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EXPLAINING THE VARIANCE OF PRICE DIVIDEND RATIOS

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ABSTRACT

This paper presents a bound on the variance of the price-dividend ratio and a decomposition of the variance of the price-dividend ratio into components that reflect variation in expected future discount rates and variation in expected future dividend growth. Unobserved discount rates needed to make the variance bound and variance decomposition hold are characterized, and the variance bound and variance decomposition are tested for several discount rate models, including the consumption based model, and models based on interest rates plus a constant risk premium.

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Explaining the Variance of Price Dividend Ratios

The contrast between the volatility of stock prices and the smoothness of dividends and discount rate measures is a long standing apparent puzzle. Many authors have quantitatively assessed whether the volatility of prices is too large to be justified by variation in dividends or discount rates, This literature is summarized in Flood and Hodrick (1988) and Gilles and LeRoy (1987).

Starting from the model that stock prices are equal to the expected present value of dividends, this paper derives two tests of the variance of price dividend ratios. The first is a *variance bound* in the tradition of LeRoy and Porter (1981), Shiller (1981), Grossman and Shiller (1981), Marsh and Merton (1986), Kleidon (1986), West (1987, 1988), LeRoy and Parke (1988) and Durlauf and Hall (1988). Variance bound exploit the idea that the variance of prices should be less than the variance of the *ex-post* present value of dividends, since the variance of the expected value of a random variable is always less than the variance of the random variable itself.

The second is a *variance decomposition*. This idea was first presented in LeRoy and Porter (1981), and is the basis of Campbell and Shiller's (1989) work. The basic idea is derived by regressing both sides of price = present value on a variable observed at time t. The regression coefficient of price on the variable should then equal an appropriately weighted sum of regression coefficients of future dividends and discount rates on that variable. This idea is used here to derive a decomposition of the variance of the price dividend ratio into terms reflecting the covariance of the price dividend ratio with future discount rates and future dividend growth rates.

The form of both tests conforms to the time series desiderata of the recent literature in this area: dividend growth rates and discount rates are assumed to be stationary, rather than presuming linear trends or stationary first differences of dividends; the time-series properties of dividend growth and discount rates are not otherwise restricted, rather than imposing parsimonious time series models. The calculations do not use a terminal price, but instead are functions of the moments of dividend growth and the price-dividend ratio. This avoids some of the sampling problems associated with earlier tests (See Flavin (1981), Flood and Hodrick (1988)).

This paper also presents an approximate model relating price dividend ratios to future dividends and future discount rates that is amenable to the same multi-asset and varying discount rate methodology pioneered by Hansen and Sigleton, and now universally used to study returns, as opposed to the single-asset constant discount rate model common to date in price or price dividend empirical work.

The previous work closest to the variance bound is that of LeRoy and Parke (1988). The variance bound presented here extends LeRoy and Parke's to include arbitrary dividend growth and discount rate processes rather than a random walk in dividend growth and constant discount rates. Also, the present value model has predictions for the mean price dividend ratio as well as the variance bound, and the tests presented here exploit predictions about both moments together, where LeRoy and Parke considered only the variance bound.

The previous work closest to the variance decomposition is that of Campbell and Shiller (1988). The most important distinction between this paper and Campbell and Shiller's is that the model here uses a second order Taylor approximation, where Campbell and Shiller used only a first order approximation. As a result, terms in the variance and covariance of dividend growth and discount rates enter the equation for the mean price dividend ratio, with the crucial implication that the mean discount rate is not identifiable from the mean price dividend ratio. In Campbell and Shiller's approximation, these terms did not enter the equation for the mean price dividend ratio, so they could solve their equation for the mean discount rate, giving a value approximately equal to the mean return.

The possibility that the mean discount rate is different from the mean return is desirable. Since there should be *one* discount rate process that prices *all* assets (for example, $\rho u'(c_{t+1})/u'(c_t)$), its mean must be different than the mean return on some assets. It turns out to make a practical difference as well, in that the variance bound and decomposition are sensitive functions of the mean discount rate.

Why examine prices, or price-dividend ratios, rather than returns? After all, a present value model incorporates the information in an Euler equation model of returns, so it includes whatever restrictions flow from that Euler equation.

The first answer is that a present value model for prices or price-dividend ratios also includes a transversality condition that rules out "rational bubbles," and a condition that the present value is finite that rules out "sunspot" solutions to the return model. Because of this extra content, a present value model of prices or price-dividend ratios can evaluate whether large *unpredictable* swings in prices such as the October 1987 crash are justified by subsequent changes in dividends and discount rates. This question *cannot* be addressed by examining returns alone: Euler equations place no limit on the variance of either returns or prices.

The second answer is that, even if one is not interested in testing for bubbles or sunspots, so that Euler equation and present value models have exactly the same statistical content, the behavior of prices or price-dividend ratios provides useful diagnostics for particular discount rate models. For example, this paper finds that discount rate models based on consumption growth or interest rates plus a constant risk premium fail because price-dividend ratios do not correctly forecast long term movements in those variables: a high price-dividend ratio should, and does not, forecast low interest rates and consumption growth rates, as it successfully forecasts low returns. While the same *statistical* rejection of either discount rate model could come from a rejection of a one period Euler equation using a particular weighted combination of 15 years of past price-dividend ratios as instruments, the present value model makes it clear *why* those discount rate models fail.

Statistical rejections of present value models have been interpreted as support for three rough categories of alternatives:

1) There is *no* discount rate process that can rationalize the volatility of prices. In the absence of arbitrage opportunities, this can only occur if there is a speculative bubble or sunspot in stock prices, which in turn occur if and only if the price dividend ratio or present value are nonstationary.

2) No *reasonable* discount rate process can rationalize the volatility of prices. Here it is admitted that there may *exist* unobserved discount rate processes that can reconcile the volatility of prices with the smoothness of dividends (since it is not claimed that either a bubble or sunspot drives the volatility of stock prices), but it is claimed that any discount rate process that works must have extreme statistical properties, such as high variance. It is considered unlikely that any discount rate process based on fundamentals (for example, consumption growth raised to a reasonable risk aversion coefficient) can have these extreme properties. For example, Poterba and Summers (1989) summarize their results by a calculation that if discount rates ("required returns") have a half-life of 2.9 years, they must have a standard deviation of 5.8%. They find it "difficult to think of risk factors that could account for such variation in required returns" (p.51). I will use the term "fads" to denote this alternative-- that there is no speculative bubble, but expected returns vary over time in a way that is clearly unrelated to fundamentals (e.g., consumption growth) in the larger economy.

3) *Particular discount rate models* are rejected. Most of the variance bound literature rejects a model in which discount rates are constant over time; Grossman and Shiller (1981) argue that the volatility of prices is too high to be explained by discount rates generated from consumption data with a particular family of utility functions; Campbell and Shiller (1988) try on a variety of models.

To address each of the three categories of potential rejections of the present value model, the variance bound and variance decomposition derived in this paper are each used in three ways,

1) They are calculated with no assumptions on discount rates beyond stationarity.[1] These calculations allow us to test whether there is *any* discount rate process that can explain the volatility of price dividend ratios.

2) To assess whether the implied discount rates are reasonable, mean-standard deviation frontiers are calculated for the discount rate processes that must be invoked to make the mean price dividend ratio, variance bound and variance decomposition hold. These can be compared with the discount rates from a variety of models to see if they are "reasonable." This methodology is in the spirit of Hansen and Jagannathan (1989); puzzles that asset price or return data seem to require high standard deviations of discount rates also include Grossman and Shiller (1981), Mehra and Prescott (1985), Cochrane (1988b), and Poterba and Summers (1988) and West (1988).

3) With specific models for discount rates, the variance bound can be further restricted, and the the variance decomposition can be tested--we can check whether all variance of the price-dividend ratio is in fact accounted for by changing expectations of future dividend growth and discount rates. Models considered below include constant discount rates, discount rates generated by a constant risk premium plus the treasury bill rate, a long term government rate and a corporate bond rate, discount rates equal to returns, and discount rates generated by the consumption based model with constant relative risk aversion utility.

1. The Present Value Model

The null hypothesis of this paper is that stock prices are equal to the expected present value of future dividends

$$(1) \qquad P_t = E_t \left(\sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} \gamma_{t+k} \right) d_{t+j} \right) .$$

where P_t denotes the ex-dividend price at time t, E_t denotes expectation conditional on information at time t, d_t stand for dividends, and γ_t stands for the ex-post (stochastic) discount factor. (A summary of the notation is presented in the appendix.) With nominal dividend and price data, γ_t is interpreted as a nominal discount factor or marginal rate of substitution between dollars at t and dollars at t-1; with real data γ_t is interpreted as a real discount factor or marginal rate of substitution between goods at t and t-1.

This form of the present value model nests others considered in the literature as special cases. It is derived by recursive substitution of the Euler equation

(2) $$1 = E_t (\gamma_{t+1} R_{t+1}) \quad \text{where } R_{t+1} = (P_{t+1} + d_{t+1})/P_t ,$$

together with a transversality condition and a condition that the sums converge, described in the appendix.

The present value model (1) is inappropriate for statistical analysis if dividend growth rates rather than levels are stationary, because in this case both sides of (1) are nonstationary. To transform it to a relation that can hold between stationary variables, divide both sides of (1) by dividends and express the result in terms of discount rates and dividend growth rates:

(3) $$\frac{P_t}{d_t} = E_t \sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} \gamma_{t+k} \right) \frac{d_{t+j}}{d_t} = E_t \sum_{j=1}^{\infty} \prod_{k=1}^{j} (\gamma_{t+k}\, \eta_{t+k})$$

where $\eta_t = d_t/d_{t-1}$.

The discount factor γ_t and dividend growth rate η_t are assumed strongly stationary. Since (3) expresses the price-dividend ratio as a time-invariant continuous function of stationary variables, the price dividend ratio is also stationary, if the right hand side is finite almost surely. This is assumed to be the case, and first and second moments of all variables are also assumed to exist, so that the price dividend ratio, dividend growth and discount rates are all covariance stationary.

The next task is to characterize the moments of the price-dividend ratio. Equation (3) can be rewritten as

(4) $$\frac{P_t}{d_t} = E_t \sum_{j=1}^{\infty} \exp\left(\sum_{k=1}^{j} (n_{t+k} - g_{t+k}) \right)$$

where $g_t = -\ln(\gamma_t)$ (the - sign is introduced so that discount rates g_t will be positive), $n_t = \ln(\eta_t)$ denotes the dividend growth rate.

Taking expectations of both sides of (4), the mean price-dividend ratio is

(5) $$E\left(\frac{P}{d} \right) = E\left(\sum_{j=1}^{\infty} \exp \sum_{k=1}^{j} (n_{t+k} - g_{t+k}) \right)$$

(The t subscripts are deleted when they are not necessary to indicate timing. $E(P/d) = E(P_t/d_t)$). Since the variance of the *ex ante* price-dividend ratio must be less than the variance of the ex-post price-dividend ratio ($\text{var}(E_t(X)) \le \text{var}(X)$ for any random variable X) the variance bound is

(6) $$\text{var}\left(\frac{P}{d} \right) \le \text{var}\left(\sum_{j=1}^{\infty} \exp \sum_{k=1}^{j} (n_{t+k} - g_{t+k}) \right).$$

Many variance bounds tests form the ex-post prices or price-dividend ratios (the terms inside the brackets in (5) and (6)), and then take their sample variance to evaluate (6). This procedure usually requires many terms of the sum, often more than the sample size, so it is common to use a terminal price or price-dividend ratio. If this is fixed, it introduces sampling problems (Kleidon(1986), Flavin(1981), Flood and Hodrick(1988)). If terminal price data are used, the "present value" becomes only an iterated Euler equation, which is better estimated by filtering the instruments rather than iterating the Euler equation (see Hansen and Hodrick (1989)) However, *forecasts* of dividend growth and discount rates, or their correlations, should die off much more quickly than their ex-post values, so (5) and (6) might be well approximated with relatively few terms of the sums if we could move the expectations inside the sums, or if the mean price dividend ratio and variance bound can be expressed in terms of the moments of dividend growth and discount rates.

To do this, an approximate model is derived by taking a second order

Taylor expansion of the terms inside the brackets in (5) and (6) with respect to dividend growth and discount rates, about their unconditional means, and then taking the expectations inside the brackets, to yield moments of the price dividend ratio in terms of means, variances and covariances of dividend growth and discount rates. The derivation is presented in the appendix. The approximate model is:

$$(7)\quad \frac{P_t}{d_t} = \frac{\Omega}{1-\Omega} + \frac{\Omega}{2(1-\Omega)^2}\sum_{j=-\infty}^{\infty}\Omega^{|j|}\operatorname{cov}(n_t - g_t,\ n_{t-j} - g_{t+j}) + \frac{1}{1-\Omega}E_t\left(\sum_{j=1}^{\infty}\Omega^j(\tilde{n}_{t+j} - \tilde{g}_{t+j})\right)$$

Where $\tilde{n}_t = n_t - E(n)$, $\tilde{g}_t = g_t - E(g)$, and Ω is defined as

$$(8)\qquad \Omega = e^{E(n) - E(g)}\ .$$

Taking the expected value of (7), we obtain the mean price dividend ratio

$$(9)\qquad E\left(\frac{P}{d}\right) = \frac{\Omega}{1-\Omega} + \frac{\Omega}{2(1-\Omega)^2}\sum_{j=-\infty}^{\infty}\Omega^{|j|}\operatorname{cov}(n_t - g_t,\ n_{t-j} - g_{t+j})$$

Note that (7) and (9) embody the hope used to motivate them: the covariances and forecasts of discount rates and dividend growth may die off much more quickly than their ex-post values, so relatively few terms of the sums may be needed.

