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INTEGRABILITY AND GENERALIZED SEPARABILITY

Thibault Fally

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ABSTRACT

This paper examines demand systems where the demand for a good depends only on its own price, consumer income, and a single aggregator synthesizing information on all other prices. This generalizes directly-separable preferences where the Lagrange multiplier provides such an aggregator. As indicated by Gorman (1972), symmetry of the Slutsky substitution terms implies that such demand can take only one of two simple forms. Conversely, here we show that only weak conditions ensure that such demand systems are integrable, i.e. can be derived from the maximization of a well-behaved utility function. This paper further studies useful properties and applications of these demand systems.

Thibault Fally
Department of Agricultural and Resource Economics
University of California, Berkeley
301 Giannini Hall
Berkeley, CA 94720-3310
and NBER
fally@berkeley.edu

1 Introduction

The integrability problem, which consists in characterizing demand systems that can be rationalized and derived from utility functions, has long been a central issue in economic theory. Its earliest contributions date from Antonelli (1886), with applications to various fields, including micro and macroeconomics, econometrics, industrial organization and international trade. Theorists have provided broad sufficient and necessary conditions for demand patterns to be integrable, notably Hurwicz and Uzawa (1971), who provide conditions based on the Slutsky substitution matrix, which must be symmetric and negative semi-definite for all prices and income levels.

While very general, the Hurwicz and Uzawa (1971) integrability conditions lack practicality. Perhaps a consequence is that applied theorists and practitioners have often focused on less general cases to ensure both tractability and rationality. In particular, one often focuses on directly-separable, indirectly-separable or quasi-linear preferences. An attractive feature of these preferences is that demand depends only on a few variables, namely consumer income, a good's own price, and a single aggregator (scalar) that is itself a function of the vector of prices and income. Such an aggregator can be, for instance, a price index (e.g. with constant elasticity of substitution preferences) or the marginal utility of money (directly-separable preferences). These preferences, however, have properties that may be undesirable and too restrictive in terms of income and price effects. For instance, quasi-linear preferences suppress income effects, while direct separability implies that income elasticities and price elasticities are proportional across goods ("Pigou's law"), a testable prediction that has been rejected by Deaton (1974).

In this paper, we characterize more general demand systems that retain a key practical property of these widely-used demand systems: the existence of a single price aggregator, a feature that is very useful to welfare analysis and models of monopolistic and oligopolistic competition. Following Pollak (1972), we refer to such a demand systems as "generalized separable" whereby demand satisfies:

$$q_i = q_i(p_i/w, \Lambda) \tag{1}$$

for each good i, where w refers to consumer income (total outlays), p_i the price of good i, and Λ is a scalar (aggregator) function of all prices and income. Without providing a complete proof, Gorman (1972, 1995) indicates that such demand system can take either of two main forms[1] if

[1]There are other cases that can be ruled out under additional restrictions on price sensitivity.

we impose the Slutsky matrix to be symmetric:

$$q_i = \frac{D_i(F(\Lambda)p_i/w)}{H(\Lambda)} \qquad \text{or:} \qquad q_i = A_i(\Lambda)(p_i/w)^{-\sigma(\Lambda)} \tag{2}$$

where, in both cases, Λ is a scalar that is adjusted so that the budget constraint is satisfied, and can thus be defined as an implicit function of prices and income. As little is known about these demand systems, they have not been used in the applied literature in spite of their usefulness.[2]

The objectives of this paper are twofold. First, I provide conditions under which demand with generalized separability have to take these two forms (i.e. a formal statement of Gorman's claims mentioned above). These functional forms however do not imply that these demand systems are well defined and integrable, as shown with counter-examples in each case. The second and main contribution is to show that any demand function that takes the form in equation (2) can be integrated under weak restrictions. These restrictions ensure that the Slutsky substitution matrix not only is symmetric but also negative semi-definite, or equivalently, a quasi-concave and well-defined utility function.

In the first case, where demand satisfies $q_i = D_i(F(\Lambda)p_i/w)/H(\Lambda)$, integrability is guaranteed if D_i is monotonically decreasing in p_i, and demand q_i is decreasing in Λ. We will refer to this case as a "generalized Gorman-Pollak" demand system, which generalizes the demand systems mentioned above. It corresponds to directly-additive utility when $H(\Lambda)$ is constant and indirectly-additive utility when $F(\Lambda)$ is constant. This also generalizes the results of Matsuyama and Ushchev (2017), who focus on homothetic demand where $F(\Lambda) = 1/H(\Lambda) = \Lambda$. Income and price elasticities both depend on the functional form chosen for D_i, which can be very flexible; demand and price shifters $H(\Lambda)$ and $F(\Lambda)$ also influence income effects and depend flexibly on the price aggregator.

In the second case, with common price elasticities across goods and $q_i = A_i(\Lambda)(p_i/w)^{-\sigma(\Lambda)}$, integrability requires that the demand shifters $A_i(\Lambda)$ increase quickly enough in Λ to ensure that the associated utility is well defined. In that case, there is a one-to-one mapping between Λ and utility. Notice that the price elasticity $\sigma(\Lambda)$ does not have to remain constant or monotonic across indifference curves; it can increase or decrease with Λ, i.e. indifference curves can become flatter or more convex as income goes up, as long as they do not cross. We will refer to that case as "generalized non-homothetic CES". This second case features Allen-Uzawa substitution elasticities that do not vary across goods but may vary with the demand aggregator Λ, and generalizes implicitly-additive utility functions previously defined by (Comin et al., 2015) who impose a constant elasticity of substitution $\sigma(\Lambda) = \sigma$. Relative to Gorman-Pollak demand, this

[2] There are a few recent exceptions, including Bertoletti and Etro (2017a) for the first case, Comin et al. (2015) and Matsuyama (2015) for the second case with homogeneous $\sigma(\Lambda) = \sigma$.

case allows for more flexible income patterns, but less flexible price effects.[3]

The demand systems in both cases yield various applications. They are particularly useful in the case of monopolistic competition. In the limit where firms have small market shares, they choose their price by taking as given other prices and quantities. It is then practical to have a single industry-wide indicator Λ that uniquely determines the locus of the demand curve for a good with respect to its own price.[4]

With the first type of demand system, the generalized Gorman-Pollak form, the price aggregator Λ can be interpreted as an index of tightness of the budget constraint,[5] or alternatively as an index of the toughness of competition in a model with firms. It may shift downward or upward along demand curves as income and competition grows, with flexible implications for markups. These demand functions encompass most examples from Mrázová and Neary (2013), e.g. bi-power and inverse bi-power demand functions, or Weyl and Fabinger (2013), e.g. Bulow-Pfleiderer demand. With iso-elastic functions H and D_i, they coincide with the self-dual addilog demand systems (Houthakker, 1965) and extend the constant relative income elasticities (CRIE) used for instance in Fieler (2011) and Caron et al. (2014). The generalized Gorman-Pollak form can also generate choke prices (as demand D_i for a good i goes to zero) which can be expressed as a simple function of income and the price aggregator, with a functional form that is again more flexible than commonly used in macroeconomics and international trade. In particular, this form can be used to generalize the results of Bertoletti and Etro (2017b) and Bertoletti et al. (2016) in which the choke price is proportional to income.

Applications of the second type of demand, with generalized non-homothetic CES, remain very tractable and empirically relevant, because they allow for heterogeneous price elasticities across the income distribution. Several studies (such as Handbury, 2013 and Faber and Fally, 2017) based on expenditure surveys and scanner data have shown that price elasticities vary significantly with income. Handbury (2013) and Faber and Fally (2017) model income effects in the elasticity of substitution by relying on a numeraire good. With generalized non-homothetic CES, we can instead generate such relationship between income and the elasticity of substitution through utility, without relying on a numeraire, while retaining a common elasticity $\sigma(\Lambda)$ across goods for a given income level.

The paper is also related to a vast literature studying functional forms restrictions of utility

[3]Note that implicitly-additive utility depends on a single aggregator only in the case where the Allen-Uzawa elasticity of substitution is common across goods. In other cases, such as Kimball (1995), two aggregators are needed. See also Preckel et al. (2005) for another form of implicitly additive utility and generalization of CES preferences.

[4]Bertoletti and Etro (2017a) recent work formalizes this insight and covers this demand system as an example.

[5]Aggregator Λ is implicitly determined by the budget constraint and its role is very similar to the Lagrange multiplier under directly-separable preferences.

and demand systems. Ligon (2016) focuses on cases where the aggregator Λ corresponds to the Lagrange multiplier associated with the budget constraint, and shows that $\Lambda-$separability implies directly-additive utility with even more specific functional forms. Nocke and Schutz (2017) study the ("quasi-") integrability of quasi-linear demand systems, i.e. without income effects. Atkin et al. (2018) examine quasi-separable preferences (following Gorman, 1970, 1995) and applications to welfare estimation using Engel curves. The discussion on the existence of aggregators also mirrors the restrictions associated with the rank of a demand system (Gorman, 1981; Lewbel, 1991, 2010; LaFrance and Pope, 2006; Lewbel and Pendakur, 2009). The rank of a demand system corresponds to the number of vectors and homothetic price aggregators needed to recover Engel curves (see Lewbel, 1991). Here, the single aggregator Λ is generally not homogeneous of degree one in prices (it also depends on income) and the two demand systems studied here do not have restrictions in terms of rank. Finally, Blackorby et al. (1978) study functional forms implied by various definitions of separability, and find that the same functional structure as with generalized non-homothetic CES is obtained when imposing stronger forms of separability that imply equality among Allen-Uzawa elasticities of substitution.

The remainder of the paper proceeds as follows. Section 2 examines the functional forms imposed by generalized separability. Section 3 provides sufficient conditions to ensure integrability for each type of demand. Section 4 discusses key properties of these demand systems. Section 5 concludes by discussing various applications.

2 Functional Forms under Generalized Separability

Additively-separable utility allows us to obtain demand as a simple function of a good's own price p_i and a single aggregator, the Lagrange multiplier. While practical, both direct and indirect separability put strong constraints on the structure of demand, such as a tight relationship between price elasticity and income elasticity, with for instance the adverse consequence that preferences with constant elasticity of substitution (CES) are the only directly-separable and indirectly-separable preferences that are homothetic.

In an attempt to generalize the concept of separability, Gorman (1972) and Pollak (1972) define generalized separability as demand that would take the form:

$$q_i = q_i(p_i/w, \Lambda) \tag{3}$$

where q_i refers to demand for good i (quantity) and Λ is implicitly defined by the budget constraint:

$$\sum p_i q_i = p_i q_i(p_i/w, \Lambda) = w$$

i.e. such that total expenditures equal a fixed revenue w. Note that, generally, Λ is not a Lagrange multiplier, except for the case where demand can be derived from a directly-additive separable utility (Ligon 2016).

In an unpublished note by Gorman (printed in Gorman, 1995) mentioned by Pollak (1972), Gorman indicates that a demand system defined as above needs to take specific forms in order to satisfy Slutsky's symmetry condition. With a few additional restrictions, this result can be formulated as follows:[6]

Proposition 1 *If demand satisfies the following conditions:*

i) generalized separability (equation 3);

ii) there are at least four goods,

iii) holding Λ constant, $(p_i/w)q_i(p_i/w, \Lambda)$ is not constant over the range of prices p_i

Then demand can be written as either:

$$\begin{aligned} \text{case 1:} \quad q_i(p_i/w, \Lambda) &= \frac{D_i(F(\Lambda)p_i/w)}{H(\Lambda)} && \text{for all goods } i \text{ and all } p_i, w, \Lambda \\ \text{case 2:} \quad q_i(p_i/w, \Lambda) &= A_i(\Lambda)(p_i/w)^{-\sigma(\Lambda)} && \text{for all goods } i \text{ and all } p_i, w, \Lambda \\ + \text{ case 2':} \quad q_i(p_i/w, \Lambda) &= A_i\Lambda^{\rho}(p_i/w)^{-\sigma} && \text{for } \underline{\text{all but one}} \text{ good } i \end{aligned}$$

or a combination of cases 2 and 2' depending on Λ. In all cases, Λ is implicitly defined such that budget constraint is satisfied, i.e. such that $\sum_i q_i(p_i/w, \Lambda)p_i = w$.

The proof of Proposition 1 is rather tedious. A key step is to show that the symmetry of the Slutsky matrix implies, with a few exceptions, that either the cross-elasticity of demand w.r.t to the normalized price p_i/w and Λ is constant across all goods (this corresponds to case 2 above) or that the ratio of elasticities $\frac{\epsilon_{\Lambda i} - \epsilon_{\Lambda i}}{\epsilon_{yi} - \epsilon_{yi}}$ is identical across all pairs of goods (conditional on Λ), where $\epsilon_{\Lambda i}$ denotes the demand elasticity of good i w.r.t Λ and ϵ_{yi} w.r.t own normalized prices p_i/w. Solving differential equations implied by these constraints, we can then show that the functional forms above are the only possible demand functions. Symmetry of the Slutsky matrix is the only constraint that is imposed on substitution patterns, and the negative semi-definiteness is not yet guaranteed.

[6] Gorman's sketch of proof had many shortcuts, as he himself noted: "Throughout this paper I have talked as if my claims were definitely proven. Of course this is not so: my arguments are far from rigorous" (Gorman, 1995). Here I impose somewhat stronger assumptions on the form of demand and price effects in order to avoid a few inelegant cases. In particular, the assumption that expenditure shares are not just a function of Λ allows me to avoid what Gorman calls "the abnormal case".

Since the third case is relatively interesting and elegant (CES for all but one good), the remainder of the paper focuses on cases 1 and 2, setting aside case 2'. Note that there may be alternative functional forms under generalized separability if we allow for price-insensitive expenditures shares, which Gorman calls "abnormal" goods. Assumption iii) allows us to exclude such cases.

Finally, note that functional forms are unique up to a constant term and a monotonic transformation of Λ:

Proposition 2 *Uniqueness of functional forms, except for the CES case:*

> *Case 1: $H(\Lambda)$ and $F(\Lambda)$ are uniquely determined by demand patterns, up to a constant term and a strictly-monotonic transformation of Λ*
>
> *Case 2: $A_i(\Lambda)$ and $\sigma(\Lambda)$ are uniquely defined by demand patterns, up to a strictly-monotonic transformation of Λ*

To prove this result, in case 1), note that price effect depend tightly on ε_{Di} (see sections 3.2 and 3.3), hence price elasticities can be used to determine D_i. One can then identify functions F and H by examining variations in ε_{Di} depending on income (except in the CES case).[7] In case 2), σ corresponds to price elasticities and $A_i(\Lambda)$ can be determined by examining income expansion path. Note that for any function H, F, A_i and σ, one would obtain the same demand patterns after the change in variable $\Lambda' = g(\Lambda)$ with any one-to-one mapping g.

In the first case, note that we can express the inverse demand system as:

$$p_i/w = (1/F(z))\ D_i^{-1}(H(z)q_i) \tag{4}$$

where z is implicitly defined as described in equation (8) as function of the vector of consumption. When $F(z)H(z)$ is constant and preferences are homothetic, this aggregator z is homogeneous of degree one in quantities q_i. This inverse demand formulation highlights the symmetric role of H vs. F. While H is the demand shifter in the direct demand function (equation 4), now F is the demand shifter in the inverse demand function.

[7] Price and income elasticities depend on the ratio of ε_F and ε_H which is uniquely determined once we know these elasticities. Taking a strictly monotonic transformation of Λ does not change this ratio and yields the same consumption patterns (assuming that this new aggregator is also such that the budget constraint is satisfied).

3 Integrability

3.1 Integrability of Generalized Gorman-Pollak Demand

Let us now examine the reciprocal of Proposition 1. Under which conditions are these demand systems integrable, i.e. can be derived from a rational utility-maximizing consumption behavior? These functional forms, imposed by the symmetry of the Slutsky matrix, do not necessarily correspond to rational consumer behavior (see counter-examples in Appendix B). However it turns out that only weak additional conditions are sufficient to guarantee that the demand systems described in Proposition 1 are integrable.

