamental to move outside a specified band for the fundamental. As we shall see this will imply a well defined band for the exchange rate.<sup>8</sup>

Hence we assume that there are lower and upper bounds for the fundamental,  $f$  and  $f$ , such that the fundamental fulfills

$$(2.5) \quad f \leq f(\tau) \leq f.$$

With interventions affecting the money supply, the stochastic process for the fundamental obeys

$$(2.6) \quad df(\tau) = dm(\tau) + dv(\tau),$$

where we now let  $dm$  represent the effect of foreign exchange market interventions on domestic money supply. Velocity is assumed to follow a Brownian motion with drift  $\mu$  and instantaneous standard deviation  $\sigma$  (this obviously involves conditions on the components in (2.3),

$$(2.7) \quad dv(\tau) = \mu d\tau + \sigma dz(\tau),$$

where  $z(\tau)$  is a Wiener process with  $E[dz] = 0$  and  $E[dz^2] = d\tau$ . Inside the band, there are no interventions,  $dm = 0$ , and the fundamental follows

$$(2.8) \quad df(\tau) = dv(\tau) = \mu d\tau + \sigma dz(\tau).$$

<sup>7</sup> A bubble to (2.1) is a stochastic process  $B(\tau)$  that can be added to the solution (2.4) and produce another solution. Such a bubble must obviously fulfill  $B(\tau) = \alpha E_\tau[dB(\tau)]/d\tau$ , that is, it has an expected growth rate equal to  $1/\alpha$ . Any stochastic process of the form  $dB(\tau) = [B(\tau)/\alpha]d\tau + \sigma(\tau)dz(\tau)$  will do (with  $z(\tau)$  a Wiener process, and where  $\sigma(\tau)$  could be a function of  $B(\tau)$ ). A special case is the deterministic exponential  $B(\tau) = B(0)\exp(\tau/\alpha)$ . Since  $\alpha > 0$  in the present model, the expected value of a bubble would grow indefinitely.

<sup>8</sup> As Flood and Garber (1989) and Froot and Obstfeld (1989a) have emphasized, it is not sufficient to just specify that interventions will occur when the exchange rate reaches the edges of the band. This is because there are several different ways to intervene to defend a given exchange rate band. In order to have a determinate equilibrium it is therefore necessary to specify exactly how the interventions are undertaken.

As argued in Svensson (1989), if an exchange rate band is announced together with the rule that interventions will be infinitesimal and undertaken at the edges of the band only, a unique fundamental band can be inferred from the exchange rate band also if the fundamental band is not explicitly announced.

At the edges of the band, there are *infinitesimal* interventions to prevent the fundamental from moving outside the band.<sup>9</sup> These interventions can be represented by lower and upper 'regulators,'  $L$  and  $U$ , such that

$$(2.9) \quad dm(\tau) = dL(\tau) - dU(\tau),$$

where  $dL$  and  $dU$  are nonnegative,  $dL$  represents increases in money supply and is positive only if  $f = f$ , and  $dU$  represents reductions in money supply and is positive only if  $f = \bar{f}$ . Once the fundamental moves inside the band, the interventions cease. This implies that the fundamental is a *regulated* Brownian motion.<sup>10</sup>

The saddle-path solution to the differential equation (2.1) under these circumstances will be a twice differentiable function of the fundamental,  $e(\tau) = e(f(\tau))$ . More precisely, it is given by

$$(2.10) \quad e(f) = f + \alpha\mu + A_1 \exp(\lambda_1 f) + A_2 \exp(\lambda_2 f),$$

where  $\lambda_1$  and  $\lambda_2$  are the roots to the characteristic equation in  $\lambda$ ,

$$(2.11) \quad (\alpha\sigma^2/2)\lambda^2 + \alpha\mu\lambda - 1 = 0,$$

and where the constants  $A_1$  and  $A_2$  are determined from the much discussed "smooth pasting" conditions,

$$(2.12a) \quad e_f(f) = e_f(\bar{f}) = 0,$$

that the function  $e(f)$  should be flat at the edges of the fundamental band.<sup>11</sup> The smooth pasting conditions and (2.10) then imply

$$(2.12b) \quad 1 + A_1 \lambda_1 \exp(\lambda_1 f) + A_2 \lambda_2 \exp(\lambda_2 f) = 0 \text{ and}$$

$$(2.12c) \quad 1 + A_1 \lambda_1 \exp(\lambda_1 \bar{f}) + A_2 \lambda_2 \exp(\lambda_2 \bar{f}) = 0,$$

from which the constants  $A_1$  and  $A_2$  can be solved.<sup>12</sup>

<sup>9</sup> Flood and Garber (1989) extend the target zone model to include finite interventions.

<sup>10</sup> See Harrison (1985) for the theory of regulated Brownian motion.

<sup>11</sup> See Krugman (1988), Dixit (1989b) and Dumas (1989b) for further discussion of the smooth pasting conditions. As emphasized by Dumas (1989b), the smooth pasting conditions here is really an infinitesimal value-matching condition, arising for any given infinitesimal regulator. This condition should be distinguished from "higher-order contact" conditions resulting from optimizing.

<sup>12</sup> Application of Ito's lemma on  $E[de(f)]/d\tau$  in (2.1) results in the ordinary differential

The function  $e(f)$  thus derived can be shown to be increasing, and the exchange rate will have lower and upper bounds given by

$$(2.13) \quad \underline{e} = e(f) \text{ and } \bar{e} = e(f).$$

The familiar S-shaped graph of the exchange rate function is shown in *Figure 1*. The smooth pasting conditions imply that the exchange rate curve is tangent to the edges of the exchange rate band. The parameters are  $\mu = 0$ , zero fundamental drift;  $\sigma = .1$ , which corresponds to an instantaneous standard deviation of the exchange rate of 10 percent per year under free float; and  $\alpha = 3$ , which corresponds to a money demand interest rate elasticity of .3 with a 10 percent nominal interest rate. The bounds for the fundamental in *Figure 1* are  $\pm 9.4$  percent ( $f = -f = .094$ ), which under zero fundamental drift corresponds to symmetric bounds for the exchange rate equal to  $\pm 1.5$  percent ( $\bar{e} = \underline{e} = .015$ ), the width of the Swedish exchange rate band. This is the set of parameters for which all diagrams below will be constructed.

### 3. The Interest Rate Differential

Let  $i^*(\tau; t)$  denote the foreign nominal interest rate on a pure discount foreign currency bond purchased at time  $\tau$ , with term (time to maturity)  $t$ , that is, maturing at time  $\tau + t$ ,  $t \geq 0$ . These foreign interest rates are given for our small open economy. The foreign interest rates may be stochastic processes (derived, say, as in Cox, Ingersoll and Ross (1985)) or deterministic — as long as they are exogenous to our small open economy. Let

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equation  $e(f) = f + \alpha \mu e_f(f) + \alpha \sigma^2 e_{ff}(f)/2$ . The solution to this differential equation is (2.10).

In the case of a zero fundamental drift and a symmetric fundamental band,  $\mu = 0$  and  $f = -f$ , there is a neat symmetric solution. Then the roots  $\lambda_1$  and  $\lambda_2$  can be written  $\lambda_1 = -\lambda < 0$  and  $\lambda_2 = \lambda > 0$ , where  $\lambda = \sqrt{2\alpha}/\sigma$ . The constants  $A_1$  and  $A_2$  fulfill  $A_1 = A > 0$  and  $A_2 = -A < 0$ , with  $A = 1/[2\lambda \cosh(\lambda f)]$ . The target zone exchange rate can then be written as  $e(f) = f - \sinh(\lambda f)/[\lambda \cosh(\lambda f)]$ . (We recall that the hyperbolic sine and cosine fulfill  $\sinh(x) = [\exp(x) - \exp(-x)]/2$  and  $\cosh(x) = [\exp(x) + \exp(-x)]/2$ .)

us also assume that we can disregard any risk premium between home and foreign nominal bonds, and assume uncovered interest rate parity.<sup>13</sup> Let  $i(f, \tau; t)$  denote the nominal interest rate on a home currency pure discount bond, purchased at time  $\tau$  with the fundamental  $f(\tau)$  equal to  $f$ , and maturing at time  $\tau + t$ ,  $t > 0$ . Then we assume the following approximate form of uncovered interest rate parity:

$$(3.1) \quad i(f, \tau; t) - i^*(\tau; t) = \frac{E[e(f(\tau + t)) \mid f(\tau) = f] - e(f)}{t}$$

The difference between the home term- $t$  interest rate and the foreign term- $t$  interest rate equals the expected change in the (log) exchange rate (the expected depreciation until maturity) divided by the time to maturity. Since the exchange rate and fundamental are Markov processes, the right-hand side of (3.1) depends only on the level of the fundamental at the time of purchase,  $f$ , and the term,  $t$ , not on the time of purchase of the bond,  $\tau$ . Therefore, the interest rate differential, the left-hand side of (3.1), which we will denote by  $\delta(f; t)$ , also depends only on  $f$  and  $t$ . Since both sides of (3.1) are independent of the purchase time, we can set the purchase time equal to zero and define the interest rate differential as

$$(3.2a) \quad \delta(f; t) = \frac{E[e(f(t)) \mid f(0) = f] - e(f)}{t}, \quad t > 0.$$

The interest rate differential for instantaneous bonds,  $\delta(f; 0)$ , will then be given by

$$(3.2b) \quad \delta(f; 0) = \lim_{t \rightarrow 0+} \delta(f; t) = E[de(f(\tau))]/d\tau.$$

The instantaneous interest rate differential equals the expected rate of change, the drift, of the exchange rate.

The interest rate differential for very long terms approach zero, since the numerator on the right-hand side in (3.2a) is bounded,

<sup>13</sup> The risk premium should be very small in narrow target zones, as shown in Svensson (1989, 1990), so uncovered interest arbitrage should be a good approximation.

Note that it is the *foreign exchange* risk premium for the same domestic and foreign term that is assumed to be small. There may very well be significant term premia between foreign currency bonds of different terms. This would show up in the exogenous foreign interest rates  $i^*(\tau; t)$ . With uncovered interest parity the same risk premia would then show up between home currency bonds of different terms.

$$(3.2c) \quad \delta(f; \infty) = \lim_{t \rightarrow \infty} \delta(f; t) = 0.$$

### 3.1 Free Float

Before further consideration of a target zone, let us briefly consider a free float, meaning an exchange rate regime without any interventions.

First we specify what the exchange rate is under a free float. We assume that under a free float the money supply  $m$  is held constant at, for simplicity, a zero level,  $m = 0$ . Then the fundamental is simply equal to velocity,  $f = v$ . Velocity and hence the fundamental follow the Brownian motion (2.8). This implies that the probability distribution of  $f(s)$  at time  $s$ , conditional upon  $f(\tau) = f$  at time  $\tau$ ,  $s \geq \tau$ , is normal with mean  $f + \mu(s - \tau)$  and variance  $\sigma^2(s - \tau)$ . By directly integrating (2.4) we see that the solution to the exchange rate equation (2.1) is simply

$$(3.3) \quad \hat{e}(\tau) = f(\tau) + \alpha\mu.$$

It then follows from (3.2) that the interest rate differential fulfills

$$(3.4) \quad \hat{\delta}(f; t) = \mu, \quad t \geq 0.$$

The interest rate differential is equal to the constant expected rate of change of the exchange rate, the expected rate of depreciation of the home currency, and it is independent of the term and the fundamental.

