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A CONTINUOUS-TIME APPROACH

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Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach
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ABSTRACT

We recast the Aiyagari-Bewley-Huggett model of income and wealth distribution in continuous time. This workhorse model – as well as heterogeneous agent models more generally – then boils down to a system of partial differential equations, a fact we take advantage of to make two types of contributions. First, a number of new theoretical results: (i) an analytic characterization of the consumption and saving behavior of the poor, particularly their marginal propensities to consume; (ii) a closed-form solution for the wealth distribution in a special case with two income types; (iii) a proof that there is a unique stationary equilibrium if the intertemporal elasticity of substitution is weakly greater than one; (iv) characterization of “soft” borrowing constraints. Second, we develop a simple, efficient and portable algorithm for numerically solving for equilibria in a wide class of heterogeneous agent models, including – but not limited to – the Aiyagari-Bewley-Huggett model.

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Introduction

One of the key developments in macroeconomics research over the last three decades has been the incorporation of explicit heterogeneity into models of the macroeconomy. Fueled by the increasing availability of high-quality micro data, the advent of more powerful computing methods as well as rising inequality in many advanced economies, such heterogeneous agent models have proliferated and are now ubiquitous. This is a welcome development for a number of reasons. First, it opens up the door to bringing micro data to the table in order to empirically discipline macro theories. Second, macroeconomists often want to analyze the welfare implications of particular shocks or policies. This is impossible without asking “who gains and who loses?”, that is, distributional considerations often cannot be ignored. Third, models with heterogeneity often deliver strikingly different aggregate implications than do representative agent models, for example with respect to monetary and fiscal policies.¹

Despite the continuously increasing popularity of macroeconomic models with rich heterogeneity, the literature has suffered from a dearth of theoretical and analytical results. Little is known about the properties of consumption and saving behavior in the presence of borrowing constraints, those of the resulting wealth distribution, and equilibrium uniqueness (or lack thereof). Instead, most studies rely on purely numerical analyses to characterize the implications of such theories. But even such computational approaches are often difficult and costly, particularly if the question at hand requires solving for the economy’s transition dynamics or if the model features non-differentiabilities or non-convexities.

In this paper we make some progress on these issues by recasting the standard incomplete market model of [Aiyagari \(1994\)](#), [Bewley \(1986\)](#) and [Huggett \(1993\)](#) in continuous time. Our main contributions are twofold. First, we prove a number of new theoretical results about this workhorse model.² Second, we develop a simple, efficient and portable algorithm for numerically solving both stationary equilibria and transition dynamics of a wide class of heterogeneous agent models, including – but not limited to – the Aiyagari-Bewley-Huggett model.

Both types of contributions make use of an important property: when recast in continuous time, heterogeneous agent models boil down to systems of two coupled partial differential equations. The first of these is a Hamilton-Jacobi-Bellman (HJB) equation for the optimal choices of a single atomistic individual who takes the evolution of the distribution and hence prices as given. And the second is a Kolmogorov Forward (KF) equation characterizing

¹[Deaton \(2016\)](#) succinctly summarizes the second and third reasons: “While we often must focus on aggregates for macroeconomic policy, it is impossible to think coherently about national well-being while ignoring inequality and poverty, neither of which is visible in aggregate data. Indeed, and except in exceptional cases, macroeconomic aggregates themselves depend on distribution.”

²Of course and as is well-known, the unadorned Aiyagari-Bewley-Huggett model is not sufficiently rich to be an empirically realistic theory of income and wealth distribution. Understanding its theoretical properties is nevertheless important, simply because it forms the backbone of much of modern macroeconomics.

the evolution of the distribution, given optimal choices of individuals.³ In the context of the Aiyagari-Bewley-Huggett model, the HJB equation characterizes individuals’ optimal consumption and saving behavior given a stochastic process for income; and the KF equation characterizes the evolution of the joint distribution of income and wealth. The two equations are coupled because optimal consumption and saving depend on the interest rate which is determined in equilibrium and hence depends on the wealth distribution. We start with a particularly parsimonious case: a [Huggett \(1993\)](#) economy in which idiosyncratic income risk takes the form of exogenous endowment shocks that follow a two-state Poisson process and in which individuals save in unproductive bonds that are in fixed supply. Later in the paper, we extend many of our results to more general stochastic processes and to an [Aiyagari \(1994\)](#) economy in which individuals save in productive capital and income takes the form of labor income.