Suppose that demand is given by:

$$q_i = \frac{D_i(F(\Lambda)p_i/w)}{H(\Lambda)} \tag{5}$$

where Λ is implicitly determined by the budget constraint $\sum_i p_i D_i(F(\Lambda)p_i/w)/H(\Lambda) = w$, which can be rewritten:

$$H(\Lambda) \;=\; \sum_i (p_i/w)\, D_i(F(\Lambda)p_i/w) \tag{6}$$

Let us denote by $\varepsilon_{Di} = \frac{\partial \log D_i}{\partial \log x}$ the elasticity of D_i in its argument, and $\varepsilon_H = \frac{\partial \log H}{\partial \log \Lambda}$ and $\varepsilon_F = \frac{\partial \log F}{\partial \log \Lambda}$ the elasticity of H and F in Λ.

To ensure integrability, we impose the following sufficient regularity restrictions:

Regularity assumptions [**A1**] on functions D_i and H:

i) D_i is differentiable, $\varepsilon_{Di} < 0$ unless $D_i = 0$

ii) H and F are differentiable and $\varepsilon_F\, \varepsilon_{Di} < \varepsilon_H$ for all i, Λ and p_i/w

iii) For any set of normalized prices p_i/w, equation (6) admits a solution in Λ.

Note that instead of [A1] ii) we could assume that $\varepsilon_F\, \varepsilon_{Di} - \varepsilon_H$ has the same sign for all i, Λ and p_i/w. Assuming that this difference is negative is without loss of generality as we can always make the change in variable $\Lambda' = 1/\Lambda$ to invert its sign for all goods and prices. Assumptions i) and ii) imply that the solution in Λ to equation (6) is always unique, but they are also needed to show that utility is quasi-concave and that the Slutsky substitution matrix is negative semi-definite. However, condition ii) on elasticities does not ensure that there exists a Λ to satisfy the budget constraint, so we need to also impose condition iii). Note also that condition iii) could be implied by stricter conditions on elasticities (see Appendix C for practical examples).

Under these conditions, we obtain:

Proposition 3 *If H and D_i satisfy the regularity conditions [A1], the demand described in equations (5) and (6) is integrable, i.e. can be derived from a utility function.*

I provide two alternative proofs in Appendix. First, we can reconstruct a utility function that depends on an implicitly-defined aggregator.[8] We can show that demand can be derived from the maximization of:

$$U(q) = \sum_i \int_{x=x_{i0}}^{x=H(z)q_i} D_i^{-1}(x)dx \; - \; \int_{z_0}^{z} H'(z)F(z)dz \tag{7}$$

for arbitrary z_0, $x_{0i} \geq 0$ and where z is itself a function of the consumption vector x implicitly defined by:

$$F(z) = \sum_i q_i D_i^{-1}(H(z)q_i) \tag{8}$$

As a slight abuse of notation, we define $D_i^{-1}(0) = a_i$ if $D_i(y) = 0$ for all $y \geq a_i$ (which yields a choke price) and $D_i^{-1}(x) = 0$ for all $x \geq b_i$ if $D_i(0) = b_i$. Regularity conditions [A1] are needed to ensure that equation (8) always has a unique solution in z and that the utility function is quasi-concave. Note that equation (8) can be seen as a first-order condition such that the derivative of the expression above for U has a zero derivative in z, and such that:

$$\frac{\partial U}{\partial q_i} = D_i^{-1}(H(z)q_i) \tag{9}$$

The two aggregators Λ (function of prices and income) and z (function of quantities q) coincide for the optimal consumption basket.[9]

An alternative proof of Proposition 3 checks that the Slutsky substitution matrix is negative semi-definite under these restrictions, so that we can apply the integrability theorem of Hurwitz and Uzawa (1971). Thanks to Proposition 1, we already know that it is symmetric but this does not ensure semi-definite negativity. As one could expect, the conditions ensuring the semi-definite negativity of the Slutsky matrix are the same as those providing the quasi-concavity of the utility function above.

[8] This utility representation was pointed out by Gorman (1987) with a more restrictive formulation and no formal proof that such utility function is well defined and quasi-concave. Gorman formulated this as a maximization: $U = \max_z \{\sum_i u_i(zq_i) - \Phi(z)\}$ but this approach omits very useful cases (e.g. cases with $0 < \beta < 1$ in applications 3, 4, 9 and 10 in Section 4) where the second order condition of this maximization is not satisfied yet the utility function remains quasi-concave with z implicitly defined by equation 8.

[9] But note that z is defined as a function of the vector of quantities while Λ is a function of the vector of prices and income.

Next, an important concern is whether the set of conditions [A1] can be relaxed, but I argue here that all are needed. First, the demand system would clearly not be well defined if it does not have a solution in equation (6), so condition iii) is unavoidable. It is possible to impose more simple conditions to ensure existence, but such condition would be less general. Second, restriction ii) is the simplest and more direct way to insure that the equation defining the price aggregator has a unique solution. It is required for good i for a given level of prices when a good i has a sufficiently large expenditure share. In Appendix B, I provide an example with two goods where restrictions i) and iii) are met but the Slutsky matrix is no longer negative semi-definite when $\varepsilon_F \, \varepsilon_{Di} - \varepsilon_H$ does not have the same sign for the two goods. Finally, restriction i) ensures that we have a negative effect of prices on demand when the expenditure share of a good is small (a positive price effect would not be rational for small expenditure shares). Inverting D_i is also needed in equations (8) and (7) to retrieve utility.

The case $F(\Lambda) = 1/H(\Lambda)$ coincides with Matsuyama and Ushchev (2017) for homothetic preferences. With the change in variable: $F(\Lambda) = 1/H(\Lambda) = w\Lambda'$ (where Λ' is again implicitly determined by the budget constraint), a demand system such that $q_i = \Lambda' w D_i(\Lambda' p_i)$ can be rationalized as long as $\varepsilon_{Di} + 1$ has the same sign for all goods (which is equivalent to A1-ii). If $\varepsilon_{Di} + 1$ is negative, we define Λ' instead by $w/\Lambda' = F(\Lambda) = 1/H(\Lambda)$.

In general, Proposition 3 does not require $F(\Lambda)$ to be monotonic in Λ. If $F'(\Lambda) > 0$, an increase in Λ (tightness of the budget constraint) leads to a downward shift in the partial demand curve D_i. When $F'(\Lambda) < 0$, we would instead have an upward shift in D_i, which needs to be compensated by a large enough decrease in the demand shifter $H(\Lambda)$. If $F(\Lambda)$ can be inverted (which is satisfied for most applications provided in Section 4), note that we can reformulate demand as:

$$q_i = \widetilde{H}(\Lambda w) D_i(\Lambda p_i) \qquad \text{s.t.:} \qquad \widetilde{H}(\Lambda w) \sum_i p_i D_i(\Lambda p_i) = w \tag{10}$$

with the change in variable: $\Lambda' = F(\Lambda)/w$ and the transformation $1/\widetilde{H}(\cdot) = H(F^{-1}(\cdot))$. In this case, condition [A1] ii) and iii) are automatically satisfied if there exists $\epsilon > 0$ such that $\varepsilon_{\widetilde{H}} + \varepsilon_{Di} < -\epsilon$ for all p_i, w and Λ.

3.2 Integrability of Generalized Non-Homothetic CES

Now, consider the second case of Proposition 1. Let us assume that demand is given by:

$$p_i q_i = w \left(G_i(\Lambda) \, p_i / w \right)^{1-\sigma(\Lambda)} \tag{11}$$

where Λ is an implicit function of the vector of normalized prices p_i/w such that the budget constraint is satisfied:

$$\sum_i (G_i(\Lambda)p_i/w)^{1-\sigma(\Lambda)} = 1 \tag{12}$$

To ensure integrability, we impose the following sufficient regularity restriction [A2]:

Regularity assumptions [A2] For each Λ, we have $\sigma(\Lambda) \neq 1$ and *either* one of the following two conditions:

i) $\sigma(\Lambda)$ is weakly increasing in Λ and $G_i(\Lambda)$ is strictly increasing in Λ

ii) $\sigma(\Lambda)$ is decreasing in Λ and, for each Λ_0, there exists $\alpha_i > 0$ such that $\sum_i \alpha_i = 1$ and such that $G_i(\Lambda)\, \alpha_i^{\frac{1}{\sigma(\Lambda)-1}}$ is strictly increasing in Λ in a neighborhood of Λ_0

When both $\sigma(\Lambda)$ and $G_i(\Lambda)$ are all differentiable, condition ii) can be rewritten after solving for the minimum α_i that would satisfy this monotonicity condition. Condition ii) is formally equivalent to imposing:[10]

$$\sum_i \exp\left(\frac{(\sigma(\Lambda)-1)^2 G_i'(\Lambda)}{\sigma'(\Lambda)G_i(\Lambda)}\right) < 1 \tag{13}$$

See Appendix for the proof of equivalence.

We obtain the following proposition for generalized non-homothetic CES:

Proposition 4 *Suppose that demand can be written as in equation (11) where G_i and σ are continuous and where Λ is implicitly defined by (12). This demand system is integrable if conditions [A2] are satisfied. Under [A2], demand can be derived from a utility function that is implicitly defined by:*

$$\sum_i (q_i/G_i(U))^{\frac{\sigma(U)-1}{\sigma(U)}} = 1 \tag{14}$$

which has a unique solution in U, with $\Lambda = U$ for the demand q_i described above.

The constant elasticity case $\sigma(\Lambda) = \sigma$ corresponds to implicitly additive utility as in (Comin et al., 2015). This is not equivalent to the standard CES since, even in that case, non-trivial

[10]In general, note that condition ii) need not hold for *any* set of α_i's, it is sufficient that it holds for a single set of α_i's. In particular, using $\alpha_i = 1/N$ (where N denotes the number of goods), a sufficient condition is that $G_i(\Lambda)\, N^{\frac{1}{1-\sigma(\Lambda)}}$ strictly increases in Λ.

income effects through the demand shifter $G_i(\Lambda)$ allow for very flexible Engel curves. The main contribution of this proposition is to generalize to variable elasticity of substitution.

The proof of Proposition 4 mainly consists in showing that Λ is well-defined, i.e. that the budget constraint has a unique solution in Λ, and that utility is also uniquely defined by equation 14. As the more general case allows for varying curvature of indifference curves, one needs to ensure in particular that these indifference curves do not cross.

The proof proceeds as follows. First we show in a lemma that $\left[\sum_i \alpha_i x_i^\rho\right]^{\frac{1}{\rho}}$ is monotonically increasing in ρ if $\sum_i \alpha_i = 1$ (a consequence of Jensen's inequality). This allows us to obtain comparative statics in the exponent in equations (14) and (12). We can then show that the solutions to these equations are unique, for a given set of income and prices, or quantities. Once we have uniqueness, it is easy to verify the quasi-concavity of the utility function (as in Comin et al., 2015). The last step is to check that this utility maximum problem does yield the demand system described above.

Again, as for Proposition 3, a potential concern is whether restrictions [A2] are necessary. When neither condition i) or ii) is satisfied, neither the demand system described above nor the utility in Proposition 3 is well defined. Counter-examples in the Appendix further illustrate the role of each condition, showing that equations (12) and (14) admit multiple solutions in Λ and U if conditions i) and ii) are not satisfied. Incidentally, this shows that monotonicity in demand shifters $G_i(\Lambda)$ is not sufficient.[11]

One should also point out why we need different conditions depending on whether $\sigma(\Lambda)$ decreases or increases with Λ. In the first case, where $\sigma(\Lambda)$ increases with Λ, indifference curves become flatter as we move away from the origin (with increases in income and Λ). In that case, indifference curves are most likely to cross around the intercepts (when only one good is consumed). Monotonicity in $G_i(\Lambda)$ is then sufficient to ensure that indifference curves do not cross. In the second case, where the elasticity of substitution $\sigma(\Lambda)$ decreases with Λ, the indifference curves are more curved as we move away from the origin. In this case, indifference curves are most likely to be close to each other and intersect in the middle.

[11] We can also have $\sigma(\Lambda) = 1$ for a discrete number of limit cases.

4 Properties: Welfare, price and income effects

4.1 Generalized Gorman-Pollak

Price elasticities. In this case, the price elasticity of Marshallian demand is:

$$\frac{\partial \log q_i}{\partial \log p_j} = \varepsilon_{Di} \,.\, \mathbb{1}_{(i=j)} - \frac{W_j(1+\varepsilon_{Dj})(\varepsilon_H - \varepsilon_F \varepsilon_{Di})}{\varepsilon_H - \varepsilon_F \bar{\varepsilon}_D}$$

where W_j is the expenditure share of good j and $\mathbb{1}_{(i=j)}$ is a dummy equal to one when $i = j$. Given our restriction $\varepsilon_H > \varepsilon_F \varepsilon_{Di}$, the cross-price elasticity ($i \neq j$) is positive if and only if $\varepsilon_{Dj} < -1$. The own-price elasticity is always negative, which rules out Giffen goods.

The own price elasticity is mainly determined by the shape of function D_i when that good has a negligible market share:

$$\frac{\partial \log q_i}{\partial \log p_i} \approx \varepsilon_{Di}$$

Since we impose very few constraints on ε_{Di} the patterns of price elasticities can be very flexible.

Shifters and choke prices. For applications to oligopolistic and monopolistic competition, a key determinant of the toughness of competition is the position along demand curves. Here this is reflected in the price shifter $F(\Lambda)/w$ as well as the demand shifter $H(\Lambda)$. Either one is constant with directly or indirectly-separable preferences.

The demand system can be accommodated to yield choke prices (also called reservation prices), i.e. a price threshold p_i^* such that $D_i(F(\Lambda)p_i/w) = 0$ for all $p_i \geq p_i^*$, which arises as soon as $D_i(y) = 0$ for large enough y's.[12]

Of particular interest is how choke prices depend on income. Income is irrelevant with homothetic preferences, but this prediction contradicts substantial evidence (e.g. Hummels and Klenow 2007) showing that richer consumer buy a greater variety of goods and that richer countries import a larger variety of products. A special case corresponds to indirectly-additive preferences (H is constant). In this case, choke prices are linear in income, as documented by Bertoletti et al (2017). In the general case, the choke price depends on both income and the toughness of competition, captured by Λ:

$$p_i^* = \frac{a_i w}{F(\Lambda)}$$

[12] Choke prices are particularly useful in international trade to explain why less efficient firms are less likely to export to a specific market (without having to rely on export fixed costs) and to obtain gravity equations as shown in Melitz and Ottaviano (2008) and Arkolakis et al (2015) among others.

where $a_i = \min\{D_i^{-1}(0)\}$ is the level above which demand is null.

Income effects. The income elasticity of good i is:

$$\frac{\partial \log q_i}{\partial \log w} = 1 + \frac{(\varepsilon_H + \varepsilon_F)(\bar{\varepsilon}_D - \varepsilon_{Di})}{\varepsilon_H - \varepsilon_F \bar{\varepsilon}_D} \tag{15}$$

Using this expression, one can see that homotheticity implies that either $\varepsilon_H = -\varepsilon_F$ or $\varepsilon_{Di} = \bar{\varepsilon}_D$ for all i.

As pointed out by Pigou (1910) and Deaton (1974), own-price elasticities and income elasticities are tightly linked when demand is derived from a directly-additive utility (which corresponds to the case where $\varepsilon_H = 0$). Here we obtain: $\frac{\partial \log q_i}{\partial \log w} = \frac{\varepsilon_{Di}}{\bar{\varepsilon}_D}$ with directly separable utility. With $\varepsilon_H \neq 0$, the relationship between income elasticity and price elasticity is muted and we obtain a demand system that is more in line with Deaton (1974) empirical results. Note that $\frac{\partial \log q_i}{\partial \log w} - 1$ has the same sign as $\bar{\varepsilon}_D - \varepsilon_{Di}$ when $\varepsilon_F + \varepsilon_H > 0$, and flipped otherwise. This property can seem attractive, as empirical evidence indicates that price-elastic goods are not necessarily more income elastic.