Since for any X, $\operatorname{var}(E_t(X)) \le \operatorname{var}(X)$, (7) implies the variance bound

$$(10)\quad \operatorname{var}\left(\frac{P}{d}\right) \le \frac{1}{(1-\Omega)^2}\operatorname{var}\left(\sum_{j=1}^{\infty}\Omega^j\,(n_{t+j} - g_{t+j})\right)$$

$$= \frac{\Omega}{(1-\Omega^2)(1-\Omega)^2}\sum_{j=-\infty}^{\infty}\Omega^{|j|}\operatorname{cov}(n_t - g_t,\ n_{t+j} - g_{t+j})$$

The variance and covariance term is the same in (9) and (10). Substituting out that term, the content of the pair (9) and (10) can be summarized by (9) together with

$$(11)\qquad \operatorname{var}\left(\frac{P}{d}\right) \le \frac{2\Omega}{1-\Omega^2}\left(E\left(\frac{P}{d}\right) - \frac{\Omega}{1-\Omega}\right)$$

This is a variance bound that holds for *arbitrary* discount rates.

Premultiplying (7) by P_t/d_t and taking expectations yields

$$(12) \quad \mathrm{var}\left(\frac{P}{d}\right) = \frac{1}{1-\Omega}\,\mathrm{cov}\left(\frac{P_t}{d_t}, \sum_{j=1}^{\infty}\Omega^j n_{t+j}\right) + \frac{1}{1-\Omega}\,\mathrm{cov}\left(\frac{P_t}{d_t}, \sum_{j=1}^{\infty}\Omega^j \; -g_{t+j}\right).$$

This is a decomposition of the variance of the price dividend ratio into price-dividend forecasts of future discount rates and dividend growth rates. It is not an orthogonal decomposition, so terms less than 0 and greater than 100% are possible. High price dividend ratios may be associated with low future dividend growth if they are also associated with *much* lower discount rate growth.

Interpreting the approximate present value model.

The mean and variance diverge to infinity as the mean discount rate E(g) approaches the mean dividend growth rate E(n), or as $\Omega \to 1$. The appendix shows that this is true for the exact present value model as well (precisely that the *ex post* present value converges if and only if E(g) > E(n)). The Taylor expansion that generates the approximate model was taken about the mean log discount rate and dividend growth rate E(g) and E(n), rather than (say) the mean discount factor $E(\gamma)$ and gross dividend growth rate $E(d_t/d_{t-1})$, so that the exact and approximate model would diverge at the same point.

The weights on future discount rates and dividend growth rates in the present value (7) decline as $\Omega^j = e^{-j(E(g)-E(n))}$, for an "effective discount rate" equal to the mean discount rate *less* the mean dividend growth rate. For example, if one adopts the common benchmark model of a constant real 5% discount rate, the equally weighted portfolio has a mean dividend growth rate of 4.65% so the rate at which information about future dividends and discount rate movements are downweighted in (7) is only 0.35%. Thus even with a 5% real discount rate, events in the very far off future can and should affect the price dividend ratio today.

Consider the case that discount rates and dividend growth are both white and uncorrelated at leads and lags. Then the expected price-dividend ratio (9) reduces to

$$E\left(\frac{p}{d}\right) = \frac{\Omega}{1-\Omega} + \frac{\Omega}{2(1-\Omega)^2}\left(var(g) + var(n) - 2cov(g,n) \right)$$

The first term is the exact value of the price-dividend ratio in a certainty world with constant dividend growth E(n) and constant discount rates E(g) and $\Omega = e^{E(n)-E(g)}$. The second term includes an adjustment for covariance of dividends with discount rates. If two assets have the same mean dividend growth, but one has greater covariance with discount rates, that asset will have a lower price-dividend ratio and hence a higher average return.

The mean price-dividend ratio responds only to very long run properties of discount rates and dividend growth rates. Since Ω is near 1, the term in the variance and covariance terms in (9) is nearly the spectral density at frequency zero of dividend growth less discount rates, or (equivalently) the variance of a very long moving average of dividend growth minus discount rates--(9) can be written

$$(9') \qquad E\left(\frac{p}{d}\right) = \frac{\Omega}{1-\Omega} + \frac{1-\Omega^2}{2\Omega(1-\Omega)^2} var\left(\sum_{j=1}^{\infty} \Omega^j \left(n_{t+j} - g_{t+j}\right)\right).$$

In particular, if discount rates are constant and log dividend *levels* are stationary, then the spectral density at frequency 0 of dividend growth is 0. In this case the term in the variances and covariances of dividend growth of the mean price dividend ratio (9) or (9′) and also the variance bound (10) is nearly but not quite 0. In consequence, if dividends are assumed stationary about a linear trend, the variance bound is very close to zero, practically *by construction*.

Two features of the variance decomposition (12) are noteworthy. First, since the sums start at j=1, only *predictable* changes in the discount rate or dividend growth rate can explain variation in the price-dividend ratio. If the discount rate and dividend growth rate are pure white noise, the price dividend ratio should be a *constant*. (This is also seen directly in the approximate model (7).)

Second, the variance decomposition is a special case of a moment condition that can be used for more general tests than those presented in this paper. Premultiplying (7) by a vector of mean 0 variables $\tilde{Z}_t$ and taking expectations, we obtain

$$E\left(\tilde{Z}\,\frac{P}{d}\right) = \frac{1}{1-\Omega}\,E\left(\tilde{Z}_t \sum_{j=1}^{\infty} \Omega^j \,(n_{t+j} - g_{t+j})\right)$$

These moment conditions can be use to estimate parameters of the discount rate process and can be tested with a vector of instruments, a discount rate process (e.g. $g_t = -\ln(\rho u'(c_t)/u'(c_{t-1}))$ and a vector of asset prices and corresponding dividends.

These moment conditions can also be written as a restriction on long horizon regression coefficients in the style of Fama and French (1988b). Let the first element of $\tilde{Z}_t$ be the price dividend ratio, and premultiply by the variance covariance matrix of $\tilde{Z}_t$, yielding

$$\beta\left(\frac{1}{1-\Omega}\sum_{j=1}^{\infty}\Omega^j(n_{t+j}-g_{t+j}) \;\middle|\; \begin{bmatrix} P_t/d_t \\ \tilde{Z}_t \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where $\beta(y|x)$ denotes the OLS regression coefficient of y on x. Since Ω is near 1, the left hand variable is approximately a long horizon dividend growth or discount rate.

Last, note that time-varying covariances are missing from the approximate model (7) and hence its moments. They would require *third* moments $E(Z_t g_{t+j} n_{t+k})$, which are ignored by the second order Taylor expansion.

2. Discount Rate Models

The following discount rate models are considered.

1) Discount rate equals a constant. This model is included because it is widely tested in the variance bounds and variance decomposition literature, and it helps to understand how the more plausible models work.

It predicts no risk premia--expected returns on all assets are the same.

2) Discount rate = reference return plus a risk premium that is constant over time. One way to obtain a proxy for expected discount rate variation is to assume that discount rates are generated by a reference return r_t^0 (a constant, or the rate of return on a bond portfolio) plus an unpredictable (but not necessarily mean 0) homoskedastic random variable[2] ϵ_t,

$$(13) \qquad g_t = r_t^0 + \epsilon_t \quad ;$$

$$E_t(\epsilon_{t+1}) = E(\epsilon_{t+1}),\ Cov_t(\epsilon_{t+1}Z) = Cov(\epsilon_{t+1}Z) \text{ for all } Z.$$

With this model the terms in future discount rates in the variance decomposition (12) can be measured using the reference return, since

$$Cov(P_t/d_t,\ g_{t+j}) = Cov(P_t/d_t,\ r_{t+j}^0)$$

Note that the model (13) does not identify the mean discount rate. $E(\epsilon)$ need not be 0, and because of the term in the variance of ϵ, neither the equation for the mean price-dividend ratio nor the one period return equation can be solved for the mean discount rate.[3]

3) Discount rate = return. In this model, the discount rate is equal to the return itself, $g_t = r_t$. This model solves the Euler equation for a single security by construction, both ex ante and ex post:[4] with $g_t = r_t$, $\gamma_t = R_t^{-1}$ so $1 = E_{t-1}(R_t^{-1}R_t) = R_t^{-1}R_t$.

This does not mean that the model discount rate = return is without content for the price-dividend ratio, because the sums could not converge or the transversality condition could fail. For example, if there is a speculative bubble in prices, then the variance of the price-dividend ratio is infinite, but the price-dividend ratio need have no forecast power for either dividend growth or future returns, violating the variance decomposition (12), and the variance bound (10) can be 0. Thus the model discount rate = return is included below as a test of bubbles or sunspots, and as a check on the accuracy of the Taylor approximation used to derive the moments of the price dividend ratio, together with the fact that only a finite number of covariances can be included in the calculations.

If several assets are considered together, the model discount rate = return does not automatically hold for returns. It is not necessarily true that asset i is correctly priced by the inverse of asset j's return. In particular, the statement that all assets can be priced by the inverse of the market return is a version of the capital asset pricing model, derivable from logarithmic utility, and so has content in a multiple asset case.

4) *Consumption based discount rates.* With utility

$$U = E\sum_{t=0}^{\infty} \rho^t u(c_t) = E\sum_{t=0}^{\infty} \rho^t \frac{c_t^{1-\alpha} - 1}{1 - \alpha}$$

the real discount rate is

$$(14) \qquad m_t = -\ln\left(\rho u'(c_t)/u'(c_{t-1})\right) = -\ln(\rho) + \alpha \ln(c_t/c_{t-1}).$$

The nominal discount rate is composed of that real rate and inflation π_t,

$$g_t = m_t + \pi_t .$$

3. Variance Bounds

Table 1 presents summary statistics for the NYSE value and equally weighted portfolios.

Note that mean dividend growth is higher for the equally weighted portfolio, and note the large standard errors of estimation of mean dividend growth. Since the estimates of the formulas for the mean-price dividend ratio, its variance bound and the variance decomposition are sensitive functions of $\Omega = \exp(E(n)-E(g))$, this standard error is a large component of the estimation uncertainty in the calculations that follow. Table 1 also includes the autocorrelations of dividend growth and estimates of partial sums of autocorrelations. This sum can be used to estimate the random walk component of log dividends and is therefore a diagnostic for the presence of a unit root in log dividends (see Cochrane (1988a)). The values of the summed autocorrelations are consistent with the presence of a unit root in log dividends, as assumed. However, they also indicate some serial correlation in real dividend growth, which may thus not be well modelled as a pure random walk.

A. **Bounds on the variance of the price-dividend ratio with no restrictions on discount rates.**

Fig. 1 presents the mean price dividend ratio (9) and variance bound (10) calculated with constant discount rates. As the mean discount rate rises, both the mean and variance bound decline. In part, this is due to the down weighting of high autocovariances of dividend growth as the mean discount rate rises and hence Ω declines. The largest part of the decline in Fig. 1 is due to the effect of the leading terms in Ω of the two formulas.

Typical variance bounds tests with constant discount rates (such as Leroy and Parke (1988)) pick a value for the mean discount rate, such as the mean return or the value that satisfies the mean price dividend ratio, and see whether the variance bound is satisfied at that value. Fig. 1 shows that this procedure will not lead to a violation so long as the chosen mean discount rate is less than about 10.5%. In particular, Fig. 1 shows that the the variance bound is not violated at the mean discount rate implied by the mean price dividend ratio.

The variance bound with no restrictions on discount rates (11) exploits both the mean price dividend equation (9) and the variance bound (10) together. Table 2 presents calculations of the variance bound (11) for a variety of assumed values for the mean discount rate. Fig. 2 illustrates these calculations for the value weighted portfolio.

Since the substitution that produced (11) works for constant discount rates as well as any other discount rate process, this bound may also be used to test the constant discount rate model. In this case, the mean price dividend ratio (9) may be solved for the mean discount rate. These values of the mean discount rate are marked with "var(g)=0" or "var(m)=0" in the tables and figures, according to the use of nominal or real dividend growth and hence the assumption of constant nominal or real discount rates. Thus the variance bound with the constant discount rate model is again equation (11), but evaluated only at the particular mean discount rate that solves the mean price dividend ratio equation. As shown in table 2 and fig. 2, the variance bound with constant discount rates is satisfied by the point estimates for all the portfolios.

The different appearance of the variance bound (11) in table 2 or fig. 2 and the bound (10) in fig. 1 is due to the fact that the mean price dividend ratio equation (9) is always implicitly satisfied in using the bound (11), where it is ignored in using (10) or in Fig. 1. In particular, this explains why the lowest possible value of the mean discount rate is higher, and why the bound slopes up rather than down with increasing mean discount rates.

The variance terms in (9) or (11) must be positive, so the mean price-dividend ratio can be no lower than $\Omega/(1-\Omega)$. This gives rise to the lower limit for the mean discount rate:

$$E\left(\frac{p}{d}\right) \geq \frac{\Omega}{1-\Omega} \quad \text{or } \Omega \leq \frac{E\left(\frac{p}{d}\right)}{1 + E\left(\frac{p}{d}\right)} \quad \text{or } E(g) \geq E(n) - \ln\left(\frac{E\left(\frac{p}{d}\right)}{1 + E\left(\frac{p}{d}\right)}\right)$$

Denote the value of E(g) that satisfies this equation with equality (the leftmost points in Fig. 1) as $E(g)_{min}$. This is tighter restriction on mean discount rates than the restriction that the mean discount rate exceed the mean dividend growth rate ($\Omega < 1$) in Fig. 1.

To see why the bound rises with mean discount rate rather than falling, start at the mean discount rate $E(g)_{var(g)=0}$, at which the mean price dividend ratio equation (9) is exactly satisfied with no variance in discount rates. Referring to (9) or fig. 1, if the mean discount rate increases, the mean price-dividend ratio decreases. Therefore, a higher variance of discount rates must be implicitly assumed, to increase the right hand side of the mean discount rate (9). But this also increases the right hand side of the variance bound (10), by exactly the same amount. For this reason, the variance bound (11) increases to the right of $E(g)_{var(g)=0}$ in Fig. 1. Conversely, as the mean discount rate decreases from $E(g)_{var(g)=0}$, the mean price-dividend ratio increases. Therefore, to match the mean price dividend ratio (9), we must implicitly assume a process for discount rates with increasing variance that is positively correlated with dividend growth rates, in order to lower the variance and covariance term in (9). But this also lowers the value of the variance and covariance term on the right hand side of the variance bound (10), so the variance bound decreases. At the mean discount rate $E(g)_{min}$, $E(P/d) = \Omega/(1-\Omega)$ so discount rates must be *equal* to dividend growth rates to *eliminate* the variance and covariance term in the mean price-dividend ratio (9). But doing this also eliminates that term on the right side of the variance bound (10), so the variance bound (11) goes to 0 by construction.