### 3.2 Target Zone

Next we shall consider the target zone exchange rate regime, described by the stochastic process for the fundamental, the regulated Brownian motion (2.2) and (2.5)-(2.9), and the stochastic process for the exchange rate, described by (2.10)-(2.12). For zero term, the instantaneous interest rate differential is by (3.2b) and (2.1) easy to compute as

$$(3.5) \quad \delta(f; 0) = [e(f) - f]/\alpha.$$

For positive terms, it is more difficult to compute the interest rate differentials (3.2a). The difficulty lies in computing the expected exchange rate at maturity on the right-hand

side of (3.2a),

$$(3.6) \quad h(f;t) = E[\alpha(f(t)) \mid f(0) = f].$$

This is difficult since the exchange rate is a complicated nonlinear heteroscedastic stochastic process, with variable drift and instantaneous standard deviation.

Application of a finite-horizon variant of Harrison's (1985) infinite-horizon version of the Feynman-Kac formula for regulated Brownian motion (see appendix for details) shows, after some manipulations, that the function  $h(f;t)$  defined in (3.6) will be the solution to the partial differential equation,

$$(3.7a) \quad h_f(f;t) = \mu h_f(f;t) + (1/2)\sigma^2 h_{ff}(f;t), \quad f \leq f \leq f', \quad t \geq 0,$$

with the initial condition

$$(3.7b) \quad h(f;0) = \alpha(f), \quad f \leq f \leq f',$$

and the derivative boundary conditions

$$(3.7c) \quad h_f(f;t) = 0 \quad \text{and} \quad h_{ff}(f;t) = 0, \quad t \geq 0.$$

Equation (3.7a) is a so-called parabolic partial differential equation, similar to those that arise in option pricing problems (see Merton (1973)). The initial condition (3.7b) is obvious. The derivative boundary conditions (3.7c) are smooth pasting conditions like (2.12a). (For  $\mu$  equal to zero the partial differential equation is identical to the so-called heat equation in physics which describes thermal diffusion; see Churchill and Brown (1987).)

The intuition for the smooth pasting conditions for the expected maturity exchange rate is similar to, although somewhat more involved than, the smooth pasting conditions for the exchange rate: For a fixed future date  $T$ , define the the stochastic process  $F(f(\tau),\tau)$  according to

$$(3.8) \quad F(f,\tau) = h(f;T-\tau), \quad \tau \leq T.$$

That is,  $F(f(\tau),\tau)$  is the expectation at *calendar time*  $\tau$ , given the value of the fundamental at time  $\tau$ , of the exchange rate at a fixed future date  $T$  ( $h(f;t)$  is a function of term). Let us express (3.7) in terms of  $F(f,\tau)$ . It is shown in the appendix that (3.8) and Ito's lemma implies that (3.7a) is equivalent to

$$(3.9a) \quad E[dF(f(\tau), \tau)] = 0, \quad f \leq f(\tau) \leq \bar{f}, \quad \tau \leq T,$$

that is, the expected change in  $F(f(\tau), \tau)$  is zero. The initial condition (3.7b) is now equivalent to a terminal condition,

$$(3.9b) \quad F(f, T) = e(f).$$

Since by (3.8)  $F_f(f, \tau) = h_f(f; T - \tau)$ , the smooth pasting conditions (3.7c) for  $h(f; t)$  are equivalent to the smooth pasting conditions

$$(3.9c) \quad F_f(f, \tau) = 0, \quad \text{and} \quad F_f(\bar{f}, \tau) = 0, \quad \tau \leq T,$$

for  $F(f, \tau)$ .

Let us now propose that there exists a solution  $h(f; t)$  to (3.7a) and (3.7b) that does not fulfill the smooth pasting conditions (3.7c) for all  $t \geq 0$ . Then there also exists a solution  $F(f, \tau)$  fulfilling (3.9a) and (3.9b) but not the smooth pasting conditions (3.9c). This solution is also the (mathematical, but not economically meaningful) solution to (3.9a) and (3.9b) under a free float when no interventions are undertaken and the fundamental is allowed to drift freely outside the fundamental band. (We then interpret the function  $e(f)$  in (3.9b) as referring to the mathematical extension of the exchange rate function (2.10), with (2.11) and (2.12), to values of the fundamental outside the fundamental band.) Suppose that  $F(f, \tau)$  instead of fulfilling the smooth pasting condition is increasing at the upper edge of the fundamental band,  $F_f(\bar{f}, \tau) > 0$ , for a particular  $\tau < T$ . Such an  $F(f, \tau)$  is illustrated in *Figure 2* by the dashed curve, increasing at point  $Y$  at the upper edge of the fundamental band. Under a free float, at point  $Y$  the fundamental can either decrease and move inside the band or increase and move outside the band. We know that  $F(f, \tau)$  fulfills (3.9a) for  $f = \bar{f}$  under a free float. If we impose the band, interventions prohibit the fundamental from taking values above  $\bar{f}$ . Then, at point  $Y$ , the fundamental can either stay the same or decrease, never increase. Since  $F(f, \tau)$  is increasing, it follows that it must now be the case that  $E[dF(f, \tau)] < 0$ , in violation of (3.9a). Hence the proposed  $F(f, \tau)$  and  $h(f; t)$  cannot be a solution with the band. The only case when  $F(f, \tau)$  and  $h(f; t)$  can be a solution both with and without the band is when they are neither increasing or decreasing at the edges of the band, that is,

the smooth pasting conditions (3.7c) and (3.9c) must be fulfilled. Such an  $F(f, \tau)$  is illustrated by the solid curve through point  $X$  in Figure 2. This completes the discussion of the intuition of the smooth pasting conditions.

The partial differential equation (3.7a) with initial condition (3.7b) and boundary conditions (3.7c) can be solved in (at least) two different ways. First, it has an analytical Fourier-series solution, which can be derived with methods described in for instance Churchill (1963). This series involve a summation of infinitely many terms, and in practice this analytical solution has to be computed numerically as a summation of a truncated series. The Fourier-series solution is presented in the appendix. Second, the partial differential equation can be solved in a direct numerical way, using for instance the so-called explicit method described by Gerald and Wheatley (1989). This method of solution is also presented in the appendix. It is reassuring that both methods of computing the solution give the same result.

The numerical solution to the expected maturity exchange rate  $h(f; t)$  is illustrated in Figures 3a-c, for a zero fundamental drift. *Figure 3a* shows the expected maturity exchange rate as a function of the fundamental, for terms equal to 0, 1, 3, 6, 12, and 60 months. For zero term the expected exchange rate, of course, coincides with the current exchange rate, the solid curve. For positive terms the expected maturity exchange rate has a similar but flatter S-shape, still increasing in the fundamental and flat at the edges. The expected maturity exchange rate lies between the current exchange rate and the exchange rate's unconditional mean. (With a symmetric band and zero fundamental drift, the exchange rate's unconditional mean is zero.) The further into the future the maturity, the closer the expected maturity exchange rate is to the unconditional mean. This is seen also in *Figure 3b*, which shows the expected maturity exchange rate as a function of the term for given levels of the fundamental equal to zero,  $\pm$  half the fundamental band, and  $\pm$  the full fundamental band (that is,  $f$  equals 0 %,  $\pm 4.7$  %, and  $\pm 9.4$  %). The solid curve corresponds to the fundamental and the exchange rate being at their lower edges. The expected exchange rate five years ahead or more is practically equal to the unconditional

mean.

Figure 3c shows a three-dimensional graph of the expected maturity exchange rate as a function of both the fundamental and the term.

Given the behavior of the expected maturity exchange rate  $h(f;t)$ , it is then straightforward to compute the interest rate differential for positive terms as

$$(3.10) \quad \delta(f;t) = \frac{h(f;t) - e(f)}{t}, \text{ for } t > 0.$$

The interest rate differential for a given positive term can be found in Figure 3a, first, by taking the vertical signed difference between the corresponding expected maturity exchange rate curve and the current exchange rate curve and, second, by dividing the difference by the term. The interest rate differential is illustrated in Figures 4a-d.

Figure 4a shows the interest rate differential as a function of the fundamental, for the same terms as in Figure 3a, that is, 0, 1, 3, 6, 12, and 60 months. The solid curve is the instantaneous (zero-term) interest rate differential,  $\delta(f;0)$ , which coincides with the drift of the exchange rate and the expected instantaneous depreciation of the currency, and which can easily be computed according to (3.5). The properties of the instantaneous interest rate differential are extensively examined in Svensson (1989). Here it is only necessary to note that it is decreasing in the fundamental. In the left and lower part of the fundamental and the exchange rate bands, that is when the currency is strong, expectations of future interventions to increase the money supply imply expectations of a depreciation of the currency and hence a positive interest rate differential. Conversely, in the right and upper part of the fundamental and exchange rate band, expectations of future interventions to reduce the money supply imply expectations of an appreciation and a negative interest rate differential. The instantaneous interest rate differential does not fulfill the smooth pasting conditions.<sup>14</sup>

<sup>14</sup> The *instantaneous* interest rate differential is, in the case with zero drift and symmetric bands, by footnote 11 and (3.5) given by  $\delta(f;t) = -\sinh(\lambda f)/[\alpha\lambda\cosh(\lambda f)]$ . The interest rate differential's derivative with respect to the fundamental is then given by  $\delta_f(f;t) = -\cosh(\lambda f)/[\alpha\cosh(\lambda f)]$ , and the derivative at the edges are  $\delta_f(f;t) = \delta_f(f;t)$

For positive terms the interest rate differential curves are flatter than for zero term, except for very short terms and in the middle of the band. In contrast to the instantaneous interest rate differential, positive-term interest rate differentials fulfill the smooth-pasting conditions and are flat at the edges of the band.<sup>15</sup> This implies that the derivative of the interest rate differential with respect to the fundamental,  $\delta_f(f;t)$ , is discontinuous at  $t = 0$ . *Figure 4b* shows the interest rate differential as a function of the term for the same levels of the fundamental as in *Figure 3b* (zero,  $\pm$  half the fundamental band, and  $\pm$  the full fundamental band). The solid curve corresponds to the fundamental and exchange rate being at the lower edges of their bands. The interest rate differential is decreasing in the term, except for very short terms and intermediate levels of the fundamental, where it is approximately flat. For long terms the expected maturity exchange rate is close to its unconditional mean, and the absolute value of the interest rate differential is approximately the distance between the exchange rate's unconditional mean and its current level, divided by the term, which will be decreasing in the term. For shorter terms, the difference between the exchange rate's expected maturity value and current value (the vertical difference between the corresponding curves in *Figure 3a*) is increasing in the term, which effect goes the other way. Except for very short terms and intermediate levels of the fundamental, the effect of dividing by the term dominates.

We also see that for very long terms, the interest rate differential approaches zero, in accordance with (3.2c).

*Figure 4c* shows a three-dimensional graph of the interest rate differential as a function of both the fundamental and the term.

*Figure 4d* shows the interest rate differential as a function of the exchange rate rather than the fundamental, for given terms. We note that for positive terms, the relationship is approximately linear, and more so for longer terms.

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$$= -1/\alpha < 0.$$

<sup>15</sup> The interest rate differential fulfills the smooth pasting conditions for positive maturities, since by (3.13)  $\delta_f = (h_f - e_f)/t$  and both  $h(f;t)$  and  $e(f)$  fulfill the smooth pasting conditions.

We see that we get very specific conclusions about the properties of the interest rate differential for different terms. These specific properties certainly invite empirical testing of the target zone model, in particular the relations between the term structure of interest rate differentials and the exchange rate depicted in Figure 4d. We shall return to Figure 4d when we look at some data. But first we shall deal with devaluation risk.