We prove four new theoretical results about the Aiyagari-Bewley-Huggett model. *First*, we provide an analytic characterization of the consumption and saving behavior of the poor. We show that, under natural assumptions,⁴ an individual’s saving policy function behaves like $-\sqrt{2\nu a}$ in the vicinity of the borrowing constraint where a is her wealth in deviations from this constraint and ν is a constant that depends on parameters. Equivalently, her consumption function behaves like her total income plus $\sqrt{2\nu a}$. This characterization implies that: (i) individuals necessarily hit the borrowing constraint in finite time after a long enough sequence of low income shocks, (ii) we have an intuitive characterization of the speed ν at which an individual does so as well as her *marginal propensity to consume* (MPC) out of a windfall income gain. This MPC is higher the lower is the interest rate relative to the rate of time preference, the more willing to intertemporally substitute individuals are, or the higher is the likelihood of getting a high income draw; it is non-monotone in the income received in low-income states (e.g. unemployment benefits). Understanding the theoretical determinants of MPCs is, of course, important for a large body of applied work.⁵ *Second*, we derive an analytic solution for the wealth distribution for a special case with two income types. This analytic solution provides a clean characterization of various properties of the wealth distribution, particularly the behavior of its left and right tails. For example, a direct corollary

³[Lasry and Lions \(2007\)](#) have termed such systems “Mean Field Games” in analogy to the continuum limit often taken in statistical mechanics and physics, e.g. when solving the Ising model. We build on their earlier work. Also see [Guéant, Lasry, and Lions \(2011\)](#) and [Cardialaguet \(2013\)](#) and the references cited therein. The “Kolmogorov Forward equation” is also often called “Fokker-Planck equation.” Because the term “Kolmogorov Forward equation” seems to be somewhat more widely used in economics, we will use this convention throughout the paper. But these are really two different names for the same equation.

⁴Namely that either the borrowing constraint is tighter than the natural borrowing constraint or the coefficient of absolute risk aversion is bounded as consumption approaches zero (or both).

⁵The distribution of MPCs determines, for example, the efficacy of fiscal stimulus (e.g. [Kaplan and Violante, 2014](#); [Hagedorn, Manovski, and Mitman, 2017](#)), the transition mechanism of monetary policy (e.g. [Auclert, 2017](#); [Kaplan, Moll, and Violante, 2016](#)), the effect of a credit crunch or house price movements on consumer spending (e.g. [Guerrieri and Lorenzoni, 2017](#); [Berger, Guerrieri, Lorenzoni, and Vavra, 2015](#)) and the extent to which inequality affects aggregate demand (e.g. [Auclert and Rognlie, 2016, 2017](#)).

of individuals hitting the borrowing constraint in finite time is that the wealth distribution features a Dirac point mass at this constraint. *Third*, we prove existence and uniqueness of a stationary equilibrium for general utility functions and income processes under the intuitive condition that the intertemporal elasticity of substitution (IES) $-u'(c)/(u''(c)c)$ is weakly greater than one for all consumption levels c .⁶ Without a uniqueness result the economy could, in principle, be subject to poverty traps and history dependence.⁷ *Fourth*, we consider “soft borrowing constraints”, i.e. a wedge between borrowing and saving rates, and characterize their implication for saving behavior and the wealth distribution. This form of constraint can explain the empirical observation that wealth distributions typically have a spike at zero net worth and mass both to the left and the right of zero.