Thus there is always a region of low mean discount rates in which the point estimate of the variance bound is violated. The value of the mean discount rate at which the variance bound (11) is just satisfied (where the variance bound and the sample variance intersect in Fig. 1) is denoted $E(g)_{bound}$ and marked "Var(P/d)=bound" in the tables and figures. However, even in the region in which the point estimate of the variance bound (11) is below the point estimate of the variance of the price-dividend ratio, it is never even one standard error below, so a statistical test would not reject the present value model for any value of the mean discount rate at which it can be constructed (above $E(g)_{min}$).

The variance bound is the same for real and nominal portfolios, since it is only a function of the dimensionless price dividend ratio and $\Omega = e^{E(n)-E(g)} = e^{E(nr)-E(m)}$. The only difference between real and nominal portfolios is the value of the mean discount rate at which the mean price dividend ratio is satisfied $E(g)_{var(g)=0}$ and $E(g)_{var(m)=0}$. The bound is satisfied for both mean discount rates, for both portfolios.

The shape of the variance bound is the same for the real equallyweighted portfolio. The results for the different portfolios differ most noticeably in the values of mean discount rate. Since the mean equally weighted dividend growth rate is higher (7.7%) than the mean value weighted dividend growth rate (3.9%), correspondingly higher values of the mean discount rate must be used to keep the mean price-dividend ratio finite or equal to its sample value.

B. Variance bounds with discount rate models

Since only the mean discount rate enters in the variance bound (11), the only way a discount rate model can restrict the variance bound is if it provides enough information to restrict the mean discount rate through the mean price dividend ratio (9). For example, the constant discount rate model had one free parameter, E(g), which could be estimated by the mean price dividend equation. The constant risk premium or interest rate plus constant risk premium discount rate models together with the mean price dividend equation (9) do not restrict the mean discount rate, and so do not restrict the variance bound beyond the values presented above with no assumptions on discount rates.

Discount rate = return

With the model discount rate = return, the return is used in the place of discount rates in the mean price-dividend ratio (9) and variance bound (10) or (11). This calculation is both a test of the accuracy of the Taylor approximation and the truncation to 15 covariances, and a test for bubbles or sunspots, whose absence is the only ingredient of the present value model with discount rate equal to return. The result is given in Table 4. For the value weighted portfolio, the predicted mean is a bit lower than the actual mean, due to the term in the variance and covariances, which is about half the size it should be (the term $\Omega/(1-\Omega) = 18.59$, the term in the variances and covariances should contribute the rest). The predicted mean for the equally weighted portfolio is closer to the sample mean. However, the difference between sample and predicted means is less than about one standard deviation for both portfolios. The variance bound is above the sample variance in both cases.

Consumption based discount rates

Given values for risk aversion α and the subjective discount factor ρ, consumption based discount rates (14)can be used in the mean price-dividend ratio (9) and the variance bound (10). These variance bounds are presented in Table 5. Since there are two free parameters (α and ρ), the mean price-dividend ratio equation (9) cannot be solved for a unique mean discount rate and thus restrict the variance bound beyond the values calculated with no restrictions on discount rates. Therefore, table 5 presents the variance bound for a variety of assumed mean subjective discount factors ρ, from .95 to .99. For each ρ, a risk aversion parameter α is estimated by the mean price dividend equation (9), and then the variance bound is estimated at the mean discount rate corresponding to the assumed ρ and estimated α.

The implied mean discount rates are quite constant over this range of subjective discount factors ρ, so there is little variation in the bound, which is satisfied by the point estimates for both portfolios. Since constant discount rates were not rejected, the only way consumption based discount rates could be rejected is if consumption growth was strongly positively correlated with dividends, so that the mean price dividend ratio equation (9) is satisfied at a low mean discount rate, in the region where the variance bound is violated. That the consumption based model does not give rise to a violation just reflects a sufficiently low correlation between consumption growth and dividend growth.

4. Decomposing the Variance of the Price/Dividend Ratio

A. Variance decomposition with unobserved discount rates, constant discount rates, and discount rate = interest rate plus constant risk premium models.

Table 3 presents calculations of the variance decomposition (12). For these discount rate models, the mean price dividend ratio equation (9) only restricts the variance decomposition by placing the lower bound $E(g) \geq E(g)_{min}$ on the mean discount rate.

The columns of table 3 marked "n" and "nr" give the real and nominal dividend growth terms of (17) respectively. The columns marked "g" and "m" give 100% less the dividend growth terms, and thus represent the fraction of the variance of the price dividend ratio that must be explained by correlation of the price dividend ratio with future discount rates.

Note first the effect of the mean discount rate. As the mean discount rate is increased, the covariance of the price dividend ratio with dividend growth in the far future counts less, so the contribution of the dividend growth term declines with increasing E(g). More importantly, the leading term $1/(1-\Omega)$ lowers the size of the dividend growth term as the mean discount rate is increased. Conversely, as the mean discount rate declines, the term $1/(1-\Omega)$ grows without bound. Therefore, so long as the dividend growth term in (12) is positive, there will be some mean discount rate at which *all* variance of the price-dividend ratio is attributed to the dividend term. However, that mean discount rate may be lower than the value $E(g)_{min}$ which is the minimum value consistent with the mean price dividend ratio.

Thus, the model discount rate = constant + constant risk premium can be rejected if either the dividend growth term in (12) is negative, or if the mean discount rate that required is lower than $E(g)_{min}$. In the model discount rate=constant, the mean price dividend ratio equation can be solved for the unique mean discount rate value $E(g)_{var(g)=0}$, so that model can be rejected if 100% of the variance of the price dividend ratio is not accounted for by dividend growth at the mean discount rate $E(g)_{var(g)=0}$.

For the value weighted portfolio, the dividend growth terms are small, and more than two standard errors away from 100% at $E(g)_{min}$ (15% ± 25% nominal, -37% ±11% real), $E(g)_{var(g)=0}$ (11% ± 19%), and $E(g)_{var(m)=0}$ (-33% ± 9%), so the real and nominal discount rate = constant and constant plus constant premium models are statistically rejected. In fact none of the value weighted divided growth terms are significantly greater than 0, and the real dividend growth terms are significantly *negative*.

For the equally weighted portfolio, the dividend growth terms are positive, larger, and less than two standard errors from 100% at $E(g)_{min}$ (73% ± 49% nominal, 39% ± 48%real), $E(g)_{var(g)=0}$ (55% ± 39%), and $E(g)_{var(m)=0}$

(30% ± 39%). Thus the real or nominal discount rate - constant and constant plus constant risk premium models are not statistically rejected for the equally weighted portfolio. However, the standard errors are large enough that we cannot reject that the dividend growth terms are zero either.

The columns marked tb, gb, cb and tbr gbr, cbr measure the discount rate term of (12) using the models discount rate equals treasury bill, government bond, and corporate bond rate plus a risk premium that is constant over time. Ideally, the fraction of the variance of the price dividend ratio accounted for by the discount rate proxy should be equal to the residual left over from the dividend growth term, calculated in the "g" and "m" columns.

For the nominal value-weighted portfolio, all these discount rate measures make *negative* contributions. High price dividend ratios forecast *high* nominal interest rates, rather than low nominal interest rates. A large part of this phenomenon is that a high price dividend ratio forecasts higher inflation, presented in the column marked π. The real government and corporate rates at least make positive contributions, but the real treasury bill rate still makes a negative contribution--high price dividend ratios forecast high real as well as nominal treasury bill rates.

The total percentage explained is the same in the real and nominal columns: going from nominal to real shifts the correlation of price dividend with inflation from the dividend growth term to the discount rate term. The highest total percentage explained is negative.

All these discount rate terms are about one standard deviation from 0 for a wide range of mean discount rates, with the exception of inflation, which is about two standard deviations above zero, and the nominal t-bill rate, which is more than two standard deviations below 0. Thus we can easily reject that any of these discount rate proxies make the large positive contributions to explaining the variance of the value weighted price dividend ratio required for mean discount rates above $E(g)_{min}$.

The picture is roughly similar for the equally weighted portfolio. The contributions of the nominal discount rate terms are all negative; the contributions of the real discount rate terms are all slightly positive, but the corporate and government returns make lower contributions (11 and 13% at $E(g)_{min}$) than they did to the value weighted price dividend ratio, and all the positive contributions are less than one standard deviation from 0. The total variance explained is never more than 39% + 19% = 58%, using the treasury bill rate and $E(g) = E(g)_{min}$, and the real treasury bill rate at 19% ± 23% is the largest positive contribution of any interest rate term.

B. Variance decomposition with discount rate = return and consumption based discount rates.

i) Discount rate = return and market return

Table 4 presents both terms of the variance decomposition (12) with discount rates equal to returns on the portfolios themselves. (The dividend term is the same as in table 3, at a mean discount rate equal to mean return). The variance of value weighted price-dividend ratio is 10% nominal dividends / 86% nominal returns and -28% real dividends / 124% real returns. A total of 97% of the variance of the price-dividend ratio is accounted for, which is a measure of the accuracy of the approximation, and an indication of the absence of bubbles. The variance of the equally weighted price-dividend ratio is 57% nominal dividends / 50% nominal discount rate, and 31% real dividends / 77% real discount rate, for a total of 108%.

Table 4 also presents a decomposition of the equally weighted price-dividend ratio, using the value weighted return as the discount rate. The variance attributed to dividends is the same as in table 3 at the value of mean discount rate equal to the expected return on the *value* weighted portfolio. The dividend terms are 245% (nominal) or 127% (real). The covariances with the value weighted return add another 127% (nominal) or 392% (real), for a total of 519%. Of course, these point estimates are subject to large standard errors, as seen in the table. The reasons such a large percentage is accounted for are that the mean value weighted return is lower than the mean equally weighted return so Ω is higher, together with the fact that the equally weighted price-dividend ratio is a good forecaster of the value weighted return.

iv) Consumption based discount rates

In the consumption based model, the discount rate is measured by (14), so the covariance of the price-dividend ratio with discount rates by

$$\mathrm{cov}(P_t/d_t, -m_{t+j}) = \mathrm{cov}(P_t/d_t, -\alpha \ln(c_{t+j}/c_{t+j-1})) .$$

Table 5 presents the variance decomposition (12) using this consumption based model of discount rates. As before, the mean price dividend ratio equation (9) can be used to infer one parameter of the utility function, so α is estimated by that equation for various values of ρ.

All the consumption-based discount rate contributions are negative, and many are more than two standard deviations below 0. A high price dividend ratio forecasts *higher* future real and nominal consumption growth. It should forecast *lower* future consumption growth: any wealth effects of a high stock price should be incorporated into consumption immediately, and then consumption *growth* should be lower, as discount rates are lower.

5. Bounds on the mean and standard deviation of discount rates.

Are the unobserved discount rate processes that satisfy the mean price dividend ratio, the variance bound, and the variance decomposition "reasonable," or must they have unusual time series processes suggestive of "fads?" To address this question, this section computes the minimum standard deviation of discount rates required to satisfy the mean price dividend ratio, the variance bound, and the variance decomposition. The problem is: for a given mean discount rate, find the discount rate process (choose its variance, autocovariances, and cross covariances with dividend growth and the price dividend ratio) with minimum variance such that the expected price-dividend ratio (9), the variance bound (10) or (11), and/or variance decomposition (12) are satisfied.

The form of the solution to this problem is discount rate process that is a function of the dividend growth and price dividend ratio process:

$$g_t = \alpha(L)n_t + \beta(L)(P_t/d_t).$$

($\alpha(L)$ and $\beta(L)$ may be two sided.) Adding a noise term uncorrelated with dividend growth and price dividend ratio just adds standard deviation without helping to attain the constraints. The problem is then to find the form of $\alpha(L)$ and $\beta(L)$ that minimize the variance of g subject to the constraints. It is a straightforward but algebraically unpleasant Lagrangian minimization, and so is presented in the appendix. The results are presented in table 6 and illustrated in fig. 3. and fig. 4

Examine first the bound on the standard deviation of nominal discount rates that satisfy the mean price dividend ratio alone. It has a global minimum at $E(g) = E(g)_{var(g)=0}$ and $var(g) = 0$. This value of the mean discount rate is defined as the value at which the mean price dividend ratio equation (9) is satisfied with constant discount rates, so the minimum standard deviation of discount rates had better be 0 at this mean discount rate. As explained in conjunction with the variance bound, a discount rate process with higher variance must be assumed to keep the mean price dividend ratio satisfied to the right of $E(g)_{var(g)=0}$, and this is reflected in the rising frontier. The frontier also rises to the left of $E(g)_{var(g)=0}$. In this region, discount rate processes with higher variances that are positively correlated with dividend growth rates must be assumed in order to maintain the mean price dividend ratio constraint. In the limit that $E(g) = E(g)_{min}$, so $E(P/d) = \Omega/(1-\Omega)$, discount rates must be equal to dividend growth and hence $var(g) = var(n)$, as shown in the "$E(g)_{min}$" rows of Table 6. For mean discount rates lower than $E(g)_{min}$, no discount rate process is consistent with the mean price dividend ratio equation, so the bound rises to infinity.

Next, examine the bound on the standard deviation of discount rates that satisfy the variance decomposition. The variance decomposition allows us to infer the regression coefficient of discount rates g_t on the variable $\sum_{j=1}^{\infty} \Omega^j$

$(\frac{P}{d})_{t-j}$, observable at time t-1. (The variance of the price dividend ratio less the dividend growth term in (12) leaves $\sum_{j=1}^{\infty} \Omega^j \ \mathrm{cov}(\frac{P}{d}, g_{t+j}) = \mathrm{cov}\left(g_t, \sum_{j=1}^{\infty} \Omega^j \ (\frac{P}{d})_{t-j}\right)$. This regression coefficient leads to an obvious lower bound on the variance of discount rates: it must be higher than the variance of the forecast using this variable. The appendix shows that the resulting bound is tight, and is thus the greatest lower bound of standard deviations of discount rates that satisfy the variance decomposition.