#### 4. Devaluation risk

So far the target zone exchange rate regime has been assumed to be perfectly credible. Let us now extend our analysis of the term structure of interest rate differentials by allowing for a devaluation/realignment risk.

There are several ways of modeling devaluations/realignments. We follow Svensson (1989) and model devaluations as reoccurring with some given constant probability, regardless of where in the band the exchange rate lies. This will allow a simple analytic solution of the exchange rate equation. Also, some real world devaluations seem indeed to occur when the exchange rate is in the interior of the band.<sup>16</sup> More precisely we model devaluations as occurring according to a Poisson process  $N(\tau)$  with constant parameter  $\nu > 0$ , meaning that during the interval  $d\tau$  the process will take at least one jump of unity with probability  $\nu d\tau + o(d\tau)$  and remain unchanged with probability  $(1 - \nu d\tau) + o(d\tau)$ .<sup>17</sup> Then  $N(\tau)$ , taking integer values, can be interpreted as the number of devaluations that have occurred up to and including time  $\tau$ .

A devaluation is then modeled as a simultaneous shift of the *same* magnitude  $g$  in the lower and upper bounds for the fundamental *as well as* in the money supply. (A positive  $g$  corresponds to a net devaluation, a negative to a net revaluation.) Then a devaluation

<sup>16</sup> For instance, when Sweden devalued in September 1981 and October 1982 the Krona's value was above previous minimum values.

See Miller and Weller (1988, 1989a) and Bertola and Caballero (1989) for an analysis of repeated realignments that occur with a given probability at the edge of the band.

<sup>17</sup> We let  $o(d\tau)$  denote terms of order higher than  $d\tau$ , that is,  $\lim_{d\tau \rightarrow 0} o(d\tau)/d\tau = 0$ .

maintains the fundamental's position relative to the fundamental band. More precisely, the lower and upper bounds for the fundamental are functions  $f(N)$  and  $f(N)$  of the number of devaluations given by

$$(4.1) \quad f(N) = f_0 + gN \text{ and } f(N) = \bar{f}_0 + gN,$$

where  $f_0$  and  $\bar{f}_0$  are constants. After  $N$  devaluations, the fundamental is restricted to fulfill

$$(4.2) \quad f(N) \leq f \leq f(N).$$

The upper and lower bounds thus change according to

$$(4.3) \quad df = g dN \text{ and } df = g dN,$$

where  $dN$  is unity with probability  $\nu dt$  and zero with probability  $(1 - \nu dt)$ . Similarly, the money supply is now given by the process  $m = L - U + gq$ , that is,

$$(4.4) \quad dm = dL - dU + g dN,$$

where  $L$  and  $U$  are the lower and upper regulators described in section 2.

The exchange rate still fulfills the exchange rate equation (2.1) (we continue to disregard the risk premium), but it will now be a function  $\bar{e}(f, N)$  of both the fundamental and the number of devaluations. It is easy to show (see Svensson (1989)) that the exchange rate equation is given by

$$(4.5) \quad \bar{e}(f, N) = f + \alpha\mu + \alpha\nu g + A_1 \exp[\lambda_1(f - gN)] + A_2 \exp[\lambda_2(f - gN)],$$

where  $\lambda_1$  and  $\lambda_2$  are the roots of the characteristic equation (2.11) and the constants  $A_1$  and  $A_2$  are given by (2.12) with  $f_0$  and  $\bar{f}_0$  substituted for  $f$  and  $f$ .

Comparing (4.4) for  $N = 0$  with the solution without devaluation risk, (2.10), we realize that the only modification of the exchange rate function is the addition of the term  $\alpha\nu g$ ,  $\alpha$  times the rate of expected devaluation. Indeed, the exchange rate functions with and without devaluation risks are related by

$$(4.6) \quad \bar{e}(f, N) = e(f - gN) + gN + \alpha\nu g.$$

In Figure 1, for instance, the only modification is that the curve corresponding to the target zone exchange rate is shifted up by  $\alpha\nu g$ . When a devaluation occurs, the curve is shifted each time up and to the right by the magnitude  $g$ , hence every point on the curve

is shifted on a 45 degree line northeast in the figure.

Let  $\bar{\delta}(f, N; t)$  denote the term- $t$  interest rate differential, given the current level of the fundamental and number of devaluations. It is given by

$$(4.7) \quad \bar{\delta}(f, N; t) = \frac{\bar{h}(f, N; t) - \bar{e}(f, N)}{t},$$

where

$$(4.8) \quad \bar{h}(f, N; t) = E[\bar{e}(f(t), N(t)) \mid f(0) = f, N(0) = N]$$

is the expected maturity exchange rate. We observe that the expected maturity exchange rate can be written

$$(4.9) \quad \begin{aligned} \bar{h}(f, N; t) &= E[e(f(t) - gN) \mid f(t) = f, N(t) = N] + E[gN(t) \mid N(0) = N] + \alpha \nu g = \\ &= h(f - gN; t) + gN + \nu g t + \alpha \nu g. \end{aligned}$$

The first equality follows from (4.6) and the fact that the expected maturity exchange rate can be decomposed into the expected movement inside the band absent any devaluations (the first term on the right-hand side of the first equality) and the expected devaluations (the second term on the right-hand side of the first equality). The second equality follows from the definition of  $h(f; t)$  in (3.6) in the absence of a devaluation risk, and from the properties of a Poisson process.

Substitution of (4.9) and (4.6) into (4.7) gives

$$(4.10) \quad \bar{\delta}(f, N; t) = \frac{h(f - gN; t) + \nu g t - e(f - gN)}{t} = \delta(f - gN; t) + \nu g,$$

where the second equality follows from the definition of the interest rate differentials (3.10) in the absence of any devaluation risk.

Hence, with a devaluation risk the only modification of the interest rate differentials is that a constant equal to the rate of expected devaluation,  $\nu g$ , is added for each term. Otherwise, the interest rate differentials depend on the term and the fundamental relative to the current fundamental band as without devaluation risk. In particular, for very long terms the interest rate differential does not approach zero but the rate of expected devaluation,

$$(4.11) \quad \bar{\delta}(f, N; \omega) = \nu g.$$

In Figures 4a-d, the only modification is that all graphs are shifted up by  $\nu g$ .

### 5. Empirical Results: Sweden 1986-1988 <sup>18</sup>

Sweden has had a unilateral exchange rate target zone since 1977. The band for the krona is specified in terms of an index for a currency basket of trade-weighted currencies, with double weight for the dollar. The krona was devalued 1981 and 1982 by 10 and 16 percent, respectively. At the last devaluation the central parity was set at 132. The width of the band was initially kept secret at  $\pm 2.25$  percent. During the Spring of 1985 there were considerable capital out-flows, the krona depreciated, and the index rose above 132. This may have caused increased uncertainty about the exchange rate band. Interest rate differentials rose to very high levels. In June 1985 the band was reduced to  $\pm 1.5$  percent and publicly announced. During the Fall of 1985 the interest rate differentials fell considerably and capital flows stabilized.<sup>19</sup> During the winter 1989-1990, after considerable nominal wage increases, interest rates rose dramatically and devaluation rumors started to circulate.

I have chosen to examine the period February 1986-October 1989 because this appears to be a rather stable period for the Swedish target zone, after the unrest of 1985 and the regime shift of June 1985 and before the unrest of the winter 1989-1990.

Monthly data (last day of month quotations) of the exchange rate index, Swedish Treasury Bill interest rates, and Euro interest rates have been collected for terms 1, 3, 6, 12 and 60 months. From the Euro interest rates a foreign interest rate index has been computed with the same weights as in the currency basket. The interest rate differentials are then the difference between the Swedish interest rate and the foreign interest rate

<sup>18</sup> I thank Nils Gottfries for many discussions on this section (and exempt him from any responsibility).

<sup>19</sup> See Hörngren and Viotti (1985) and Ringström (1987) for details.

index. (The sample of 60 month Euro interest rates is incomplete. Some national bond interest rates have been added. The corresponding currencies make up only about 50 percent of the currency basket.)

The exchange rates and the interest rate differential data is plotted for the period February 1986-February 1990 (months numbered 2-50) in *Figure 5a*. The exchange rate is measured in percentage deviation from the central parity, and the interest rate differentials are measured in percent per year. *Figure 5b* shows the interest rate differentials for 1, 3, 6, 12 and 60 months shifted up 0, 1, 2, 3 and 4 percent, respectively. Summary statistics for the period February 1986-October 1989 are collected in *Table 1*. The correlation coefficients are displayed in *Table 2*.

We initially note in *Figure 5* and *Table 1* that the exchange rate during 1986-89 fluctuated between the lower edge (-1.5%) and the middle of the band (0%) and never went far into the upper half of the band. This raises the possibility that the Riksbank during the period actually defended an implicit band consisting of the lower half of the explicit band.<sup>20</sup> We will return to this possibility below. With regard to the issue of any drift in the fundamental, a positive drift would imply that the exchange rate were most of the time away from the lower edge of the band. In *Figure 5* the exchange rate is fairly evenly distributed in the lower part of the band, which is not inconsistent with a zero fundamental drift. Let us tentatively conclude that the hypothesis of a zero drift cannot be rejected.

Let us recall *Figure 4d*, which summarizes the theoretically derived relations between the exchange rate and the interest rate differential for different terms, with zero fundamental drift and in the absence of a devaluation risk. In the theory, the mean of the interest rate differentials is zero, absent any devaluation risk. In the data, we see in *Figure 5* and *Table 1* that the mean is positive and between 1.2 and 2 for all interest rate differentials, alerting us to the possibility of a devaluation risk. In the theory, the interest

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<sup>20</sup> See Hörngren and Viotti (1985), Ringström (1987) and Klein (1989a) on implicit bands inside the official band.

rate differentials are negatively correlated with the exchange rate. In the data, Table 2 indeed displays a clear negative correlation between the exchange rate and the differentials.

In the theory, the band for each interest rate differential should be decreasing in the term, as is apparent from Figure 4d. In the data, Table 1 shows that the interest rate differentials' minimum is increasing in the term (except for the 60-month interest rate differential), and that the maximum is decreasing in the term. Hence, the interest rate differential bands defined this way are indeed decreasing in the term. In the theory, the (unconditional) standard deviations of the interest rate differentials are decreasing in the term. We see in in Table 1 that this is also the case for the data. Even if the differences in the standard deviations may not be significant, we note that there are no reversals in the order.

It has often been observed that the variability of long term interest rates is less than the variability of short term interest rates (see Shiller and McCulloch (1987)). Here we see a similar result with respect to the interest rate *differentials*. The similarity is only superficial, though. The theoretical explanation for the result on the differentials is completely independent of the variability of interest rates; it has only to do with the variability of expected future exchange rates.

Hence, we see that the theoretical results on the interest rate differentials are indeed consistent with some summary statistics of the data. Let us however look a bit closer at the data. In Figure 4d, the theoretical relations between the interest rate differentials and the exchange rate are fairly linear (especially for longer terms). Let us consider a linear approximation to the relation the interest rate differentials and the exchange rate, for given terms  $t$ :

$$(5.1) \quad \delta(\tau; t) = a(t) + b(t)\epsilon(\tau) + \epsilon(\tau; t).$$

The coefficients  $b(t)$  are negative and increasing in term, according the theory. Absent fundamental drift, the constants  $a(t)$  should be zero if there is no devaluation risk. We take the error terms  $\epsilon(\tau; t)$  to have zero mean and to be uncorrelated with the exchange

rate. Then it is appropriate to do a linear regression of the interest rate differentials on the exchange rate, for each term, in order to estimate  $a(t)$  and  $b(t)$ .