Homotheticity. This demand system is homothetic if and only if $H(\Lambda)F(\Lambda)$ is constant or if it is CES. This corresponds to the case studied by Matsuyama and Ushchev (2017). In this case, without loss of generality we can assume that $F(z) = 1/H(z) = z$. We obtain that a utility representation given by:

$$U(q) \ = \ \log(z) \ + \ \sum_i \int_{x=x_{i0}}^{x=q_i/z} D_i^{-1}(x)dx \tag{16}$$

where z is such that $\sum_i (q_i/z) D_i^{-1}(q_i/z) = 1$, and x_{i0} are constant terms.

Indirect utility. Here we draw from Pollak (1972) to show that indirect utility can be expressed as:

$$V(p, w) = \widetilde{V}(p, w, \Lambda) = \sum_i \int_{(p_i/w)F(\Lambda)}^{y_{i0}} D_i(y)dy \ + \ \int_{\Lambda_0}^{\Lambda} F'(l)H(l)dl \tag{17}$$

where $\Lambda = \Lambda(p, w)$ is implicitly defined as above, and where y_{i0} and Λ_0 are constant terms. Conveniently, we can verify that $\frac{\partial \widetilde{V}}{\partial \Lambda} = 0$, hence we obtain simple expressions for marginal (indirect) utility from income and price changes. Taking the derivative w.r.t income, one can interpret the product of the two shifters as the marginal utility of income:

$$\frac{\partial V(p,w)}{\partial w} = \frac{F(\Lambda)H(\Lambda)}{w} \tag{18}$$

This expression can also be useful to compute equivalent and compensating variations, implicitly defined such that $V(p', w - CV) = V(p, w)$ and $V(p, w + EV) = V(p', w')$.

Additivity When are these preferences directly or indirectly additive? Recall that preferences are directly additive if utility can be written as $U(q) = f(\sum_i u_i(q_i))$ where f and $\{u_i\}$ are scalar functions. Preferences are indirectly additive if indirect utility can be written as $V(p, w) = g(\sum_i v_i(p_i/w))$ where g and $\{v_i\}$ are scalar functions.

From equation (7), we can conclude that utility is directly-additive if and only if H is constant or demand is CES. From equation (17) for indirect utility, we can see that we have indirect additivity if and only if F is constant or demand is CES.

4.2 Generalized Non-Homothetic CES

Price effects. In the second case, when demand corresponds to equation (11), price effects are more simple. The price elasticity of the Hicksian demand corresponds to $\sigma(\Lambda)$, since we can also interpret Λ as utility (Proposition 3).

Income effects. In the second case, demand corresponding to equation (11) yields even more flexible income effects. Changes in $G_i(\Lambda)$ in Λ need not be related to changes in $\sigma(\Lambda)$. Starting with the special case where $\sigma(\Lambda) = \sigma$ is constant, which corresponds to Hanoch (1975), the effect of income on Λ is such that:

$$\frac{\partial \log \Lambda}{\partial \log w} = \frac{1-\sigma}{\bar{\varepsilon}_G} \tag{19}$$

where $\bar{\varepsilon}_G$ is an average of elasticities $\varepsilon_{Gi} = \frac{\Lambda G_i'(\Lambda)}{G_i(\Lambda)}$ weighted by expenditures shares. We obtain the income elasticity:

$$\frac{\partial \log q_i}{\partial \log w} = \sigma + (1-\sigma)\left(\frac{\varepsilon_{Gi}}{\bar{\varepsilon}_G}\right) \tag{20}$$

Good i is income-elastic if $\sigma < 1$ and $\varepsilon_{Gi} > \bar{\varepsilon}_G$ or if $\sigma > 1$ and $\varepsilon_{Gi} < \bar{\varepsilon}_G$. In the more general case where $\sigma(\Lambda)$ is not constant, function G_i plays a similar role and dictates income effects. Moreover, depending on how income affects Λ (with depends on both the sign of $\sigma'(\Lambda)$ and whether $\sigma(\Lambda)$ is smaller than unity), one can have σ increase or decrease with income.

One constraint, however, links the price elasticity and the income elasticity. Both in the cases where $\sigma(\Lambda)$ is fixed or increasing in Λ, the price elasticity imposes a lower bound on income elasticities of demand:[13]

$$\frac{\partial \log q_i}{\partial \log w} > \sigma(\Lambda) \qquad \text{if } \sigma(\Lambda) < 1$$
$$\frac{\partial \log q_i}{\partial \log w} < \sigma(\Lambda) \qquad \text{if } \sigma(\Lambda) > 1$$

Homotheticity. The demand system in (11) is homothetic only in the CES case, as one can see for instance in expression (20).

5 Concluding remarks

Economists have often focused on demand systems where prices are conveniently summarized by a single aggregator, and where demand depends solely on such an aggregator, total expenditures and a good's own price ("generalized separability", following the terminology of Pollak 1972). Here I show that such a demand system can take only one of two forms when price effects are not trivial. This result was already known by Pollak (1972) and Gorman (1972) but not formally demonstrated and is not well known today in spite of its usefulness. Furthermore, I show that these two types of demand systems are integrable (i.e. can be derived from well-behaved utility functions) under fairly mild regularity restrictions that guarantee the semi-definite negativity of the Slutsky substitution matrix (and quasi-concavity of utility).

The first case of demand allows for very flexible price effects but more restricted income effects. This is particularly useful in industrial organization where income effects are not the main focus. The second case of demand allows for very flexible income effects (highly flexible Engel curves) but more restricted price effects. Allen-Uzawa substitution elasticities have to be constant across goods to ensure the symmetry of the Slutsky matrix, but here I show that they may vary (increase or decrease) with utility and thus vary indirectly with income.

There can be numerous applications and uses of such demand systems, spanning different fields. I provide additional examples in Appendix C. I describe cases of Gorman-Pollak forms where functions H and F are iso-elastic, which provides simple generalizations of directly-separable and indirectly-separable preferences, and cases where functions D_i are iso-elastic, which coincide with self-dual addilog demand systems (Houthakker 1965). Other examples described in the appendix generalize demand systems featured in Mazrova and Neary (2013), such as cases where functions D_i take a bi-power form, and the use of non-homothetic CES.

[13] When $\sigma(\Lambda)$ decreases in Λ, the income elasticity exceeds $\sigma(\Lambda)$ if and only if $\exp\left(\frac{(\sigma(\Lambda)-1)^2 G_i'(\Lambda)}{\sigma'(\Lambda) G_i(\Lambda)}\right) < W_i$. This need not be satisfied, even if inequality (13) is imposed.

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Appendix A – Proofs of Propositions 1, 3 and 4

Proof of Proposition 1

Consider the demand system:

$$q_i = (w/p_i).W_i(p_i/w, \Lambda) \tag{21}$$

where W_i denotes expenditure shares as a function of normalized prices p_i/w (demand is homogeneous of degree zero in prices and income) and Λ (by assumption), and where Λ is implicitly determined by the budget constraint:

$$\sum_i W_i(p_i/w, \Lambda) = 1 \tag{22}$$

We denote normalized prices by $y_i \equiv p_i/w$ such that $q_i = (1/y_i).W_i(y_i, \Lambda)$ and with Λ an implicit function of the vector of y_i such that: $\sum_j W_j(y_j, \Lambda) = 1$.

For $i \neq j$, the Slutsky substitution coefficient is:

$$\begin{aligned}
s_{ij} &= \frac{\partial q_i}{\partial p_j} + q_j \frac{\partial q_i}{\partial w} \\
&= \frac{w}{p_i}\frac{\partial W_i}{\partial \Lambda}\left[\frac{\partial \Lambda}{\partial p_j} + q_j \frac{\partial \Lambda}{\partial w}\right] - \frac{q_j}{w}\frac{\partial W_i}{\partial y_i} - \frac{q_j q_i}{w} \\
&= \frac{\partial \log W_i}{\partial \log \Lambda}\left[\frac{q_i}{p_j}\frac{\partial \log \Lambda}{\partial \log p_j} + \frac{q_i q_j}{w}\frac{\partial \log \Lambda}{\partial \log w}\right] - \frac{q_j q_i}{w}\frac{\partial \log W_i}{\partial \log y_i} - \frac{q_j q_i}{w} \\
&= \frac{q_i \epsilon_{\Lambda i}}{p_j}\frac{\partial \log \Lambda}{\partial \log p_j} + \frac{q_i q_j \epsilon_{\Lambda i}}{w}\frac{\partial \log \Lambda}{\partial \log w} - \frac{q_j q_i \epsilon_{yi}}{w} - \frac{q_j q_i}{w}
\end{aligned}$$

where we denote:

$$\epsilon_{yi} \equiv \frac{\partial \log W_i}{\partial \log y_i} \qquad \text{and} \qquad \epsilon_{\Lambda i} \equiv \frac{\partial \log W_i}{\partial \log \Lambda}$$

To compute the derivatives of Λ in w and p_i, we differentiate (22) w.r.t w, which gives:

$$\frac{\partial \Lambda}{\partial w}\left[\sum_i \frac{\partial W_i}{\partial \Lambda}\right] - \left[\sum_i \frac{p_i}{w^2}\frac{\partial W_i}{\partial y_i}\right] = 0$$

$$\iff \frac{\partial \log \Lambda}{\partial \log w}\left[\sum_i W_i \epsilon_{\Lambda i}\right] - \left[\sum_i W_i \epsilon_{yi}\right] = 0$$

We obtain:

$$\frac{\partial \log \Lambda}{\partial \log w} = \frac{\sum_i W_i \epsilon_{yi}}{\sum_i W_i \epsilon_{\Lambda i}} = \frac{\overline{\epsilon_y}}{\overline{\epsilon_\Lambda}}$$

where $\overline{\epsilon_y}$ and $\overline{\epsilon_\Lambda}$ denotes the expenditure-weighted averages of ϵ_{yi} and $\epsilon_{\Lambda i}$.

Similarly, differentiating (22) w.r.t price p_j, we get:

$$\frac{\partial \Lambda}{\partial p_j}\left[\sum_i \frac{\partial W_i}{\partial \Lambda}\right] + \frac{1}{w}\frac{\partial W_j}{\partial y_j} = 0$$

$$\iff \frac{\partial \log \Lambda}{\partial \log p_j}\left[\sum_i W_i \epsilon_{\Lambda i}\right] \;+\; W_j \epsilon_{yj} = 0$$

which gives:

$$\frac{\partial \log \Lambda}{\partial \log p_j} = -\frac{W_j \epsilon_{yj}}{\overline{\epsilon_\Lambda}}$$

Incorporating the expressions for the derivatives of Λ, the Slutsky coefficients become:

$$\begin{aligned}
s_{ij} &= -\frac{q_i \epsilon_{\Lambda i}}{p_j}\frac{W_j \epsilon_{yj}}{\overline{\epsilon_\Lambda}} + \frac{q_i q_j \epsilon_{\Lambda i}}{w}\frac{\overline{\epsilon_y}}{\overline{\epsilon_\Lambda}} - \frac{q_j q_i \epsilon_{yi}}{w} - \frac{q_j q_i}{w} \\
&= -\frac{q_i q_j \epsilon_{\Lambda i}}{w}\frac{\epsilon_{yj}}{\overline{\epsilon_\Lambda}} + \frac{q_i q_j \epsilon_{\Lambda i}}{w}\frac{\overline{\epsilon_y}}{\overline{\epsilon_\Lambda}} - \frac{q_j q_i \epsilon_{yi}}{w} - \frac{q_j q_i}{w} \\
&= \frac{q_i q_j}{w \overline{\epsilon_\Lambda}}\left[-\epsilon_{yj}\epsilon_{\Lambda i} + \epsilon_{\Lambda i}\overline{\epsilon_y} - \epsilon_{yi}\overline{\epsilon_\Lambda} - \overline{\epsilon_\Lambda}\right]
\end{aligned}$$

Rearranging, one can see that the Slutsky substitution matrix is symmetrical if and only if, for all $i \neq j$:

$$\epsilon_{yi}\epsilon_{\Lambda j} - \epsilon_{\Lambda j}\overline{\epsilon_y} + \epsilon_{yj}\overline{\epsilon_\Lambda} = \epsilon_{yj}\epsilon_{\Lambda i} - \epsilon_{\Lambda i}\overline{\epsilon_y} + \epsilon_{yi}\overline{\epsilon_\Lambda}$$

Subtracting $\epsilon_{yj}\epsilon_{\Lambda j}$ on both sides, rearranging and factorizing, this can be rewritten:

$$(\epsilon_{yj} - \overline{\epsilon_y})(\epsilon_{\Lambda j} - \epsilon_{\Lambda i}) = (\epsilon_{yj} - \epsilon_{yi})(\epsilon_{\Lambda j} - \overline{\epsilon_\Lambda})$$

This holds for any pair of goods i and j. Picking any three goods i, j, k from the consumption basket, we have:

$$\begin{aligned}
(\epsilon_{yj} - \overline{\epsilon_y})(\epsilon_{\Lambda j} - \epsilon_{\Lambda i}) &= (\epsilon_{yj} - \epsilon_{yi})(\epsilon_{\Lambda j} - \overline{\epsilon_\Lambda}) \\
(\epsilon_{yi} - \overline{\epsilon_y})(\epsilon_{\Lambda i} - \epsilon_{\Lambda k}) &= (\epsilon_{yi} - \epsilon_{yk})(\epsilon_{\Lambda i} - \overline{\epsilon_\Lambda}) \\
(\epsilon_{yk} - \overline{\epsilon_y})(\epsilon_{\Lambda k} - \epsilon_{\Lambda j}) &= (\epsilon_{yk} - \epsilon_{yj})(\epsilon_{\Lambda k} - \overline{\epsilon_\Lambda})
\end{aligned}$$

Adding up, the average terms $\overline{\epsilon_y}$ and $\overline{\epsilon_\Lambda}$ disappear and we obtain:

$$\epsilon_{\Lambda i}\epsilon_{yj} + \epsilon_{\Lambda k}\epsilon_{yi} + \epsilon_{\Lambda j}\epsilon_{yk} = \epsilon_{\Lambda j}\epsilon_{yi} + \epsilon_{\Lambda i}\epsilon_{yk} + \epsilon_{\Lambda k}\epsilon_{yj}$$

Subtracting $\epsilon_{\Lambda j}\epsilon_{yj}$ on both sides and factorizing, we obtain:

$$(\epsilon_{\Lambda i} - \epsilon_{\Lambda j})(\epsilon_{yj} - \epsilon_{yk}) = (\epsilon_{\Lambda j} - \epsilon_{\Lambda k})(\epsilon_{yi} - \epsilon_{yj}) \tag{23}$$

Let us denote by $\mathcal{Y}(\Lambda)$ the set of feasible y (vector of normalized prices for which the aggregator takes a value Λ):

$$\mathcal{Y}(\Lambda) = \left\{ y\,;\;\; s.t. \sum_i W_i(y_i, \Lambda) = 1 \right\}$$

and denote by $\mathcal{Y}_i(\Lambda)$ its projection onto the i axis, i.e. the set of feasible y_i for a given Λ.

For a given Λ, to satisfy equation (23) for any triplet of goods, three cases arise naturally:

- case 1A: $\epsilon_{yj} = \epsilon_{yi}$ for all goods, for each $y \in \mathcal{Y}(\Lambda)$.
- case 1B: $\epsilon_{yj} = \epsilon_{yi}$ for all but one good $i0$, for all $y \in \mathcal{Y}(\Lambda)$.
- case 2: there exists three goods $i \neq j \neq k$ such that $\epsilon_{yi} \neq \epsilon_{yj} \neq \epsilon_{yk}$ for some $y \in \mathcal{Y}(\Lambda)$.