In addition to these results, which are new also relative to the existing discrete-time literature, we extend some useful existing discrete-time results and concepts to continuous time. First, we adapt a number of results from [Aiyagari \(1994\)](#), e.g. that the wealth distribution has a finite upper bound and that a stationary equilibrium exists. Second, we characterize the saving behavior of the wealthy and show that, with constant relative risk aversion (CRRA) utility, consumption and saving policy functions become linear for high wealth ([Benhabib, Bisin, and Zhu, 2015](#); [Benhabib and Bisin, 2016](#)). Third, we show how to define in continuous time marginal propensities to consume and save over discrete time intervals. This is not obvious and, at the same time, important for bringing the model to the data. Finally, a methodological contribution of our paper is to show how to handle borrowing constraints in continuous time: conveniently, the borrowing constraint never binds in the interior of the state space and only shows up in a boundary condition.⁸ The consumption first-order condition always holds with equality, thereby sidestepping any complications due to “occasionally binding constraints.” Many of our proofs exploit this fact.

As already mentioned, our second main contribution is the development of a simple, efficient and portable numerical algorithm for computing a wide class of heterogeneous agent models. The algorithm is based on a finite difference method and applies to the computation of both stationary and time-varying equilibria.⁹ We explain this algorithm in the context of the Aiyagari-Bewley-Huggett model. But the algorithm is, in fact, considerably more

⁶In addition we assume for the uniqueness result that individuals cannot borrow. A key step in our proof is an important result by [Olivi \(2017\)](#). Contemporaneous work by [Light \(2017\)](#) derives a uniqueness result under more restrictive assumptions in a discrete-time setting: he proves uniqueness for an Aiyagari economy with CRRA utility and Cobb-Douglas production (as well as no borrowing) under the assumption that the constant IES is greater than one.

⁷For the same reasons, one of the first result that every graduate student learns is that the neoclassical growth model – the representative-agent counterpart to the Aiyagari model – features a unique steady state.

⁸This is in contrast to discrete-time formulations where there is typically a critical level of wealth, strictly bigger than the borrowing constraint, such that the constraint binds for all lower levels of wealth.

⁹Our numerical method is based on [Achdou and Capuzzo-Dolcetta \(2010\)](#) and [Achdou \(2013\)](#) but modified to handle the particular features of heterogeneous agent models, in particular borrowing constraints. [Candler \(1999\)](#) has previously used a finite difference method to solve HJB equations arising in economics and we discuss the relation to his work in Section 3.7.

general and applies to *any* heterogeneous agent model with a continuum of atomistic agents (and without aggregate shocks). In Section 4 we demonstrate the algorithm’s generality by applying it to other theories that feature non-convexities, a fat-tailed wealth distribution and multiple assets. Codes for these applications (and many more) are available from <http://www.princeton.edu/~moll/HACTproject.htm> in Matlab as well as Python, Julia and C++.

The first step of the algorithm is to solve the HJB equation for a given time path of prices. The second step is to solve the KF equation for the evolution of the joint distribution of income and wealth. Conveniently, after having solved the HJB equation, one obtains the time path of the distribution essentially “for free,” i.e. with very few lines of code. This is because the KF equation is the “transpose problem” of the HJB equation.¹⁰ The third step is to iterate and repeat the first two steps until an equilibrium fixed point for the time path of prices is found. For the first step, we make use of the theory of “viscosity solutions” to HJB equations (Crandall and Lions, 1983), and the corresponding theory for their numerical solution using finite difference methods (Barles and Souganidis, 1991). While our paper can be read without knowledge of the theory of viscosity solutions, Online Appendix C.1 provides an “economist-friendly” introduction and lists a number of relatively accessible references.