OLS regressions reveal low Durbin-Watson statistics and hence possible serial correlation of the error terms. I have therefore used Hansen and Hodrick's (1980) version of GMM (General Methods of Moment) together with Newey and West's (1987) modification which makes sure that the estimate of the variance-covariance matrix is positive semidefinite. This method allows serially correlated and heteroscedastic errors. In the particular case considered here, the GMM estimator coincides with the OLS estimator, whereas the standard errors are computed differently.<sup>21</sup>

The results from the GMM regressions are reported in *Table 3*. The estimates of the constants and the coefficients of the exchange rate are displayed in columns (2) and (3). I have also run iterated GLS regressions under the assumption that error terms are AR(1) (using a modified Cochrane-Orcutt procedure which includes the first observation). The GLS regressions give similar results and are not reported here.

The estimates of the constants are all significantly positive. Under the maintained hypothesis of zero fundamental drift, we can therefore reject the hypothesis of no devaluation risk. The estimates of the coefficients for the exchange rate are negative as in the theory. They are significantly less than zero except for a 60 months term. The coefficients are decreasing in the term, as in the theory. Even if each decrease in that slope is not significant, we note that there are no reversals in the order. The regression lines are plotted in *Figure 6*. Except for the differing intercepts, the similarity between the theoretical *Figure 4b* and the empirical *Figure 6* is striking. The steepest line corresponds to a 1 month term, the second steepest line to a 3 month term, etc., exactly as in the theory. The slope of the 60-month term line is not significantly different from zero. In the theory it should be very flat.

Let us go on and discuss the estimates of the constant terms in the regressions. A

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<sup>21</sup> I am grateful to John Campbell and Bernard Dumas for suggesting the use of GMM. Further details on GMM are given in Hansen (1980), Cumby, Huizinga and Obstfeld (1983) and the survey by Hodrick (1987).

constant in the regression equation might (according to (4.10) and the theoretical analysis in section 4) be interpreted as an estimate of an average devaluation risk, measured as the expected rate of devaluation, the probability per unit of time of a devaluation times the size of the devaluation. (For instance, a 10 percent probability per year of a 20 percent devaluation results in an expected rate of devaluation of 2 percent.) The constants correspond to the intercepts of the regression lines in Figure 6, that is, the levels of the regression lines for a zero exchange rate. Looking at the intercepts in Figure 6, we see that the devaluation risk is about 1.2 percent.

Suppose, however, that the exchange rate band has actually been only the lower half of the official  $\pm 1.5$  percent band. Then the relevant estimate of the average devaluation risk is not the intercept in the middle of the official band, at a zero exchange rate, but the level of the regression lines in the middle of the lower half of the band, that is for an exchange rate equal to  $-0.75$  percent. Then we see from Figure 6 that the average devaluation risk is about 1.9 percent for a 1-12 month horizon, and about 1.2 percent for a 60 month horizon. We recall that the sample of 60-month Euro interest rates is incomplete, which may be one reason why that differential appears to be an outlier.

A considerable proportion of the total variability of the interest rate differentials remains unexplained by the exchange rate and is instead explained by serial correlation in the error terms. (The R-squared of the regressions is between .2 and .4, for 1-12 months term, see column (4) in Table 3.) A rather reasonable interpretation of these error terms is that they are serially correlated devaluation risks. It makes sense that the perceived devaluation risk might vary over the period, and it also makes sense that it would be serially correlated. The estimates of the constants referred to above are then the averages of these time-varying devaluation risks. If time-varying devaluation risks actually enter the error terms, it is important that they are uncorrelated with the exchange rate. Otherwise the estimates of the coefficients  $b(t)$  are biased. Since the exchange rate has been rather stable in the strong half of its band during this period, it does not seem impossible that any devaluation risks actually have been uncorrelated with the exchange

rate.

Any non-zero correlation between devaluation risks and the exchange rate is bound to be positive rather than negative (increased devaluation risk should if anything depreciate the currency). Such positive correlation would make the estimates of the slopes of the regression lines biased towards less negative slopes and more positive slopes. Even with such possible bias the estimated slopes are still negative and significantly different from zero (except for a 60 months term). Any bias would therefore actually seem to strengthen our results.

Miller and Weller (1988, 1989a) and Bertola and Caballero (1989) have developed models where devaluations occur with a given probability at the upper edge of the exchange rate band. In such models there may arise a positive relation between the interest rate differential and the exchange rate.<sup>22</sup> Whereas there is some evidence for such a positive relation for Franc-Deutschmark and Lire-Deutschmark interest rate differentials (see Bertola and Caballero (1989) and Bodner (1989)) our results show that there is no evidence for this in Swedish data from the period February 1986 - October 1989. Our results on a negative relation between interest rate differentials and the exchange rate indicate that any devaluation risk is at most weakly and possibly not at all correlated with the exchange rate. It may very well be the case that the Swedish target zone has operated quite differently from the French and Italian target zones in the European Monetary System.

Another interpretation of the serially correlated error terms is that they are shadow taxes, or transactions costs, due to remaining exchange controls. Swedish exchange controls have been gradually dismantled during the last few years, and were (almost) completely abolished only in July 1989. The effects of the exchange controls in the last few years have been much discussed. The general view seems to be that they have hardly been binding during the period and probably had very little effect.

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<sup>22</sup> As shown by Bertola and Caballero (1989) the positive relation occurs when the exchange rate function has a reversed S-shape.

A third interpretation of the serially correlated error terms is that they might be time-dependent serially correlated foreign exchange risk premia, as discussed in the survey by Hodrick (1987) and by Hörngren and Vredin (1988). However, for relatively small target zones (like the Swedish one) Svensson (1989, 1990) finds that in theory such risk premia should be insignificant, at least with limited devaluation risks.

I conclude from this look at the data that, although the error terms need further explanation, and the theory need further elaboration (for instance to include more sophisticated devaluation risks than the ones considered in section 4), the theory and the Swedish data for the period examined are consistent to a considerable extent, for instance as summarized in Figures 4d and 6.

## 6. Summary and Conclusions

We have computed the term structure of interest rate differentials in a narrow target zone, as a function of the fundamental's and the exchange rate's position in their bands. The computation has been done in two different ways, via an analytical Fourier-series method and via a direct numerical method. Both methods give the same results. In diagrams we have illustrated both the qualitative and quantitative behavior of the interest rate differentials.

The interest rate differentials' qualitative dependence on the *position in the band* is quite intuitive. For a given term, the interest rate differential is decreasing in the fundamental and the exchange rate. In the lower part of the exchange rate and fundamental bands, that is, when the currency is strong, expectations of future interventions to increase money supply imply an expected future depreciation, which in turn imply positive interest rate differentials; vice versa in the upper part of the fundamental and exchange rate bands. In contrast to the instantaneous interest rate differential, finite-term interest rate differentials fulfill smooth pasting conditions at the edges of the band. The relation between interest rate differentials and log exchange rates

is rather linear, especially for longer terms.

The interest rate differentials' qualitative dependence on the *term* is also quite intuitive. The interest rate differential is determined by expected depreciation until maturity, divided by the term. For a given position in the fundamental and exchange rate bands, for medium and long terms the interest rate differential's absolute value is decreasing in the term, since the expected depreciation until maturity is bounded by the exchange rate band and not much affected by changes in the term.

We have also shown how devaluation/realignment risks can be incorporated. For the particular assumptions used here, for instance that the devaluation risk is independent of the exchange rate's position in the band, a devaluation risk simply implies a constant upward shift, equal to the expected rate of devaluation, of all interest rate differentials.

An examination of Swedish data shows broad consistency between some implications of the theory and the data. The theory's prediction that the interest rate differentials are decreasing in the exchange rate, less so for longer terms, and that the interest rate differentials' bands and standard deviations are decreasing in the term are all confirmed in the data. On the other hand, a limited proportion of the total variability of interest rate differentials are explained by variations in the exchange rate.

As possible factors explaining the remaining variability of interest rate differential and serial correlation of error terms, we have noted serially correlated devaluation risks, remaining exchange controls, and risk premia. The possibility of distinguishing these will be subject to further theoretical and empirical work.

That interest rate differentials are decreasing in the exchange rate supports the assumption that devaluation risks in the Swedish target zone do not increase when the Krona depreciates within the band. In this respect the Swedish target zone may differ from how the French and Italian target zones have operated within the European Monetary System, where it appears that interest rate differentials increase when the Franc and Lira depreciate relative to the Mark, as discussed by Bertola and Caballero (1989) and Bodner (1989).

In the introduction we remarked that the theory of the term structure of interest rate *differentials* developed here is a complement and not a substitute to established theories of the term structure of interest *rates*. One way to combine the two would be to use the relations derived in this paper as additional restrictions in empirical tests of the established theories of the term structure of interest rates.

Let us also briefly comment upon some real or only apparent limitations of the analysis. A most obvious limitation is perhaps the simplifying assumptions about the behavior of velocity shocks, and the assumption about infinitesimal marginal interventions only. These two assumptions imply that the fundamental follows a regulated Brownian motion with reflecting barriers, that is, with constant drift and instantaneous standard deviation inside the band. Intramarginal interventions, that is, interventions inside the band, can be incorporated to a considerable extent, though. If they lead to a process for the fundamental with mean reversion, or more generally to any process with a variable drift that depends on the level of the fundamental,  $\mu(f)$ , the exchange rate function  $e(f)$  can always be solved numerically.<sup>23</sup> The numerical solution to the exchange rate function can then be used in our procedure to solve the interest rate differentials numerically. Our partial differential equation (3.7a) is a bit more complicated, with a variable  $\mu$ , but it can always be solved with the direct numerical method should the Fourier-series method become cumbersome. I believe that the relation between interest rate differentials and the exchange rate illustrated in Figure 4d is rather robust and should appear also with other specifications. Clearly, data on central bank interventions would be extremely helpful in deciding which stochastic process is the best representation of central bank behavior.<sup>24</sup>

Finite interventions rather than infinitesimal can be handled and the exchange rate function can then still be solved analytically, as demonstrated by Flood and Garber (1989)

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<sup>23</sup> Actually, if the drift function  $\mu(f)$  is linear, an analytical solution to the exchange rate function exists, namely the so-called confluent hypergeometric function (see Dumas (1988) and Froot and Obstfeld (1989b)).

<sup>24</sup> Mundaca (1989) uses intervention data to model interventions of Norges Bank and the effect of interventions on exchange rate volatility in the Norwegian target zone.

and Bertola and Caballero (1989). Our numerical method of determining interest rate differentials would, it seems, still work; only the smooth pasting conditions would be replaced by so called value-matching conditions (see Dixit (1989b) and Dumas (1989b) for a discussion of the relation between smooth pasting and value-matching conditions). Again, data on central bank interventions would be extremely helpful in deciding whether finite or infinitesimal interventions are the best approximation.