In table 3, for portfolios with a positive dividend growth term, there was a mean discount rate at which 100% of the variance of the price dividend ratio was accounted for by dividend growth, and the bound shows a global minimum standard deviation of discount rates equal to 0 at these values. (This is most visible in the nominal equally weighted portfolio.) At higher mean discount rates, the required standard deviation of discount rates rises steadily. Here the $1/(1-\Omega)$ term is shrinking, so the weighted sum of the covariances of price dividend ratios with future discount rates must increase. Also, the weights on far future discount rates are decreasing, so that more variance of the discount rates are required. Below the standard deviation = 0 mean discount rate, the bound rises slightly. At a mean discount rate equal to the mean dividend growth rate, $\Omega=1$, all the formulas blow up, no lower mean discount rate is allowed, and the bound rises to infinity at this point.

The variance bound is an inequality constraint, and so only adds the information that the mean discount rate must be greater than the value $E(g)_{\text{bound}}$ at which var(P/d)=bound, indicated in the table.

The bound that includes both the mean and the variance decomposition is tighter than the intersection of the mean and variance decomposition bounds. That is because the mean bound is a singular function of dividend growth ($g_t = \alpha(L)n_t$), while the variance decomposition bound is a singular function of the price dividend ratio ($g_t = \beta(L)(P_t/d_t)$). For example, at the global minimum of the mean bound, the process g=constant satisfies that bound. But a process $g_t = \beta(L)(P_t/d_t)$ is required to satisfy the variance decomposition, and this process does not satisfy the mean bound.

The standard errors of the "Mean" and "both" bounds become very large at the lower limits of mean discount rates, but then decline rapidly with increasing mean discount rate. At the high discount rates, the standard errors are a great deal smaller than the estimated coefficients.

The bounds on discount rate processes that satisfy the mean, variance decomposition and variance bound separately all prescribe lower limits on the mean discount rate. (The large estimation uncertainty near this lower limit reflects estimation uncertainty about exactly what the lower limit is.) These bounds on the mean discount rate can be surprisingly high: the mean real discount rate must be as much as 10% to satisfy the equally weighted portfolio results. However, each of these bounds admits a discount rate process with zero *variance* for a mean discount rate near the lower bound, so the traditional puzzle of the *variance* of discount rates is not present in satisfying the mean price dividend ratio, the variance bound, or the variance decomposition taken alone.

When the mean, variance bound, and variance decomposition are taken together, a bound on the standard deviation appears along with bounds on the mean: the value weighted portfolio requires a standard deviation of real discount rates somewhat above 5%, while the equally weighted requires somewhat above 9%. These minima are about two standard errors above 0, but the standard errors increase much faster than the minimum standard deviations for smaller mean discount rates.

Are discount rates in these mean-standard deviation frontiers "reasonable?" To get some idea, fig. 3 and fig. 4 also present the mean and standard deviation of consumption based discount rates. The utility parameters are a subjective discount factor $\rho = .98$ and a variety of risk aversion coefficients α below 7. Changing the subjective discount factor ρ simply shifts the curve to the left or right. As the figure shows, the consumption based discount rates have means and standard deviations in the required regions with a range of "reasonable" parameters. The minimum risk aversion coefficient required to get in the point estimates of the regions is about 3.

This only argues that the required mean and standard deviation of discount rates is "reasonable" compared to the discount rates generated by the standard consumption model. This is *not* a test of the consumption model, as it fails to generate other moments consistent with the price-dividend model. For example, table 5 showed that forecasts of consumption growth from price dividend ratios had the wrong sign.

As another way to display some characteristics of the required discount rates that are inconsistent with the consumption based model, Table 5 calculates the first order autocorrelation coefficient of the variance minimizing discount rate process. For most mean discount rates, this autocorrelation is very near 1. There is a trade off between autocorrelation and variance: by adding white noise to the standard deviation minimizing discount rate processes, we can decrease their autocorrelation by burying the required predictable components in noise, at the expense of higher and potentially "puzzling" standard deviation.

The required predictability is only a puzzle together with bounds on the mean discount rate. At very low discount rates, the standard deviation of the predictable (which is equal to actual) discount rate that satisfies the variance decomposition alone is quite small. We are only forced to have larger predictable components (around 3-4% annual standard deviation) by the lower limits on the mean discount rate imposed by the mean price dividend ratio and the variance bound.

Thus the required discount rates are "reasonable" from the usual criterion of their standard deviation, but are quite unlike consumption based discount rates in their autocorrelation or persistence, and their predictability from price dividend ratios. This requirement for a predictable component of discount rates may appear as a sharper puzzle for consumption based discount rates than bounds on the standard deviation of discount rates.[5]

5. Concluding Remarks

This paper tried to answer the three questions raised by the volatility of price dividend ratios together with the autocorrelation structure and predictability of dividend growth from price dividend ratios: 1) Is there *any* discount rate process that is consistent with the volatility of the price dividend ratio, or is its volatility an indication of "bubbles" or "sunspots?" 2) Is there a *reasonable* discount rate process that explains the volatility of the price dividend ratio and 3) Do particular discount rate *models* account for the volatility of the price dividend ratio, including a constant, constant plus constant risk premium, interest rate plus constant risk premium, consumption based discount rates, and discount rate=return

This paper finds no evidence that there is *no* discount rate process that explains the variance of price-dividend ratios. The mean price dividend ratio, the bound on the variance of the price dividend ratio, and the decomposition of the variance of the price dividend ratio into forecasts of future dividend growth and discount rates were all satisfied using returns for discount rates. (The only ingredient of these tests is the lack of bubbles or sunspots, as discount rate=return satisfies the Euler equation by construction.) Furthermore, the global maximum of the variance bound with no restrictions on discount rates was well above the sample variance of the price dividend ratio

The particular models studied yielded mixed results. The variance bound was satisfied for all the discount rate models including a constant discount rate. In contrast to the extensive literature on variance bounds, the mean price dividend ratio together with the variance decomposition proved more difficult to reconcile here.

For the value weighted portfolio, price dividend ratios provided very small forecasts of future nominal dividend growth, and forecast negative future real dividend growth. The very small mean discount rates required to explain all the variance of the price dividend ratio by nominal dividend growth were ruled out by the mean price dividend equation. Thus, the constant discount rate model and constant plus constant risk premium model were rejected for the value weighted portfolio. Furthermore, the interest rate and consumption based discount rate models did not provide sufficiently strong forecasts of discount rates from price dividend ratios to account for the variance of the price dividend ratio together with these weak dividend growth forecasts. In fact, when the estimates are even more than one standard error from 0, high price dividend ratios are associated with *higher* future interest rates and consumption growth, so the discrepancy is one of *sign* as well as magnitude.

The equally weighted price dividend ratio provided stronger forecasts of its dividend growth, so that the constant, constant plus risk premium and interest rate models could not be statistically rejected. However, standard errors were large enough that the hypotheses that the dividend growth and discount rate forecasts are zero could not be rejected either. The nominal interest rate contributions and all the consumption based discount rate contributions to explaining the variance of the price dividend ratio were again negative, while the real interest rate forecasts contributed only small amounts, less than 19%. The greater success of the decomposition for the equally weighted portfolio is thus entirely due to better price dividend forecasts of dividend growth, rather than any better forecasts of discount rates in these models.

The failure of these discount rate models indicates that the needed movement in expected discount rates predicted by the price-dividend ratio (and found in forecasts of returns) is a movement in *risk premia*, such as changes in the slope of the mean-standard deviation frontier or the conditional variance of discount rates, rather than movement in a relatively riskless reference return, which would be a movement in the intercept of the mean-standard deviation frontier.

To assess whether the required unobserved discount rates are reasonable, mean standard deviation frontiers for discount rates that satisfy the variance bound, variance decomposition and mean price dividend ratio were calculated. The required discount rates have *standard deviations* that seem "reasonable," at least as compared with those predicted by the consumption based model with "reasonable" discount factors and risk aversion coefficients. However, the required discount rates are more *predictable* and *autocorrelated* than consumption growth rates. Also, the bounds included relatively high lower bounds on the *mean* discount rate.

In summary, the results of this paper suggest that there *exists* a discount rate process that, together with dividend growth, rationalizes the variance of the dividend price ratio: there are neither bubbles nor sunspots and the variance of the price-dividend ratio *can* be attributed to changing expectations of future discount rates and dividend growth; it suggests that the discount rate process has some "reasonable" characteristics: its standard deviation can be lower than that of returns or consumption growth multiplied by a reasonable risk aversion coefficient, and some "puzzling" characteristics--it must be predictable and autocorrelated, as returns are, but consumption growth seems not. The challenge posed by these results is thus exactly that of Euler equation models of returns, namely to find an observable proxy for the discount rate other than returns themselves, a proxy that can be used to connect discount rate variation to events in the real economy.

These results can be extended by considering several assets together as well as more instruments than just the price dividend ratio. This would join the time series implications studied here and in the rest of the variance bound literature with the cross sectional implications of Euler equation studies. Since all assets should be priced by the same discount rate, this may make it possible to infer more about the discount rate process. Sharper bounds on the discount rate will likely emerge. For example, the mean discount rate must be larger than the mean dividend growth rate on each asset separately, so inclusion of assets with high dividend growth rates may rule out the low range of mean discount rates required for many of the tests in this paper.

Appendix.

2. Existence and stationarity of the price dividend ratio and present value, bubbles and sunspots.

To derive the present value model, express the Euler equation (2) as

$$\frac{p_t}{d_t} = E_t\left(\gamma_{t+1}\frac{p_{t+1}}{d_{t+1}}\frac{d_{t+1}}{d_t} + \gamma_{t+1}\frac{d_{t+1}}{d_t}\right)$$

Iterating this equation with $\eta_{t+1} = d_{t+1}/d_t$, we obtain the present value model

$$(3) \qquad \frac{p_t}{d_t} = E_t\sum_{j=1}^{\infty}\prod_{k=1}^{j}(\gamma_{t+k}\,\eta_{t+k})$$

if the associated transversality condition

$$\lim_{j\to\infty} E_t \prod_{k=1}^{j}(\gamma_{t+k}\,\eta_{t+k})\frac{p_{t+j}}{d_{t+j}} = 0$$

holds.

A discount factor that satisfies the Euler equation exists trivially for a single asset: $\gamma_t = R_t^{-1}$, or $\gamma_t = R_t/E_{t-1}(R_t^2)$ for example. Hansen and Richard (1987) show the existence of a single discount factor that prices any *group* of asset returns under weak conditions, essentially that the price of a portfolio equal the value of its constituent securities. Hence, there *exists* a present value relation (there exists a discount factor such that (3) holds) under those conditions, and the transversality condition. In addition, since the price dividend ratio is finite, the present value must be finite almost surely. The latter two conditions are thus the extra content of a price-dividend model over a return model.

Assume that the discount factor and dividend growth are strongly stationary and positive a.s. Since the price-dividend ratio (3) is a time invariant function of stationary variables it too is stationary, if it is finite almost surely. A necessary condition that the price-dividend ratio be finite is that the ex-post price-dividend ratio

$$\frac{p_t}{d_t}^{ex} = \sum_{j=1}^{\infty}\prod_{k=1}^{j}(\gamma_{t+k}\,\eta_{t+k}) = \sum_{j=1}^{\infty}\exp\sum_{k=1}^{j}(n_{t+k} - g_{t+k}) = \sum_{j=1}^{\infty}\exp\sum_{k=1}^{j}w_{t+k}$$

is finite a.s.. Therefore,

Proposition 1: E(n) < E(g) or E(w) < 0 are necessary and sufficient for the ex-post price-dividend ratio to be finite a.s., and hence stationary.

Proof: By the weak law of large numbers, sample means of any strongly stationary random variable converge to the population means almost surely. With T = sample size, this means

$$\forall\, \delta > 0\ \exists\, T^* > 0 \text{ s.t. } \forall\, T > T^*,\quad \frac{1}{T}\sum_{k=1}^{T} w_k - E(w) < \delta$$

Therefore,

$$\forall\, \delta > 0\ \exists\, T^* > 0 \text{ s.t. } \forall\, T > T^*,\quad \sum_{k=1}^{T} w_k < T\,(\delta + E(n-g)) \quad \text{a.s.}.$$

If E(w) < 0, pick $\delta = -E(w)/2$, so

$$\exists\, T^* > 0 \text{ s.t. } \forall\, T > T^*,\quad \sum_{k=1}^{T} w_k < T/2\ E(w) < 0 \quad \text{a.s.}$$

Write the ex-post price-dividend ratio as

$$\frac{P_t^{ex}}{d_t} = \sum_{j=1}^{T^*} \exp \sum_{k=1}^{j} w_{t+k} + \sum_{j=T^*}^{\infty} \exp \sum_{k=1}^{j} w_{t+k} <$$

$$\sum_{j=1}^{T^*} \exp \sum_{k=1}^{j} w_{t+k} + \sum_{j=T^*}^{\infty} e^{jE(w)/2} = \sum_{j=1}^{T^*} \exp \sum_{k=1}^{j} w_{t+k} + \frac{e^{T^* E(w)/2}}{1 - e^{E(w)/2}} < \infty .$$

The same argument shows that if E(g) < E(n), or E(w) > 0, then the ex-post price-dividend ratio is infinite a.s. ■

For the price-dividend ratio to be finite, it is not sufficient that the ex-post price-dividend ratio be finite, because the expectation may not exist. For example, suppose w is lognormally distributed. Then, E(exp(w)) = exp(E(w) + 1/2var(w)), so

$$E(P/d) = E\left(\sum_{j=1}^{\infty} \exp \sum_{k=1}^{j} w_{t+k}\right) = \sum_{j=1}^{\infty} \exp\left(j\, E(w) + 1/2 \text{ var } \left(\sum_{k=1}^{j} w_{t+k}\right)\right).$$

Now,

$$\lim_{j\to\infty} \frac{1}{j} \text{var}\left(\sum_{k=1}^{j} w_{t+k}\right) = S_w(0)$$

where $S_w(0)$ denotes the spectral density of w at frequency 0. Thus, if w is lognormally distributed, the expected price-dividend ratio is finite if and only $E(w) + S_w(0) < 0$, which is more stringent than $E(w) < 0$. For this reason the logic of the text is to *assume* that the price-dividend ratio, dividend growth and discount rates are all stationary with first and second moments, and to point out that this is consistent with the present value formula (3). $E(g) > E(n)$ is *necessary* by not *sufficient* for price-dividend stationarity.