Case 1A is easier to consider. Recall that, in general, ϵ_{yj} is a function of y_j and Λ. The goal is to show that

- $W_i(y_i, \Lambda) = A_i(\Lambda) y_i^{\epsilon(\Lambda)}$ for all $y_i \in \mathcal{Y}_i(\Lambda)$ for some functions $A_i(\Lambda)$ and $\epsilon(\Lambda)$.
- $\mathcal{Y}_i(\Lambda) = \left\{ y_i \ ; \ A_i(\Lambda) y_i^{\epsilon(\Lambda)} < 1 \right\}$

Suppose that $y^* \in \mathcal{Y}(\Lambda)$, i.e. such that $\sum W_i(y_i^*, \Lambda) = 1$ and denote $\epsilon(\Lambda) = \epsilon_{yj}(y_j, \Lambda)$. Also suppose for now that $\epsilon(\Lambda)$ is strictly positive (the same proof applies to the other case after a simple change in variable). Define $A_i(\Lambda)$ such that $W_i(y_i^*, \Lambda) = A_i(\Lambda) y_i^{*\epsilon(\Lambda)}$.

Denote by $(\underline{y_i}, \overline{y_i})$ the maximum interval included in $\mathcal{Y}_i(\Lambda)$ containing y_i^* and such that, for each $y_i \in (\underline{y_i}, \overline{y_i})$, there exists $\tilde{y} \in \mathcal{Y}(\Lambda)$, with $\tilde{y}_j \in (\underline{y_j}, \overline{y_j})$ in each of its argument, with $\tilde{y}_i = y_i$, and such that $W_j(\tilde{y}_j, \Lambda) = A_j(\Lambda)(\tilde{y}_j)^{\epsilon(\Lambda)}$.

By contradiction, suppose that this is not the case: suppose that for a good $i = I$ we have $\underline{y_I} > 0$. We need to show that it is then possible to construct a new vector y' such that $W_j(y'_j, \Lambda) = A_j(\Lambda)(y'_j)^{\epsilon(\Lambda)}$ and $\sum_j A_j(\Lambda)(y'_j)^{\epsilon(\Lambda)} = 1$, and such that $y'_I < \underline{y_I}$.

Since we deal with bounded intervals, we can construct a series $y^{(n)}$ that satisfies these four conditions:
i) $W_j(y_j^{(n)}, \Lambda) = A_j(\Lambda)(y_j^{(n)})^{\epsilon(\Lambda)}$;
ii) $\sum_j W_j(y_j^{(n)}, \Lambda) = 1$;
iii) each term converge to a finite (possibly zero) value denoted y_j^∞;
iv) and such that $y_I^{(n)}$ converges to $\underline{y_I} > 0$.

By continuity (with the abuse of notation $W_i(0, \Lambda) = 0$ in the limit case), we must have $W_j(y_j^\infty, \Lambda) = A_j(\Lambda)(y_j^\infty)^{\epsilon(\Lambda)}$ and $\sum_j W_j(y_j^\infty, \Lambda) = 1$. Denote by K_0 the set of goods k such that $y_k^\infty = 0$. If K_0 is empty, pick a good $k \neq I$. Since $\underline{y_I} > 0$, we know that such a good k satisfies $\underline{y_k} < 1/A_k$. Next, since $\underline{y_I} > 0$ and $\underline{y_k} < 1/A_k$ for $k \in K_0$, and since the derivative of W_I is non-zero at $\underline{y_I}$, we can find small enough but positive $\nu_I > 0$ and $\nu_k > 0$ (for $k \in K_0$) such that:

$$W_I(y_I^\infty - \nu_I, \Lambda) + \sum_{k \in K_0} W_k(y_k^\infty + \nu_k, \Lambda) + \sum_{j \notin K_0, k \neq I} W_j(y_j^\infty, \Lambda) = 1$$

Moreover, given that the derivative of W_i is strictly positive on each interval $(y_I^\infty - \nu_I, y_I^\infty)$ and $(y_k^\infty, y_k^\infty + \nu_k)$, we can construct a continuum of other vectors y on such intervals that satisfy the same condition as above. For all these vectors, the elasticity ν_{yj} must be kept constant so we must have: $W_j(y_j^\infty, \Lambda) = A_j(\Lambda)(y_j^\infty)^{\epsilon(\Lambda)}$. This contradicts the assumption that $\underline{y_I} > 0$ and proves that $\underline{y_i} = 0$ for all i.

Conversely, one can show that $\overline{y_i} = 1/A_i$ otherwise one can construct a new vector y' such as y^∞ with a i^{th} argument strictly above $\overline{y_i}$ if $\overline{y_i} < 1/A_i$.

Here we have imposed $\epsilon(\Lambda) > 0$. The same arguments can be applied to the case with $\epsilon(\Lambda) < 0$ with the change in variable $y'_i = 1/y_i$.

Case 1B Here we assume that all but one good $i0$ has a common elasticity $\epsilon(\Lambda)$. Applying the same arguments as in the previous case, we obtain that $W_j(y_j, \Lambda) = A_j(\Lambda) y_j^{\epsilon(\Lambda)}$ for all y_j such that $A_i(\Lambda) y_i^{\epsilon(\Lambda)} < 1 - s_{i0}$ where s_{i0} is the minimum expenditure share of good $i0$ for which its elasticity differs from other goods.

On these intervals, we have:

$$\epsilon_{\Lambda j}(y_j,\Lambda)-\epsilon_{\Lambda k}(y_k,\Lambda)=\left(\frac{\epsilon_{\Lambda i0}-\epsilon_{\Lambda j}}{\epsilon_{yi0}-\epsilon_{yj}}\right)(\epsilon_{yj}-\epsilon_{yk})=0$$

which holds for any two goods $j,k\neq i0$, we obtain that they also share the same elasticity with respect to Λ: $\epsilon_{\Lambda j}(y_j,\Lambda)=\epsilon_{\Lambda k}(y_k,\Lambda)=\rho(\Lambda)$. Hence these elasticities do not depend on y_i or y_k. This implies that both ϵ and ρ are constant, and that for any good $k\neq i$ on these intervals we must have:

$$W_k(y_k,\Lambda)=A_k\Lambda^\rho y_k^\epsilon \qquad\qquad k\neq i0$$

Case 2 is more involved. Pick any three goods i,j,k from the consumption basket. From equation (23), we have:

$$(\epsilon_{\Lambda i}-\epsilon_{\Lambda j})(\epsilon_{yj}-\epsilon_{yk})=(\epsilon_{\Lambda j}-\epsilon_{\Lambda k})(\epsilon_{yi}-\epsilon_{yj})$$

In this case, there exists at least three goods $i\neq j\neq k$ such that $\epsilon_{yj}\neq\epsilon_{yi}\neq\epsilon_{yk}$ for some $y\in\mathcal{Y}(\Lambda)$. For these goods, we obtain:

$$\frac{\epsilon_{\Lambda i}-\epsilon_{\Lambda j}}{\epsilon_{yi}-\epsilon_{yj}} = \frac{\epsilon_{\Lambda j}-\epsilon_{\Lambda k}}{\epsilon_{yj}-\epsilon_{yk}} \tag{24}$$

$$\frac{\epsilon_{\Lambda i}-\epsilon_{\Lambda j}}{\epsilon_{yi}-\epsilon_{yj}} = \frac{\epsilon_{\Lambda k}-\epsilon_{\Lambda i}}{\epsilon_{yk}-\epsilon_{yi}} \tag{25}$$

Notice that the left-hand side of both equations potentially depend on y_i and y_j but the right-hand side of equation (24) does not depend on y_i and the right-hand side of equation (25) does not depend on y_j. Since there are at least four goods (by assumption) and the expenditure share of these other goods vary with prices for a given Λ, we obtain that equations (24) and (25) hold for a neighborhood of y_i, y_j and y_k (these can vary over that neighborhood while holding Λ constant).

Denote by $f(\Lambda)=\frac{\epsilon_{\Lambda i}-\epsilon_{\Lambda j}}{\epsilon_{yi}-\epsilon_{yj}}$ the left-hand side of equations (24) and (25).

Picking any good h instead of k, equation (23) also applies and yields:

$$\epsilon_{\Lambda j}(y_j,\Lambda)-\epsilon_{\Lambda k}(y_h,\Lambda)=f(\Lambda)\left(\epsilon_{yj}(y_j,\Lambda)-\epsilon_{yh}(y_h,\Lambda)\right) \tag{26}$$

Given that there are at least four goods, this should hold over a non-trivial range of y_h (same argument as above). Taking the derivative w.r.t. y_h, we get:

$$\frac{\partial\epsilon_{\Lambda h}}{\partial\log y_h}=f(\Lambda)\,\frac{\partial\epsilon_{yh}}{\partial\log y_h} \tag{27}$$

Given the symmetry of the cross-derivative (we assume that demand is twice differentiable), this can be rewritten:

$$\frac{\partial\epsilon_{yh}}{\partial\log\Lambda}=f(\Lambda)\,\frac{\partial\epsilon_{yh}}{\partial\log y_h} \tag{28}$$

By continuity, the conditions required for this equation hold over a neighborhood of Λ. Take reference Λ_0 and consider the function: $M(y,\Lambda)=\epsilon_{yh}(y,\Lambda)-\epsilon_{yh}(yF(\Lambda),\Lambda_0)$ where $F(\Lambda)=\exp\left(\int_{\Lambda_0}^{\Lambda}f(t)\frac{dt}{t}\right)$. Note that $F(\Lambda_0)=1$ hence $M(y,\Lambda_0)=0$. For other Λ, we obtain:

$$\frac{\partial M}{\partial\log\Lambda} = \frac{\partial\epsilon_{yh}(y,\Lambda)}{\partial\log\Lambda}-\frac{\partial\log F}{\partial\log\Lambda}\,\frac{\partial\epsilon_{yh}(yF(\Lambda),\Lambda_0)}{\partial\log y_h}$$

$$
\begin{aligned}
&= \frac{\partial \epsilon_{yh}(y,\Lambda)}{\partial \log \Lambda} - f(\Lambda)\,\frac{\partial \epsilon_{yh}(yF(\Lambda),\Lambda_0)}{\partial \log y_h} \\
&= 0
\end{aligned}
$$

Hence, $M(y,\Lambda) = 0$ over the neighborhood where equation (28) holds. This implies:

$$\epsilon_{yh}(y_h,\Lambda) = \epsilon_{yk}(y_h F(\Lambda),\Lambda_0)$$

Now, for a given reference point y^* and Λ_0, let us construct D_i as:

$$D_h(y_h) = W_i(y_h^*,\Lambda_0)\ \exp\left[\int_{y_h^*}^{y_h} \epsilon_{yh}(y,\Lambda_0)dy\right]$$

Integrating over y from a reference point y^* in the region where this equality holds, we obtain that demand can be written as:

$$
\begin{aligned}
W_h(y_h,\Lambda) &= W_h(y_h^*,\Lambda)\ \exp\left[\int_{y_h^*}^{y_h} \epsilon_{yh}(y,\Lambda)\frac{dy}{y}\right] \\
&= W_h(y_h^*,\Lambda)\ \exp\left[\int_{y_h^*}^{y_h} \epsilon_{yh}(yF(\Lambda),\Lambda_0)\frac{dy}{y}\right] \\
&= W_h(y_h^*,\Lambda)\ \exp\left[\int_{y_h^* F(\Lambda)}^{y_h F(\Lambda)} \epsilon_{yh}(y,\Lambda_0)\frac{dy}{y}\right] \\
&= W_h(y_h^*,\Lambda)\,.\,\frac{D_h(y_h F(\Lambda))}{D_h(y_h^* F(\Lambda))} \\
&= H_h(\Lambda)\,D_h(y_h F(\Lambda))
\end{aligned}
$$

where H_h is a function of Λ defined as:

$$H_h(\Lambda) \equiv \frac{D_h(y_h^* F(\Lambda))}{W_h(y_h^*,\Lambda)}$$

for which, by definition of D_h, we have: $H_h(\Lambda_0) = \frac{D_h(y_h^* F(\Lambda_0))}{W_h(y_h^*,\Lambda_0)} = \frac{D_h(y_h^*)}{W_h(y_h^*,\Lambda_0)} = 1$.

Examining elasticities w.r.t. prices and Λ, we can then show that H_h is in fact identical across goods. To check this, with $W_h(y_h,\Lambda) = D_h(y_h F(\Lambda))/H_i(\Lambda)$, we have:

$$\epsilon_{yh} = \epsilon_{Dh}$$

where ϵ_{Dh} denotes the elasticity of D_h, while the elasticity in Λ is:

$$\epsilon_{\Lambda h} = f(\Lambda)\,\epsilon_{Dh} - \epsilon_{Hh}$$

where ϵ_{Hh} denotes the elasticity of H_h in Λ. Thanks for condition (26), we obtain that for any two goods h and k:

$$
\begin{aligned}
\epsilon_{Hk} - \epsilon_{Hh} &= \epsilon_{\Lambda h} - \epsilon_{\Lambda k} - f(\Lambda)\,(\epsilon_{Dh} - \epsilon_{Dk}) \\
&= \epsilon_{\Lambda h} - \epsilon_{\Lambda k} - f(\Lambda)\,(\epsilon_{yh} - \epsilon_{yk}) \\
&= 0
\end{aligned}
$$

This implies that $H_h(\Lambda)$ and $H_k(\Lambda)$ remain proportional for any two goods k and h. Given that $H_h(\Lambda_0) = H_k(\Lambda_0) = 1$, H_h must be identical across all goods h.

Hence, demand in case 1 can be written for any good h as:

$$W_h(y_h, \Lambda) = D_h(y_h F(\Lambda)) \, / \, H(\Lambda)$$

Combinations of cases: Locally, for a given Λ and around it, one must be in one of these three cases. A remaining question is whether demand can be a mixture of these three cases as Λ varies. To finish the proof of Proposition 1, we show that we cannot combine case 2 with cases 1A and 1B. Hence the functional form of case 2 needs to hold globally across all Λ's, while we can potentially have a combination of 1A and 1B across Λ.

Combination of cases 2+1A Here we show that we cannot have a combination of cases 1A and 2 globally. First, note that for a given Λ, case 1 and 2 are mutually exclusive by definition. Hence, if we have a mixture of cases 1 and 2, it must occurs along different Λ's. By contradiction, suppose that there exists Λ^* such that, at least locally,

$$\begin{aligned} W_i(y_i, \Lambda) &= D_i(F(\Lambda)y_i) \, / \, H(\Lambda) \qquad && \text{if } \Lambda < \Lambda^* \\ W_i(y_i, \Lambda) &= A_i(\Lambda) y_i^{1-\sigma(\Lambda)} && \text{if } \Lambda > \Lambda^* \end{aligned}$$

By continuity, at the limit where $\Lambda = \Lambda^*$, we must have:

$$\frac{\partial \log D_i(F(\Lambda^*)y)}{\partial \log y} = 1 - \sigma(\Lambda^*)$$

Since it must hold for any i and any y, it implies that $F^*(y) = 0$ and that $1 - \sigma(\Lambda^*) = 0$, which contradicts our assumption that $W_i(y_i, \Lambda)$ is not locally constant across y_i for any given Λ.

Combinations of cases 1+2B Here we show that we cannot have a combination of cases 1B and 2 globally, using the same arguments as above. Note again that for a given Λ, case 1B and 2 are mutually exclusive by definition. Hence, if we have a mixture of cases 1B and 2, it must occurs along different Λ's.

By contradiction, suppose that there exists Λ^* such that, at least locally, such that for all but one good we have:

$$\begin{aligned} W_i(y_i, \Lambda) &= D_i(F(\Lambda)y_k)/H(\Lambda) \qquad && \text{if } \Lambda < \Lambda^* \\ W_i(y_i, \Lambda) &= \Lambda^{-\rho} A_i y_i^{1-\sigma} && \text{if } \Lambda > \Lambda^* \end{aligned}$$

Again, by continuity, at the limit where $\Lambda = \Lambda^*$, we must have:

$$\frac{\partial \log D_i(F(\Lambda^*)y)}{\partial \log y} = 1 - \sigma$$

Since it must hold for any i and any y, it implies that either demand is CES or $F^*(y) = 0$ and that $1 - \sigma(\Lambda^*) = 0$. This contradicts the assumption that $W_i(y_i, \Lambda)$ is not locally constant across y_i for any given Λ.