Continuous time imparts a number of computational advantages relative to discrete time. As explained in more detail in Section 3.1, these relate to the handling of borrowing constraints, the numerical solution of first-order conditions and the fact that continuous-time problems with discretized state space are, by construction, very “sparse.” These computational advantages are reflected in the algorithm’s efficiency. Just to give a flavor, computing the stationary equilibrium of Huggett’s economy with two income types and 1,000 grid points in the wealth dimension using Matlab takes around a quarter second on a Macbook Pro laptop computer. At the same time, the algorithm is simple. Implementing it requires only some basic knowledge of matrix algebra and access to a software package that can solve sparse linear systems (e.g. Matlab). Finally, the algorithm is portable. For example, it applies without change to problems that involve non-differentiabilities and non-convexities. These are difficult to handle with standard discrete-time methods.¹¹ In contrast, viscosity solutions and finite difference methods are *designed* to handle non-differentiable and non-convex problems. To illustrate this, we use the same algorithm to compute equilibria of an economy in which the interplay of indivisible housing and mortgages with a down-payment constraint cause a non-convexity which can result in individual poverty traps and multiple stationary distributions (Section 4.3).

Besides hopefully being useful in their own right, our paper’s contributions also constitute the foundation for a number of generalizations that go beyond the setup that we

¹⁰More precisely, the differential operator in the KF equation is the adjoint of the differential operator in the HJB equation. The adjoint of an operator is the infinite-dimensional analogue of a matrix transpose.

¹¹Because first-order conditions are no longer sufficient and standard envelope theorems do not apply.

consider in the present paper. First, building on the finite difference method developed here, [Ahn, Kaplan, Moll, Winberry, and Wolf \(2017\)](#) develop a computational method for solving heterogeneous agent models with aggregate uncertainty in addition to idiosyncratic risk (as in [Den Haan, 1997](#); [Krusell and Smith, 1998](#)). Second, [Kaplan, Moll, and Violante \(2016\)](#) incorporate multiple assets and kinked adjustment costs. Third, [Olivi \(2017\)](#) leverages our continuous-time formulation to obtain powerful comparative statics and sufficient statistics in incomplete markets models – results we build on in our uniqueness proof. Fourth, [Parra-Alvarez, Posch, and Wang \(2017\)](#) discuss how to identify and estimate continuous-time Aiyagari-Bewley-Huggett models. Fifth, [Nuño and Moll \(2017\)](#) and [Nuño and Thomas \(2017\)](#) devise a method for computing social optima in such environments. Sixth, [Shaker Akhtekhane \(2017\)](#) extends our computational approach to economies with heterogeneous firms à la [Hopenhayn \(1992\)](#). Seventh, [Ahn \(2017\)](#) computes equilibria in economies with a large number of individual state variables (four or more) by marrying our finite difference method with a sparse grid approach ([Bungartz and Griebel, 2004](#); [Gerstner and Griebel, 2010](#)). All of these generalizations build on the tools developed in the present paper.

A large theoretical and quantitative literature studies environments in which heterogeneous households are subject to uninsurable idiosyncratic shocks. See [Heathcote, Storesletten, and Violante \(2009\)](#), [Guvenen \(2011\)](#), [Quadrini and Ríos-Rull \(2015\)](#) and [Krueger, Mitman, and Perri \(2015\)](#) for recent surveys, and the textbook treatment in [Ljungqvist and Sargent \(2004\)](#). All of these are set in discrete time.

Much fewer papers have studied equilibrium models with heterogeneous households in continuous time.¹² All of these papers make “just the right assumptions” about the environment being studied so that equilibria can be solved explicitly (or at least characterized tightly).¹³ In contrast, our aim is to develop tools for solving and analyzing models that do not permit closed-form solutions. Our methods apply as long as the model under consideration can be boiled down to an HJB equation and a KF equation, a feature shared by a wide class of heterogeneous agent models. These two approaches are clearly complimentary: on the one hand, having explicit solutions is often extremely valuable for gaining intuition; on the other hand, restricting attention to environments for which these can be found may represent a sort of “analytic straitjacket” for some applications and the availability of more