A bubble term is defined as the expression on the left hand side of the transversality condition, if it is not zero. Denoting it P_t^b/d_t, the bubble obeys

$$\frac{P_t^b}{d_t} = \lim_{j\to\infty} E_t \prod_{k=1}^{j} (\gamma_{t+k}\, \eta_{t+k}) \frac{P_{t+j}}{d_{t+j}} = E_t \left(\gamma_{t+1}\, \eta_{t+1} \frac{P_{t+1}^b}{d_{t+1}} \right)$$

Hence if γ and η are stationary, the bubble *return* $(P_{t+1}^b/d_{t+1})/(P_t^b/d_t)$ may be stationary, but the bubble P_t^b/d_t will not be stationary. *If there is a bubble, the price dividend ratio is not stationary*, and the *variance of the price-dividend ratio is infinite*.

Sunspots occur when $E(n_t) > E(g_t)$ so that the present value does not converge, but the Euler equation still holds. Starting from (2), define ϵ_t by

$$P_{t-1} = \gamma_t P_t + \gamma_t d_t + \epsilon_t d_{t-1}$$

so that $E_{t-1}(\epsilon_t) = 0$. Solving for P_t/d_t

$$\frac{P_t}{d_t} = \frac{1}{\gamma_t \eta_t} \frac{P_{t-1}}{d_{t-1}} - 1 - \frac{1}{\gamma_t \eta_t} \epsilon_t .$$

Iterating backwards,

$$\frac{P_t}{d_t} = \lim_{j\to\infty} \prod_{k=0}^{j-1} \frac{1}{\gamma_{t-k}\eta_{t-k}} \frac{P_{t-j}}{d_{t-j}} - \sum_{j=0}^{\infty} \prod_{k=0}^{j-1} \frac{1}{\gamma_{t-k}\eta_{t-k}} - \sum_{j=0}^{\infty} \prod_{k=0}^{j-1} \frac{1}{\gamma_{t-k}\eta_{t-k}} \epsilon_{t-j}$$

Following the same logic as before, these backward sums converge if and only if $E(g) < E(n)$. Such backward solutions of the Euler equation are called "sunspots" (Flood and Hodrick (1988)) because variability in the price-dividend ratio is in part due to the shock ϵ_t, which may have no relation to fundamentals.

Thus the absence of both bubbles and sunspots are necessary conditions for stationarity of the price dividend ratio and the present value formula. If there is a sunspot, the price-dividend ratio itself could be stationary, but, since the present value is infinite, the price dividend ratio need not forecast dividend growth or discount rates as predicted by the variance decomposition (12), nor satisfy the variance bound (10) or (11) and the mean price-dividend formula (9). If there is a bubble, the present value may be finite and stationary, but the price dividend ratio is nonstationary with infinite variance, so again, the variance bound and decomposition need not hold.

Since the price and/or present value are *not* stationary under the bubble or sunspot alternatives, sample moments do not converge to (infinite or undefined) population moments. This means that the test statistics are valid only under the null, and that their power against bubble or sunspot alternatives is an open question.

If we rule out sunspots, the price dividend ratio is finite and nonstationary if and only if there is a bubble, so a test for the stationarity of the price dividend ratio would ideally capture all the testable content of a bubble. However, unit roots tests have little power against the alternative of long swings in the price dividend ratio, which can be generated by by long swings in discount rates. Thus, although bubbles and time varying discount rates (whether due to "fads" or due to "fundamentals") are distinct hypotheses in population, because bubbles generate price dividend ratios with unit roots and time varying discount rates generate stationary price dividend ratios, these alternatives are not distinguishable in a finite sample, since long swings in discount rates generate long swings in price dividend ratios against which unit roots tests have arbitrarily low power.

3. Derivation of the approximate present value model.

Starting from the present value model (3), premultiplying by any variable Z_t observed at time t and taking expectations, we obtain

$$E\left(Z_t \frac{P_t}{d_t}\right) = E\left(Z_t \sum_{j=1}^{\infty} \exp \sum_{k=1}^{j} w_{t+k} \right)$$

where $w_t = n_t - g_t$. Construct a second order Taylor expansion of the expression in the brackets, with respect to w_{t+j}, j = 1, 2, ..., and Z_t about their means E(w) and E(Z). With $\tilde{w}_t = w_t - E(w)$, that Taylor expansion is:

(A.3.1)

$$Z_t \sum_{j=1}^{\infty} \exp \sum_{k=1}^{j} w_{t+k} \cong \frac{Z_t \Omega}{1-\Omega} + \frac{Z_t}{1-\Omega} \sum_{j=1}^{\infty} \Omega^j \tilde{w}_{t+j} + \frac{1}{2} \frac{E(Z)}{1-\Omega} \sum_{j=1}^{\infty} \Omega^j \left(\tilde{w}_{t+j}^{\,2} + 2 \sum_{k=1}^{\infty} \Omega^k \tilde{w}_{t+j} \tilde{w}_{t+j+k} \right)$$

Where $\Omega = e^{E(w)} = e^{E(n)-E(g)}$.

With Z = 1, we obtain an expression for the mean price dividend ratio,

$$(9) \qquad E\left(\frac{P}{d} \right) = \frac{\Omega}{1-\Omega} + \frac{\Omega}{2(1-\Omega)^2} \left[\text{var}(w) + 2 \sum_{j=1}^{\infty} \Omega^j \text{cov}(w_t, w_{t+j}) \right].$$

Then, taking the expected value of (A.3.1) for any Z, and with $\tilde{Z} = Z - E(Z)$

$$E\left(\tilde{Z} \frac{P}{d}\right) = \frac{1}{1-\Omega} E\left(\tilde{Z}_t \sum_{j=1}^{\infty} \Omega^j \tilde{w}_{t+j} \right)$$

Since this must hold for any variable Z_t, we have

$$(A.3.2) \qquad \frac{P_t}{d_t} - E\left(\frac{P}{d} \right) = \frac{1}{1-\Omega} E_t \sum_{j=1}^{\infty} \Omega^j \tilde{w}_{t+j} .$$

(9) and (A.3.2) together give the model (7) presented in the text.

4. Accuracy of approximation

(A section checking the accuracy of the Taylor approximation was deleted to fit the NBER page limit. It is available from the author.)

5. Finding the mean standard deviation frontier for discount rates.

The problem is: min var(g_t), subject to the mean (9) and variance decomposition (12). The variance decomposition will be presented for any variable Z_t rather than just the price dividend ratio. The constraints (9) and (12) can be rewritten as

$$\left(E\left(\frac{P}{d} \right) - \frac{\Omega}{1-\Omega} \right) \frac{2\Omega(1-\Omega)^2}{1-\Omega^2} = X_1 = \text{var}\left(\sum_{j=1}^{\infty} \Omega^j \, (n_{t+j} - g_{t+j}) \right)$$

$$\text{cov}\left(Z_t, \sum_{j=1}^{\infty} \Omega^j n_{t+j}\right) - (1-\Omega)\ \text{cov}(Z, \tfrac{P}{d}) = X_2 = \text{cov}\left(Z_t, \sum_{j=1}^{\infty} \Omega^j\ g_{t+j}\right) .$$

The problem is most easily solved in the frequency domain,

$$\min\ \frac{1}{\pi} \int_0^{\pi} S_g(\omega)\ d\omega \quad \text{subject to}$$

$$X_1 = \frac{1}{\pi} \int_0^{\pi} |h(e^{-i\omega})|^2\ S_{n-g}(\omega)\ d\omega \qquad \text{and} \quad X_2 = \frac{1}{2\pi} \int_0^{\pi} S_{z,\ h(L)g}(\omega)\ d\omega$$

where

$$h(L) = \sum_{j=1}^{\infty} \Omega^j\ L^{-j} \qquad \text{so} \quad h(e^{-i\omega}) = \frac{\Omega e^{i\omega}}{1 - \Omega e^{i\omega}} .$$

and $S_x(\omega)$ $S_{x,y}(\omega)$ denote spectral and cross spectral densities.

The variance-minimizing process is singular, there is an α and β such that $g_t = \alpha(L)n_t + \beta(L)z_t$. With this form for the g process and using $S_{x,h(L)y} = h(e^{i\omega})S_{x,y}$, the problem becomes

$$\min_{\{\alpha,\beta\}}\ \frac{1}{\pi} \int_0^{\pi} d\omega\ \ |\alpha|^2 S_n + |\beta|^2 S_z + \alpha\beta^* S_{nz} + \alpha^*\beta S^*_{nz} \quad \text{subject to}$$

$$X_1 = \frac{1}{\pi} \int_0^{\pi} d\omega\ |h|^2 \left(|1-\alpha|^2 S_n + |\beta|^2 S_z - (1-\alpha)\beta^* S_{nz} - (1-\alpha)^*\beta S^*_{nz} \right)$$

$$X_2 = \frac{1}{2\pi} \int_0^{\pi} d\omega\ \ (h\alpha S_{nz} + h^*\alpha^* S^*_{nz}) + (h\beta + h^*\beta^*) S_z$$

where the $(e^{-i\omega})$ notation is suppressed following α, β, h and the spectral densities.

The first order conditions to this Lagrangian maximization yield

$$\alpha = \frac{\delta_1 |h|^2}{1 + \delta_1 |h|^2} , \quad \beta = \frac{\delta_2 \quad h^*}{2\ (1 + \delta_1 |h|^2)} .$$

Where δ_1 and δ_2 are Lagrange multipliers on the two constraints. Substituting, the minimum variance of discount rates is

$$\text{var}(g_t) = \frac{1}{\pi}\int_0^{\pi} d\omega \frac{|h|^2}{(1+\delta_1|h|^2)^2}\left(\delta_1^2|h|^2 S_n + \frac{\delta_2^2}{4} S_z - \frac{\delta_1\delta_2}{2}(hS_{nz} + h^*S^*_{nz})\right)$$

(A.5.1)

where the constraints δ_1 and δ_2 are found from the constraint equations

(A.5.2) $$X_1 = \frac{1}{\pi}\int_0^{\pi} d\omega \frac{|h|^2}{(1+\delta_1|h|^2)^2}\left(S_n + \frac{\delta_2^2}{4}|h|^2 S_z + \frac{\delta_2}{2}(hS_{nz} + h^*S^*_{nz})\right)$$

(A.5.3) $$X_2 = \frac{1}{2\pi}\int_0^{\pi} d\omega \frac{|h|^2}{1+\delta_1|h|^2}\left(\delta_1(hS_{nz} + h^*S^*_{nz}) - \delta_2 S_z\right).$$

The second constraint can be solved for δ_2 as a function of δ_1:

(A.5.4) $$\delta_2 = \left(2\pi X_2 - \int_0^{\pi} d\omega \frac{|h|^2}{1+\delta_1|h|^2}\,\delta_1(hS_{nz} + h^*S^*_{nz})\right) \Big/ \int_0^{\pi} d\omega \frac{|h|^2}{1+\delta_1|h|^2} S_z$$

To calculate the minimum standard deviation of discount rates for a given mean discount rate the following procedure is followed: The spectral densities S_n, S_z, S_{nz} are constructed using the first fifteen covariances, down weighted as in Newey-West (1987) to ensure that the spectral densities are positive. (A.5.4) is used to substitute for δ_2 in (A.5.2), and then a value of δ_1 that satisfies (A.5.2) is searched for, performing the integrals numerically. With the resulting values of δ_1 and δ_2, (A.5.1) is evaluated to give the minimum variance of discount rates.

The case $\delta_1=0$, which corresponds to the same minimization imposing only the second constraint, yields a natural interpretation as a regression of discount rates on a variable observed at time t. In this case, the minimum variance reduces to

$$\text{var}(g_t) = \frac{1}{\pi}\int_0^{\pi} d\omega \frac{\delta_2^2}{4}|h|^2 S_z;\ \text{with } X_2 = -\frac{1}{2\pi}\int_0^{\pi} d\omega\, \delta_2 |h|^2 S_z.$$

Solving the constraint for δ_2 and substituting in the variance, we obtain

$$\text{var}(g_t) = \frac{1}{\pi}\int_0^{\pi} d\omega\, |h|^2 S_z \left(-X_2 \Big/ \frac{1}{\pi}\int_0^{\pi} d\omega\, |h|^2 S_z\right)^2$$

$$= \left[\mathrm{cov}\left(z_t, \sum_{j=1}^{\infty} \Omega^j n_{t+j}\right) - (1-\Omega)\ \mathrm{cov}(z, \tfrac{P}{d})\ \right]^2 / \mathrm{var}\left(\sum_{j=1}^{\infty} \Omega^j z_{t-j}\right)$$

$$= \left[\mathrm{cov}\left(g_t, \sum_{j=1}^{\infty} \Omega^j z_{t-j}\right) / \mathrm{var}\left(\sum_{j=1}^{\infty} \Omega^j z_{t-j}\right)\right]^2 \mathrm{var}\left(\sum_{j=1}^{\infty} \Omega^j z_{t-j}\right).$$

This is the variance of the fitted value of the regression of g_t on $\sum_{j=1}^{\infty} \Omega^j z_{t-j}$. Thus, the minimum variance of discount rates compatible with the variance decomposition only can also be found by inferring the OLS regression coefficient of discount rates g_t on the variable $\sum_{j=1}^{\infty} \Omega^j z_{t-j}$.