## Appendix

### A.1 The Expected Time to Hit the Edges of the Band

The fundamental is a regulated Brownian motion fulfilling (2.5)-(2.9). Using results in Harrison (1985), it is not difficult to find the expected time  $E[T(f)]$  for the fundamental to hit the first time either of the edges, starting at the level  $f$ ,  $f \leq f \leq f$ . It is given by

$$(A.1a) \quad E[T(f)] = \frac{(f - f)(f - f)}{\sigma^2} \quad \text{for } \mu = 0, \text{ and}$$

$$(A.1b) \quad E[T(f)] = \left\{ \frac{1 - \exp[-2\mu(f - f)/\sigma^2]}{1 - \exp[-2\mu(f - f)/\sigma^2]} (f - f) - (f - f) \right\} / \mu \quad \text{for } \mu \neq 0.$$

Consider the zero drift case. With  $f = -f = .094$  and  $\sigma = .1$  per year,  $E[T(f)]$  is about 10.6 months for  $f = 0$ .

### A.2 The Finite-Horizon Feynman-Kac Formula for Regulated Brownian Motion

We use a finite-horizon variant of Harrison's (1985, p. 83) infinite-horizon Feynman-Kac formula for regulated Brownian motion:

Consider a regulated Brownian motion  $Z_\tau$  ( $\tau$  is calendar time) with lower and upper bounds  $a$  and  $b$ ,

$$Z_\tau = X_\tau + L_\tau - U_\tau \in [a, b].$$

The stochastic process  $X_\tau$  is a Brownian motion with drift  $\mu$  and instantaneous variance  $\sigma^2$ , and  $L_\tau$  and  $U_\tau$  are the lower and upper regulators, respectively. Let  $u(x, \tau)$  and  $g(x, \tau)$  be given twice continuously real-valued functions  $[a, b] \times R \rightarrow R$  and let  $T$ ,  $\eta$ ,  $c$  and  $r$  be given constants. Assume  $g_x(a, T) = c$  and  $g_x(b, T) = r$ . Consider the integral

(A.2)

$$F(x, \tau) = E \left\{ \int_{s=\tau}^T e^{-\eta(s-\tau)} [u(Z_s, s) ds - c dL_s + r dU_s] + e^{-\eta(T-\tau)} g(Z_T, T) \mid Z_\tau = x \right\},$$

for  $a \leq x \leq b$  and  $\tau \leq T$ . Then the finite-horizon Feynman-Kac formula for regulated Brownian motion says that, under some regularity conditions,  $F(x, \tau)$  is the unique

solution to the partial differential equation

$$(A.3a) \quad DF(x, \tau) - \eta F(x, \tau) + u(x, \tau) = 0, \quad a \leq x \leq b, \quad \tau \leq T,$$

with the terminal condition

$$(A.3b) \quad F(x, T) = g(x, T)$$

and the derivative boundary conditions (smooth pasting conditions)

$$(A.3c) \quad F_x(a, \tau) = c \quad \text{and} \quad F_x(b, \tau) = \tau, \quad \tau \leq T.$$

Here  $D$  denotes the differential operator

$$DF(x, \tau) = F_\tau(x, \tau) + \mu F_x(x, \tau) + (1/2)\sigma^2 F_{xx}(x, \tau).$$

If we restrict  $\eta$  to be positive, let  $T$  go to infinity, delete the function  $g(\cdot)$  and the terminal condition, assume that the function  $u(\cdot)$  does not depend on  $\tau$ , observe that then  $F(\cdot)$  will not depend on  $\tau$ , and for convenience set  $\tau = 0$ , we get Harrison's (1985, p. 83) formulation (with  $t = s$ ,  $\lambda = \eta$ ,  $a = 0$ , and  $h(x) = F(x, t)$ ).

If we instead let  $a$  and  $b$  go to minus and plus infinity, respectively, delete the regulators and the smooth pasting conditions (that is, we consider an unregulated Brownian motion,  $Z_\tau = X_\tau$ ), we get the formulation in Duffie (1988, p. 226) (with  $t = \tau$ ,  $V(x, t) = F(x, t)$ ,  $\rho(x, t) = \eta$ ,  $\mu(x, t) = \mu$ , and  $\sigma(x, t) = \sigma$ ).

In order to solve (3.6), we can define, for a given constant  $T$ ,

$$F(f, \tau) = E[e(f_T) \mid f_\tau = f], \quad \tau \leq T.$$

That is, in (A.2) we have  $Z_\tau = f_\tau$ ,  $x = f$ ,  $a = f$ ,  $b = f$ ,  $u(\cdot) \equiv 0$ , and  $\eta = c = r = 0$ .

Then it follows from (A.3) that  $F(f, \tau)$  is the unique solution to the partial differential equation

$$(A.4a) \quad F_\tau(f, \tau) + \mu F_f(f, \tau) + (1/2)\sigma^2 F_{ff}(f, \tau) = 0,$$

with terminal condition

$$(A.4b) \quad F(f, T) = e(f)$$

and derivative boundary conditions

$$(A.4c) \quad F_f(f, \tau) = 0 \quad \text{and} \quad F_{ff}(f, \tau) = 0,$$

for  $f \leq f \leq f$  and  $\tau \leq T$ .

We now set the term  $t = T - \tau$  for  $\tau \leq T$ , that is,  $t \geq 0$ . We then define  $h(f, t) \equiv$

$F(f, T - t)$ , in which case we have  $h_f(f, t) \equiv F_f(f, T - t)$  and  $h_t(f, t) \equiv -F_T(f, T - t)$ . Then (3.7) follows from (A.4).

By Ito's lemma we have  $E[dF(f(\tau), \tau)] = DF(f(\tau), \tau)d\tau$ . Therefore, (A.4a) is equivalent to

$$E[dF(f(\tau), \tau)] = 0,$$

that is, (3.9a).

### A.3 The Analytical Solution to the Partial Differential Equation (3.7)

The analytical Fourier-series solution to the partial differential equation (3.7) can be derived with methods described in, for instance, Churchill and Brown (1987). For  $\mu = 0$ , the problem is identical to the boundary value problem for the temperature  $h(f; t)$  in an infinite slab of material, bounded by the planes  $f = \underline{f}$  and  $f = \bar{f}$ , when the slab's faces are insulated and the initial temperature distribution is the prescribed function  $e(f)$  (see Churchill and Brown (1987, p. 36-42)).<sup>25</sup>

Let  $a = (\bar{f} - \underline{f})/\pi$ ,  $x = (f - \underline{f})/a$ ,  $\bar{e}(x) = e(ax + \underline{f})$ , and  $g(x, t) = h(ax + \underline{f}; t)$ . Then (3.7) implies the partial differential equation for  $g(x, t)$

$$(A.5a) \quad g_t = \bar{\mu}g_x + \bar{\sigma}^2 g_{xx}/2,$$

$$(A.5b) \quad g(x, 0) = \bar{e}(x), \text{ and}$$

$$(A.5c) \quad g_x(0, t) = g_x(\pi, t) = 0,$$

for  $0 \leq x \leq \pi$  and  $t \geq 0$ , with  $\bar{\mu} = \mu/a$  and  $\bar{\sigma} = \sigma/a$ . Separation of variables,  $g(x, t) = X(x)T(t)$ , leads to two ordinary differential equations,

$$(A.6) \quad T_t(t) + \lambda T(t) = 0$$

and a version of the so-called Sturm-Liouville problem,

$$(A.7a) \quad X_{xx}(x) + \bar{\theta}X_x(x) + \lambda(\bar{\theta}/\bar{\mu})X(x) = 0,$$

$$(A.7b) \quad X_x(0) = X_x(\pi) = 0,$$

where  $\bar{\theta} = 2\bar{\mu}/\bar{\sigma}^2$  and  $\lambda$  is a constant.

There is a countable infinity of real numbers  $\lambda_0, \lambda_1, \lambda_2, \dots$ , *eigenvalues*, of the

<sup>25</sup> I am grateful to Harald Lang for showing me how to compute the solution.

parameter  $\lambda$  for each of which the Sturm-Liouville problem has a solution not identically zero fulfilling (A.7). The corresponding solutions  $X_0(x), X_1(x), \dots$  are called *eigenfunctions*. The eigenfunctions are orthogonal with weight factor  $\exp(\theta x)$ , the integrating factor of this Sturm-Liouville problem.

The differential equation (A.6) for  $T(t)$  is solved by  $T(t) = \exp(-\lambda t)$  for each  $\lambda$ . Then any solution to the partial differential equation (A.5) can then be written as the Fourier series

$$g(x, t) = \sum_{n=0}^{\infty} c_n X_n(x) \exp(-\lambda_n t),$$

where the constants  $c_n$  are determined by the initial condition (A.5b) according to

$$c_n = \int_0^{\pi} \exp(\theta x) \bar{e}(x) X_n(x) dx / \int_0^{\pi} \exp(\theta x) |X_n(x)|^2 dx.$$

Given this, with a fair amount of algebra it can be shown that the solution to (3.7) can be written (with  $\theta = 2\mu/\sigma^2$ ):

$$(A.8a) \quad h(f; t) = \sum_{n=0}^{\infty} c_n y_n(f) \exp(-\lambda_n t), \quad f \leq f \leq f, \quad t \geq 0,$$

where

$$(A.8b) \quad y_0(f) = 1,$$

$$(A.8c) \quad y_n(f) = \exp[-\theta(f-f)/2] \{2n \cos[n(f-f)/a] + \theta a \sin[n(f-f)/a]\}, \quad n \geq 1,$$

$$(A.8d) \quad c_0 = \int_f^f e(f) df \frac{1}{f-f}, \quad \text{for } \mu = 0,$$

$$(A.8e) \quad c_0 = \int_f^f \exp(\theta f) e(f) df \frac{\theta}{\exp(\theta f) - \exp(\theta f)}, \quad \text{for } \mu \neq 0,$$

$$(A.8f) \quad c_n = \int_f^f \exp[\theta(f-f)] e(f) y_n(f) df \frac{1}{4(f-f)\lambda_n a^2/\sigma^2}, \quad n \geq 1,$$

$$(A.8g) \quad \lambda_0 = 0,$$

and

$$(A.8h) \quad \lambda_n = (n^2/a^2 + \theta^2/4)\sigma^2/2 > 0, \quad n \geq 1.$$

For  $\mu = 0$  this solution coincides with the solution reported in Churchill and Brown

(1987, p. 41)).

We see that  $h(f;t) \rightarrow c_0$  for  $t \rightarrow \infty$ . Indeed,  $c_0$  is the unconditional expected exchange rate. When  $\mu = 0$ , the unconditional (that is, ergodic) probability distribution is uniform, whereas for  $\mu \neq 0$  it is truncated exponential, with density function  $\bar{\varphi}(f) = \theta \exp(\theta f) / [\exp(\theta f) - \exp(\theta f)]$ ,  $f \leq f \leq f$  (see Harrison (1985)).

Since the function  $e(f)$  in our case by (2.10) is a sum of linear and exponential terms, the integrals in (A.8d-f) are exact and  $c_0$  and  $c_n$  can actually be solved explicitly. (The solutions are not reported here.) When  $e(f)$  is such that explicit solutions of  $c_0$  and  $c_n$  cannot be found, (A.8d-f) can still easily be computed numerically.

We note that it follows from (A.8) that  $h(f;t)$  can be written

$$(A.9a) \quad h(f;t) = \int_f^f e(g) \varphi(f,g;t) dg,$$

where

$$(A.9b) \quad \varphi(f,g;t) = \bar{\varphi}(g) + \frac{\exp[\theta(f-f)]}{4(f-f)} \sum_{n=1}^{\infty} \frac{y_n(f) y_n(g)}{\lambda_n a^2 / \sigma^2} \exp(-\lambda_n t),$$

$$(A.9c) \quad \bar{\varphi}(g) = 1/(f-f) \text{ for } \mu = 0,$$

$$(A.9d) \quad \bar{\varphi}(g) = \theta \exp(\theta g) / [\exp(\theta f) - \exp(\theta f)] \text{ for } \mu \neq 0.$$

Here  $\varphi(f,g;t)$  is the transition density of  $f(t) = g$  at time  $t$ , given  $f(0) = f$  at time 0.

