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THE FACTOR CONTENT OF EQUILIBRIUM EXCHANGE RATES

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This paper has evolved over several years and has benefited enormously from comments on earlier iterations by Robert Hodrick, Angelo Melino, Ken West, Jeremie Banet, Niels Pedersen, Hans- Helmut Kotz, Shaowen Luo, Ryan Liu, Andy Pham and seminar participants at the New York Fed, Chicago Fed, St. Louis Fed , Columbia, Swiss National Bank, Bank of England, and the Bundesbank. As will be evident, this paper owes an enormous debt to Backus, Foresi, and Telmer (2001) in particular and more generally to the research program of David Backus, a peerless economist and even better friend who is sorely, sorely missed. The views expressed herein are those of the author and do not necessarily reflect the views of the National Bureau of Economic Research.

The author has disclosed a financial relationship of potential relevance for this research. Further information is available online at http://www.nber.org/papers/w24735.ack

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ABSTRACT

This paper develops framework to estimate and interpret the factor content of equilibrium real exchange rates. The framework – which builds on Backus, Foresi, and Telmer (2001) and Ang Piazzesi (2003) – respects the restrictions imposed by stochastic discount factors that generate standard, no arbitrage, essentially affine term structure models of inflation indexed bond yields in a home and a foreign country. We derive a sufficient set of parameter restrictions on the SDFs that deliver a stationary real exchange rate that is linear in the factors that govern the evolution of the SDFs. Our model implies that both the real exchange rate, and the ex ante real exchange rate risk premium at any horizon are linear functions of a "home" and "foreign" factors and that inflation indexed bond yields are functions of these factors as well as a "global" factor that accounts for the observed correlation in bond yield levels across countries. Home and foreign factors in turn are simple linear functions of the level slope and curvature factors extracted from home and foreign yield curves a la Litterman and Scheinkman (1991). We find that a real exchange rate risk premium accounts for about half the variance of the dollar – pound real exchange rate and that this risk premium if fully accounted for by the traditional level, slope, and curve curve factors in the UK linkers curve. We find that a home factor accounts for about 40 percent of the variance of the real exchange rate, and that this home factor is fully accounted for by the US specific component of the LS level factor in the US TIPs curve.

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A data appendix is available at http://www.nber.org/data-appendix/w24735

1. Introduction

This paper develops framework to estimate and interpret the factor content of equilibrium real exchange rates . The framework respects the restrictions on bond yields and the real exchange rate imposed by stochastic discount factors that generate standard, no arbitrage, essentially affine term structure models of inflation indexed bond yields in a home and a foreign country. It builds on the pioneering contributions of Backus, Foresi, and Telmer (2001] and Ang and Piazzesi (2003) but extends the literature along several dimensions that are of empirical and practical relevance. We show that , in general, global equilibrium in a world comprised of countries with stochastic discount factors that generate affine term structures will in general generate a non stationary real exchange rate that is a non linear function of the state variables (factors) that govern the evolution of the SDFs. We derive parameter restrictions on the SDFs that can generate a stationary real exchange rate which is a linear function of the SDF factors and show how our system can be cast and estimated as a standard state space model that accounts for the observed joint dynamics of inflation indexed bond yields in a home and foreign country and the level of the real exchange rate. With the parameter estimates obtained from that state space model, we use the results of BFT and AP to recover the deep parameters of the home and foreign SDFs that are consistent with the joint dynamics of inflation indexed bond yields *and the level of the real exchange rate* we observe in the data. An important feature of the data is that benchmark (10 year maturity) bond yields are highly correlated across countries. One of the factors in our model is a " global " factor that accounts for this correlation and that loads equally in the home and foreign SDFs (as in the BFT (2001) , page 291; see also Lustig et. al. (2011) for a related framework that can account for cross country variation in nominal exchange rate expected returns). As a result, in our specification, the global rates factor is "differenced" out and is not relevant for accounting for the real exchange rate or the real exchange rate risk premium. The other two factors in our model are a "home " and a "foreign" factor that are necessary to account for the bond yield differential between the home and foreign country and the level of the real exchange rate. Our framework also implies that any ex ante real exchange rate risk premium that is present in the data must also be spanned by these home and foreign factors.

But what are these home and foreign factors? In our model the "home" factor can be written as a linear combination of three factors : the Litterman Scheinkman (1991] "slope and "curve" factors extracted via eigenvector decomposition from the home country yield curve plus the *home country specific component* of the LS level factor. The "foreign " factor can be decomposed in a similar fashion.

Thus in its most general form our model predicts that up to six factors could be required to account for the observed time series for bond yields in two counties and their real exchange rate and real exchange rate risk premium. However, we show that as an empirical matter in our US UK data set , a much more parsimonious factor structure can account for the data. We find that the "home factor" has a correlation of 0.99 with the US specific component of the LS level factor extracted from the US TIPs yield curve. We find that a stochastic risk premium accounts for about half the variance of the real exchange rate , and that this risk premium can be fully accounted for by the UK slope , curve , and UK specific component of the LS level factor. Finally, combining these two findings, we show that a special case of our model which cannot be rejected empirically has a simple three factor structure that can fully account for the real exchange rate and exchange rate risk premium time series in our data set.

In the theoretical derivation and empirical implementation of our model, we study a three variable system comprised benchmark inflation indexed bond yields in a home and a foreign country and the CPI real exchange rate between those two countries. The reason for focusing on US "TIPS" and UK "linkers" is that these are the two deepest markets for inflation indexed bonds with the longest available time series (the US began issuing TIPS in 1997; the UK in the 1980s). But more importantly, the theoretical linkage among real exchange rates and inflation indexed bond yields is very tight, whereas the linkage among real exchange rates and nominal bond yields is much more complicated as it must account for expected long horizon inflation as well the covariance between long horizon realized inflation and the nominal SDF (see Engel (1996;2011); Campbell, Shiller, Viceira (2009)). Letting $Z_{t,1}$ denote the real SDF , the nominal SDF satisfies $M_{t,1} = Z_{t,1} P_t/P_{t+1}$ (Cochrane (2009)). As our framework estimates all the relevant parameters of $Z_{t,1}$ from data on inflation indexed bond yields and the real exchange rate, it can also be used to study real exchange rates and nominal bond yields given a parametric specification of the stochastic evolution of inflation in each country. But that is a project for future research.

1. The Model

Consider the following three equation block comprised of a real exchange rate and inflation indexed bond yields in a home and a foreign country. This is a close cousin of a model studied by Backus Foresi and Telmer (2001;page 293) that we embed in the essentially affine framework for yield curve modeling pioneered by Ang and Piazzesi (2003). The home real stochastic discount factor (SDF) is given by

$$(1) \qquad \ln Z_{t,1} = -\delta + a(1-\rho)h_t + k(1-\rho)x_t - aae_{t+1} - k\varepsilon_{t+1} - \lambda h_t e_{t+1} - \psi x_t \varepsilon_{t+1} - \chi f_t \xi_{t+1} - 0.5v^z_t$$

The foreign real SDF is given by

$$(2) \qquad \ln Z^*_{t,1} = -\delta^* + b(1-\rho) f_t + k(1-\rho)x_t - bb\xi_{+1} - k\varepsilon_{t+1} - \lambda h_t e_{t+1} - \psi x_t \varepsilon_{t+1} - \chi f_t \xi_{t+1} - 0.5v^{z^*}_t$$