6. Data description

The treasury bill, government bond index, corporate bond index and CPI are from the SBBI database. The real per capita consumption series is the same as in Campbell and Shiller (1988) and is described there. Price-dividend and dividend growth data are based on the CRSP value weighted and equally weighted portfolio returns, with and without dividends, converted to annual frequencies. Thus P_t is the December 31, year t-1 closing price and dividends in year t are the monthly dividends, brought forward from the end of the month in which they are paid to December 31 by reinvestment at the

7. Standard Errors

All the statistics of the paper can be expressed as functions $f(\mu)$, where y_y is a vector of means, variances and covariances of price dividend ratios, dividend growth and discount rates that converges to its expected value μ. These statistics are estimated by using sample moments μ_T in place of the population moment μ. Hansen (1985) shows that $T^{1/2}\left(f(\mu_t)-f(\mu)\right)$ converges in distribution to a normally distributed random vector with mean 0 and covariance matrix $\nabla f(\mu)' V_0 \nabla f(\mu)$, where

$$V_0 = \lim_{J\to\infty} \sum_{j=-J}^{J} \left(1 - \frac{|j|}{J}\right) C(j), \text{ and } C(j) = E\left[(y_t-\mu)(y_{t-j}-\mu)'\right] \quad .$$

∇f is calculated analytically where possible, and otherwise as the numerical derivative of the procedures that calculate test statistics as a function of sample moments. J=5 is used throughout the tables.

Table 1. Summary Statistics for value and equally weighted NYSE portfolios, annual data 1927-1987

	Value weighted			Equally weighted		
	Mean	Std. dev.	Variance	Mean	Std. dev.	Variance
Nom. dividend growth (%)	3.89	13.67		7.73	20.31	
standard errors	1.83	1.93		2.99	4.24	
Real dividend growth (%)	0.81	13.39		4.65	19.35	
standard errors	1.35	1.43		2.55	3.76	
Price/dividend ratio	23.15	5.88	34.55	26.00	7.57	57.37
standard errors	1.69	0.64	7.56	1.99	1.05	15.87

Autocorrelation $\rho(j)$ of dividend growth and sums $1 + 2\sum_{k=1}^{j} \frac{j-k+1}{j+1}\rho(k)$

Portfolio		lag j 1	2	3	4	5	10	15
Nominal	$\rho(j)$	0.03	0.06	0.03	-0.07	-0.11	-0.09	-0.00
value	sum	1.03	1.07	1.11	1.11	1.07	0.95	0.94
weighted	s.e.	0.15	0.22	0.28	0.33	0.35	0.44	0.54
Real	$\rho(j)$	-0.16	-0.10	-0.04	0.01	-0.06	-0.08	0.03
value	sum	0.84	0.72	0.64	0.60	0.55	0.32	0.30
weighted	s.e.	0.12	0.15	0.16	0.18	0.18	0.15	0.17
Nominal	$\rho(j)$	0.28	0.07	-0.09	-0.28	-0.29	-0.15	0.03
equal	sum	1.28	1.42	1.44	1.35	1.18	0.88	0.67
weighted	s.e.	0.19	0.30	0.37	0.40	0.39	0.41	0.38
Real	$\rho(j)$	0.15	0.00	-0.10	-0.22	-0.26	-0.14	0.10
equal	sum	1.15	1.21	1.18	1.08	0.92	0.63	0.51
weighted	s.e.	0.17	0.25	0.30	0.32	0.31	0.29	0.29

Note to table 1:

Sum standard errors are calculated using the Bartlett formula s.e. = $(4j/3T)^{1/2}$ x sum. Calculation of all other standard errors is described in the appendix.

Table 2. Bounds on the variance of price-dividend ratios.

Value Weighted				Equally Weighted				
Assumed mean discount rate nom. E(g)	real E(m)	Variance bound (Actual= 34.55)	Standard error	Assumed mean discount rate nom. E(g)	real E(m)	Variance bound (Actual= 57.37)	Standard error	Note on mean discount rate
8.12	5.04	0.00	35.74	11.50	8.42	0.00	50.07	E(g)min
8.13	5.05	1.32	35.65	11.51	8.43	1.85	49.95	var. bound
8.22	5.14	12.61	34.84	11.60	8.52	17.64	48.79	violated
8.42	5.34	34.55	33.14	11.87	8.79	57.37	45.59	bound=Var P/d
8.51	5.43	43.66	32.38	12.31	9.23	101.84	41.31	
8.51	5.43	43.66	20.77	12.31	9.23	101.84	97.64	var(m)=0
8.95	5.87	76.66	29.29	12.36	9.28	106.12	40.84	
8.95	5.87	76.66	28.71	12.36	9.28	106.12	84.98	var(g)=0
9.00	5.92	79.77	28.96	13.00	9.92	142.75	36.06	
10.00	6.92	119.05	23.68	14.00	10.92	168.10	30.69	
11.00	7.92	134.61	19.94	20.00	16.92	149.05	18.78	
12.00	8.92	139.32	17.15					
13.00	9.92	138.82	15.02					
14.00	10.92	135.76	13.35					
20.00	16.92	107.71	8.23					

Note to table 2:

"Assumed mean discount rate E(g) and E(m)" give assumed values of the mean nominal and real discount rate, respectively, in annual percent units. "Variance bound" gives a calculation of the variance bound (11). "Standard error" gives the standard error of (var(P/d) - variance bound). Data are annual, 1927-1987.

Notes on mean discount rate: "E(g) min" gives the minimum value of the mean discount rate, at which $E(P/d) = \Omega/(1-\Omega)$. "Variance bound violated" notes that the point estimate of the variance bound is violated in these rows. "var(g) = 0" and "var(m) = 0" note the values of the mean discount rate that solve the mean price dividend ratio equation (9) with constant discount rates and nominal or real dividend growth. The first row at this mean discount rate gives the standard error of (var(P/d)-variance bound); the second row at this mean discount rate gives the standard error including the uncertainty of estimating the constant discount rate, and thus provides a test of the constant discount rate model. "bound = Var(P/d)" marks the mean discount rate at which the point estimate of the variance bound is equal to the point estimate of var(P/d).

Table 3. Percent of variance of price dividend ratio due to dividends and discount rates

1. Value weighted portfolio

Mean discount rate		Nominal dividends and discount rate						Real dividends and discount rates					Notes on mean discount rate
E(g)	E(m)	n	g	tb	gb	cb	π	nr	m	tbr	gbr	cbr	
3.89	0.81	∞	$-\infty$	$-\infty$	$-\infty$	$-\infty$	∞	$-\infty$	∞	$-\infty$	∞	∞	E(n)
5.00	1.92	87	13	-354	-150	-154	262	-175	275	-92	112	108	
6.00	2.92	40	60	-171	-68	-70	126	-86	186	-45	58	56	
7.00	3.92	24	76	-106	-39	-41	78	-54	154	-28	39	38	
8.00	4.92	16	84	-74	-25	-26	54	-38	138	-20	29	28	
8.12	5.04	15	85	-71	-24	-25	52	-37	137	-19	28	27	$E(g)_{min}$
8.42	5.34	13	87	-65	-21	-22	47	-34	134	-17	26	25	var(P/d)=bound
8.52	5.44	13	87	-63	-21	-21	46	-33	133	-17	25	24	var(m)=0
8.95	5.87	11	89	-55	-18	-18	40	-29	129	-15	23	22	var(g)=0
9.00	5.92	11	89	-55	-17	-18	40	-29	129	-15	23	22	
9.13	6.05	10	90	-53	-16	-17	38	-28	128	-14	22	21	E(r)
10.00	6.92	8	92	-42	-12	-13	31	-23	123	-12	18	18	
12.00	8.92	4	96	-27	-6	-7	19	-15	115	-8	13	13	
15.00	11.92	2	98	-16	-2	-2	11	-9	109	-5	9	9	
20.00	16.92	0	100	-8	-0	-0	5	-5	105	-2	5	5	

Standard errors

E(g)	E(m)	n	g	tb	gb	cb	π	nr	m	tbr	gbr	cbr	Notes
5.00	1.92	127	127	109	102	105	132	49	49	71	150	160	
6.00	2.92	61	61	52	50	51	64	24	24	35	74	78	
7.00	3.92	38	38	32	31	33	40	15	15	22	47	50	
8.00	4.92	26	26	22	22	23	28	11	11	16	33	35	
8.12	5.04	25	25	22	21	22	27	11	11	16	32	34	$E(g)_{min}$
8.42	5.34	23	23	20	19	20	25	10	10	14	29	31	var(P/d)=bound
8.52	5.44	22	22	19	19	20	24	10	10	14	28	30	var(m)=0
8.95	5.87	19	19	17	17	18	22	9	9	12	25	27	var(g)=0
9.00	5.92	19	19	17	17	17	21	9	9	12	25	27	
9.13	6.05	18	18	16	16	17	21	8	8	12	24	26	E(r)
10.00	6.92	15	15	13	13	14	17	7	7	10	20	21	
12.00	8.92	9	9	8	9	9	11	5	5	7	13	14	
15.00	11.92	5	5	5	5	6	7	4	4	4	8	9	
20.00	16.92	2	2	2	3	3	4	2	2	2	4	5	

(Table 3 Continues)

(Table 3 continued)

2. Equally weighted portfolio

Mean discount rate		Nominal dividends and discount rate						Real dividends and discount rates					Notes on mean discount rate
E(g)	E(m)	n	g	tb	gb	cb	π	nr	m	tbr	gbr	cbr	
7.73	4.65	∞	-∞	-∞	-∞	-∞	∞	∞	-∞	∞	∞	∞	E(n)
9.00	5.92	273	-173	-59	-89	-81	132	142	-42	73	43	51	
10.00	6.92	139	-39	-30	-45	-41	66	73	27	37	22	26	
11.00	7.92	88	12	-19	-28	-26	41	47	53	23	14	16	
11.50	8.42	73	27	-15	-23	-21	34	39	61	19	11	13	$E(g)_{min}$
11.87	8.79	64	36	-14	-20	-18	30	34	66	16	10	11	bound=var(P/d)
12.00	8.92	62	38	-13	-19	-18	29	33	67	16	9	11	
12.23	9.15	57	43	-12	-18	-16	26	31	69	14	9	10	E(r)
12.32	9.24	56	44	-12	-17	-16	26	30	70	14	9	10	var(m)=0
12.38	9.30	55	45	-12	-17	-16	25	30	70	14	8	10	var(g)=0
13.00	9.92	46	54	-10	-14	-13	21	25	75	11	7	8	
15.00	11.92	28	72	-6	-8	-8	12	16	84	6	4	4	
20.00	16.92	11	89	-2	-3	-3	4	7	93	2	1	1	
						Standard errors							
9.00	5.92	160	160	92	60	67	56	157	157	81	72	80	
10.00	6.92	86	86	48	31	34	30	84	84	42	37	41	
11.00	7.92	57	57	31	20	22	19	57	57	27	24	27	
11.50	8.42	49	49	26	17	18	16	48	48	23	20	22	$E(g)_{min}$
11.87	8.79	44	44	23	15	16	14	43	43	20	18	20	bound=var(P/d)
12.00	8.92	42	42	22	14	16	14	42	42	19	17	19	
12.23	9.15	40	40	20	13	15	13	39	39	18	16	18	E(r)
12.32	9.24	39	39	20	13	14	13	39	39	18	16	17	var(m)=0
12.38	9.30	38	38	20	13	14	13	38	38	17	15	17	var(g)=0
13.00	9.92	33	33	17	11	12	11	33	33	15	13	14	
15.00	11.92	22	22	10	7	7	7	22	22	9	8	9	
20.00	16.92	11	11	5	3	3	3	12	12	4	4	4	

"Mean discount rate" gives the assumed mean discount rate, in annual percent units. g=nominal, m=real. The quantities in the columns "Nominal.." and "Real dividends and discount rates" give the percent of the variance of the price dividend ratio attributable to nominal or real dividend growth (n, nr), unobserved nominal or real discount rates (g, m; these columns are 100-the n or nr columns), nominal and real treasury bill (tb, tbr), government bond (gb, gbr), corporate bond (cb, cbr) returns, and inflation (π). These are calculated by equation (12).

Notes on mean discount rate: "E(n)" marks the mean discount rate equal to the mean dividend growth rate, at which $\Omega = 1$ and all the formulas diverge to ∞ or -∞. "$E(g)_{min}$" marks the mean discount rate at which $E(P/d) = \Omega/(1-\Omega)$. Below this value, the mean discount rate equation is violated. "Var(m) = 0" and "var(g) = 0 " mark the mean discount rate at which the mean price dividend formula (9) holds with constant discount rates, and nominal or real dividend growth. "Var(P/d)=bound" marks the mean discount rate at which the variance bound (11) holds exactly. "E(r)" marks the mean discount rate equal to the mean log return.

Table 4. Mean price-dividend ratio, variance bound and variance decomposition with discount rate = return.

1. Mean price dividend ratio and variance bound

	Value Weighted			Equally Weighted		
	mean	variance	bound	mean	variance	bound
Sample	23.15	34.55		26.00	57.37	
Standard error	1.69	7.56		1.99	15.87	
Predicted	20.84		43.00	25.85		91.86
Standard error	1.15		22.00	3.51		77.81
S.e., sample-pred.	1.96			4.31		

2. Variance decomposition

		nominal dividends	nominal return	real dividends	real return	total
Value Weighted	Percent of Var(P/d)	10.48	86.06	-27.92	124.46	96.54
	Standard error	18.33	17.38	8.42	28.53	
Equally Weighted	Percent of Var(P/d)	57.32	50.81	30.87	77.27	108.14
	Standard error	39.68	16.31	39.40	18.38	
EW, disc. rate=VW	Percent of Var(P/d)	245.24	274.53	127.31	392.46	519.77
	Standard error	144.83	68.74	141.51	75.54	

Note to table 4:

"Sample" gives the mean and variance of the price dividend ratio, 1927-1987. The "predicted mean" is calculated from equation (9) with returns in the place of discount rates. The "predicted" "bound" is calculated from equation (11). The percent of the variance of the price-dividend ratio accounted for by dividends and returns are calculated by equation (12). "Total" is the total variance accounted for.

Table 5. Variance bound and variance decomposition using consumption based discount rates.