¹²See for example [Jovanovic \(1979\)](#), [Moscarini \(2005\)](#), [Alvarez and Shimer \(2011\)](#), [Moll \(2014\)](#), [Stokey \(2014\)](#), [Vindigni, Scotti, and Tealdi \(2014\)](#), [Jones and Kim \(2014\)](#), [Jones \(2015\)](#), [Toda and Walsh \(2015\)](#), [Benhabib, Bisin, and Zhu \(2016\)](#), [Cao and Luo \(2017\)](#) and [Kasa and Lei \(2017\)](#). [Luttmer \(2007, 2011, 2015\)](#) and [Benhabib, Perla, and Tonetti \(2017\)](#) analyze theories with heterogeneous producers. These papers, like ours, all study economies with a continuum of heterogeneous agents giving rise to a system of coupled HJB and KF equations. In contrast, other papers study environments with a finite number of heterogeneous agents (typically equal to two). For example, see [Scheinkman and Weiss \(1986\)](#) and applications of their framework by [Conze, Lasry, and Scheinkman \(1993\)](#) and [Lippi, Ragni, and Trachter \(2013\)](#).

¹³Similarly, there are also several discrete-time approaches for retaining tractability in environments with heterogeneous households (e.g. [Bénabou, 2002](#); [Krebs, 2003](#); [Heathcote, Storesletten, and Violante, 2014](#)).

general methods may prove useful in such contexts.

One other paper by [Bayer and Wälde \(2010\)](#) also studies a continuous-time version of an Aiyagari-Bewley-Huggett model. The main differences between their paper and ours are: (i) they analyze a partial equilibrium framework whereas we consider a general equilibrium framework, and (ii) we develop a numerical algorithm for solving both stationary and time-varying equilibria, thereby actually “operationalizing” the analysis of this class of models.^{14,15} Also closely related, [Rocheteau, Weill, and Wong \(2015\)](#) propose an elegant alternative general equilibrium model with incomplete markets in continuous time. Market incompleteness in their framework stems from lumpy consumption expenditure shocks (e.g. health events) rather than idiosyncratic income risk. As a result their model features only one individual state variable and many results can be derived in closed form. The tradeoff is that their theory is further from the standard Aiyagari-Bewley-Huggett model that forms the backbone of much of modern macroeconomics.¹⁶

Section 1 lays out our continuous-time version of the workhorse macroeconomic model of income and wealth distribution in the parsimonious form due to [Huggett \(1993\)](#). Section 2 contains our new theoretical results. Section 3 describes our computational algorithm for both stationary and time-varying equilibria and discusses computational advantages relative to existing discrete-time methods. Section 4 demonstrates the algorithm’s generality by applying it to a number of other economies. Section 5 concludes.

1 The Workhorse Model of Income and Wealth Distribution in Macroeconomics

To explain the logic of our approach in the simplest possible fashion, we present it in a context that should be very familiar to many economists: a general equilibrium model with incomplete markets and uninsured idiosyncratic labor income risk as in [Aiyagari \(1994\)](#), [Bewley \(1986\)](#) and [Huggett \(1993\)](#). We first do this in the context of an economy in which individuals save in unproductive bonds that are in fixed supply as in [Huggett \(1993\)](#). We

¹⁴Additionally, Bayer and Wälde study a version of the model with the “natural borrowing constraint” implying that individuals never actually hit that constraint. Another difference is that they characterize individuals’ saving behavior in terms of a differential equation for its consumption policy function. In contrast, we do this in terms of the HJB equation for the value function, thereby allowing us to take advantage of the theory of viscosity solutions. Of course the two approaches are related and their equation for consumption can be derived from the envelope condition of the HJB equation.

¹⁵Also [Lise \(2013\)](#) studies heterogeneous households in continuous time but in partial equilibrium.

¹⁶[Wang \(2007\)](#) proposes an elegant continuous-time Aiyagari-Bewley-Huggett model that can be solved analytically but at the cost of making two non-standard assumptions on preferences: that individuals have CARA utility and that current discount rates are increasing in past consumption (in the absence of the second assumption, CARA utility implies exploding wealth inequality and no stationary wealth distribution exists).

later consider different ways of closing the model.