Here h_t (f_t) is a 'home' (foreign) factor with innovation e_t (ξ_t) and x_t is a global factor with innovation ε_t and v_t denotes conditional variance. The evolution of the log of the CPI real exchange rate under complete markets is, by Backus and Smith (1993) , given by $\Delta q_{t+1} = z^*_{t,1} - z_{t,1}$

$$(3) \qquad \Delta q_{t+1} = \delta - \delta^* + a(\rho-1)h_t - b(\rho-1)f_t + aa\lambda v_h h_t - bb\chi v_f f_t + aae_{t+1} - bb\xi_{t+1} + 0.5(aa^2 v_h - bb^2 v_f)$$

where q_t is the deviation of the real exchange rate from its unconditional mean. In what follows we assume that $\delta^* - \delta = 0.5(aa^2 v_h - bb^2 v_f)$ which is required for a stationary real exchange rate. If markets are incomplete, a version of (3) can continue to hold with $\Delta q_{t+1} = z^*_{t,1} - z_{t,1} + \Lambda_{t+1}$ with the stochastic deviation of real exchange rate returns from complete markets Λ_{t+1} a linear function of e_{t+1} and ξ_{t+1} (Brandt, Cochrane, Santa Clara (2006);BFT (2001)). This essentially affine model assumes that the state variables h_t f_t and x_t are independent, normal, with zero mean and evolve according to first order auto regressions with parameter ρ (factor specific rho's are easily handled). This implies that, in general, the backward solution for the real exchange rate has a unit root. Suppose that these three equations are imbedded in a larger general equilibrium model in which the real exchange rate is stationary with constant unconditional mean. Then level of the real exchange relative to mean rate must be determined by forward solution to equation (3) $E_t \Delta q_{t+1} = a(\rho-1)h_t - b(\rho-1)f_t + aa\lambda v_h h_t - bb\chi v_f f_t$

Using the method of undetermined coefficients the unique, stationary forward solution is

(4) $q_t = \{a + aa\,\lambda(\rho\text{-}1)^{-1}\}h_t - \{b + bbX(\rho\text{-}1)^{-1}\}f_t$

Note that this solution (4) reproduces the arbitrage dynamics for Δq_{t+1} if and only if aa = $a + aa\,\lambda(\rho\text{-}1)^{-1}$ and $bb = b + bbX(\rho\text{-}1)^{-1}$. Thus, given ρ which is pinned down by the physical dynamics of the state variables, of the three SDF parameters a,aa, and λ only two can be independent if the equilibrium real exchange rate is stationary (and similarly for b,bb, and χ). In what follows we assume these parameter restrictions are satisfied. Thus if equations (1) to (3) are part of a general equilibrium model in which the CPI real exchange rate is stationary, the admissible parameters of the home and foreign SDF are restricted . Later we give an example of a simple two country general equilibrium model which satisfies these restrictions and generates SDF's and a real exchange rate which take the above essentially affine structure.

The essentially affine model has become the workhorse for studying bond yields because it implies that yields and bond risk premia are linear in the factors even when the factors themselves are conditionally homoscedastic. However, in general the EAM does not deliver a log real exchange that is linear in the state but that instead, is quadratic in the state. For example, if the 'foreign' shock ξ_t is absent from the home SDF and the' home' shock e_t is absent from the foreign SDF we will have $\Delta q_{t+1} = a(\rho\text{-}1)h_t - b(\rho\text{-}1)f_t + aa\lambda h_t - bb\chi f_t + aae_{t+1} - bb\xi_{t+1} + 0.5\,(\lambda h_t)^2 - 0.5\,(\chi f_t)^2$.

The inflation indexed short rate at home (relative to its mean) satisfies $exp - r_t - \delta = E_t\,(Z_{t+1})$ which implies $r_t = a(\rho\text{-}1)h_t + k(\rho\text{-}1)x_t$ and similarly, $r^* = b(\rho\text{-}1)f_t + k(\rho\text{-}1)x_t$. Since our SDF 's are of the essentially affine class with short rates that are linear in factors, inflation linked bond yields of any maturity will be linear in the factors. Let R_t denote the yield on a benchmark inflation linked bond with 10 years to maturity expressed as a deviation from its unconditional mean. Given the structure of our model, the loading on the 'global' factor x_t for R_t and for R^*_t will be identical . We will normalize this common loading to unity. As we will show below, this normalization will facilitate interpreting our h_t and f_t factors as functions of traditional level , slope, and curvature yield curve factors in the home and foreign countries.

Based upon the above considerations, we will estimate the following state space model on a monthly US and UK data set beginning in 1997:7 with the first issuance of TIPS by the US

$$
\begin{aligned}
& R_t = A\, h_t + x_t \\
& R^*_t = B\, f_t + x_t \\
(5)\qquad & q_t = aa\, h_t - bb\, f_t \\
& h_t = \rho\, h_{t-1} + e_t \\
& f_t = \rho\, f_{t-1} + \xi_t \\
& x_t = \rho\, x_{t-1} + \varepsilon_t
\end{aligned}
$$

The first three equations are the measurement equations and the last three equations are the transition equations . Under our assumptions, the model directly estimates the deep SDF parameters aa and bb. Moreover, from AP (2003; Appendix A) and the parameter restrictions discussed above that insure stationarity, we have A = A(aa, ρ, v_h, λ) and B = B(bb , ρ, v_f, χ) which are defined by non-linear recursions that can be solved numerically for λ and χ conditional on the estimates of the other parameters. We impose $v_h = v_f = v$ and set $v^{0.5} = 1/12$ (as we are using monthly data) and leave v_x unrestricted. Left for future research is adjusting for the relative illiquidity of inflation indexed bonds and extending our framework in the direction of Abrahams, Adrian, Crump, and Moench (2016) to jointly price nominal bonds, inflation indexed bonds, and real exchange rates. Also, beyond the scope of this paper is a multi – country extension a la Hodrick and Vassalou (2002) to study real exchange rates and inflation indexed bond yields in a panel of countries. From (3) and the expressions for r_t and r^*_t

$$(6)\qquad E_t\, \Delta q_{t+1} = r_t - r^*_t + aa\lambda v_h h_t - bb\chi v_f f_t$$

with

$$(7)\qquad \vartheta_{t,1} = aa\lambda v_h h_t - bb\chi v_f f_t$$

the one period ex ante deviation from real uncovered interest parity for inflation linked bonds.

Our decomposition isolates a common global factor in inflation indexed bond yields that satisfies the property that the common bond yield factor is "differenced out" when accounting for real exchange rates. BFT, and more recently Lustig et. al. (2011), consider a similar structure for home and foreign nominal interest rates and nominal exchange rate *returns*. BFT work with an extension of the Cox, Ingersoll, and Ross (1985) 'square root' model and show that a common factor – common coefficient SDF specification for modeling nominal interest rates and exchange rate depreciation implies sample paths with negative nominal interest rates that are inconsistent with a square root specification. What is new here is exploiting the level information in the real exchange rate, respecting the parameter restrictions that insure it, and taking full advantage of the affine structure and the advent of a global market in inflation indexed bonds to study a time series decomposition of the month by month evolution of the real exchange rate and expected excess returns to currency speculation.

The level of the real exchange rate as well as the ' Fama premium ' for the CPI real exchange rate in a world with inflation indexed bonds will be linear the h_t and f_t factors, but what are these factors? The most popular yield curve factor model is the model of Litterman and Scheinkman (1991) in which the level slope and curvature factors are identified from an eigenvector decomposition of the unconditional variance covariance matrix of bond yields at the different points along the yield curve. Following Bliss (1997), the tradition is to achieve identification of the LS factors by assuming that orthogonal factors have a unit variance. This is obviously arbitrary. An alternative identification which is convenient for our purposes (and which allows for direct comparison with the decomposition proposed by Diebold and Li (2006) as well as industry practice) is to assume factors are orthogonal with common variance that results in a unit loading on the "level" factor in the decomposition of the benchmark 10 year yield. This is also consistent with industry practice which defines the 'level' or duration factor equal to the yield at the 10 year point on the yield curve. This normalization of variance can always be done for any one factor at any one point on the yield curve and preserves factor orthogonality. With this version of an LS decomposition, we will have $R_t = L_t + D_s S_t + D_c C_t$ with D_s and D_c the loadings on the US slope and curve factors of the 10 year TIP yield and $R^*_t = L^*_t + D^*_s S^*_t + D^*_c C^*_t$ for the UK linker. Comparing with (5) we see immediately that $h_t = (L_t - x_t)/A + (D_s S_t + D_c C_t)/A$ and $f_t = (L^*_t - x_t)/B + (D_s S^*_t + D_c C^*_t)/B$. Substituting into (5) we can write the equilibrium real exchange rate as an exact function of *six* factors : the slope and curve factors for each country plus *the country specific component* of the home and foreign level factors.