1. Value weighted portfolio

Utility Parameters		Mean discount rate		Variance Decomposition % var(P/d) due to					Std. errors				Variance Bound Var(P/d)=35.28	
ρ	α	E(g)	E(m)	n	g	nr	m	total	n	g	nr	m	bound	s.e.
0.99	2.96	8.72	5.63	13	-67	-35	-20	-55	16	35	10	17	38.6	152.3
0.98	2.30	8.71	5.62	13	-63	-35	-15	-50	16	32	10	13	37.5	140.5
0.97	1.65	8.71	5.62	13	-59	-35	-11	-46	16	29	10	10	37.4	131.5
0.96	0.99	8.72	5.63	13	-54	-35	-7	-42	16	25	10	6	38.4	126.6
0.95	0.33	8.74	5.65	13	-50	-35	-2	-37	16	22	10	2	40.4	126.2

2. Equally weighted portfolio

Utility Parameters		Mean discount rate		Variance Decomposition %var(P/d) due to					Std. errors				Variance Bound Var(P/d)=58.30	
ρ	α	E(g)	E(m)	n	g	nr	m	total	n	g	nr	m	bound	s.e.
0.99	5.83	13.20	10.11	48	-59	25	-37	-11	34	30	34	26	132.9	134.7
0.98	5.12	13.10	10.01	50	-56	26	-33	-7	35	27	35	23	127.6	139.9
0.97	4.40	13.01	9.91	51	-53	27	-29	-2	36	25	36	21	122.3	145.3
0.96	3.68	12.92	9.83	52	-49	27	-25	3	37	22	36	18	117.2	151.5
0.95	2.96	12.85	9.76	53	-46	28	-20	8	37	20	37	14	112.2	159.1

Note to table 5:

Discount rates are generated by the consumption based model: the real discount rate is calculated by $m_t = -\ln(\rho) + \alpha\ln(c_t/c_{t-1})$ and the nominal discount rate is $g_t = m_t + \pi_t$ (π_t=inflation). For a given choice of ρ, α is calculated to satisfy the mean price dividend ratio (9) with this real discount factor m_t and real dividend growth. The variance decomposition gives the percent of the variance of the price dividend ratio due to dividends and discount rates, calculated from the decomposition (12). The variance bound is calculated by (11). The standard errors of the variance bound take the preference parameters ρ and α as fixed, and include the uncertainty of estimating E(m) from consumption growth rates.

Table 6. Minimum standard deviation of discount rates required to satisfy the mean price dividend ratio, variance bound, and variance decomposition.

1. Nominal Value Weighted Portfolio

Mean disc. rate	Constraint									Notes on mean discount rate
	Mean P/d			Var. decomp.			Both			
E(g)	$\sigma(g)$	s.e.	ρ	$\sigma(g)$	s.e.	ρ	$\sigma(g)$	s.e.	ρ	
3.89				0						E(n)
4.00				0.01	0.20	1.00				
6.00				0.91	0.57	1.00				
8.00				2.64	0.83	0.99				
8.12	13.56		0.03	2.76	0.85	0.99				$E(g)_{min}$
8.12	10.79	10813	0.28	2.76	0.85	0.99				
8.13	7.56	49.87	0.65	2.78	0.85	0.99				
8.41	1.37	2.17	1.00	3.08	0.90	0.99				Var(P/d)-bound
8.50	1.02	1.56	1.00	3.17	0.91	0.99	18.97	27.81	0.20	
8.60	0.69	1.64	1.00	3.28	0.93	0.99	13.48	15.18	0.49	
8.87	0.00	2.23	1.00	3.59	0.98	0.99	7.99	5.34	0.86	var(g)-0
9.00	0.29	1.35	1.00	3.74	1.00	0.99	7.01	4.23	0.91	
9.13	0.56	1.30	1.00	3.89	1.03	0.99	6.43	3.64	0.93	E(r)
10.00	2.21	1.17	1.00	4.97	1.20	0.99	5.52	2.25	0.98	
12.00	5.98	1.21	1.00	7.84	1.66	0.99	7.88	1.41	0.99	
15.00	12.38	1.37	1.00	13.08	2.43	0.99	13.97	1.27	1.00	
20.00	25.15	1.79	1.00	24.25	3.81	0.98	27.00	1.67	1.00	

2. Real Value Weighted Portfolio

Mean disc. rate	Constraint									Notes on mean discount rate
	Mean P/d			Var. decomp.			Both			
E(m)	$\sigma(m)$	s.e.	ρ	$\sigma(m)$	s.e.	ρ	$\sigma(m)$	s.e.	ρ	
0.81										E(nr)
0.92				0.15	0.10	1.00				
2.92				1.40	0.54	1.00				
4.92				3.25	1.04	0.99				
5.04	13.28		-0.15	3.38	1.07	0.99				$E(m)_{min}$
5.05	6.60	72.42	0.42	3.39	1.07	0.99				
5.06	4.72	37.84	0.65	3.40	1.08	0.99				
5.10	1.89	10.35	0.94	3.44	1.09	0.99				
5.20	0.57	2.28	1.00	3.55	1.11	0.99				
5.33	0.00	1.21	1.00	3.70	1.15	0.99				var(m)-0
5.35	0.06	1.43	1.00	3.72	1.15	0.99	18.16	1144	0.15	$E(m)_{bound}$
5.40	0.21	1.12	1.00	3.78	1.17	0.99	12.39	25.48	0.49	
5.50	0.47	1.02	1.00	3.89	1.19	0.99	8.28	8.69	0.81	
5.92	1.38	1.04	1.00	4.38	1.30	0.99	5.48	2.55	0.97	
6.05	1.62	1.02	1.00	4.53	1.34	0.99	5.33	2.21	0.97	E(rr)
6.92	3.20	0.96	1.00	5.64	1.58	0.99	5.70	1.84	0.99	
8.92	6.93	1.02	1.00	8.53	2.13	0.99	8.78	1.59	0.99	
11.92	13.32	1.21	1.00	13.77	2.98	0.99	15.13	1.69	1.00	
16.92	26.12	1.66	1.00	24.90	4.43	0.98	28.30	2.22	1.00	

(Table 6 continued)

3. Nominal Equally Weighted Portfolio

Mean disc. rate	Constraint									Notes on mean discount rate
	Mean P/d			Var. decomp.			Both			
E(g)	σ(g)	s.e.	ρ	σ(g)	s.e.	ρ	σ(g)	s.e.	ρ	
7.73										E(n)
8.00				0.32	0.67	1.00				
9.00				0.04	1.06	1.00				
10.00				0.83	0.96	0.99				
11.00				2.04	0.73	0.99				
11.50	20.15		0.26	2.76	0.64	0.99				$E(g)_{min}$
11.50	18.60	192917	0.36	2.76	0.64	0.99				
11.50	17.79	42631	0.41	2.76	0.64	0.99				
11.60	7.50	10.17	0.89	2.91	0.63	0.99				
11.87	3.03	2.33	0.99	3.32	0.62	0.99				var(P/d)=bound
12.00	2.23	1.99	0.99	3.53	0.63	0.99	30.78	66.79	0.10	
12.10	1.80	1.88	0.99	3.70	0.63	0.99	25.83	17.28	0.25	
12.20	1.45	1.84	1.00	3.86	0.65	0.99	22.42	13.57	0.35	
12.77	0.00	6.07	1.00	4.85	0.79	0.99	13.40	7.20	0.69	var(g)=0 nom.
13.00	0.48	1.83	1.00	5.26	0.87	0.99	11.93	6.41	0.76	
15.00	4.38	1.80	1.00	9.35	1.93	0.98	10.93	4.64	0.93	
20.00	15.76	1.87	1.00	22.75	5.29	0.97	23.02	6.64	0.96	

4. Real Equally Weighted Portfolio

Mean disc. rate	Constraint									Notes on mean discount rate
	Mean P/d			Var. decomp.			Both			
E(m)	σ(m)	s.e.	ρ	σ(m)	s.e.	ρ	σ(m)	s.e.	ρ	
4.65				0						E(nr)
4.92				0.16	0.60	1.00				
6.92				1.20	0.94	0.99				
7.92				2.45	0.87	0.99				
8.42	19.19		0.14	3.18	0.88	0.99				$E(g)_{min}$
8.43	13.70	2252	0.50	3.19	0.88	0.99				
8.50	6.54	12.58	0.86	3.29	0.89	0.99				
8.60	3.60	5.98	0.96	3.45	0.90	0.99				
8.79	1.78	2.61	0.99	3.74	0.93	0.99				var(P/d)=bound
8.92	1.18	2.56	1.00	3.96	0.95	0.99	24.63	22.85	0.24	
9.12	0.56	2.50	1.00	4.29	0.99	0.99	17.25	12.05	0.50	
9.35	0.00	5.60	1.00	4.67	1.05	0.99	13.23	8.02	0.67	var(m)=0
9.92	1.18	2.37	1.00	5.69	1.25	0.99	9.62	5.19	0.86	
10.92	3.07	2.30	1.00	7.63	1.70	0.99	9.24	4.14	0.93	
11.92	5.00	2.26	1.00	9.76	2.24	0.98	10.59	4.01	0.95	
16.92	16.38	2.28	1.00	23.07	5.45	0.97	23.14	6.11	0.97	

(Table 6 continued)

Note to table 6:

"Mean discount rate" gives the assumed mean nominal or real discount rate, in annual percent units. "Constraint" indicates which constraint is imposed. "$\sigma(g)$" or "$\sigma(m)$" gives the minimum standard deviation of nominal or real discount rates that satisfy the given constraint. "s.e." gives standard errors of the standard deviation of discount rates. "ρ" gives the first order autocorrelation coefficient of the variance minimizing discount rate.

To satisfy the "Mean" constraint, the discount rate g together with data on dividend growth rates n and price dividend ratios P/d must satisfy

$$(9) \qquad E\left(\frac{P}{d}\right) = \frac{\Omega}{1-\Omega} + \frac{\Omega}{2(1-\Omega)^2} \sum_{j=-\infty}^{\infty} \Omega^{|j|} \operatorname{cov}(n_t - g_t,\ n_{t-j} - g_{t+j})$$

To satisfy the "Var" constraint, it must satisfy the variance decomposition

$$(12) \quad \operatorname{var}\left(\frac{P}{d}\right) = \frac{1}{1-\Omega} \operatorname{cov}\left(\frac{P_t}{d_t}, \sum_{j=1}^{\infty} \Omega^j n_{t+j}\right) + \frac{1}{1-\Omega} \operatorname{cov}\left(\frac{P_t}{d_t}, \sum_{j=1}^{\infty} \Omega^j\ -g_{t+j}\right).$$

Notes on mean discount rate:

"E(n)" or "E(nr)" marks the mean discount rate E(g) or E(m) equal to the mean dividend growth rate, at which $\Omega = 1$ and all the formulas diverge to ∞ or $-\infty$. "$E(g)_{min}$" and "$E(m)_{min}$" mark the mean discount rate at which $E(P/d) = \Omega/(1-\Omega)$; it is the minimum value at which the mean constraint can hold. "var(g) = 0" marks the mean discount rate at which the mean price dividend constraint holds with constant discount rates. "var(P/d)=bound" marks the mean discount rate at which the variance bound (12) holds exactly. "E(r)" marks the mean log return. The "var(g)=0" value of the mean discount rate is slightly different from that in other tables because the covariances are down weighted here to insure a positive definite spectral density matrix.

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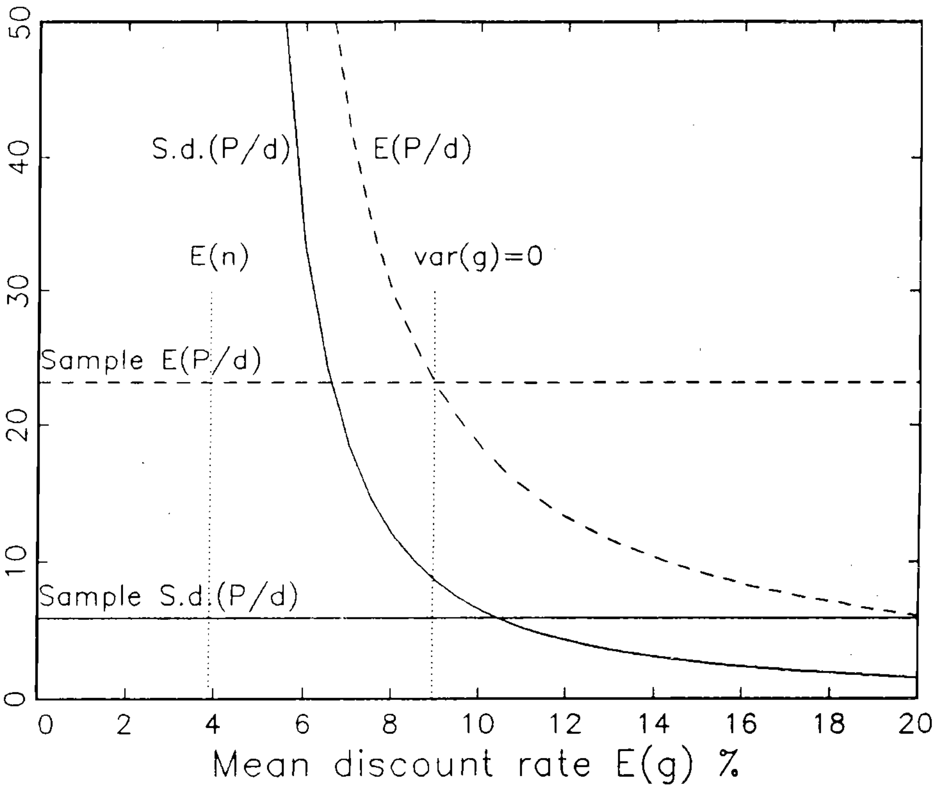

Fig. 1. Mean price dividend ratio and variance bound with constant discount rates, nominal value weighted NYSE portfolio 1927-1987.

For each assumed value of the mean discount rate E(g), the dashed curve reports the predicted mean price dividend ratio,

$$E\left(\frac{P}{d}\right) = \frac{\Omega}{1-\Omega} + \frac{\Omega}{2(1-\Omega)^2} \sum_{j=-15}^{15} \Omega^{|j|} \operatorname{cov}(n_t, n_{t+j})$$

where $\Omega = e^{E(n)-E(g)}$, n = log nominal dividend growth. The solid curve reports the variance bound

$$\operatorname{var}\left(\frac{P}{d}\right) \leq \frac{\Omega}{(1-\Omega^2)\,(1-\Omega)^2} \sum_{j=-15}^{15} \Omega^{|j|} \operatorname{cov}(n_t, n_{t+j})$$

in standard deviation units. The horizontal lines "E(P/d)" and "S.d.(P/d)" give the sample mean and standard deviation for reference. "E(n)" marks the mean discount rate equal to mean dividend growth, "var(g)=0" marks the mean discount rate at which the mean price dividend ratio equals its sample value.