Proof of Proposition 3

Define $\widetilde{U}(q,z)$ as:

$$\widetilde{U}(q,z) = \sum_i u_i(H(z)q_i) \; - \; \int_{z_0}^{z} F(z)H'(z)dz$$

where:

$$u_i(q_i) = \int_{q=0}^{q_i} D_i^{-1}(x)dx$$

and $u_i' = D_i^{-1}$. Recall that D_i is strictly decreasing unless $D_i = 0$. As noted in the text, as an abuse of notation, we define $D_i^{-1}(0) = a_i$ if $D_i(y) = 0$ for all $y \geq a_i$ (which yields a choke price) and $D_i^{-1}(x) = 0$ for all $x \geq b_i$ if $D_i(0) = b_i$.

In turn, we want to define z as an implicit function of q such that:

$$F(z) = \sum_i q_i u_i'(H(z)q_i) \tag{29}$$

We proceed in three steps. First we show that equation (29) admits a solution $z(q)$ for each q and that this solution is unique. Second we show that utility defined as $U(q) = \widetilde{U}(q, z(q))$ is well-behaved and quasi-concave. Finally, we show that maximizing U leads to the demand function in the text, and that the single aggregator Λ is also well defined.

Step 1: Implicit function $z(q)$. Here we show that for any vector q of consumption, there is a z such that equation (29) holds.

With restrictions [A1] iii), we have assumed that for each good i and each y_i, there is a Λ such that $D_i(F(\Lambda)y_i)/H(\Lambda)$ is arbitrarily small. Take $y_i = 1/(Nq_i)$ where N denotes the number of goods. For each i, there is a z such that:

$$D_i(F(z)/(Nq_i))/H(z) < q_i$$

Since the left-hand side decreases with z (this is implied by restriction [A1]-ii), we can take the maximum z of all z_i's such that it holds for all goods i with a common z. This inequality is equivalent to:

$$\begin{array}{rrcl} & D_i(F(z)/(Nq_i)) & < & H(z)q_i \\ \Leftrightarrow & F(z)/(Nq_i)) & > & u_i'(H(z)q_i) \\ \Leftrightarrow & F(z)/N & > & q_i u_i'(H(z)q_i) \end{array}$$

Going from the first to second inequality above is guaranteed by the assumption that D_i decreases strictly. Adding across all goods (given that $q_i u_i'$ is positive for all goods), we obtain that for each vector q there exists a z such that:

$$F(z) > \sum_i q_i u_i'(H(z)q_i)$$

Next, with restrictions [A1] iii), we have also assumed that there is at least one good i such that, for each y_i, there is a Λ such that $y_i D_i(F(\Lambda)y_i)/H(\Lambda)$ is larger than 1. Taking $y_i = 1/q_i$, there is a $z = \Lambda$

such that:

$$\begin{array}{rcl} (1/q_i)D_i(F(z)/q_i)/H(z) & > & 1 \\ \Leftrightarrow \;\; D_i(F(z)/q_i) & > & H(z)q_i \\ \Leftrightarrow \;\; F(z)/(Nq_i)) & < & u_i'(H(z)q_i) \\ \Leftrightarrow \qquad\qquad F(z) & < & q_i u_i'(H(z)q_i) \end{array}$$

Hence, summing across goods, we also have:

$$F(z) < \sum_i q_i u_i'(H(z)q_i)$$

We have shown that for each q there is a z such that $F(z) > \sum_i q_i u_i'(H(z)q_i)$ and a z such that $F(z) < \sum_i q_i u_i'(H(z)q_i)$. By continuity, we conclude that equation (29) has a solution.

Then, using part ii) of restrictions [A1], we can see that $D_i(F(\Lambda)y_i)/H(\Lambda)$ strictly decreases with Λ. This implies that $q_i u_i'(H(z)q_i)$ also strictly decreases with z, and that the right-hand side of equation (29), $\sum_i q_i u_i'(H(z)q_i)$, decreases faster (or increases slower) than the left-hand side of equation (29), $F(z)$. Hence the solution to equation (29) is unique.

Step 2: Quasi-concavity. The second step is to show that utility defined as $U(q) = \widetilde{U}(q, z(q))$ is quasi-concave. First, we need to get the first and second derivatives.

Derivatives in z. Here we consider the properties of $z = z(q)$, the solution of equation (29). Taking the derivative of equation (29), we get:

$$\frac{\partial z}{\partial q_i}\left[F' - H'\sum_i q_i^2 u_i''\right] = u_i' + Hq_i u_i''$$

and thus:

$$\frac{\partial z}{\partial q_i} = \frac{u_i' + Hq_i u_i''}{\Delta}$$

with $\Delta \equiv F' - H' \sum_i q_i^2 u_i''$.

Showing that Δ is positive. Note that $\frac{u_i'}{u_i'' H q_i} = \varepsilon_{Di}$ and thus:

$$\begin{aligned} \Delta &= F' - H'\sum_i q_i^2 u_i'' \\ &= (F/z)\left(\varepsilon_F - \varepsilon_H \frac{\sum_i H q_i^2 u_i''}{\sum_i q_i u_i'}\right) \\ &= (F/z)\left(\varepsilon_F - \varepsilon_H \frac{\sum_i q_i u_i'(1/\varepsilon_{Di})}{\sum_i q_i u_i'}\right) \end{aligned}$$

We can see that, if $\varepsilon_{Di} < 0$ and $\varepsilon_F \varepsilon_{Di} < \varepsilon_H$ for all i, then $\varepsilon_F - \varepsilon_H\,(1/\varepsilon_{Di}) > 0$ for all i and we always have $\Delta > 0$.

This implies that the derivatives of z are always well defined, and knowing $\Delta > 0$ will be useful below.

Derivatives in U. First derivatives:

$$\frac{\partial U}{\partial q_i} \;=\; H\,u_i'(Hq_i) + \frac{\partial z}{\partial q_i}\left[H'\sum_i q_i u_i'(Hq_i) - H'F\right] \;=\; H\,u_i'(Hq_i)$$

Second derivatives:

$$\frac{\partial^2 U}{\partial q_i^2} \;=\; \frac{\partial z}{\partial q_i}\,(u_i' + Hq_iu_i'')H' \;+\; H^2\,u_i''$$
$$\frac{\partial^2 U}{\partial q_i \partial q_j} \;=\; \frac{\partial z}{\partial q_j}\,(u_i' + Hq_iu_i'')H'$$

and thus, incorporating the derivatives in z:

$$\frac{\partial^2 U}{\partial q_i^2} \;=\; (u_i' + Hq_iu_i'')^2\,H'/\Delta \;+\; H^2\,u_i''$$
$$\frac{\partial^2 U}{\partial q_i \partial q_j} \;=\; (u_i' + Hq_iu_i'')\,(u_j' + Hq_ju_j'')\,H'\,/\Delta$$

Negative semi-definiteness. To show that utility is quasi-convex, we need to show that the bordered Hessian is negative semi-definite, i.e we need to show:

$$\sum_{i,j} t_i t_j \frac{\partial^2 U}{\partial q_i \partial q_j} \;=\; \left(\sum_i t_i\,(u_i' + Hq_iu_i'')\right)^2 H'\,/\Delta \;+\; \sum_i t_i^2\,H^2\,u_i'' \;<\; 0$$

for any t_i such that:

$$\sum_i t_i \frac{\partial U}{\partial q_i} \;=\; \sum_i t_i H u_i' \;=\; 0$$

The objective function above is homogeneous of degree 2. We can renormalize the sum $\sum_i t_i(u_i' + Hq_iu_i'')$ up to any constant without loss of generalization.

First step is to find the optimal vector of t_i's that maximizes the left-hand side of the inequality above. It is equivalent to consider the maximization:

$$\max\left\{\sum_i t_i^2\,u_i''\right\}$$

under the constraint: $\sum_i t_i(u_i' + Hq_iu_i'') = \textit{constant}$ and $\sum_i t_iu_i' \;=\; 0$.

This leads to t_i proportional to:

$$t_i \;\sim\; \frac{u_i'}{Hu_i''} + \mu q_i$$

(and second-order condition is fine, objective function is concave since $u_i'' < 0$ for all i). Given that we must have $0 = \sum_i t_iu_i' = \sum_i \frac{u_i'^2}{Hu_i''} + \mu \sum_i q_iu_i'$, μ must be:

$$\mu \;=\; -\frac{\sum_i \frac{u_i'^2}{Hu_i''}}{\sum_i q_iu_i'}$$

$$
\begin{aligned}
&= -\frac{\sum_i q_i u_i' \frac{u_i'}{q_i H u_i''}}{\sum_i q_i u_i'} \\
&= -\frac{\sum_i q_i u_i' \, \varepsilon_{Di}}{\sum_i q_i u_i'} \\
&= -\bar{\varepsilon}_D
\end{aligned}
$$

where $\varepsilon_{Di} = \frac{u_i'}{q_i H u_i''}$ and $\bar{\varepsilon}_D$ is its weighted average.

Next, using the optimal $t_i = \frac{u_i'}{H u_i''} - \bar{\varepsilon}_D q_i = q_i \varepsilon_{Di} - q_i \bar{\varepsilon}_D$, a sufficient and necessary condition for negative semi-definiteness is:

$$
\left(\sum_i (q_i \varepsilon_{Di} - q_i \bar{\varepsilon}_D)(u_i' + q_i H u_i'') \right)^2 H' / \Delta \; + \; H^2 \sum_i (q_i \varepsilon_{Di} - q_i \bar{\varepsilon}_D)^2 \, u_i'' < 0
$$

Since $\Delta > 0$, this condition can be rewritten:

$$
\begin{aligned}
\left(\sum_i (q_i \varepsilon_{Di} - q_i \bar{\varepsilon}_D)(u_i' + q_i H u_i'') \right)^2 H' \;&<\; - H^2 \, \Delta \sum_i (q_i \varepsilon_{Di} - q_i \bar{\varepsilon}_D)^2 \, u_i'' \\
\Leftrightarrow \left(\sum_i q_i u_i' - \bar{\varepsilon}_D H \sum_i q_i^2 u_i'' \right)^2 H' \;&<\; - H^2 \, \Delta \sum_i (q_i \varepsilon_{Di} - q_i \bar{\varepsilon}_D)^2 \, u_i'' \\
\Leftrightarrow \left(\sum_i q_i u_i' - \bar{\varepsilon}_D H \sum_i q_i^2 u_i'' \right)^2 H' \;&<\; - H \, \Delta \left(-\bar{\varepsilon}_D \sum_i q_i u_i' + \bar{\varepsilon}_D^2 \, H \sum_i q_i^2 u_i'' \right) \\
\Leftrightarrow \left(\sum_i q_i u_i' - \bar{\varepsilon}_D H \sum_i q_i^2 u_i'' \right)^2 H' \;&<\; \bar{\varepsilon}_D \, H \, \Delta \left(\sum_i q_i u_i' - \bar{\varepsilon}_D \, H \sum_i q_i^2 u_i'' \right)
\end{aligned}
$$

Note that $\left(\sum_i q_i u_i' - \bar{\varepsilon}_D H \sum_i q_i^2 u_i'' \right)$ is negative (unless all price elasticities ε_{Di} are identical):

$$
\sum_i q_i u_i' - \bar{\varepsilon}_D H \sum_i q_i^2 u_i'' < 0 \iff \frac{\sum_i q_i u_i'}{\sum_i q_i u_i' \frac{1}{(-\varepsilon_{Di})}} < \frac{\sum_i q_i u_i' (-\varepsilon_{Di})}{\sum_i q_i u_i'}
$$

(In the second inequality, the left hand side corresponds to a harmonic average while the right-hand-side corresponds to an arithmetic average of a positive variable $-\varepsilon_{Di} > 0$).

Hence, using $\sum_i q_i u_i' - \bar{\varepsilon}_D H \sum_i q_i^2 u_i'' < 0$ and also that $\Delta \equiv F' - H' \sum_i q_i^2 u_i''$ the previous inequality is equivalent to:

$$
\begin{aligned}
\Leftrightarrow \quad & H' \left(\sum_i q_i u_i' - \bar{\varepsilon}_D H \sum_i q_i^2 u_i'' \right) > \bar{\varepsilon}_D \, H \, \Delta \\
\Leftrightarrow \quad & H' \left(\sum_i q_i u_i' - \bar{\varepsilon}_D H \sum_i q_i^2 u_i'' \right) > \bar{\varepsilon}_D \, H \left(F' - H' \sum_i q_i^2 u_i'' \right) \\
\Leftrightarrow \quad & H' \sum_i q_i u_i' > \bar{\varepsilon}_D \, H \, F'
\end{aligned}
$$

Given that $F = \sum_i q_i u_i'$, this gives:

$$\begin{aligned} &\Leftrightarrow \quad H' \, F > \bar{\varepsilon}_D \; H \, F' \\ &\Leftrightarrow \quad \bar{\varepsilon}_D \varepsilon_F < \varepsilon_H \end{aligned}$$

This holds, given that $\varepsilon_{Di}\varepsilon_F < \varepsilon_H$ is assumed in part ii) of restrictions [A1] for each good i.

Step 3: Marshallian demand and price aggregator. Maximizing $U(q)$ under the budget constraint $\sum_i p_i q_i = w$ leads to:

$$\frac{\partial U}{\partial q_i} \;=\; H(z)\, u_i'(H(z)q_i) \;=\; \mu p_i$$

where μ henceforth denotes the Lagrange multiplier associated with the budget constraint. Summing across goods, we can see that μ is such that:

$$\mu \;=\; \frac{1}{w}\sum_i \mu p_i q_i \;=\; \frac{1}{w}\sum H\, q_i\, u_i'(H q_i) \;=\; \frac{H(z)F(z)}{w}$$

(note that it only depends on quantities and prices through z). Using $H(z)\, u_i'(H(z)q_i) \;=\; \mu p_i$, we obtain:

$$u_i'(H(z)q_i) = \frac{\mu p_i}{H(z)} = \frac{F(z)p_i}{w}$$

and thus, given the definition of u_i':

$$H(z)q_i = D_i(\mu p_i / H(z)) = D_i(F(z)p_i/w)$$

and:

$$q_i = D_i(F(z)p_i/w)/H(z)$$

The final step is to show that z can be written as $z = \Lambda$ where Λ is implicitly defined as a function of all normalized prices p_i/w.

To see this, notice that q_i must satisfy the budget constraint:

$$w = \sum_i q_i p_i = \sum_i p_i D_i(F(z)p_i/w)/H(z)$$

which can be rewritten:

$$H(z) = \sum_i (p_i/w) D_i(F(z)p_i/w)$$

This equation in z has a unique solution, which we denote Λ and is a function of all p_i/w:

- To prove uniqueness, we use restriction [A1] part ii) which implies that $D_i(F(z)p_i/w)/H(z)$ is strictly decreasing in z. Hence the right-hand side of the equation above decreases strictly faster with z (or increases strictly slower) than the left-hand side.

- To prove existence (for a given set of prices and income), we use restriction [A1] part iii) which assumes that $D_i(F(z)p_i/w)/H(z)$ can be arbitrarily small and that there is at least one good for which $D_i(F(z)p_i/w)/H(z)$ is larger for unity for some z. By continuity, there must be at least one solution $\Lambda = z$ to the equation $H(z) = \sum_i (p_i/w) D_i(F(z)p_i/w)$.