1.1 Setup

Individuals There is a continuum of individuals that are heterogeneous in their wealth a and income y . The state of the economy is the joint distribution of income and wealth. Individuals have standard preferences over utility flows from future consumption c_t discounted at rate $\rho \geq 0$:

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt. \quad (1)$$

The function u is strictly increasing and strictly concave. An individual has an income y_t which is simply an endowment of the economy's final good. His wealth takes the form of bonds and evolves according to

$$\dot{a}_t = y_t + r_t a_t - c_t, \quad (2)$$

where r_t is the interest rate. Individuals also face a borrowing limit

$$a_t \geq \underline{a}, \quad (3)$$

where $-\infty < \underline{a} < 0$.¹⁷ Finally, an individual's income evolves stochastically over time. In particular, we assume that income follows a two-state Poisson process $y_t \in \{y_1, y_2\}$, with $y_2 > y_1$. The process jumps from state 1 to state 2 with intensity λ_1 and vice versa with intensity λ_2 . The two states can be interpreted as employment and unemployment so that λ_1 is the job-finding rate and λ_2 the job destruction rate. The two-state Poisson process is chosen for simplicity and Section 4 shows how the setup can be extended to more general income processes.

Individuals maximize (1) subject to (2), (3) and the process for y_t , taking as given the evolution of the equilibrium interest rate r_t for $t \geq 0$. For future reference, we denote by $g_j(a, t)$, $j = 1, 2$ the joint distribution of income y_j and wealth a .¹⁸

Equilibrium The economy can be closed in a variety of ways. We here present the simplest possible way of doing this following Huggett (1993). We assume that the only price in this economy is the interest rate r_t which is determined by the requirement that, in equilibrium,

¹⁷As discussed in detail in Aiyagari (1994), if the borrowing limit \underline{a} is less tight than the so-called "natural borrowing limit", the constraint $a_t \geq \underline{a}$ will never bind and the "natural borrowing limit" will be the effective borrowing limit. In a stationary equilibrium with $r > 0$, the "natural borrowing limit" is $a_t \geq -y_1/r$ where y_1 is the lowest income. In an equilibrium with a time-varying interest rate r_t the natural borrowing constraint is $a_t \geq -y_1 \int_t^\infty \exp(-\int_t^s r_\tau d\tau) ds$. The natural borrowing constraint ensures that a_t never becomes so negative that the individual cannot repay her debt even if she chooses zero consumption thereafter.

¹⁸This joint distribution satisfies $\int_{\underline{a}}^\infty g_1(a, t) da + \int_{\underline{a}}^\infty g_2(a, t) da = 1$, that is $g_j(a, t)$ is the *unconditional* distribution of wealth for a given productivity type $j = 1, 2$.

bonds must be in fixed supply:

$$\int_{\underline{a}}^{\infty} ag_1(a, t)da + \int_{\underline{a}}^{\infty} ag_2(a, t)da = B, \quad (4)$$

where $0 \leq B < \infty$. $B = 0$ means that bonds are in zero net supply. Alternatively, B can be positive. For instance, a government could issue debt and sell it to individuals or there could be saving opportunities abroad.¹⁹ We later consider alternative ways of closing the economy. For instance Section 4.2 assumes that wealth takes the form of productive capital hired by a representative firm so that the interest rate equals the aggregate marginal product of capital as in Aiyagari (1994).

Useful Utility Functions We have not imposed any assumptions on the utility function u besides it being strictly increasing and strictly concave. But in later parts of the paper, it will sometimes be instructive to specialize this utility function to either constant relative risk aversion (CRRA) utility

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \quad (5)$$

or to exponential utility

$$u(c) = -\theta e^{-\theta c}, \quad \theta > 0, \quad (6)$$

and we state these here for future reference.