Given this, the transition densities of the exchange rate, interest rate differentials, etc., can all be computed. The exchange rate's transition density is for instance given by

$$(A.10) \quad \varphi^e(e_0, e; t) = \varphi(f(e_0), f(e); t) / e_f(f(e)), \quad \underline{e} < e < \bar{e},$$

where  $f(e)$  denotes the inverse of  $e(f)$ .

#### A.4 The Numerical Solution to the Partial Differential Equation (3.7)

We like to find the twice continuously differentiable function  $u(x,t): [\underline{x}, \bar{x}] \times [0, \infty) \rightarrow R$  that is the solution to the parabolic partial differential equation

$$(A.11a) \quad u_t(x,t) = \mu u_x(x,t) + (1/2)\sigma^2 u_{xx}(x,t),$$

with initial condition

$$(A.11b) \quad u(x,0) = e(x)$$

(where  $e(x): [\underline{x}, \bar{x}] \rightarrow R$  is twice continuously differentiable and fulfills  $e_x(\underline{x}) = e_x(\bar{x}) = 0$ ) and the derivative boundary conditions

$$(A.11c) \quad u_x(\underline{x}, t) = 0 \quad \text{and} \quad u_x(\bar{x}, t) = 0.$$

We follow the so-called explicit method described in Gerald and Wheatley (1989, Chapter 8), where further details are discussed.

First, we define an  $I \times J$  grid for  $(x, t)$ . For a given positive integer  $I$ , let

$$x_i = (i - 1)\Delta x + \underline{x}, \quad \text{for } i = 1, \dots, I, \quad \text{with } \Delta x = (\bar{x} - \underline{x})/(I - 1).$$

For a given positive integer  $J$  and a given positive real time-step  $\Delta t$ , let

$$t_j = j\Delta t, \quad j = 0, \dots, J - 1.$$

We then define

$$u_i^j = u(x_i, t_j) \quad \text{for } 1 \leq i \leq I, \quad 0 \leq j \leq J - 1.$$

Second, we approximate the derivatives  $u_x$  and  $u_{xx}$  by central differences and the derivative  $u_t$  by a forward difference:

$$\begin{aligned} u_x(x_i, t_j) &= (u_{i+1}^j - u_{i-1}^j)/(2\Delta x), \\ u_{xx}(x_i, t_j) &= (u_{i+1}^j - 2u_i^j + u_{i-1}^j)/(\Delta x)^2, \quad \text{and} \\ u_t(x_i, t_j) &= (u_i^{j+1} - u_i^j)/\Delta t. \end{aligned}$$

Third, we substitute these approximations of the derivatives into (A.8a) and get the difference equation

$$(u_i^{j+1} - u_i^j)/\Delta t = \mu(u_{i+1}^j - u_{i-1}^j)/(2\Delta x) + (1/2)\sigma^2(u_{i+1}^j - 2u_i^j + u_{i-1}^j)/(\Delta x)^2.$$

This difference equation can be rewritten as

$$(A.12a) \quad u_i^{j+1} = (r + s)u_{i+1}^j + (1 - 2r)u_i^j + (r - s)u_{i-1}^j, \quad \text{with}$$

$$(A.12b) \quad r = \frac{\sigma^2 \Delta t}{2(\Delta x)^2} \quad \text{and} \quad s = \frac{\mu \Delta t}{2\Delta x}.$$

Fourth, the difference equation (A.9) is defined for the interior of the  $x$ -band,  $2 \leq i \leq I - 1$ , but we also need to find the difference equation for the boundaries,  $i = 1$  and  $i = I$ . We then incorporate the derivative boundary conditions by adding an  $x$ -point below and above the previous  $x$ -grid,

$$x_0 = \bar{x} - \Delta x \quad \text{and} \quad x_{I+1} = \bar{x} + \Delta x.$$

The conditions (A.8c) can then be expressed as

$$u_x(x, t_j) = (u_2^j - u_0^j)/(2\Delta x) = 0 \quad \text{and} \quad u_x(\bar{x}, t_j) = (u_{I+1}^j - u_{I-1}^j)/(2\Delta x) = 0,$$

that is,

$$(A.13) \quad u_0^j = u_2^j \quad \text{and} \quad u_{I+1}^j = u_{I-1}^j.$$

Fifth, we can now substitute (A.13) into (A.12a) for  $i = 1$  and  $i = I$  in order to define the difference equation also on the boundaries. Then we get the system

$$\begin{aligned} u_1^{j+1} &= (1 - 2r)u_1^j + 2ru_2^j; \\ u_i^{j+1} &= (r + s)u_{i+1}^j + (1 - 2r)u_i^j + (r - s)u_{i-1}^j, \quad 2 \leq i \leq I - 1; \quad \text{and} \\ u_I^{j+1} &= 2ru_{I-1}^j + (1 - 2r)u_I^j. \end{aligned}$$

We can write the system more compactly by defining the tri-diagonal  $I \times J$  matrix  $A = (a_{ij})$  such that  $a_{ij} = 0$  for all  $i$  and  $j$  except that

$$(A.14a) \quad a_{11} = 1 - 2r, \quad a_{12} = 2r;$$

$$(A.14b) \quad a_{i,i-1} = r - s, \quad a_{ii} = 1 - 2r, \quad a_{i,i+1} = r + s \quad \text{for} \quad 2 \leq i \leq I - 1; \quad \text{and}$$

$$(A.14c) \quad a_{I,I-1} = 2r, \quad a_{II} = 1 - 2r.$$

We let  $w^j$  denote the  $I$ -dimensional column vector  $(w_i^j)_{i=1}^I$ . Then the system can finally be written

$$(A.15a) \quad w^{j+1} = Aw^j \quad \text{for} \quad j = 0, 1, \dots, J - 1,$$

with the initial condition

$$(A.15b) \quad u^0 = (u_i^0)_{i=1}^I = (e(x_i))_{i=1}^I.$$

This system is straightforward to solve numerically by forward iteration. In order to ensure stability of the system,  $\Delta x$  and  $\Delta t$  must be chosen such that  $r$  defined in (A.10) fulfills

$$(A.16) \quad r \leq .5.$$

### A.5 The Interest Rate Differential's Variability

For given positive term  $t > 0$  the interest rate differential at time  $\tau$ ,  $\delta(\tau; t)$ , will be an Ito process with drift  $\mu^\delta(f; t)$  and instantaneous standard deviation  $\sigma^\delta(f; t)$ , given by the

stochastic differential equation

$$(A.17) \quad d\delta(\tau; t) = \mu^\delta(f; t) d\tau - \sigma^\delta(f; t) dz(\tau).$$

(Since we know that the interest rate differential is negatively correlated with the fundamental, we write a minus sign before the Wiener component, in order to define  $\sigma^\delta(f; t)$  as positive.) The drift inside the band for the interest rate differential is by Ito's lemma given by

$$(A.18) \quad \mu^\delta(f; t) = \mu\delta_f(f; t) + (1/2)\sigma^2\delta_{ff}(f; t).$$

The instantaneous standard deviation is given by

$$(A.19) \quad \sigma^\delta(f; t) = -\delta_f(f; t)\sigma,$$

the product of the absolute value of the derivative of the interest rate differential and the instantaneous standard deviation of velocity. For zero term, the stochastic differential equation for the instantaneous interest rate differential also includes the effect of the lower and upper regulators in (2.9). This is because the instantaneous interest rate differential does not fulfill the smooth pasting conditions.<sup>26</sup>

The interest rate differential's instantaneous standard deviation as a function of the fundamental is shown in *Figure A1*, for different terms (0, 1, 3, 6, 12, and 60 months). The solid curve corresponds to zero term. We note the discontinuity between finite and zero term at the edges, corresponding to the discontinuity of the derivative  $\delta_f(f; t)$  at  $t = 0$ , which we have discussed above. Positive-term instantaneous standard deviations are largest in the middle of the band, and fall to zero towards the edges of the band, in contrast to the zero-term instantaneous standard deviation. This is because the positive-term interest rate differentials fulfill the smooth pasting conditions, which is not the case

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<sup>26</sup> By Ito's lemma we have  $d\delta(\tau; t) = \delta_f(f; t)df + (1/2)\delta_{ff}(f; t)df^2 =$   
 $= \delta_f(f; t)(dv + dL - dU) + (1/2)\delta_{ff}(f; t)\sigma^2 d\tau =$   
 $= [\mu\delta_f(f; t) + (1/2)\sigma^2\delta_{ff}(f; t)]d\tau + \delta_f(f; t)dL - \delta_f(f; t)dU + \delta_f(f; t)\sigma dz.$

By the smooth pasting conditions, we have  $\delta_f(f; t) = \delta_f(f; t) = 0$  for positive maturities, and the terms including  $dL$  and  $dU$  vanish, which gives (A.18) and (A.19). For zero maturity, the terms including  $dL$  and  $dU$  remain.

for the zero-term interest rate differential. We also see that the instantaneous standard deviations are decreasing in the term, except for small terms and intermediate values of the fundamental.

The asymptotic (unconditional) variability of the interest rate differentials are easy to compute numerically. The asymptotic probability density function  $\bar{\alpha}(f)$  for the fundamental is derived in Harrison (1985). It is uniform with zero drift and truncated exponential with nonzero drift. Then the asymptotic probability density function for the term- $t$  interest rate differential,  $\varphi^{\delta(t)}(\delta)$ , is given by

$$(A.20) \quad \varphi^{\delta(t)}(\delta) = \bar{\alpha}(f(\delta;t)) / \delta_f(f(\delta;t)), \quad \delta(f;t) < \delta < \delta(f;t),$$

where  $f(\delta;t)$  denotes the inverse of  $\delta(f;t)$  with respect to  $f$  (because of the smooth pasting conditions, the probability density function (A.20) is only defined on the interior of the interest rate differential band). These probability density functions can easily be plotted for different terms. The interest rate differentials' asymptotic standard deviations can then be computed numerically.<sup>27</sup>

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<sup>27</sup> The *instantaneous* interest rate differential's asymptotic standard deviation can actually be computed exactly even though its probability density function can only be computed numerically (see Svensson (1989)).

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Table 1. Summary Statistics

Variable	Mean	Std Dev	Minimum	Maximum
$e$	-0.78	0.46	-1.48	0.24
$\delta(1)$	2.03	0.88	0.16	3.73
$\delta(3)$	1.90	0.76	0.22	3.36
$\delta(6)$	1.89	0.64	0.28	3.18
$\delta(12)$	1.94	0.54	0.38	3.06
$\delta(60)$	1.24	0.54	-0.34	2.17

$\delta(t)$  denotes the interest rate differential for  $t$  months term.