(8) $q_t = (aa/A)(L_t - x_t) + (aa\, D_s/A)\, S_t + (aa\, D_c/A)\, C_t - (bb/B)(L^*_t - x_t) - bb\, D^*_s/B)\, S^*_t - (bb\, D^*_c/B)\, C^*_t$

2. Empirical results

With our assumptions (1) and (2) on the SDF's, our model can written in state space form given by (5) and estimated via maximum likelihood. Our data is monthly for the period 1997:7 – 2016:3 and we use published Fed and Bank of England time series on zero coupon inflation indexed bond yields for US TIPS and UK linkers. We construct the real exchange rate from monthly data on GBP and UK and US CPIs. We fix the common autoregressive parameter $\rho = 0.98$ (results are similar if we estimate factor specific ρ's) . We set $\psi = 0$ which pins down k from the assumption of unit loadings on x_t at the 10 year point. As discussed above, our model - along with the parameter restrictions $aa = a + aa\,\lambda(\rho\text{-}1)^{-1}$ and $bb = b + bbX(\rho\text{-}1)^{-1}$ - provides a sufficient set of restrictions on the general EAM to produce a linear state space representation with a stationary real exchange rate. As we discuss below, we can talk rigorously about over or undervaluation of the real exchange rate relative to its unconditional mean and quantify how much of any deviation from long run equilibrium is accounted for by changes in risk premia , changes in term premia, and changes in 'fair value' implied by UIP. Moreover, we can decompose each of these three components into the weighted sum of a home and foreign factor which in turn are a function of the home and foreign slope, curve, and level factors respectively via equation (8). Table 1 provides the parameter estimates along with estimated standard errors.

Table 1

Parameter	*Estimate*	*St. Error*
A	-0.1642	0.0155
B	0.0733	0.0219
aa	19.03	3.61
bb	23.62	2.33
a	21.69	4.13
b	-9.68	0.95
v_x	0.00026	.000024
λ	0.0028	.0021
χ	-.0118	.0031

By construction our model attributes all the variation in the log real exchange rate to the home and foreign factors $q_t = aa\,h_t - bb\,f_t$ with the common component of the US and UK level factors, x_t , differencing out. Figure 1 plots the contribution of h_t in accounting for q_t and Figure 2 the contribution of f_t.

Figure 1: Dollar-Pound Real Exchange Rate vs Home Factor Contribution

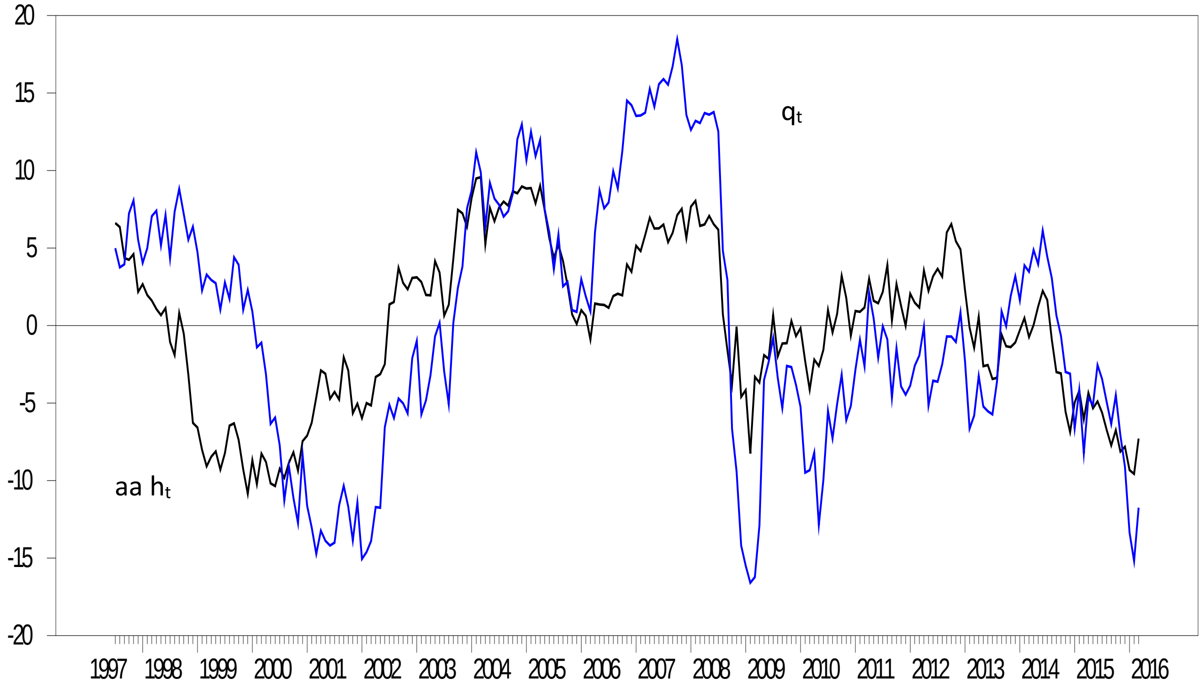
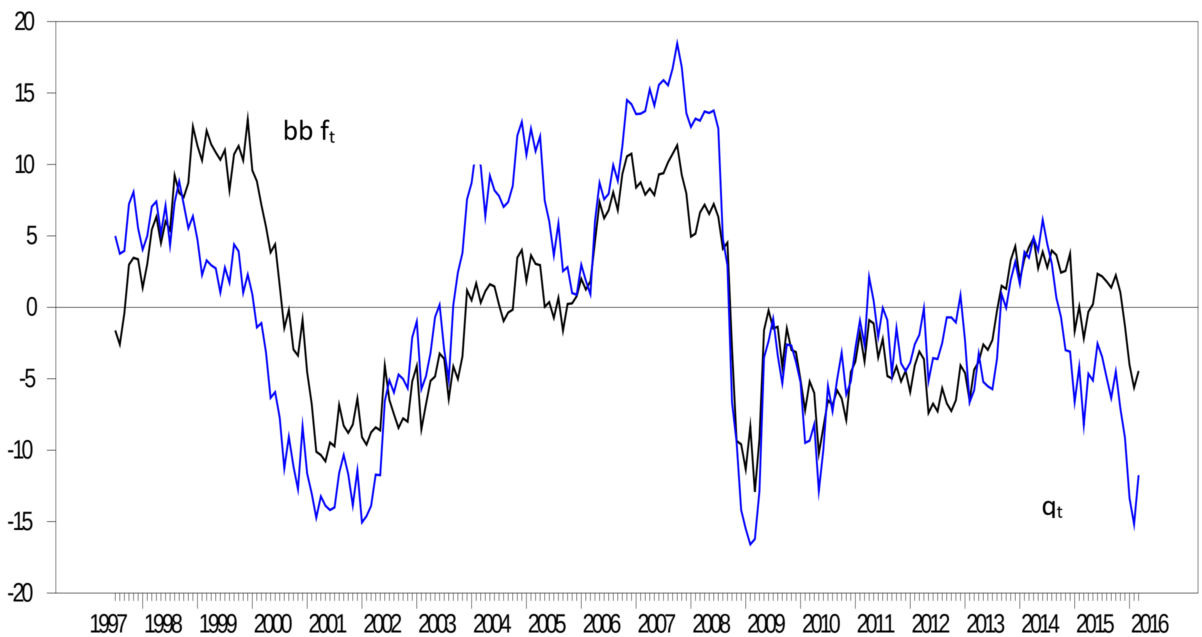