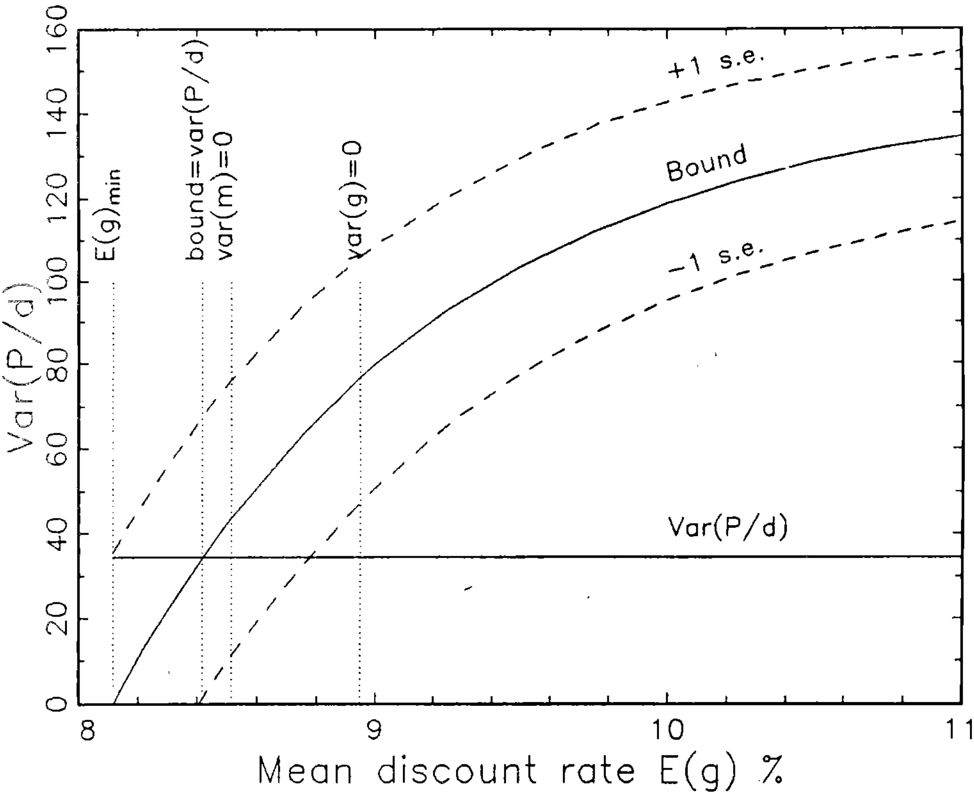

Fig. 2: Variance bound with no restrictions on discount rates, nominal value weighted NYSE portfolio 1927-1987.

At each assumed mean discount rate E(g), "bound" gives the variance bound,

$$\mathrm{var}\left(\frac{P}{d}\right) \le \frac{2\Omega}{1-\Omega^2}\left(E\left(\frac{P}{d}\right) - \frac{\Omega}{1-\Omega}\right)$$

where $\Omega = e^{E(n)-E(g)}$, n = log dividend growth. The +/- 1 standard error lines show standard errors for (var(P/d) - bound). The "Var(P/d)" line gives the sample variance of the price-dividend ratio, for reference.

"$E(g)_{min}$ " gives the minimum value of the mean discount rate, at which $E(P/d) = \Omega/(1-\Omega)$. "var(m) = 0" and "var(g) = 0" note the values of the mean discount rate that solve the mean price dividend ratio equation with constant discount rates. "Bound = Var(P/d)" marks the mean discount rate at which the point estimate of the variance bound is equal to the point estimate of var(P/d).

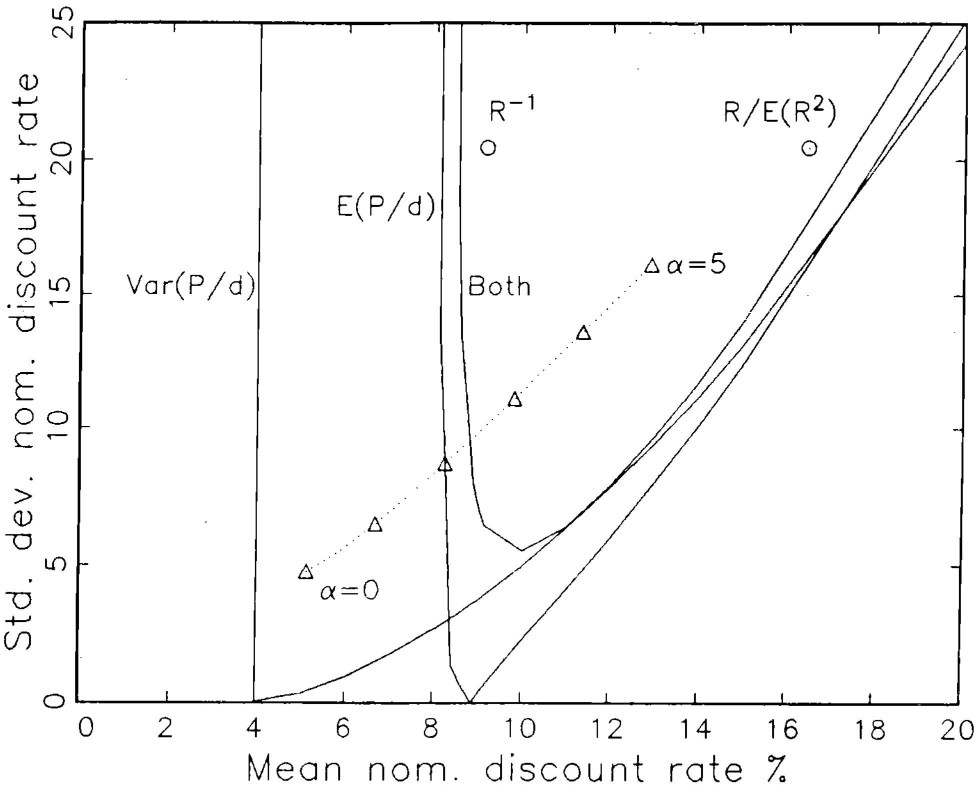

Fig. 3. Bounds on the mean and standard deviation of *nominal* discount rates that satisfy the mean price dividend ratio, variance bound and variance decomposition, value weighted NYSE 1927-1987, together with consumption based discount rates and return statistics.

The bound marked "E(P/d)" gives the minimum standard deviation of discount rates that, together with data on dividend growth rates, satisfy the sample mean price dividend ratio. The bound marked "Var(P/d)" gives the minimum standard deviation of discount rates that satisfy the variance decomposition. The bound marked "Both" gives the minimum standard deviation of discount rates that satisfy both conditions. See note to table 6 for formulas.

The line marked "$\alpha=0$" .. "$\alpha=5$" gives the mean and standard deviation of consumption based discount rates, using subjective discount factor $\rho = .98$ and the indicated risk aversion α. "R^{-1}" gives the mean and standard deviation of the log inverse return, "$R/E(R^2)$" gives the mean and standard deviation of the log of this quantity.

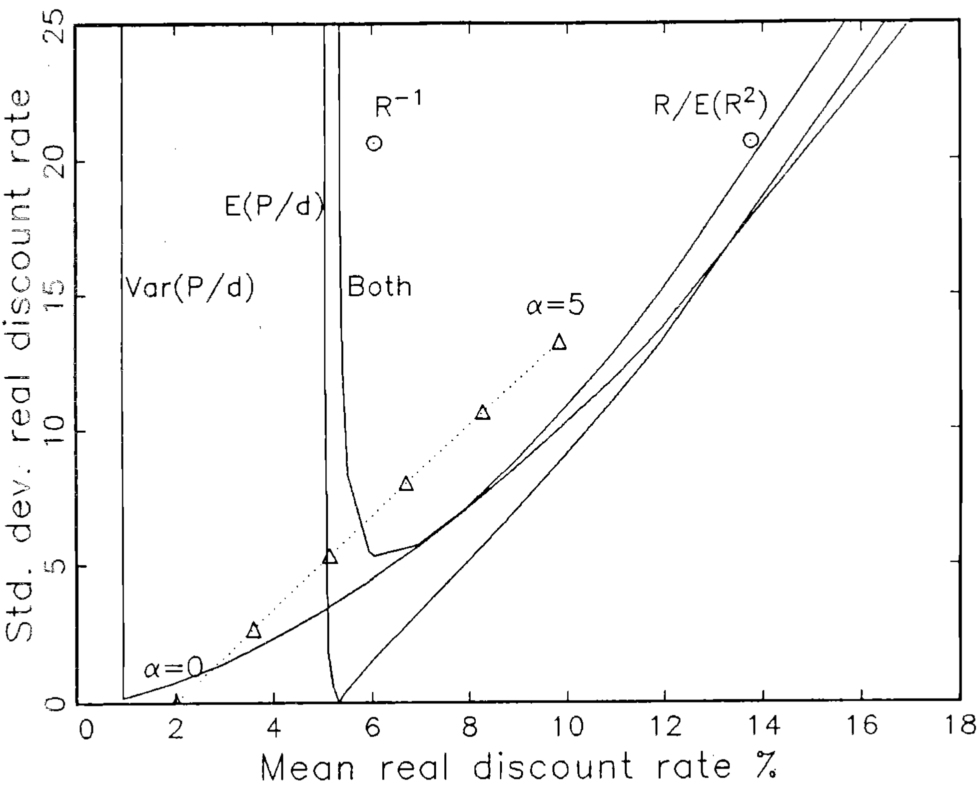

Fig. 4. Bounds on the mean and standard deviation of *real* discount rates that satisfy the mean price dividend ratio, variance bound and variance decomposition, value weighted NYSE 1927-1987, together with consumption based discount rates and return statistics.

The bound marked "E(P/d)" gives the minimum standard deviation of real discount rates that, together with data on real dividend growth rates nr, satisfy the sample mean price dividend ratio. The bound marked "Var(P/d)" gives the minimum standard deviation of real discount rates that satisfy the variance decomposition. The bound marked "Both" gives the minimum standard deviation of real discount rates that satisfy both conditions. See note to table 6 for formulas.

The line marked "α=0" .. "α=5" gives the mean and standard deviation of real consumption based discount rates, using constant relative risk aversion utility function, ρ = .98 and the indicated risk aversion = α. "R^{-1}" gives the mean and standard deviation of the log inverse return, "R/E(R^2)" gives the mean and standard deviation of the log of this quantity.

FOOTNOTES

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[1] Precisely, they are *derived* using no assumptions beyond stationarity. They are *calculated* with an extra assumption that the temporal dependence in dividend growth and discount rates is adequately captured by 15 positive and negative covariances in annual data.

[2] The model (13) generates expected returns that are equal to the reference return plus a constant risk premium when returns and discount rates are jointly lognormal and homoskedastic. (Essentially the same equations can be derived with the second order Taylor expansion used for the price-dividend ratio.) With these assumptions,

$$1 = E_{t-1}(\gamma_t R_t)$$

implies

$$0 = E_{t-1}(r_t - g_t) + 1/2 \text{ var}(r_t - g_t),$$

where $r = \ln(R)$, $g = -\ln(\gamma)$. Using (20),

$$0 = E_{t-1}(r_t - r_t^0) - E(\epsilon_t) + 1/2 \text{ var}(r_t - r_t^0) + 1/2 \text{ var}(\epsilon_t) - \text{cov}(r_t - r_t^0, \epsilon_t).$$

This equation must hold for $r = r^0$ as well, so ϵ must obey

$$E(\epsilon_t) = 1/2 \text{ var}(\epsilon_t)$$

and thus the expected return equals the expected reference return plus a risk premium that is constant over time and depends on the covariance of the return with the discount rate,

$$E_{t-1}(r_t) = E_{t-1}(r_t^0) + 1/2 \text{ var}(r_t - r_t^0) - \text{cov}(r_t - r_t^0, \epsilon_t).$$

[3] For the purposes of the variance decomposition (12), a discount rate model $g_t = r_t^0$ + constant could be used instead, and in this model the mean discount rate would be identified from the mean price dividend ratio. In this model it is also true that $\mathrm{Cov}(P_t/d_t, g_{t+j}) = \mathrm{Cov}(P_t/d_t, r_{t+j}^0)$. Like the constant discount rate, it also predicts risk premia that are constant across assets, and is not used for this reason.

Asset return information cannot be used to identify the mean discount rate. The mean nominal discount *factor* may be deduced from the mean of a nominal conditionally risk free rate (bond) as follows:

$$1 = E_t(\gamma_{t+1}R_{t+1}) \rightarrow E_t(\gamma_{t+1}) = 1/R_{t+1} \rightarrow E(\gamma) = E(1/R).$$

However, the mean discount *rate* cannot be so identified. For example, consider the pricing equation for any asset return,

$$1 = E_t(\gamma_{t+1}R_{t+1})$$

Approximating the joint distribution of γ_t R_t as a lognormal (essentially the same equation can be derived by a second order Taylor expansion as for the price-dividend ratio),

$$E_t(r_{t+1}) = E_t(g_{t+1}) + 1/2\ \mathrm{var}_t(r_{t+1} - g_{t+1})$$

where $r_{t+1} = \ln(R_{t+1})$. Hence, even in the case of a riskless rate,

$$E(r^f) = E(g) + 1/2\ \mathrm{var}(g)$$

[4] This is not the only model to satisfy the Euler equation by construction. $\gamma_t = R_t / E_{t-1}(R_t^2) + \epsilon_t$ for any ϵ_t (including $\epsilon_t = 0$) such that $E_{t-1}(\epsilon_t R_t) = 0$ also satisfies the Euler equation by construction. In fact, *all* discount rates γ_t that satisfy the Euler equation may be expressed in this way (see Hansen and Richard (1987) and Hansen and Jagannathan (1989)).

[5] Other utility functions may help resolve the puzzle. For example, Ferson and Constantinedes (1989) claim that a habit persistence utility function not only generates high variance of discount rates, but passes an Euler equation test using price-dividend ratios as instruments, in which case the discount rates may have the required predictability.