Table 2. Correlation Coefficients

Variable	$e$	$\delta(1)$	$\delta(3)$	$\delta(6)$	$\delta(12)$	$\delta(60)$
$e$	1.00					
$\delta(1)$	-0.60	1.00				
$\delta(3)$	-0.63	0.96	1.00			
$\delta(6)$	-0.63	0.92	0.98	1.00		
$\delta(12)$	-0.48	0.76	0.82	0.89	1.00	
$\delta(60)$	-0.22	0.48	0.50	0.58	0.75	1.00

Table 3. General Method of Moments Regressions of  $\delta$  on  $\epsilon$ 

(1)	(2)	(3)	(4)	(5)	(6)
Dependent Variable	Constant	Coefficient	R <sup>2</sup>	DW	Auto-correlation
$\delta(1)$	1.13** (0.21)	- 1.16** (0.19)	0.36	0.90	0.54
$\delta(3)$	1.08** (0.23)	- 1.05** (0.21)	0.40	0.72	0.64
$\delta(6)$	1.22** (0.22)	- 0.87** (0.19)	0.39	0.72	0.64
$\delta(12)$	1.50** (0.25)	- 0.56* (0.19)	0.23	0.72	0.64
$\delta(60)$	1.04** (0.29)	- 0.26 (0.23)	0.05	0.51	0.76

Standard errors are given in parenthesis. We use \*\* (\*) to denote significance at 5% (10%) level. The number of nonzero autocorrelations of the errors has been set to 8. The standard errors are not very sensitive to changes in this number. Column (6) shows the estimate of the autocorrelation of the error terms.