Figure 2: Dollar-Pound Real Exchange Rate vs Foreign Factor Contribution



As discussed earlier, the home factor h_t in our framework can be written as a linear combination of the slope factor, the curve factor, and the country specific component of the level factor extracted from the US TIPS curve: $h_t = (1/A)(L_t - x_t) + (D_S/A)S_t + (D_C/A)C_t$, with the loadings of a 10 year TIP on the slope and curve factors extracted via eigenvector decomposition (with the variance normalization described above) of the covariance matrix of 5,10, 20 year maturity TIPS yields in the Fed data set. The foreign factor can be written similarly as $f_t = (1/B)(L^*_t - x_t) + (D^*_S/B)S^*_t + (D^*_C/B)C^*_t$. In our sample, and not surprisingly, the extracted US and UK level factors are highly correlated (correlation equal 0.91). However, the country specific level factors $L_t - x_t$ and $L^*_t - x_t$, much less so (correlation equal -.18). So via the mapping (8) , what might appear to be a *two* factor model of the real exchange rate is in reality up to a *six factor* model of the real exchange rate. However, as we now show, empirically virtually all the variation in the extracted h_t factor is attributed to country specific component in the US level factor ($L_t - x_t$) and virtually none of the variation in h_t is accounted for by the slope and curve factors in the US TIPS curve.

Figure 3

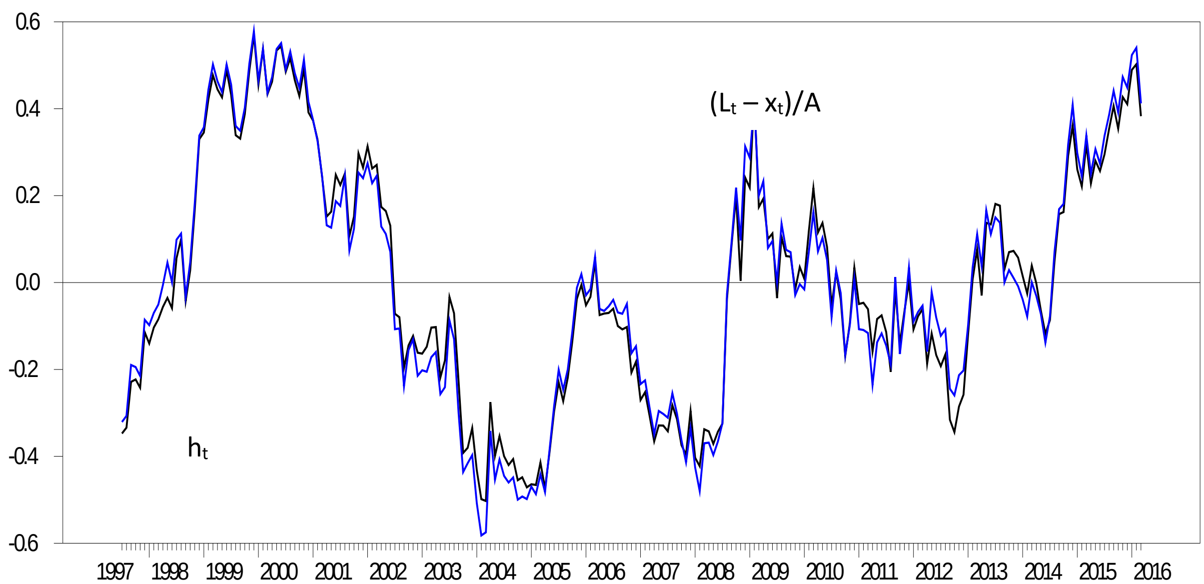
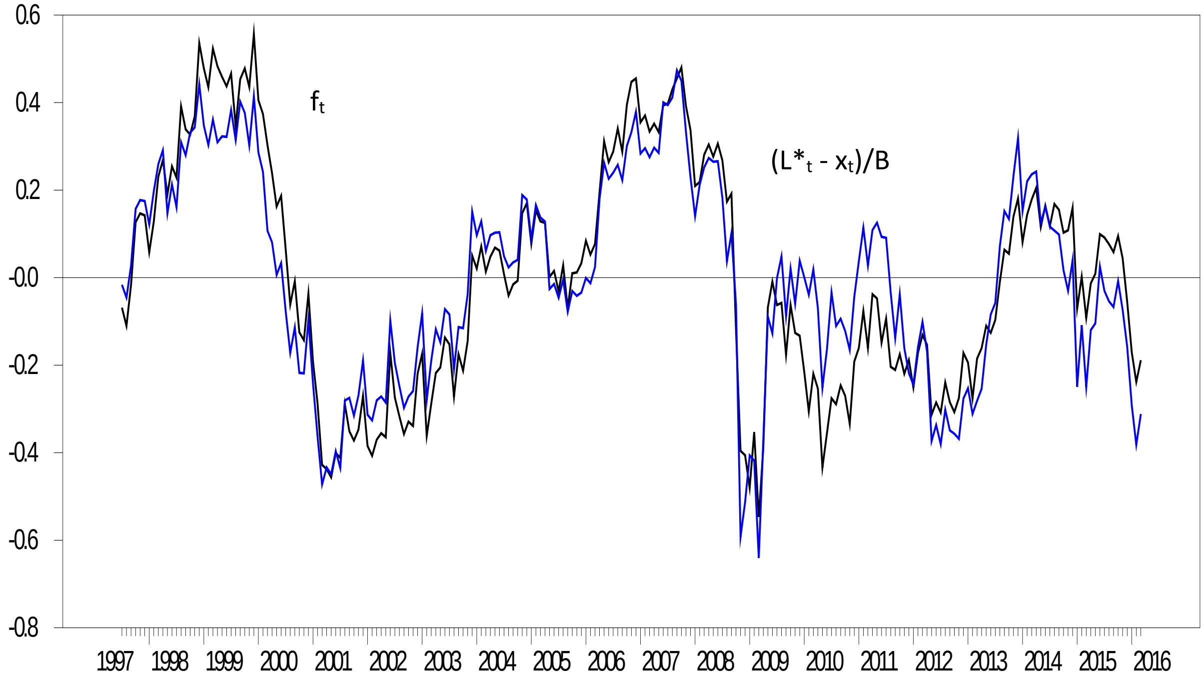


Figure 3 plots the home factor h_t against the (scaled) home specific component of the home country level factor $(L_t - x_t)/A$ where L_t is the LS first principal component of the US TIPS curve . As can be seen, virtually all the variation in the home factor (correlation 0.99) is accounted for by the home specific component of the level factor, extracted from the US TIPS curve. Figure 4 plots the foreign factor f_t against the (scaled) foreign specific component of the foreign country level factor $(L^*_t - x_t)/B$ where L^*_t is the LS first principal component of the UK linkers curve.

Figure 4



As can be seen, most of the variation (correlation is 0.93) in the f_t factor is accounted for by the UK specific component (e.g. after subtracting the global factor x_t recovered by our model) of the level factor extracted from the UK linkers curves. That said, as we will show below, to account for the time variation in long horizon dollar pound real exchange rate risk premium that we observe in the data, the UK curve and slope factors turn out to be important as well.