Alternative proof of Proposition 3 using the Slutsky Matrix

An alternative proof of proposition 3 is to show that the Slutsky matrix is symmetric and negative semi-definite, and then apply Hurwicz and Uzawa (1971) theorem. We have already proved symmetry in Proposition 1 but semi-definitiveness is yet to be checked. Consider the demand system:

$$q_i = D_i(F(\Lambda)p_i/w) \,/\, H(\Lambda) \tag{30}$$

where Λ is implicitly determined by the budget constraint $\sum_i p_i D_i(\Lambda p_i/w) \,/\, H(\Lambda) = w$, which can be rewritten:

$$\sum_i (p_i/w) D_i(F(\Lambda)p_i/w) \,/\, H(\Lambda) = 1 \tag{31}$$

Slutsky substitution coefficients. For $i \neq j$, the Slutsky term is:

$$\begin{aligned} s_{ij} &= \frac{\partial q_i}{\partial p_j} + q_j \frac{\partial q_i}{\partial w} \\ &= \frac{\partial q_i}{\partial \Lambda} \left[\frac{\partial \Lambda}{\partial p_j} + q_j \frac{\partial \Lambda}{\partial w} \right] \;-\; q_i q_j \, \frac{\varepsilon_{Di}}{w} \end{aligned}$$

where $\bar{\varepsilon}_D = \sum_i (p_i q_i/w)\varepsilon_{Di}$ and $\varepsilon_{Di} = \frac{\partial \log D_i}{\partial \log x}$. The first term is:

$$\frac{\partial q_i}{\partial \Lambda} = \frac{q_i(\varepsilon_F \varepsilon_{Di} - \varepsilon_H)}{\Lambda}$$

Then, we get the derivative of Λ w.r.t. p_j and w. From:

$$\sum_i (p_i/w) D_i(F(\Lambda)p_i/w) \,/\, H(\Lambda) = 1 \tag{32}$$

we get:

$$\begin{aligned} \frac{\partial \log \Lambda}{\partial \log p_j} &= -(p_j q_j/w) \, \frac{1 + \varepsilon_{Dj}}{\varepsilon_F \bar{\varepsilon}_D - \varepsilon_H} \\ \frac{\partial \log \Lambda}{\partial \log w} &= \frac{1 + \bar{\varepsilon}_D}{\varepsilon_F \bar{\varepsilon}_D - \varepsilon_H} \end{aligned}$$

Hence we obtain:

$$\begin{aligned} s_{ij} &= \frac{q_i(\varepsilon_F \varepsilon_{Di} - \varepsilon_H)}{\Lambda} \,.\, \Lambda \left[-(q_j/w) \, \frac{1 + \varepsilon_{Dj}}{\varepsilon_F \bar{\varepsilon} - \varepsilon_H} \;+\; (q_j/w) \, \frac{1 + \bar{\varepsilon}_D}{\varepsilon_F \bar{\varepsilon}_D - \varepsilon_H} \right] \;-\; q_i q_j \, \frac{\varepsilon_{Di}}{w} \\ &= \frac{q_i q_j}{w} \cdot \frac{(\varepsilon_F \varepsilon_{Di} - \varepsilon_H)(\bar{\varepsilon} - \varepsilon_{Dj})}{\varepsilon_F \bar{\varepsilon}_D - \varepsilon_H} \;-\; q_i q_j \, \frac{\varepsilon_{Di}}{w} \\ &= -\,\frac{q_i q_j}{w} \cdot \frac{\varepsilon_F (\varepsilon_{Di} - \bar{\varepsilon}_D)(\varepsilon_{Dj} - \bar{\varepsilon}_D)}{\varepsilon_F \bar{\varepsilon}_D - \varepsilon_H} \;-\; q_i q_j \, \frac{\varepsilon_{Dj} + \varepsilon_{Di} - \bar{\varepsilon}}{w} \end{aligned}$$

which is symmetric in i and j.

When $\varepsilon_F \neq 0$, denote $\theta_i = \varepsilon_H - \varepsilon_F \varepsilon_{Di}$, which is positive by assumption, and $\bar{\theta} = \sum_i (p_i q_i/w)\theta_i$ its weighted average across goods. It is more convenient to write the Slutsky substitution coefficient as:

$$\begin{aligned} s_{ij} &= \frac{q_i q_j}{w} \cdot \frac{(\theta_i - \bar{\theta})(\theta_j - \bar{\theta})}{\varepsilon_F \, \bar{\theta}} + \frac{q_i q_j}{\varepsilon_F w} \left(\theta_j + \theta_i - \bar{\theta} - \varepsilon_H\right) \\ &= \frac{q_i q_j}{w} \cdot \frac{\theta_i \; \theta_j}{\varepsilon_F \bar{\theta}} - \frac{q_i q_j}{w} \frac{\varepsilon_H}{\varepsilon_F} \end{aligned}$$

In turn, the diagonal coefficients of the Slutsky matrix are:

$$\begin{aligned} s_{ii} &= \frac{q_i \varepsilon_{Di}}{p_i} - \frac{q_i^2}{w} \cdot \frac{\theta_i^2}{\varepsilon_F \bar{\theta}} - \frac{q_i^2}{w} \frac{\varepsilon_H}{\varepsilon_F} \\ &= -\frac{q_i(\theta_i - \varepsilon_H)}{p_i \varepsilon_F} - \frac{q_i^2}{w} \cdot \frac{\theta_i^2}{\varepsilon_F \bar{\theta}} - \frac{q_i^2}{w} \frac{\varepsilon_H}{\varepsilon_F} \end{aligned}$$

Is the Slutsky matrix negative semi-definite? Denote $w_i = p_i q_i / w$ expenditure shares, we obtain:

$$\begin{aligned} s_{ij} \, . \, p_i p_j / w &= \frac{w_i w_j \theta_i \theta_j}{\varepsilon_F \sum_k w_k \theta_k} - w_i w_j \frac{\varepsilon_H}{\varepsilon_F} \\ s_{ii} \, . \, p_i^2 / w &= -\frac{w_i(\theta_i - \varepsilon_H)}{\varepsilon_F} + \frac{w_i^2 \theta_i^2}{\varepsilon_F \sum_k w_k \theta_k} - w_i^2 \frac{\varepsilon_H}{\varepsilon_F} \end{aligned}$$

In order to show that the Slutsky matrix is negative semi-definite, we need to show that for any vector x we have:

$$\sum_{ij} s_{ij} x_i x_j \leq 0$$

Normalizing the x's by $p_i/\sqrt{w}$, this is equivalent to showing the following inequality:

$$-\sum_i w_i \left(\frac{\theta_i - \varepsilon_H}{\varepsilon_F}\right) x_i^2 + \sum_{i,j} \frac{w_i x_i w_j x_j \theta_i \theta_j}{\sum_i w_i \theta_i} - \sum_{i,j} w_i w_j x_i x_j \left(\frac{\varepsilon_H}{\varepsilon_F}\right) \leq 0$$

$$\iff -\frac{\sum_i w_i \theta_i}{\varepsilon_F} \left[\frac{\sum_i w_i \theta_i x_i^2}{\sum_i w_i \theta_i} - \frac{(\sum_i w_i \theta_i x_i)^2}{(\sum_i w_i \theta_i)^2}\right] + \frac{\varepsilon_H}{\varepsilon_F}\left[\sum_i w_i x_i^2 - \left(\sum_i w_i x_i\right)^2\right] \leq 0$$

$$\iff -\frac{1}{\varepsilon_F} \left[\frac{\sum_i w_i \theta_i x_i^2}{\sum_i w_i \theta_i} - \frac{(\sum_i w_i \theta_i x_i)^2}{(\sum_i w_i \theta_i)^2}\right] + \frac{\varepsilon_H}{\varepsilon_F \sum_i w_i \theta_i}\left[\sum_i w_i x_i^2 - \left(\sum_i w_i x_i\right)^2\right] \leq 0$$

The terms in brackets can be interpreted as variances and are positive. We need to distinguish three cases depending on the sign of ε_H and ε_F:

- First, if $\varepsilon_F > 0$ and $\varepsilon_H \leq 0$ (and $\theta_i > 0$), each term of the sum is negative, the sum is negative and the Slutsky matrix is semi-definite negative.
- If $\varepsilon_F > 0$ and $\varepsilon_H > 0$, note that $\theta_i = \varepsilon_H - \varepsilon_F \varepsilon_{Di}$ is then larger than ε_H given that ε_{Di} is assumed to be negative. With $\varepsilon_F > 0$, the inequality above is then equivalent to:

$$\iff \left[\frac{\sum_i w_i \theta_i x_i^2}{\sum_i w_i \theta_i} - \frac{(\sum_i w_i \theta_i x_i)^2}{(\sum_i w_i \theta_i)^2}\right] - \frac{\varepsilon_H}{\sum_i w_i \theta_i}\left[\sum_i w_i x_i^2 - \left(\sum_i w_i x_i\right)^2\right] \geq 0$$

The proof then relies on Lemma 5, an auxiliary result from Matsuyama and Uchshev (2017). In this case, denote $a_i = \frac{\theta_i w_i}{\sum_j w_j \theta_j}$, $b_i = w_i$ and $\gamma = \frac{\varepsilon_H}{\sum_i w_i \theta_i}$. Since $\theta_i > \varepsilon_H$ for each i, we have $a_i > \gamma b_i$ for each i. We obtain the result that we want by applying Lemma 5, which implies the inequality above.

- Finally, if $\varepsilon_F < 0$, note that $\varepsilon_H > \theta_i$ and $\varepsilon_H > 0$ given that ε_{Di} is negative by assumption, and given that $\theta_i = \varepsilon_H - \varepsilon_F \varepsilon_{Di}$ is positive by assumption. With $\varepsilon_F < 0$, the inequality above is then equivalent to:

$$\iff \quad \left[\sum_i w_i x_i^2 \; - \; \Big(\sum_i w_i x_i \Big)^2 \right] \; - \; \frac{\sum_i w_i \theta_i}{\varepsilon_H} \left[\frac{\sum_i w_i \theta_i x_i^2}{\sum_i w_i \theta_i} \; - \; \frac{(\sum_i w_i \theta_i x_i)^2}{(\sum_i w_i \theta_i)^2} \right] \; \geq \; 0$$

The proof again relies on Lemma 5. In this case, denote $a_i = w_i$, $b_i = \frac{\theta_i w_i}{\sum_j w_j \theta_j}$, and $\gamma = \frac{\sum_i w_i \theta_i}{\varepsilon_H}$. Since $\theta_i < \varepsilon_H$ for each i, we obtain again that $a_i > \gamma b_i$ for these new a_i and b_i. Lemma 5 then implies the inequality above.

This finishes the proof that the Slutsky matrix is negative semi-definite when $\varepsilon_F \neq 0$.

What happens when $\varepsilon_F = 0$? When $\varepsilon_F = 0$, then $\varepsilon_H > 0$ by assumption, and we have instead:

$$\frac{\partial q_i}{\partial \Lambda} = -\frac{q_i \varepsilon_H}{\Lambda} \qquad ; \qquad \frac{\partial \log \Lambda}{\partial \log p_j} = (p_j q_j / w) \; \frac{1 + \varepsilon_{Dj}}{\varepsilon_H} \qquad ; \qquad \frac{\partial \log \Lambda}{\partial \log w} \; = \; - \; \frac{1 + \bar{\varepsilon}}{\varepsilon_H}$$

Hence we obtain:

$$s_{ij} \quad = \quad \frac{q_i \varepsilon_H}{\Lambda} \, . \, \Lambda \; \left[-(q_j/w) \; \frac{1 + \varepsilon_{Dj}}{\varepsilon_H} \; + \; (q_j/w) \; \frac{1 + \bar{\varepsilon}}{\varepsilon_H} \right] \; - \; q_i q_j \; \frac{\varepsilon_{Di}}{w}$$

and thus the Slutsky substitution coefficients are:

$$\begin{aligned} s_{ij} \quad &= \quad - \; q_i q_j \; \frac{\varepsilon_{Dj} + \varepsilon_{Di} - \bar{\varepsilon}}{w} \\ s_{ii} \quad &= \quad \frac{q_i \varepsilon_{Di}}{p_i} \; - \; q_i^2 \; \frac{2\varepsilon_{Di} - \bar{\varepsilon}}{w} \end{aligned}$$

Is the Slutsky matrix negative semi-definite in the special case where $\varepsilon_F = 0$? As before, we need:

$$\sum_i w_i x_i^2 \varepsilon_{Di} \; - \; \sum_{i,j} w_i w_j x_i x_j (\varepsilon_{Dj} + \varepsilon_{Di} - \bar{\varepsilon}) \; \leq \; 0$$

where w_i denotes the consumption share in good i.

This statement is successively equivalent to:

$$\begin{aligned} &\iff \sum_i w_i x_i^2 \varepsilon_{Di} \; - \; 2 \sum_{i,j} w_i w_j x_i x_j \varepsilon_{Dj} + \bar{\varepsilon} \big(\sum_i w_i x_i \big)^2 \; \leq \; 0 \\ &\iff \sum_i w_i x_i^2 \varepsilon_{Di} \; - \; 2 \big(\sum_i w_i x_i \big) \big(\sum_i w_i x_i \varepsilon_{Dj} \big) + \big(\sum_i w_i \varepsilon_{Dj} \big) \big(\sum_i w_i x_i \big)^2 \; \leq \; 0 \\ &\iff \frac{\sum_i w_i x_i^2 \varepsilon_{Di}}{\sum_i w_i \varepsilon_{Dj}} \; - \; \frac{2 (\sum_i w_i x_i) (\sum_i w_i x_i \varepsilon_{Dj})}{\sum_i w_i \varepsilon_{Dj}} + \big(\sum_i w_i x_i \big)^2 \; \geq \; 0 \end{aligned}$$

$$
\begin{aligned}
\iff \frac{\sum_i w_i \varepsilon_{Di}\Big(x_i^2 + (\sum_j w_j x_j)^2\Big)}{\sum_i w_i \varepsilon_{Dj}} &\geq \frac{2(\sum_i w_i x_i)(\sum_i w_i x_i \varepsilon_{Dj})}{\sum_i w_i \varepsilon_{Dj}} \\
\iff \frac{\sum_i w_i \varepsilon_{Di}\Big(x_i - (\sum_j w_j x_j)\Big)^2}{\sum_i w_i \varepsilon_{Dj}} + \frac{2\sum_i w_i \varepsilon_{Di} x_i (\sum_j w_j x_j)}{\sum_i w_i \varepsilon_{Dj}} &\geq \frac{2(\sum_i w_i x_i)(\sum_i w_i x_i \varepsilon_{Dj})}{\sum_i w_i \varepsilon_{Dj}} \\
\iff \frac{\sum_i w_i \varepsilon_{Di}\Big(x_i - (\sum_j w_j x_j)\Big)^2}{\sum_i w_i \varepsilon_{Dj}} &\geq 0
\end{aligned}
$$

which is satisfied if all the ε_{Dj} are negative.

Lemma 5 *(Matsuyama Ushchev, 2017) Suppose that* $a_i > \gamma b_i > 0$ *for all* $i \in \{1, ..., N\}$ *where* γ *is a positive scalar and* $\sum_i a_i = \sum_i b_i = 1$*. Define* $\boldsymbol{A} = diag\{a_1, ..., a_n\} - \boldsymbol{a}\boldsymbol{a}^T$ *and* $\boldsymbol{B} = diag\{b_1, ..., b_n\} - \boldsymbol{b}\boldsymbol{b}^T$*, two (positive semi-definite) matrices. We obtain that the matrix* $\boldsymbol{M}$*:*

$$\boldsymbol{M} = \boldsymbol{A} - \gamma \boldsymbol{B}$$

is positive semi-definite.

Proof of Lemma 5: This lemma is from Matsuyama and Ushchev (2017) which I report here again for convenience (see last part of the proof of Proposition 1 of Matsuyama and Ushchev 2017).