Figure 1

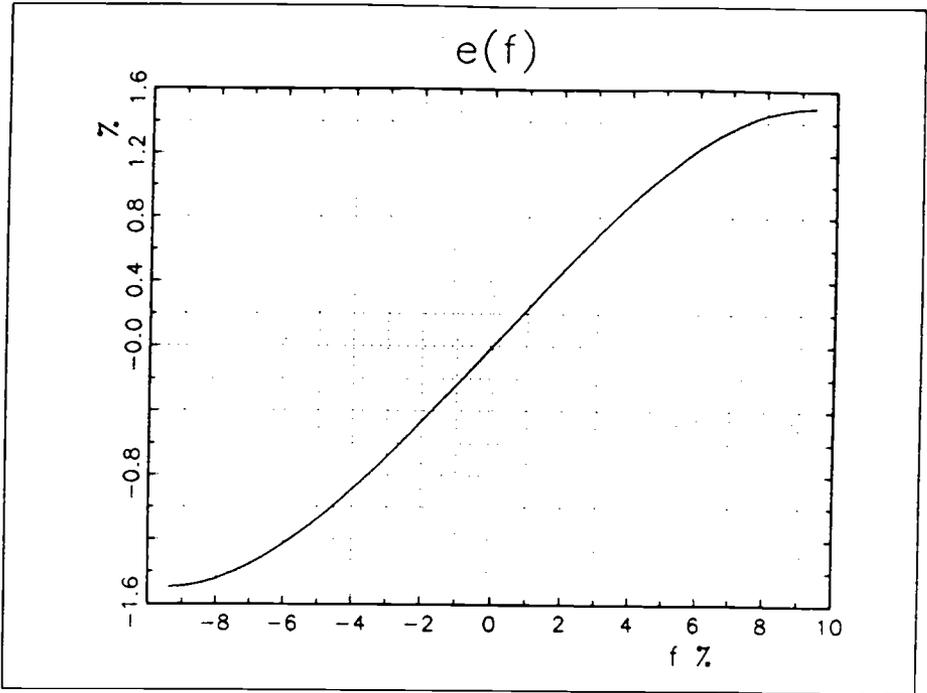


Figure 2

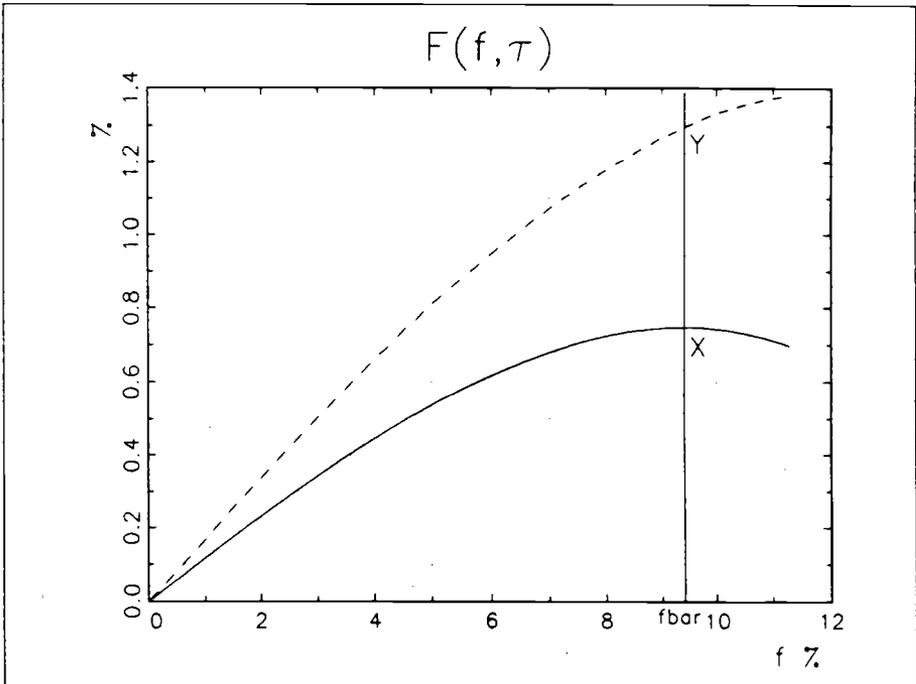


Figure 3a

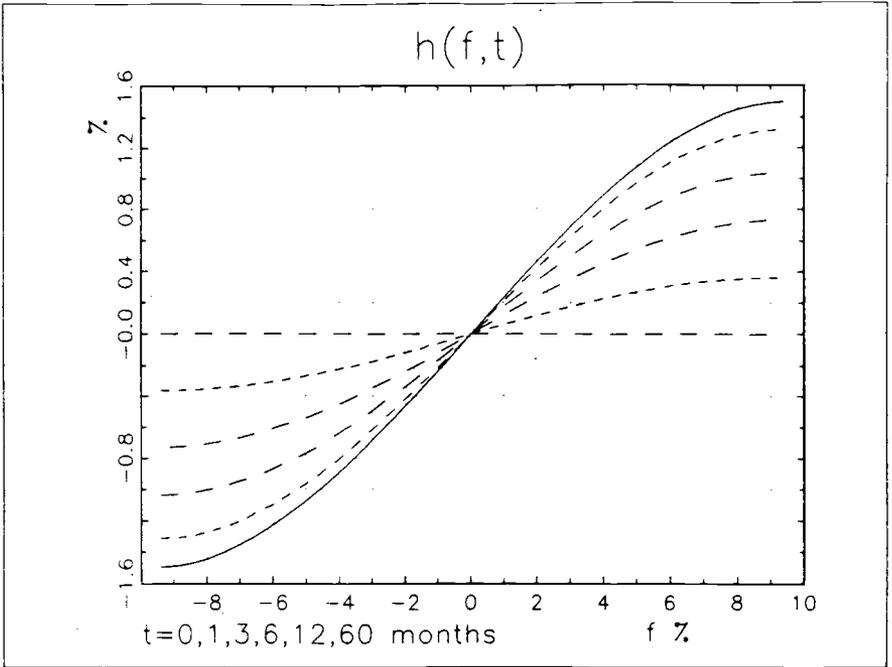


Figure 3b

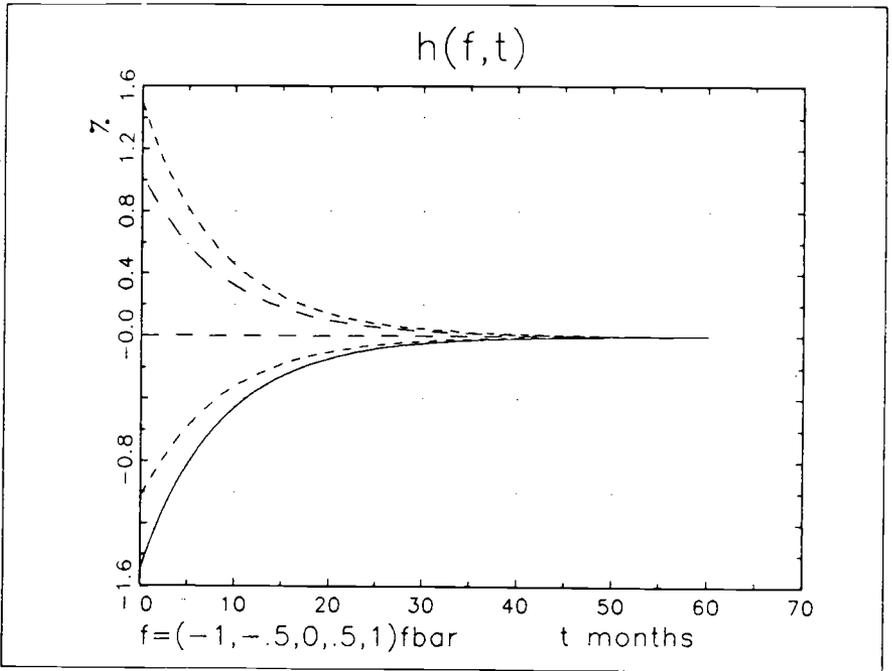


Figure 3c

$$h(f,t)$$

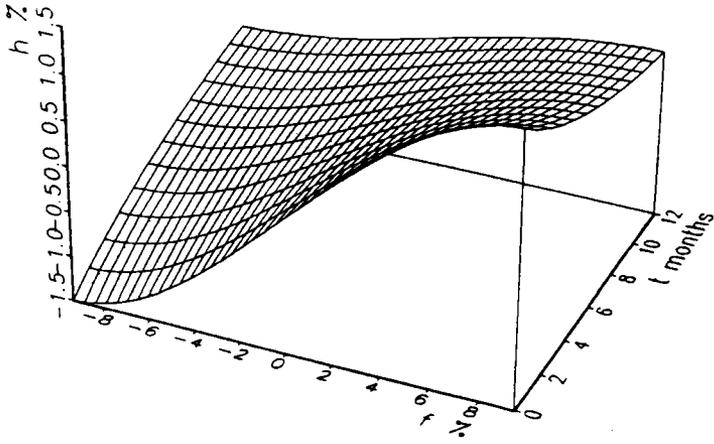


Figure 4a

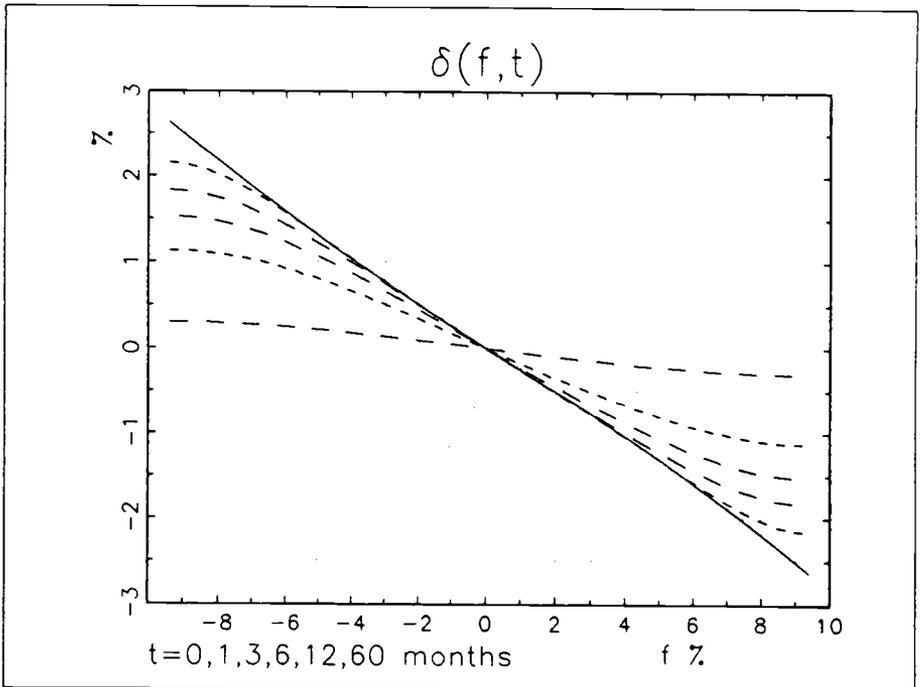


Figure 4b

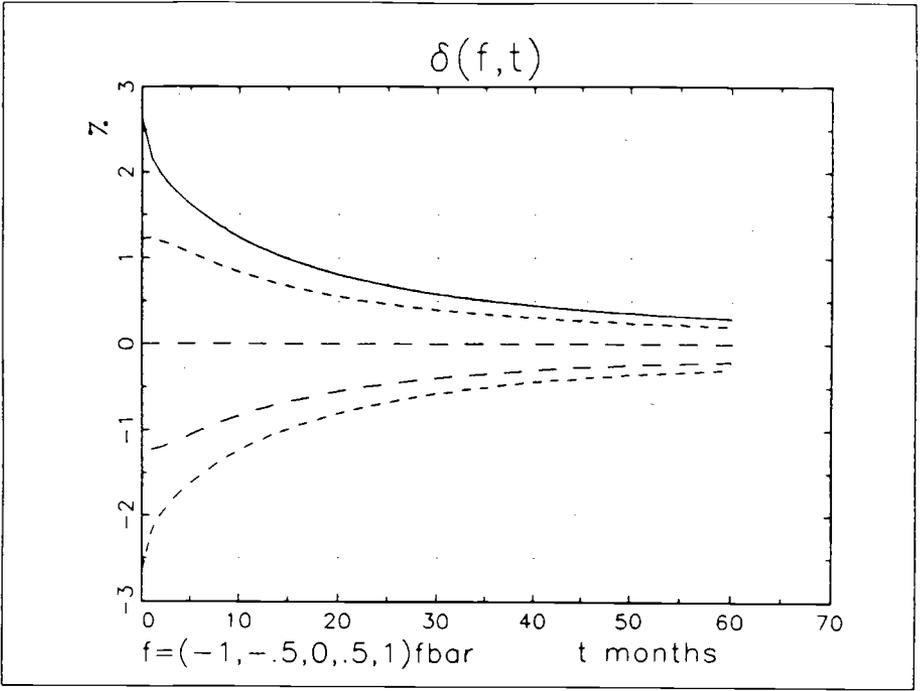


Figure 4c

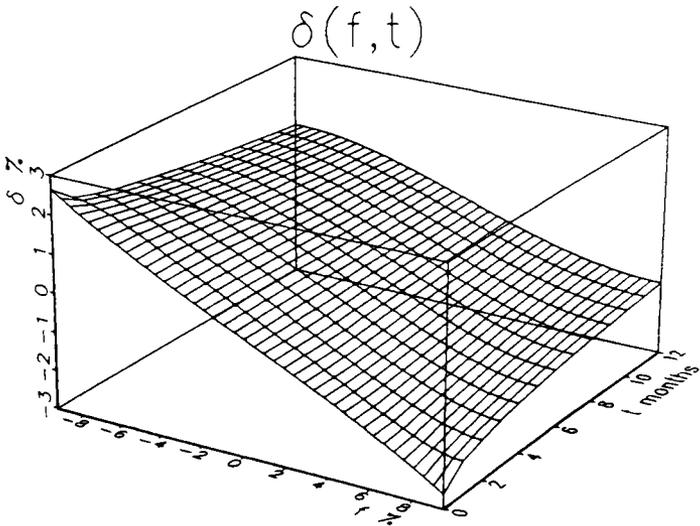


Figure 4d

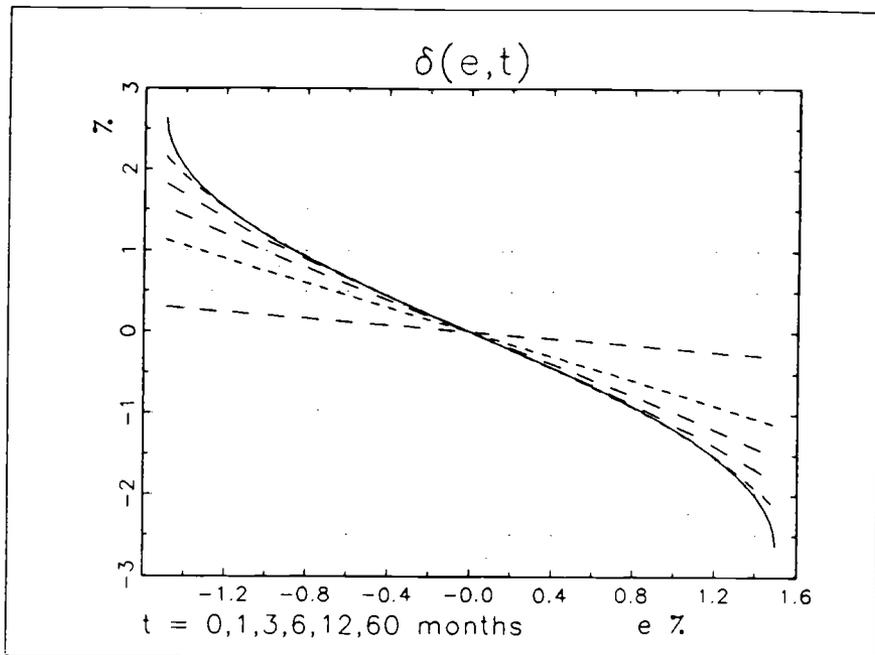


Figure 5a

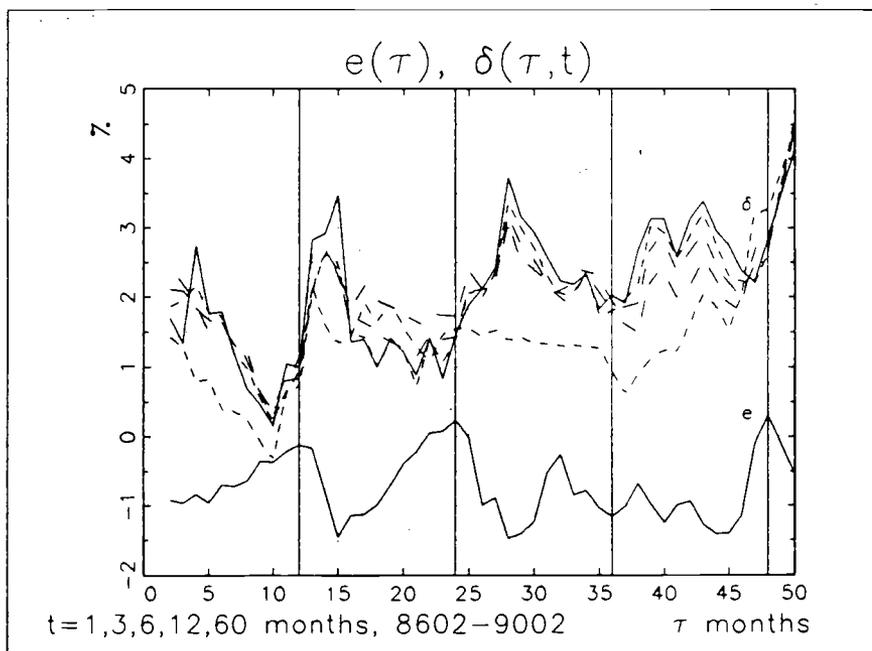


Figure 5b

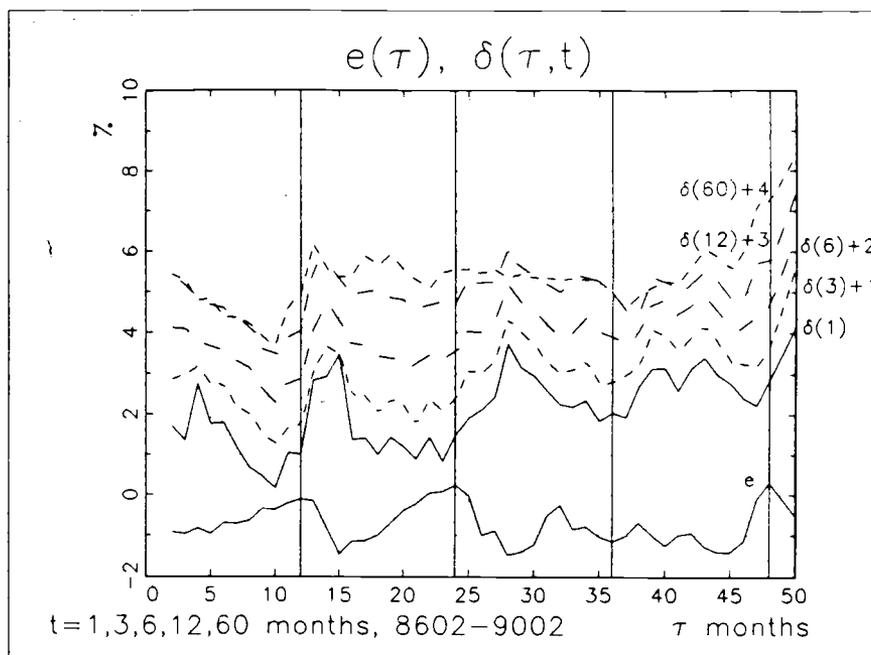


Figure 6

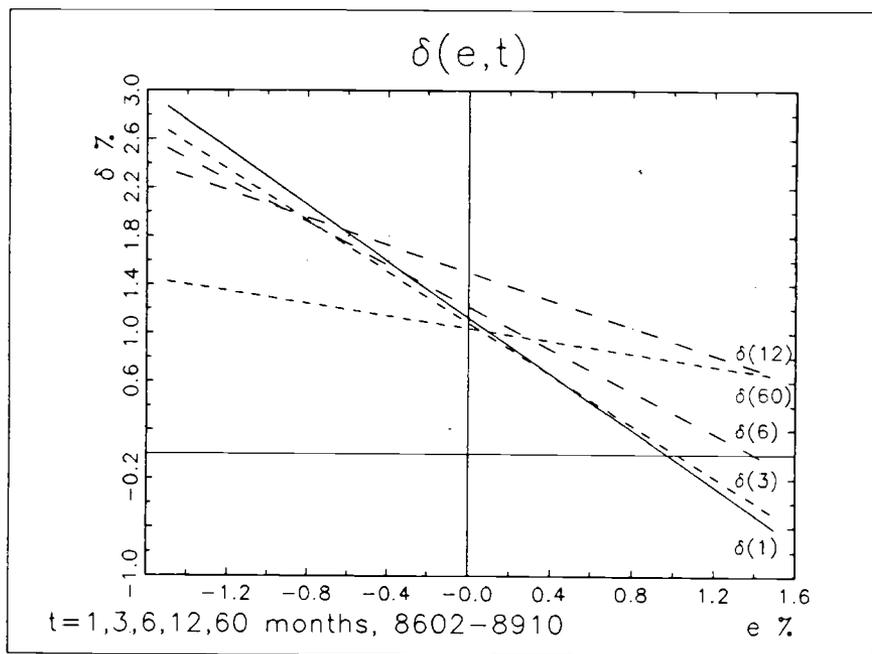


Figure A1