From the parameter estimates for aa and bb reported in Table 1, and the assumption of equal factors variances for h_t and f_t, we see that the home factor is estimated to account for about 40 percent of the variance of q_t while the foreign factors account for 60 percent of the variance. However, given the estimated standard errors, we cannot reject aa = bb which would imply that each factor accounts for half the variance of the real exchange rate. For this special case, which we will explore below, the level is of the real exchange rate is simply proportional to the difference between the home factor and the foreign factor. However, we can reject that the prices of risk parameters for the home and foreign factors, λ and χ respectively, are equal. As we discuss below, this difference will be essential to account for the risk and expected return regularities that we recover from the data.

3. The Factor Structure of Real Exchange Rates and the Real Exchange Rate Risk Premium

The expected gross real return to a US investor who buys and holds to maturity a UK linker of maturity n is simply $(E\, Q_{t+n}/Q_t) \exp n(R^*_t + \mu_{R^*})$. The known gross real return to a US investor who buys and holds to maturity a US TIP is $\exp n(R_t + \mu_R)$ Define

(14) $$\exp(\,\Theta_{t,n.} + n(\mu_{R^*} - \mu_R) + 0.5\, var_t\, q_{t+n}) = (E_t\, Q_{t+n}/Q_t) \exp n(R^*_t + \mu_{R^*} - R_t - \mu_R)$$

as the ratio of the expected long horizon return real return to a US investor who buys and holds to maturity a linker to the known real return to a US investor who buys and holds a TIP. We see that

(15) $$Q_t = \exp n(R^*_t - R_t)\, E_t\, Q_{t+n} \exp -(\,\Theta_{t,n} - 0.5\, var_t\, q_{t+n})$$

Taking logs we obtain

(16) $$q_t = n(R^*_t - R_t) + E_t\, q_{t+n} - \Theta_{t,n}$$

where $\Theta_{t,n}$ is the ex-ante n horizon mean deviation from inflation indexed UIP which differs from the n horizon real exchange rate risk premium by a constant equal to $0.5\, var_t\, q_{t+n}$. Adopting the terminology in Clarida (2012), we refer to the term $n(R^*_t - R_t) + E_t\, q_{t+n} = q^{fv}_t$ as the “fair value” of the real exchange rate: the counterfactual level of q_t at which expected long horizon returns would be equalized across US and UK linker markets.

Define $q^{UIP}_t = E_t\, \Sigma_{i=0,n-1} (r^*_{t+1} - r_{t+I}) + E_t\, q_{t+n} = [b(1-\rho^{120}) - bb\, \rho^{120}]\, f_t - [a(1-\rho^{120}) - aa\, \rho^{120}] h_t$ as the counterfactual level of the real exchange rate at time t that satisfies, up to a constant, uncovered interest parity period by period. Let $\tau_t = R_t - (1/n)\, E_t \Sigma_{i=0,n-1}\, r_{t+1}$ denote, up to a constant, the term premium in a US TIP of maturity n and define τ^*_t similarly. Substituting into (16)

(17) $$q_t = q^{UIP}_t + n(\tau^*_t - \tau_t) - \Theta_{t,n}$$

with $\tau^*_t - \tau_t = [B - (1/n)b(1-\rho^{120})]\, f_t - [A - (1/n)a(1-\rho^{120})]\, h_t$. Thus period by period we can decompose the level of the real exchange relative to its mean into the sum of three components: a UIP component, a relative term premium component, and a long horizon UIP deviation component. Note that $q^{fv}_t = q^{UIP}_t + n(\tau^*_t - \tau_t)$

Figure 5 plots q_t versus the q^{UIP}_t component: correlation with q_t = -0.18

Figure 5

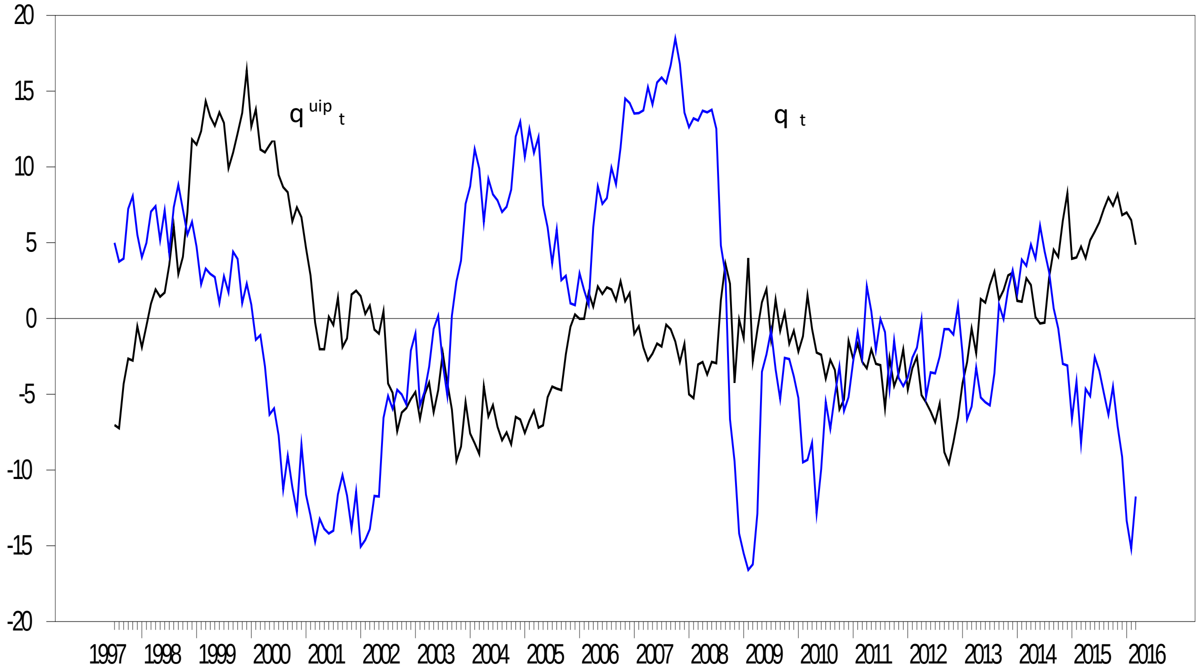


Figure 6 plots q_t against the relative term premium component: correlation = 0.31