1.2 Stationary Equilibrium

Individuals' consumption-saving decision and the evolution of the joint distribution of their income and wealth can be summarized with two differential equations: a Hamilton-Jacobi-Bellman (HJB) equation and a Kolmogorov Forward (or Fokker-Planck) equation. In a

¹⁹In the scenario with government debt we assume that this debt is financed with a lump-sum tax on all individuals and income y_t is net of this tax.

stationary equilibrium these take the form:²⁰

$$\rho v_j(a) = \max_c u(c) + v'_j(a)(y_j + ra - c) + \lambda_j(v_{-j}(a) - v_j(a)), \quad (7)$$

$$0 = -\frac{d}{da}[s_j(a)g_j(a)] - \lambda_j g_j(a) + \lambda_{-j} g_{-j}(a), \quad (8)$$

for $j = 1, 2$ and where, throughout this paper, we adopt the convention that $-j = 2$ when $j = 1$ and vice versa. The derivations of the HJB equation (7) and the KF equation (8) can be found in Appendix B. The function s_j in (8) is the saving policy function, i.e. the optimally chosen drift of wealth

$$s_j(a) = y_j + ra - c_j(a), \quad \text{where } c_j(a) = (u')^{-1}(v'_j(a)). \quad (9)$$

The domain of the two ordinary differential equations (7) and (8) is (\underline{a}, ∞) where \underline{a} is the borrowing limit in (3).

The reader may wonder why the borrowing constraint (3) does not feature in the HJB equation (7). The reason is that, in our continuous-time formulation, the borrowing constraint never binds in the interior of the state space, i.e. for $a > \underline{a}$ and as a result an undistorted first-order condition $u'(c_j(a)) = v'_j(a)$ holds everywhere.²¹ Intuitively, since wealth a is a continuously moving state variable, if it is strictly above the borrowing constraint today, it will still be strictly above the constraint an infinitesimal time interval later. Instead, the borrowing constraint gives rise to a *state constraint boundary condition*²²

$$v'_j(\underline{a}) \geq u'(y_j + r\underline{a}), \quad j = 1, 2. \quad (10)$$

To see why this is the appropriate boundary condition, note that the first-order condition $u'(c_j(a)) = v'_j(a)$ still holds at $a = \underline{a}$. The boundary condition (10) therefore implies $s_j(\underline{a}) = y_j + r\underline{a} - c_j(\underline{a}) \geq 0$, i.e. it ensures that the borrowing constraint is never violated. Online

²⁰A more compact way of writing this is to define the *Hamiltonian* $H(p) = \max_c u(c) - pc$ and to write the saving policy function as $s_j(a) = y_j + ra + H'(v'_j(a))$. The HJB and KF equations (7) and (8) can then be expressed as two non-linear partial differential equations in v_j and g_j only, that do not involve a max operator:

$$\begin{aligned} \rho v_j(a) &= H(v'_j(a)) + v'_j(a)(y_j + ra) + \lambda_j(v_{-j}(a) - v_j(a)), \\ 0 &= -\frac{d}{da}[(y_j + ra + H'(v'_j(a)))g_j(a)] - \lambda_j g_j(a) + \lambda_{-j} g_{-j}(a). \end{aligned}$$

²¹This is in contrast to discrete-time formulations where there is a set $[\underline{a}, a^*)$ with $a^* > \underline{a}$ such that type 1's borrowing constraint binds for all $a \in [\underline{a}, a^*)$ and hence the first-order condition is distorted.

²²Note that this inequality has very little to do with the inequality in discrete-time first-order conditions due to occasionally binding borrowing constraints – see e.g. equation (47) later in the paper. In fact, the two inequalities go in opposite directions. Even though both inequalities result from the presence of borrowing constraints, the logic behind them is completely different.

Appendix C.1 derives the state constraint boundary condition (10) more rigorously and makes the connection to a somewhat more general strategy for imposing state constraints used in the mathematics literature, namely to look for a *constrained viscosity solution* of (7).²³ The remainder of our paper can be read without knowledge of the theory of viscosity solutions. The KF equation (8) requires no boundary condition at \underline{a} : the state constraint is satisfied by virtue of s_j being the optimal saving policy function from the HJB equation (7).

Finally, the stationary interest rate r must satisfy the analogue of the market clearing condition (4)

$$S(r) := \int_{\underline{a}}^{\infty} a g_1(a) da + \int_{\underline{a}}^{\infty} a g_2(a) da = B. \quad (11)$$

The two ordinary differential equations (7) and (8) together with (9), (10), and the equilibrium relationship (11) fully characterize the stationary equilibrium of our economy. This system is an instance of what Lasry and Lions (2007) have called a ‘‘Mean Field Game,’’ here in its stationary form. We next turn to its time-dependent analogue, and discuss more of the properties of such systems.