One needs to show that, for each vector t of the $\mathbb{R}_{N+}$ with components $t_i \geq 0, \;\; i \in \{1, ..., N\}$:

$$t^T A\, t - \gamma \,.\, t^T B\, t \geq 0$$

Denote T_a and T_b the random variables such that: $Prob\{T_a = t_i\} = a_i$ and $Prob\{T_b = t_i\} = b_i$. Since $\sum_i a_i = \sum_i b_i = 1$, one can write each term above as a variance:

$$
\begin{aligned}
t^T A\, t &= \sum_i a_i t_i^2 - \left(\sum_i a_i t_i\right)^2 = Var(T_a) \\
t^T B\, t &= \sum_i b_i t_i^2 - \left(\sum_i b_i t_i\right)^2 = Var(T_b)
\end{aligned}
$$

Note that $Var(T_b) > 0$ unless $t_i = t_j$ for all i, j, in which case we also have $Var(T_a) = 0$ and thus $t^T A\, t - \gamma \,.\, t^T B\, t = 0$. Since $Var(T_b)$ is homogeneous of degree two and strictly positive aside from the case above, we can focus on t's such that $t^T B\, t = 1$. Under this assumption, we need to show that:

$$t^T A\, t \geq \gamma$$

Consider the maximization:

$$\max_t t^T A\, t \qquad s.t. \qquad t^T B\, t = 1$$

The maximum is attained when $At_i^* = \Lambda^* B t^*$ where Λ^* is the minimum value of the objective function, which can also be written:

$$a_i\Big(t_i^* - E[T_a^*]\Big) = \Lambda^* b_i \Big(t_i^* - E[T_b^*]\Big) \tag{33}$$

where $E[T_x]$ refers to the expectation of T_x.

The goal is to show that $\Lambda^* > \gamma$. To prove this claim by contradiction, suppose that $\Lambda^* < \gamma$. Given that $a_i > \gamma b_i$, we also have $a_i > \Lambda^* b_i$.

If $E[T_a] \leq E[T_b]$, we can see that:

$$\max t_i - E[T_a] \geq \max t_i - E[T_b] > 0$$

hence equation (33) cannot hold for $i = \arg\max t_j$.

If $E[T_a] > E[T_b]$, we can see that:

$$\min t_i - E[T_a] < \min t_i - E[T_b] < 0$$

hence, again, equation (33) cannot hold for $i = \arg\min t_j$. This yields a contradiction and concludes the proof of Lemma 5.

Proof of Proposition 4

Suppose that demand can be written:

$$q_i = G_i(\Lambda)^{1-\sigma(\Lambda)}\,(p_i/w)^{-\sigma(\Lambda)}$$

with Λ implicitly defined by $\sum_i [G_i(\Lambda)p_i/w]^{1-\sigma(\Lambda)} = 1$.

The goal is to to show that these equations:

$$\left[\sum_i (G_i(\Lambda)\,p_i/w)^{1-\sigma(\Lambda)}\right]^{\frac{1}{1-\sigma(\Lambda)}} = 1 \tag{34}$$

$$\left[\sum_i (G_i(U)/q_i)^{\frac{1-\sigma(U)}{\sigma(U)}}\right]^{\frac{\sigma(U)}{1-\sigma(U)}} = 1 \tag{35}$$

have a unique solution in Λ and U respectively. To do so, we show that the left-hand side of each of these equations strictly increase in Λ and U around the solution, showing that the left-hand side can be equal to unity only once.

We distinguish two cases, depending on whether elasticity $\sigma(\Lambda)$ increases with Λ. If the first case we assume that $G_i(\Lambda)$ strictly increases with Λ. In the second case, we impose condition ii).

1) In the first case, suppose that $\sigma(\Lambda)$ increases with Λ and that $G_i(\Lambda)$ strictly increases with Λ. The equation above in Λ is equivalent to:

$$\sum_i (G_i(\Lambda)\,p_i/w)^{1-\sigma(\Lambda)} = 1$$

If $\sigma(\Lambda) \in (0,1)$, each term $G_i(\Lambda)\,p_i/w$ in the summation increases in Λ and has to be smaller than unity. Hence, if $1-\sigma(\Lambda)$ decreases with Λ, the left-hand side of this expression is strictly increasing with Λ. The same holds if we raise the whole expression on the left-hand side to the power $\frac{1}{1-\sigma(\Lambda)}$.

If $\sigma(\Lambda) > 1$, each term $G_i(\Lambda)\,p_i/w$ in the summation increases in Λ and has to be larger than unity. Hence, if $1-\sigma(\Lambda)$ decreases with Λ (i.e. becomes more positive), the left-hand side of this expression is strictly decreasing in Λ. The inverse holds if we raise the whole expression on the left-hand side to the power $\frac{1}{1-\sigma(\Lambda)} < 0$.

Now consider the equation:

$$\sum_i \Big(G_i(U)/q_i\Big)^{\frac{1-\sigma(U)}{\sigma(U)}} = 1$$

If $\sigma(\Lambda) \in (0,1)$, the exponent $\frac{1-\sigma(U)}{\sigma(U)}$ is positive and decreases with U. The term within parenthesis increases in U. Moreover, each summation term has to be smaller than unity. Hence, as U increases, each summation term increases (strictly) with U. The same holds if we raise the whole expression on the left-hand side to the power $\frac{\sigma(U)}{1-\sigma(U)}$.

If $\sigma(\Lambda) > 1$, the exponent $\frac{1-\sigma(U)}{\sigma(U)}$ is negative and decreases with U. The term within parenthesis increases in U. Moreover, each summation term has to be larger than unity. Hence, as U increases, each summation term decreases (strictly) with U. If we raise the whole expression on the left-hand side to the power $\frac{\sigma(U)}{1-\sigma(U)}$, we obtain a strictly increasing function of U.

2) In the second case, we assume that $\sigma(\Lambda)$ decreases with Λ and that, around each solution Λ_0 of equation (34), there exists a set of α_i such that $\sum_i \alpha_i = 1$ and such that $G_i(\Lambda)\alpha_i^{-\frac{1}{1-\sigma(\Lambda)}}$ increases in Λ.

Define $K_i(\Lambda) = G_i(\Lambda)\alpha_i^{-\frac{1}{1-\sigma(\Lambda)}}$ The left-hand side of equation (34) can then be rewritten:

$$\left[\sum_i \alpha_i (K_i(\Lambda)\, p_i/w)^{1-\sigma(\Lambda)}\right]^{\frac{1}{1-\sigma(\Lambda)}}$$

To show that it strictly increases in Λ, we use Lemma 6 discussed in the next appendix section. We obtain that the left-hand side of the above equation decreases with σ, which itself decreases with Λ. Moreover, the term $K_i(\Lambda)$ strictly increases in Λ, by assumption, hence the whole left term strictly increases with Λ.

We can again use the same approach to show that the left-hand side of (35) increases strictly with U. This is equivalent to showing that the following expression strictly increases in U:

$$\left[\sum_i \alpha_i \Big(K_i(U)/q_i\Big)^{\frac{1-\sigma(U)}{\sigma(U)}}\right]^{\frac{\sigma(U)}{1-\sigma(U)}}$$

Each exponent $\frac{1-\sigma(U)}{\sigma(U)}$ increases in U and each term $K_i(U)$ strictly increases with U. With Lemma 6 again, we obtain that the whole term strictly increases with U.

Hence, in both cases, Λ and U are well defined by equations (34) and (35) which admit no more than one solution. This implicitly defines utility U as a function of q_i. It is straightforward to see that such utility function is quasi-concave in q: indifference curves have the same shape as CES indifference curves, holding $\sigma = \sigma(U)$ constant.

Consumption quantities q chosen to maximize U would satisfy the following first-order conditions:

$$\frac{(\sigma(U)-1)}{q_i\sigma(U)} \left(\frac{q_i}{G_i(U)}\right)^{\frac{\sigma(U)-1}{\sigma(U)}} = \mu p_i$$

where μ is a constant term (combination of the Lagrange multiplier associated with the equation in U and the budget constraint multiplier). To satisfy the budget constraint, $\frac{(\sigma(U)-1)\mu}{\sigma(U)}$ has to equal $1/w$. In other words, $\left(\frac{q_i}{G_i(U)}\right)^{\frac{\sigma(U)-1}{\sigma(U)}}$ corresponds to the budget share of good i in consumption baskets:

$$\left(\frac{q_i}{G_i(U)}\right)^{\frac{\sigma(U)-1}{\sigma(U)}} = \frac{(\sigma(U)-1)\mu}{\sigma(U)} p_i q_i = \frac{p_i q_i}{w}$$

This leads to the demand q_i:

$$q_i = G_i(U)^{1-\sigma(U)} \, (p_i/w)^{-\sigma(U)}$$

which is the same expression as above, with Λ corresponding to utility. Moreover, we can see that utility U is such that $\sum_i \left(\frac{q_i}{G_i(U)}\right)^{\frac{\sigma(U)-1}{\sigma(U)}} = 1$ which, using the demand for q_i just above, can be written as:

$$\sum_i [G_i(U)p_i/w]^{1-\sigma(U)} = 1$$

which is the same equation as the one determining Λ, which proves that $\Lambda = U$.

Proof of equivalence between condition ii) and inequality (13)

We mention in the text that condition ii) of Proposition 4 is equivalent to inequality (13) when both σ and G_i are differentiable.

Taking the derivative of the log of $G_i(\Lambda)\alpha_i^{-\frac{1}{1-\sigma(\Lambda)}}$ with respect to Λ, we find that it is positive if and only if:

$$\frac{G_i'(\Lambda)}{G_i(\Lambda)} \; - \; (\log \alpha_i) \cdot \frac{\partial}{\partial \Lambda} \left(\frac{1}{1-\sigma(\Lambda)} \right) > 0$$

Hence, for each good i, the minimum α_i such that it is positive is:

$$\alpha_i^* = \exp\left(\frac{(\sigma(\Lambda)-1)^2 G_i'(\Lambda)}{\sigma'(\Lambda) G_i(\Lambda)} \right)$$

One can see that inequality $\sum_i \alpha_i^* < 1$ corresponds to inequality (13) in the text.

Note: one can also verify that this condition is equivalent to imposing that $G_i(\Lambda)$ and $\sigma(\Lambda)$ are such that:

$$\left[\sum_i (G_i(\Lambda)\, p_i/w)^{1-\sigma(\Lambda)} \right]^{\frac{1}{1-\sigma(\Lambda)}}$$

increases for any set of p_i/w.

Lemma 6 *For any given set of $x_i \geq 0$ and $\alpha_i \geq 0$ such that $\sum_i \alpha_i = 1$, the following expression is monotonically increasing in $\rho \in (-\infty, +\infty)$:*

$$\Big[\sum_i \alpha_i x_i^{\rho}\Big]^{\frac{1}{\rho}}$$

Proof of Lemma 6: First, consider two values $\rho < \rho' < 0$ and consider the mapping $m(x) = x^{\frac{\rho'}{\rho}}$ which is convex in x. Jensen's inequality implies that:

$$m\Big(\sum_i \alpha_i y_i\Big) \leq \sum_i \alpha_i m(y_i)$$

and thus:

$$\Big(\sum_i \alpha_i y_i\Big)^{\frac{1}{\rho}} \leq \Big(\sum_i \alpha_i y_i^{\frac{\rho'}{\rho}}\Big)^{\frac{1}{\rho'}}$$

Choosing $y_i = [x_i]^\rho$, we obtain:

$$\Big[\sum_i \alpha_i x_i^{\rho}\Big]^{\frac{1}{\rho}} \leq \Big[\sum_i \alpha_i x_i^{\rho'}\Big]^{\frac{1}{\rho'}}$$

Second, consider two values $\rho' > \rho > 0$ and consider again the mapping $m(x) = x^{\frac{\rho'}{\rho}}$ which is now concave in x. Jensen's inequality for concave functions implies:

$$m\Big(\sum_i \alpha_i y_i\Big) \geq \sum_i \alpha_i m(y_i)$$

and thus, taking to the exponent $1/\rho < 0$, we have:

$$\Big(\sum_i \alpha_i y_i\Big)^{\frac{1}{\rho}} \leq \Big(\sum_i \alpha_i y_i^{\frac{\rho'}{\rho}}\Big)^{\frac{1}{\rho'}}$$

Choosing $y_i = [x_i]^\rho$, we obtain:

$$\Big[\sum_i \alpha_i x_i^{\rho}\Big]^{\frac{1}{\rho}} \leq \Big[\sum_i \alpha_i x_i^{\rho'}\Big]^{\frac{1}{\rho'}}$$

Note that these terms are well defined when ρ converges to zero (on both sides):

$$\lim_{\rho \to 0} \Big[\sum_i \alpha_i x_i^{\rho}\Big]^{\frac{1}{\rho}} = \prod_i x_i^{\alpha_i}$$

hence the findings above also apply to $\rho = 0$. This proves Lemma 6.

Appendix B – Counter-examples

First case with homogeneous demand shifters

Here I show that we can find a case where conditions ii) fails and where the Slutsky substitution matrix is not semi-definite negative, thus proving that condition ii) cannot be entirely waived.

Suppose that $F(\Lambda) = \Lambda$ (no problem arises when F is locally constant) and that we have two goods 1 and 2, where $\varepsilon_{D1} < \varepsilon_H$ while $\varepsilon_{D2} > \varepsilon_H$ for the other good, i.e. $\varepsilon_H \in (\varepsilon_{D1}, \varepsilon_{D2})$. In particular, to fix ideas, supposed that all elasticities are constant, with $\varepsilon_H = \frac{\varepsilon_{D2}+\varepsilon_{D1}}{2} \equiv -\kappa < 0$ and denote $\delta \equiv \varepsilon_{D2} - \varepsilon_H = \varepsilon_H - \varepsilon_{D1} > 0$. Denote by the expenditure share of product 1 as $\frac{1-\epsilon}{2}$ and the expenditure share of good 2 as $\frac{1+\epsilon}{2}$ such that $\bar{\varepsilon}_D - \varepsilon_H = \epsilon\delta$. While elasticities are constant, we can still adjust the demand shifter for each good to obtain the desired market shares (hence ϵ can be chosen independently from the elasticities).

The off-diagonal coefficients of the Slutsky substitution matrix are then:

$$s_{12}p_1p_2/w \;=\; -\frac{a_1a_2(\varepsilon_{D1}-\varepsilon_H)(\varepsilon_{D2}-\varepsilon_H)}{\bar{\varepsilon}_D - \varepsilon_H} + a_1a_2\varepsilon_H \;=\; -\frac{(1-\epsilon^2)\delta^2}{4\epsilon\delta} - \frac{(1-\epsilon^2)\kappa}{4}$$

where a_i denotes the expenditure share of good i. The diagonal coefficients are:

$$s_{11}p_1^2/w \;=\; a_1\varepsilon_{D1} - \frac{a_1^2(\varepsilon_{D1}-\varepsilon_H)^2}{\bar{\varepsilon}_D - \varepsilon_H} + a_1^2\varepsilon_H \;=-\frac{(1-\epsilon)(\kappa+\delta)}{2} + \frac{(1-\epsilon)^2\delta^2}{4\epsilon\delta} - \frac{(1-\epsilon)^2\kappa}{4}$$

$$s_{22}p_2^2/w \;=\; a_2\varepsilon_{D2} - \frac{a_2^2(\varepsilon_{D2}-\varepsilon_H)^2}{\bar{\varepsilon}_D - \varepsilon_H} + a_2^2\varepsilon_H \;=-\frac{(1+\epsilon)(\kappa-\delta)}{2} + \frac{(1+\epsilon)^2\delta^2}{4\epsilon\delta} - \frac{(1+\epsilon)^2\kappa}{4}$$

One can see that the substitution coefficients become very large as ϵ approach zero (because some of the terms have ϵ in the denominator). Moreover, if we denote by Σ the matrix with coefficients $s_{ij}p_ip_j/w$, we obtain:

$$\lim_{\epsilon\to 0+} 4\epsilon\Sigma = \begin{pmatrix} +\delta & -\delta \\ -\delta & +\delta \end{pmatrix}$$

This matrix is semi-definite positive: $x^T 4\epsilon\Sigma x = \delta^2(x_1 - x_2)^2 \geq 0$. By continuity, when ϵ is small enough, the substitution matrix with coefficient s_{ij} is semi-definite positive, which is not consistent with a rational demand system.