Figure 6

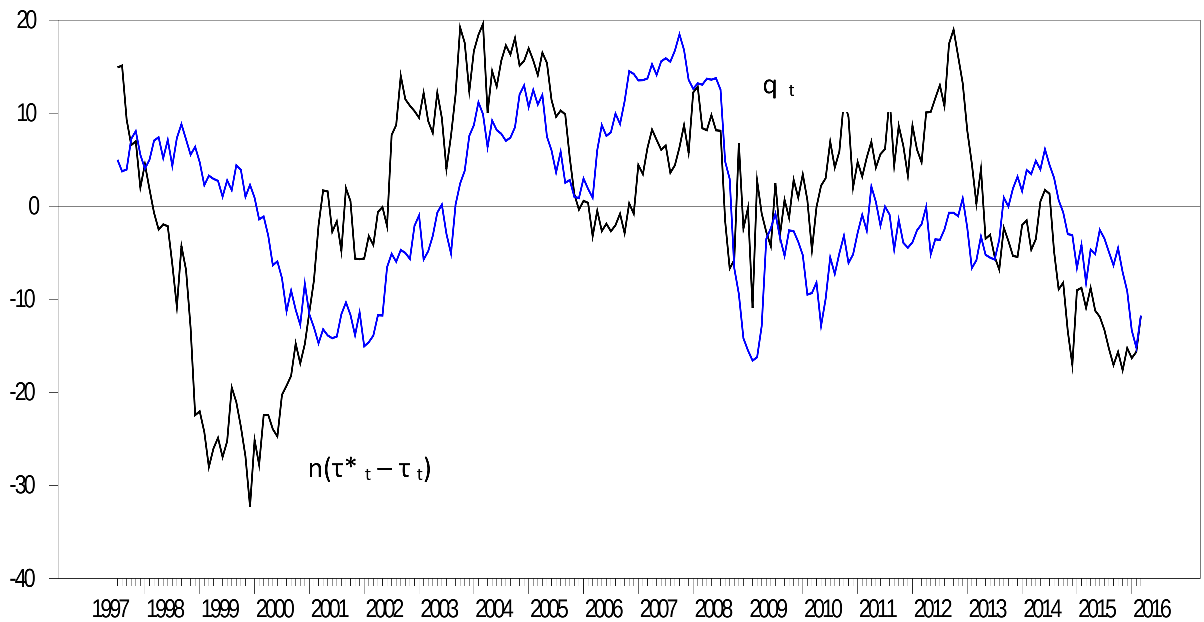
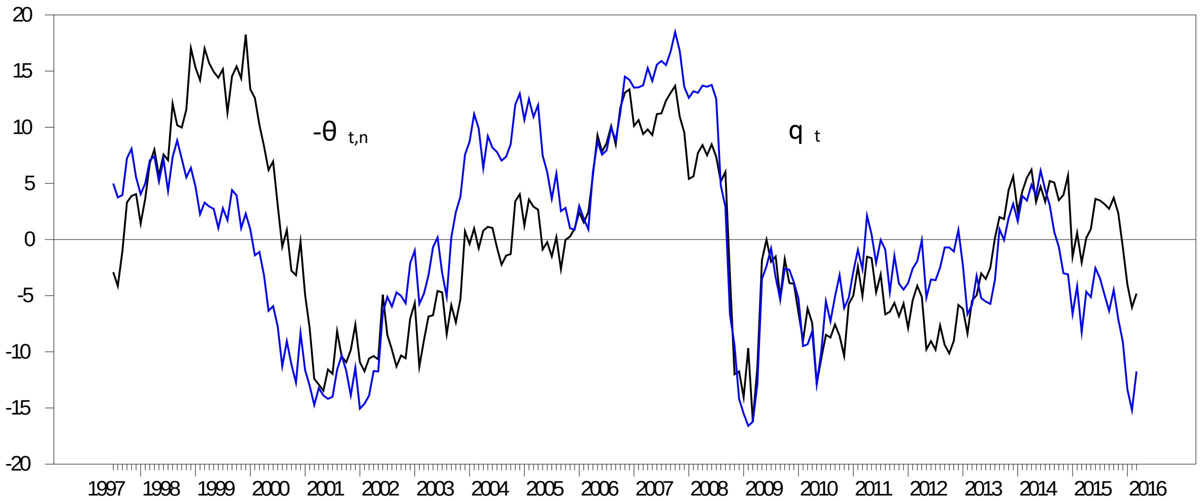


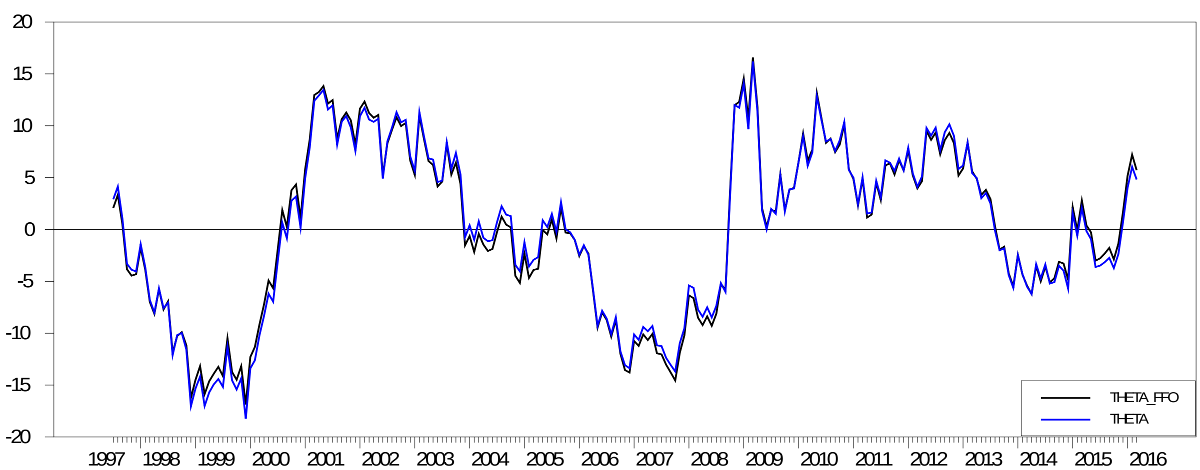
Figure 7 plots q_t against the long horizon UIP deviation component - $\Theta_{t,n}$.: correlation = 0.72

Figure 7



As with q^{uip}_t and $(\tau^*_t - \tau_t)$, we can decompose the long horizon UIP deviation component into the contributions of the h_t and f_t factor $\Theta_{t,n} = [120\, B(\chi) + bb\,(1-\rho^{120})]\, f_t - 120\; A(\lambda) + aa\,(1\text{-}\,\rho^{120})]\, h_t$ with notation to remind us that the TIPS (linkers) loading on h_t (f_t) is a function of the price of risk parameter λ (χ). Figure 8 reveals a striking feature of our data set as interpreted by our model: virtually all (correlation 0.99) of the time series variation in the long horizon dollar pound real exchange rate risk premium $\theta_{t,n}$ can be accounted for by the slope , curve and country – specific level factors extracted from the UK linkers yield curve! Define $\theta^{FFO}_{t,n} = [120\, B(\chi) + bb\,(1-\rho^{120})]\, f_t$ as the long horizon real exchange rate risk premium that is accounted for by the foreign (UK) factors only via $f_t = (1/B)\,(L^*_t - x_t) + (D^*_s/B)\, S^*_t + (D^*_c/B) C^*_t$

Figure 8



In terms of variance decomposition , we estimate that 84% of the variance of $\theta_{t,n}$ is accounted for by the UK specific level factor $L^*_t - x_t$ (Figure 10), and that 8% of the variance each is attributed to the UK slope factor (Figure 11) and curve factor (Figure 12) extracted from the UK linkers curve.

Figure 9

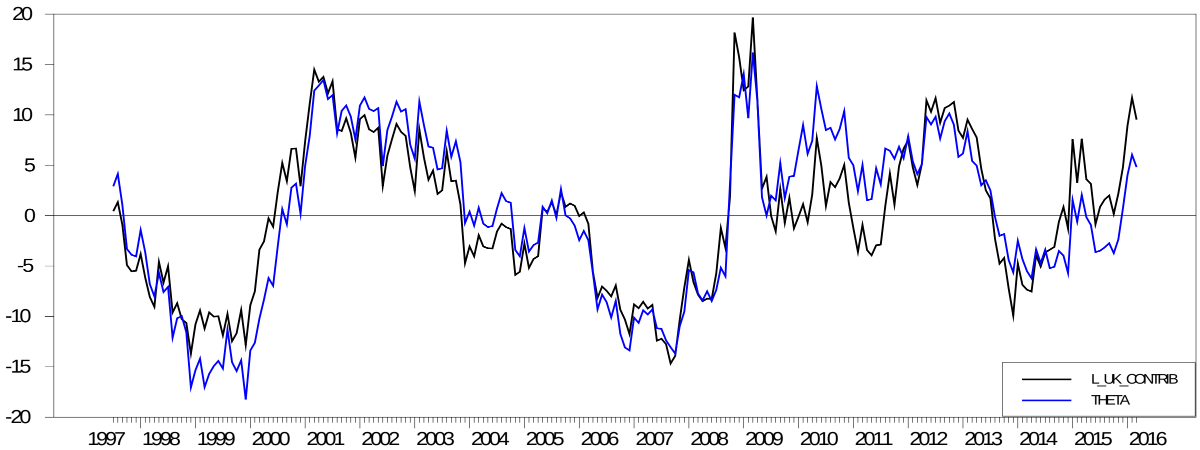


Figure 10

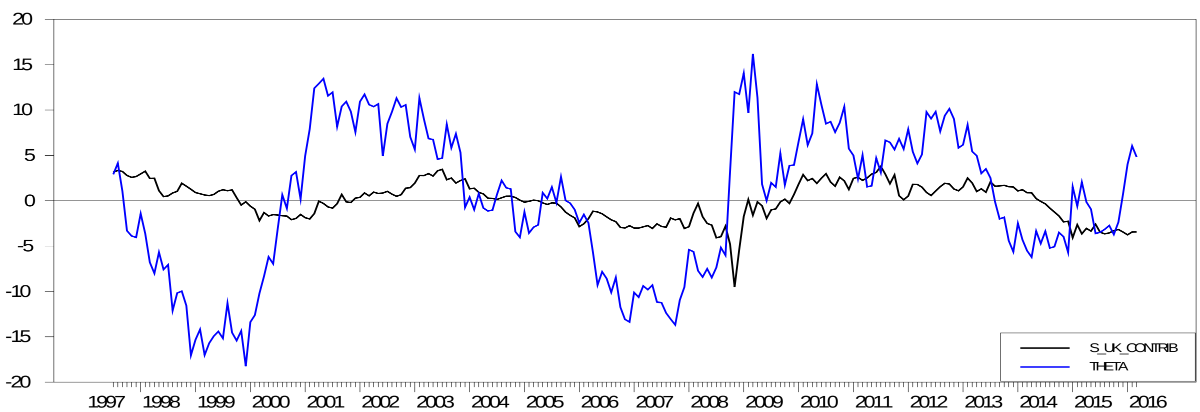
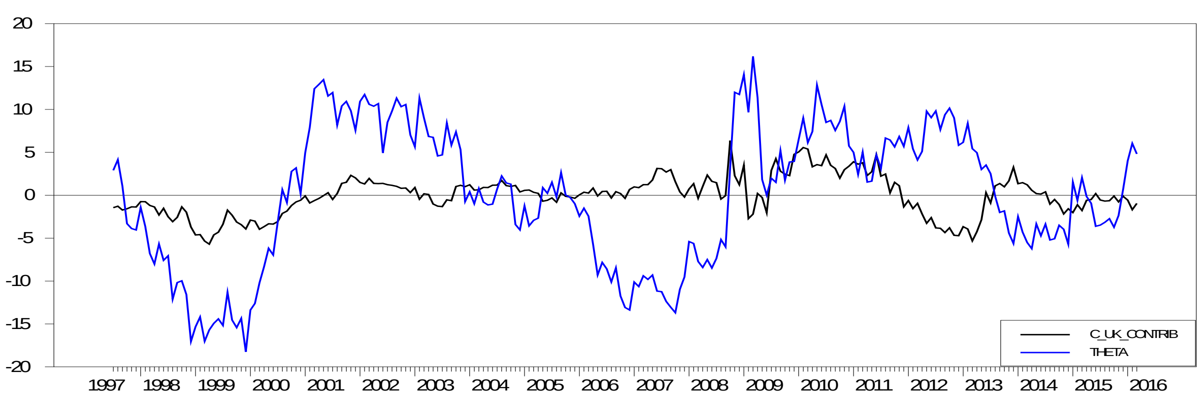


Figure 11



It is instructive to compare the quantitative decomposition of the level of the real exchange rate with the decomposition of $R^*_t - R_t$ and $\theta_{t,n}$ by plugging in our actual parameter estimates. With these estimates, we have

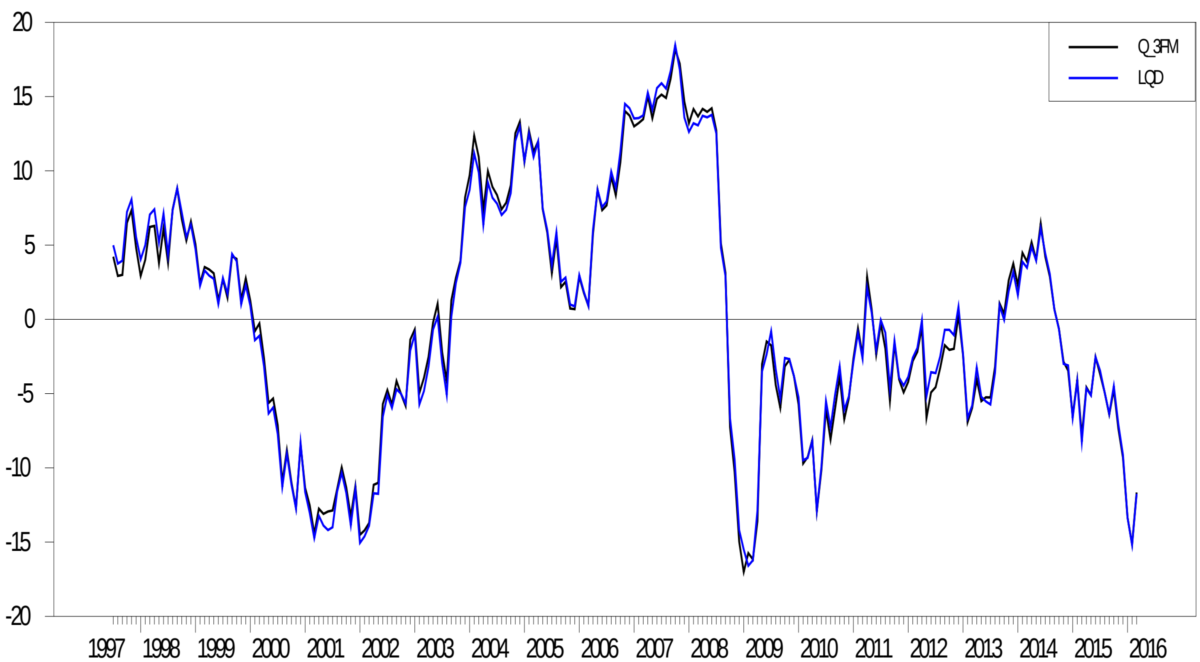
$$
(18)\qquad
\begin{aligned}
q_t &= 19.03\, h_t - 23.62\, f_t \\
n(R^*_t - R_t) &= 8.36\, f_t + 19.58\, h_t \\
\Theta_{t,n} &= 30.28\, f_t + 2.35\, h_t \\
R_t &= -0.1642\, h_t + 1.00 x_t \\
R^*_t &= 0.0733\, f_t + 1.00 x_t
\end{aligned}
$$

A (positive) shock to f_t that raises UK rates also raises the required risk premium $\Theta_{t,n}$ .If the rise in R^*_t were large enough the risk premium to a US investor could be achieved ex ante solely though adjustment in the linkers market without any necessary adjustment to the real exchange rate. But empirically, the rise in R^*_t to a rise in f_t is insufficient to deliver the high risk premium and so the dollar real exchange rate must appreciate to set up expectation of depreciation . By contrast a (negative) shock to the h_t factor which pushes up US rates appreciates the dollar pound real exchange rate , narrows the rate differential with the UK, and lowers the risk premium to invest in UK linkers. Our model is capturing the fact that over our sample, periods of rising UK real rates relative to US are associated with a weaker pound and a higher risk premium to hold UK assets, whereas periods of rising US real rates relative to UK are associated with a weaker pound and a lower risk premium to hold UK assets. Indeed, we can go further given the parameter estimates shown above: *a home shock that lowers R_t relative to R^*_t widens the rate differential in favor of the foreign country by an amount almost exactly equal to the depreciation of the real exchange rate as predicted by conditional UIP. Home shocks in turn are due almost entirely to the US specific component of the TIPS level factor.*

There is an alternative interpretation of our model which some readers may find more intuitive. In this interpretation , treat q_t as an exogenous stationary zero mean AR(1) process and recover factors h_t f_t and x_t and find parameters as in (18) that are consistent with the data (including the time series properties of q_t) and respect the restrictions implied by the SDFs (1) and (2) and the BFT exchange rate arbitrage equation (3). With this interpretation there is no factor content *per se* embed in the real exchange rate. Instead there is a factor content embed in the inflation indexes bond yields and long horizon expected real exchange rate excess returns that is consistent with the presumed exogenous evolution of the real exchange rate.