Second case with iso-elastic substitution

In this case, I provide counter-examples to show that neither Λ or U are well defined if the assumptions of Proposition 4 are not satisfied.

- First, suppose that $\sigma(\Lambda)$ increases in Λ. In this case, the elasticity of substitution increases with income and issues are more likely to arise when consumption is concentrated in one or few goods.

  When $G_i(\Lambda)$ is not monotonic in Λ for a good i, the budget constraint can be written:

  $$G_i(\Lambda)p_i/w = 1$$

  when the consumption of all other goods become negligible, i.e. when $(p_j/w)^{1-\sigma(\Lambda)} = 0$. If there exists $\Lambda_1 \neq \Lambda_2$ such that $G_i(\Lambda_1) = G_i(\Lambda_2)$, one can see that the equation above has at least two solutions when $p_i/w = 1/G_i(\Lambda_1)$.

Conversely, utility is not well defined by the implicit equation provided in Proposition 4 when G_i is not monotonic for a good. Suppose that $q_j^{\frac{\sigma(U)-1}{\sigma(U)}}$ is zero (or close to zero) for other goods j. In that case, we can see that $\left(\frac{q_i}{G_i(U)}\right)^{\frac{\sigma(U)-1}{\sigma(U)}} = 1 \Leftrightarrow G_i(U) = q_i$ has several solutions in U for some q_i if G_i is not monotonic, potentially violating the monotonicity of U w.r.t quantities.

We also need G_i' to have the same sign for all goods. If it is not the case, we can obtain situations where Λ and U are not well defined, or where U would decrease with quantities q_i for some goods.

- Counter-examples for the second case are more difficult to construct. Here we will assume here that $\sigma(\Lambda)$ and $G_i(\Lambda)$ are differentiable. Let us examine what happens when inequality (13) is not satisfied, i.e. when:

$$\sum_i \exp\left(\frac{(\sigma(\Lambda)-1)^2 G_i'(\Lambda)}{\sigma'(\Lambda)G_i(\Lambda)}\right) > 1$$

for a given $\Lambda = U_0$. In that case, we can show that it is possible to find a set of quantities q_i such that U_0 is the solution of equation (14) but where implicit utility would depend negatively on some of the quantities. This amounts to showing that the following expression:

$$\left[\sum_i \left(G_i(U)/q_i\right)^{\frac{1-\sigma(U)}{\sigma(U)}}\right]^{\frac{\sigma(U)}{1-\sigma(U)}}$$

decreases with U and for at least some of the q_i's.

Suppose that U_0 is the solution of equation (14) for a given set of q_i. We can always rearrange the q_i to match a given set of consumption shares while still having U_0 as the solution of equation (14) . In particular, choose q_i^* such that U_0 is still the solution of (14) and such that:

$$\left(G_i(U_0)/q_i^*\right)^{\frac{1-\sigma(U_0)}{\sigma(U_0)}} = \frac{1}{A}\exp\left(\frac{(\sigma(U_0)-1)^2 G_i'(U_0)}{\sigma'(U_0)G_i(U_0)}\right)$$

where $A \equiv \sum_i \exp\left(\frac{(\sigma(U_0)-1)^2 G_i'(U_0)}{\sigma'(U_0)G_i(U_0)}\right) > 1$, strictly larger than unity if condition ii) is not satisfied. Consider the function:

$$f(U,q) = \left[\sum_i \left(G_i(U)/q_i\right)^{\frac{1-\sigma(U)}{\sigma(U)}}\right]^{\frac{\sigma(U)}{1-\sigma(U)}}$$

which corresponds to the left-hand side of equation (14). One can see that the derivative in U at $U = U_0$ and $q = q^*$ is negative:

$$\begin{aligned} f_U(U_0,q^*) &= \sum_i \frac{G_i'(U_0)}{G_i(U_0)}\Big(\frac{G_i(U_0)}{q_i^*}\Big)^{\frac{1-\sigma(U_0)}{\sigma(U_0)}} + \frac{\sigma'(U_0)}{(1-\sigma(U_0))^2}\sum_i \Big(\frac{G_i(U_0)}{q_i^*}\Big)^{\frac{1-\sigma(U_0)}{\sigma(U_0)}} \log\Big(\frac{G_i(U_0)}{q_i^*}\Big)^{\frac{1-\sigma(U_0)}{\sigma(U_0)}} \\ &= \frac{\sigma'(U_0)}{(1-\sigma(U_0))^2}\log A \ < \ 0 \end{aligned}$$

while the derivative $f_{q_i}(U_0,q^*)$ in each q_i is also negative. This leads to an implicit utility function U of q that decreases with quantities.

Appendix C – Practical cases and applications

For the second case, generalized non-homothetic CES, with uniform elasticity of substitution across goods:

1. In various contexts, one has associated a lower elasticity of substitution for richer consumers (in line with empirical evidence) while keeping the practicality of CES preferences. Proposition 4 case ii) allows us to do just that. Handbury (2016) and Faber and Fally (2017) assume that the consumption of the outside good influences elasticities. Here, one can circumvent such assumption by defining utility implicitly in a similar fashion as in Hanoch (1975) and Comin et al (2015).

2. Comin et al (2015) provide an excellent application of the case with constant elasticity of substitution σ that does not depend on income. In the calibration of their model, each industry is associated with a distinct structural parameter driving income effect, while keeping constant elasticities of substitution among industries. Here we show that it can be extended to elasticities of substitution that can potentially change with real income. Moreover, these income effects in substitution would not have to be tied (in terms of functional form) to income effects in consumption shares across industries.

   For instance, one application could be to rationalize the rise of profits and fixed costs relative to variable costs. If σ decreases with utility (and thus income), growth would be associated with larger markups and larger variable profits, and under free entry with larger shares of fixed costs over total costs. If fixed costs are more intensive in capital, this would rationalize an increasing share of capital in GDP.

3. While Comin et al (2015) focus on constant elasticity $\sigma(\Lambda) = \sigma$, an opposite case would be to assume that all goods have the same shifters $G_i(\Lambda) = G(\Lambda)$. Generalized non-homothetic CES would still be non-homothetic in that case. Assuming that $\sigma(\Lambda)$ is differentiable, integrability is ensured if $G(\Lambda)$ increases sufficiently fast:

$$\frac{G'(\Lambda)}{G(\Lambda)} > \max\left\{0, -\frac{\sigma'(\Lambda)}{(\sigma(\Lambda)-1)^2}\log N\right\}$$

For the first case, generalized Gorman-Pollak form, with non-uniform elasticity of substitution:

4. A simple way to generalize both directly and indirectly-separable preferences, as well as homothetic single-aggregator (HSA) preferences, is to consider iso-elastic functions H and F such that demand can be written:

$$q_i = \Lambda^{\beta} D_i(\Lambda p_i / w)$$

   For instance, if we assume that the price elasticity of D_i is always strictly larger than unity (a common assumption under monopolistic competition to ensure finite markups), integrability is ensured as long as $\beta \leq 1$. Directly-separable preferences correspond to the special case $\beta = 0$, indirectly-separable preferences correspond to the limit case $\beta \to -\infty$, and homothetic single-aggregator to the special case $\beta = 1$.

5. Iso-elastic functions $D_i(x) = A_i x^{-\sigma_i}$, $F(\Lambda) = \Lambda$ and $H(\Lambda) = \Lambda^{-\beta}$ with $\beta < 1$ lead to another interesting case. It is equivalent to CES only if the σ_i's are identical.

   This corresponds to self-dual addilog preferences previously examined by Houthakker (1965), Pollak (1972) and just recently by Bertoletti and Etro (2017a). With direct separability (when

H is constant), we obtain preferences with "Constant Relative Income Elasticities" (Fieler 2011, Caron et al., 2017, 2014). When $\beta = 0$ (CRIE). In the latter case with $\beta = 0$, income elasticities are:

$$\frac{\partial \log q_i}{\partial \log w} = \frac{\sigma_i}{\bar{\sigma}} \tag{36}$$

where $\bar{\sigma}$ is the expenditure-weighted average of σ_j. With $\beta \neq 0$, we obtain a generalization of such preferences where income elasticities are given by:

$$\frac{\partial \log q_i}{\partial \log w} = 1 + (1-\beta)\Big(\frac{\sigma_i - \bar{\sigma}}{\bar{\sigma} - \beta}\Big) \tag{37}$$

These demand systems are non-homothetic except for the case where $\beta = 1$ (which does not necessarily imply CES) or when $\sigma_j = \sigma_i$ for all i, j.

6. Augmented bi-power form, as in Mazrova and Neary (2013)

   Consider the demand system, assuming $\sigma > \nu > 0$:

$$q_i(p_i/w, \Lambda) \; = \; \gamma(\Lambda)[p_i/w]^{-\nu} + \delta(\Lambda)[p_i/w]^{-\sigma}$$

   Under which conditions is that demand system integrable? Defining $F(\Lambda) = [\gamma(\Lambda)/\delta(\Lambda)]^{\frac{1}{\sigma-\nu}}$ and $H(\Lambda) = \delta(\Lambda)^{-1}F(\Lambda)^{-\sigma} = \gamma(\Lambda)^{-\frac{\sigma}{\sigma-\nu}}\delta(\Lambda)^{\frac{\nu}{\sigma-\nu}}$, one can recover the form of demand systems as in Mazrova and Neary (2013) by applying Proposition 3.

   To apply Proposition 3, one would need q_i to be decreasing in Λ, regardless of prices. Hence a sufficient condition is that both $\delta(\Lambda)$ and $\gamma(\Lambda)$ decrease with Λ. If those conditions are not satisfied, we can see that Λ will not be well defined.[14] Taking very low prices, we have $q_i(p_i/w, \Lambda) \approx \delta(\Lambda)[p_i/w]^{-\sigma}$ and we can see that the equation $\sum_i \delta(\Lambda)[p_i/w]^{1-\sigma} = 1$ can lead to multiple solutions in Λ if δ is not monotonic. Conversely, if we have very high prices, $q_i(p_i/w, \Lambda) \approx \gamma(\Lambda)[p_i/w]^{-\nu}$, and the equation $\sum_i \gamma(\Lambda)[p_i/w]^{1-\sigma} = 1$ can lead to multiple solutions in Λ if γ is not monotonic. Hence monotonicity in both $\gamma(\Lambda)$ and $\delta(\Lambda)$ is required to ensure that Λ is well defined.

7. The previous example builds on a symmetric case (symmetric across all goods). More generally, one can also consider asymmetric bi-power forms. With $D_i(y_i) = \gamma_i y_i^{-\nu_i} + \delta_i y_i^{-\sigma_i}$, one can obtain recoverable demand systems as long as $F(\Lambda)$ and $H(\Lambda)$ are common across all goods, i.e. if demand takes the form:

$$q_i(p_i/w, \Lambda) \; = \; \gamma_i H(\Lambda)[F(\Lambda)p_i/w]^{-\nu_i} + \delta_i H(\Lambda)[F(\Lambda)p_i/w]^{-\sigma_i}$$

   with $\frac{\partial \log H}{\partial \log \Lambda} < \min\{\nu_i, \sigma_i\}$ to ensure that Λ is well defined.

8. Similarly, one can consider inverse demand functions that are bi-power, with the same extensions as above.

9. Mrazova et al. (2017) introduce CREMR demand (with constant revenue elasticity of marginal revenue) where the own-price effects are such that the distribution of sales and productivity belong to the same family (e.g. lognormal+lognormal, Pareto+Pareto, etc.). They rationalize their demand system with a directly-additive utility function.

[14] If both δ and γ are instead increasing in Λ, we can just replace Λ by $1/\Lambda$.

Using our results, one can further generalize such demand system (though only in the superconvex case where there is no restriction on minimum quantities) with more flexible income effects while keeping the CREMR properties linked to price effects. If we specify:

$$D_i^{-1}(x) = \frac{\beta_i}{x}(x-\gamma_i)^{\frac{\sigma-1}{\sigma}}$$

with $\gamma_i < 0$, and $H(z)$ such that $\varepsilon_H \geq -1$, the following (inverse) demand systems is integrable:

$$p_i(q_i, z) = w/z\ D_i^{-1}(H(z)q_i)$$

where z is implicitly defined by the budget constraint $\sum_i \frac{p_i q_i}{w}$=1 (see Section 3), which can be written:

$$z \ = \ \sum_i \beta_i(H(z)q_i - \gamma_i)^{\frac{\sigma-1}{\sigma}}\ /\ H(z)$$

The directly additive case described in Mrazova et al. (2017) corresponds to the case where $H(z)$ is constant. The homothetic case corresponds to $H(z) = 1/z$.

10. Conditionally-linear demand:

$$q_i = (\alpha_i - \gamma_i \Lambda p_i)/H(w\Lambda)$$

is integrable as long as $\Lambda p_i \leq \frac{\alpha_i}{2\beta_i}$ and $\varepsilon_H > -1$.

This generalizes Ottaviano et al. (2002), Melitz and Ottaviano (2008) and Mayer et al. (2014) based on quasi-linear preferences. This conditionally-linear demand system nevertheless yields very simple expressions for markups in monopolistic competition when Λ is taken as given (limit case with many firms).

A practical case is to impose $H(w\Lambda) = (w\Lambda)^{-\beta}$ with $\beta < 1$, but even in this most simple case there is no fully closed-form solution since utility (direct and indirect) still depends on functions z and Λ that are implicitly defined.[15] While $z^{-\beta}q_i < \alpha_i$ and $p_i\Lambda < \alpha_i/\gamma_i$, utility and indirect utility take the form:

$$\begin{aligned} U(q) &= \frac{\beta z^{1-\beta}}{1-\beta} + \sum_i \frac{z^{-\beta}q_i(2\alpha_i - z^{-\beta}q_i)}{2\gamma_i} \\ V(p,w) &= \frac{(w\Lambda)^{1-\beta}}{1-\beta} - \sum_i \frac{p_i\Lambda(2\alpha_i - \gamma_i p_i \Lambda)}{2} \end{aligned}$$

where z and Λ are implicitly defined as the solutions of $z = \sum_i q_i(\alpha_i - z^{-\beta}q_i)/\gamma_i$ and $\sum_i (w\Lambda)^\beta p_i(\alpha_i - \gamma_i\Lambda p_i) = w$ respectively.

11. Counter-examples where demand depends on more than one aggregator: Kimball (1995) and

[15] Note that the conditions in Proposition 2 are not always satisfied if $x \leq \frac{\alpha_i}{2\gamma_i}$ and the elasticity of H is close to -1 (D_i would have an elasticity below unity in absolute value). In a monopolistic competition framework, a trick is to replace D_i by $\frac{\alpha^2}{4\gamma_i x}$ for $x < \frac{\alpha}{2\gamma_i}$. Such a function D_i would satisfy the conditions of Proposition 1 as long as the elasticity of H remains between zero and unity. In equilibrium, none of the firms would set an infinite markup, hence equilibrium prices are such that $\Lambda p_i > \frac{\alpha_i}{2\gamma_i}$ and none of the firms would end up on the non-linear portion of the demand curve.

QMOR (Feenstra 2015). QMOR demand can be expressed as:

$$q_i = \alpha u \left(\frac{p_i}{e(p)} \right)^{r-1} \left[1 - \left(\frac{p^*}{p_i} \right)^{r/2} \right]$$

which is a function of two aggregators: p^* and $e(p)$. Given that $e(u)u = w$, we also have:

$$\frac{p_i q_i}{w} = \alpha \left(\frac{p_i}{e(p)} \right)^{r} \left[1 - \left(\frac{p^*}{p_i} \right)^{r/2} \right]$$

Note that the price elasticity can be expressed as a function of p^* only.

Similarly, demand with Kimball preferences with an implicit aggregator depends in fact on two aggregators (it also depend on the Lagrange multiplier associated with the budget constraint) hence they are not a special case of the demand systems described here.