As mentioned earlier, while our framework implies that up to six factors might be necessary to account for real exchange rates and expected excess real exchange rate returns, in our US UK data set the home factor h_t is essentially proportional to the US specific level factor $(L_t - x_t)$ while the UK yield curve factors account for the long horizon real exchange rate risk premium. Also, our parameter point estimates for aa = 19.03 and bb = 23.62 differ only by one standard error. This suggest that a model comprised of the *relative country specific level factor* $(L_t - x_t)/A - (L^*_t - x_t)/B$, and the UK slope S^*_t and curve C^*_t factors should account virtually all the observed variation in q_t. We regress q_t on $(L_t - x_t)/A - (L^*_t - x_t)/B$, S^*_t and C^*t. The fitted value from this three factor model as well as q_t are shown in Figure 12. This 3 factor model accounts for 99 percent of the variance of q_t.

Figure 12



Given our finding that the long horizon risk premium $\Theta_{t,n}$ accounts for about half the variance of the mean deviation in q_t, long horizon UIP for inflation indexed bonds is rejected in our data set as is short horizon (n = 1) UIP. Indeed, via (7) the $\Theta_{t,1}$ implied by our parameter estimates is almost perfectly correlated with $\Theta_{t,n}$ and thus in turn with f_t. To get a sense of the magnitude of this violation, we regress $E_t q_{t+n} - q_t$ on a constant and $n(R_t - R^*_t)$ with n= 120 and $E_t q_{t+n} = \rho^{120} q_t$. The estimated slope coefficient is *positive* 0.40 with standard error of 0.08, results that are reminiscent of Meredith and Chen (1998) who tested long horizon UIP with data on nominal bond yields.

4. Concluding Remarks

This paper has developed framework to estimate and interpret the factor content of equilibrium real exchange rates . The framework respects the restrictions on inflation indexed bond yields and the real exchange rate imposed by stochastic discount factors that generate standard, no arbitrage, essentially affine term structure models of inflation indexed bond yields in a home and a foreign country.   We derive a sufficient set of parameter restrictions on the SDFs that deliver a stationary real exchange rate that is linear in the factors that govern the evolution of the SDFs. We cast the model in state space form and recover the deep parameters of the home and foreign SDF using the theory as set out in BFT and AP.   Our model implies that both the real exchange rate, and the ex ante real exchange rate risk premium at any horizon are linear functions of a “home” and ”foreign” factors and that inflation indexed bond yields are functions of these factors as well as a global factor that accounts for the observed correlation in bond yield levels across countries. Home and foreign factors in turn are simple linear functions of the level slope and curvature factors extracted from home and foreign yield curves a la Litterman and Scheinkman (1991). We find that a real exchange rate risk premium accounts for about half the variance of the dollar – pound real exchange rate and that this risk premium if dully accounted for by the traditional LS yield curve factors in the UK linkers curve. We find that a home factor accounts for about 40 percent of the variance of the real exchange rate, and that this home factor is fully accounted for by the US specific component of the LS level factor in the USA TIPs curve.

Appendix

A Model

Consider a two country, three good, four asset model. Consumption in each country is a Cobb Douglass aggregate of a basket comprised of a traded good – which serves as numeraire – and a country specific non- traded good. The intertemporal elasticity of substitution is 1. Supply of the traded good in each country is a stochastic dividend from a Lucas trees located in that country, and shares in these trees are held and can be traded internationally. There is also a market in home and foreign CPI linked bonds which are in zero net supply. The non-traded good in each country is produced with labor (which is in inelastic supply normalized to unity) with a technology that is linear in labor multiplied by a random productivity shifter. Thus the level of productivity is the level of non-traded output in a country. The value of non-traded output in terms of the numeraire is the nominal (wage) income of the household. Real wage income fluctuates with nominal wage income and the CPI. The CPI is a function of the relative price of non-traded goods. The real exchange rate is the ratio of the foreign CPI to the home CPI.

Let a = b equal the share of spending on non-traded goods in each country and let k = 1- a = 1- b equal the share of spending on the traded good in each country. Equity shares are initially equally allocated to home and foreign households (so, for example, home households own half the share claims on home traded good dividends and half the share claims on foreign traded good dividends). At an optimum $k(C^T_t)^{-1} = k(2X_t – C^T_t)^{-1}$ where $X_t = 0.5(T_t + T^*_t)$ is the per capita global supply of the traded good in period t. Thus $C^T_t = C^{*T}_t = X_t$ period by period which can be replicated with the initial allocation of equities. Clearly $C^N_t = H_t$ and $C^{*N}_t = F_t$ period by period with H_t the level of productivity in the home non-traded goods sector and F_t the level of productivity in the home non-traded goods sector. The relative price of non-traded goods at home is $P^H_t = (a/k)(X_t/H_t)$ and the relative price of non-traded goods abroad is $P^{*F}_t = (a/k)(X_t/F_t)$. The home CPI is given by CPI = $a^{-a}k^{-k}(P^H_t)^a$ and foreign CPI is given by CPI* = $a^{-a}k^{-k}(P^{*F}_t)^b$. The real exchange rate is given by $Q_t = (P^{*F}_t)^b(P^H_t)^{-a} = H_t^a / F^b_t$. The real stochastic discount factors in the home and foreign country and the real exchange rate are given by

$$\begin{aligned} & Z_{t,1} = e^{-\delta}(H_t)^a(X_t)^k(H_{t+1})^{-a}(X_{t+1})^{-k} \\ (12)\quad & Z^*_{t,1} = e^{-\delta^*}(F_t)^b(X_t)^k(F_{t+1})^{-b}(X_{t+1})^{-k} \\ & Q_{t+1} = H_{t+1}{}^a / F^b{}_{t+1} \end{aligned}$$

Letting $h_t = \ln H_t$, f_t, and x_t evolve as in (5), this system becomes`

$z_{t,1} = -\delta + a(1-\rho)h_t + k(1-\rho)x_t - ae_{t+1} - k\varepsilon_{t+1}$

(13) $z^*_{t,1} = -\delta^* + b(1-\rho)f_t + k(1-\rho)x_t - b\xi_{t+1} - k\varepsilon_{t+1}$

$\Delta q_{t+1} = \delta - \delta^* + a(\rho-1)h_t - b(\rho-1)f_t + a\, e_{t+1} - b\xi_{t+1}$

which is a special case of (1) – (3) with $\psi = \chi = \lambda = 0$. This example can reproduce the common plus home and foreign factor model presented above. It is also possible to extend this example by tacking on a common conditionally heteroscedastic preference shock that generates time varying risk premia while preserving the common plus home and foreign factor model presented above.

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