1.3 Transition Dynamics

Many interesting questions require studying transition dynamics, that is the evolution of the economy when the initial distribution of income and wealth does not equal the stationary distribution. The time-dependent analogue of the stationary system (7) to (11) is²⁴

$$\rho v_j(a, t) = \max_c u(c) + \partial_a v_j(a, t)(y_j + r(t)a - c) + \lambda_j(v_{-j}(a, t) - v_j(a, t)) + \partial_t v_j(a, t), \quad (12)$$

$$\partial_t g_j(a, t) = -\partial_a [s_j(a, t)g_j(a, t)] - \lambda_j g_j(a, t) + \lambda_{-j} g_{-j}(a, t), \quad (13)$$

$$s_j(a, t) = y_j + r(t)a - c_j(a, t), \quad c_j(a, t) = (u')^{-1}(\partial_a v_j(a, t)), \quad (14)$$

for $j = 1, 2$, and together with the equilibrium condition (4). We here use the short-hand notation $\partial_a v = \partial v / \partial a$ and so on, and as before s_j denotes the optimal saving policy function. The domain of the two partial differential equations (12) and (13) is $(\underline{a}, \infty) \times \mathbb{R}^+$ (though more on the time domain momentarily). The function v_j again satisfies a state constraint

²³See Soner (1986a,b) and Capuzzo-Dolcetta and Lions (1990).

²⁴As in the stationary case, there is again a more compact way of writing this system as two non-linear partial differential equations in v_j and g_j only, that do not involve a max operator:

$$\begin{aligned} \rho v_j(a, t) &= H(\partial_a v_j(a, t)) + \partial_a v_j(a, t)(y_j + r(t)a) + \lambda_j(v_{-j}(a, t) - v_j(a, t)) + \partial_t v_j(a, t), \\ \partial_t g_j(a, t) &= -\partial_a [(y_j + r(t)a + H'(\partial_a v_j(a, t)))g_j(a, t)] - \lambda_j g_j(a, t) + \lambda_{-j} g_{-j}(a, t), \end{aligned}$$

with the Hamiltonian H given by $H(p) := \max_c u(c) - pc$.

boundary condition similar to (10)

$$\partial_a v_j(\underline{a}, t) \geq u'(y_j + r(t)\underline{a}), \quad j = 1, 2. \quad (15)$$

The density g_j satisfies the initial condition

$$g_j(a, 0) = g_{j,0}(a). \quad (16)$$

The value function satisfies a terminal condition. In principle, the time domain is \mathbb{R}^+ but in practice we work with $(0, T)$ for T “large” and impose

$$v_j(a, T) = v_{j,\infty}(a), \quad (17)$$

where $v_{j,\infty}$ is the stationary value function, i.e. the solution to the stationary problem (7) to (11).

The two partial differential equations (12) and (13) together with (14), the equilibrium relationship (4) and the boundary conditions (15) to (17) fully characterize the evolution of our economy. This “Mean Field Game” has two properties that are worth emphasizing. First, the two equations (12) and (13) are coupled: on one hand, an individual’s consumption-saving decision depends on the evolution of the interest rate which is in turn determined by the evolution of the distribution; on the other hand, the evolution of the distribution depends on individuals’ saving decisions. Second, the two equations run in opposite directions in time: the Kolmogorov Forward equation *runs forward* (as indicated by its name) and *looks backwards* – it answers the question “given the wealth distribution today, savings decisions and the random evolution of income, what is the wealth distribution tomorrow?” In contrast, the Hamilton-Jacobi-Bellman equation (12) *runs backwards* and *looks forward* – it answers the question “given an individual’s valuation of income and wealth tomorrow, how much will she save today and what is the corresponding value function today?”

1.4 Classical versus Weak Solutions of HJB and KF equations

A classical solution to a PDE or ODE is a solution that is differentiable as many times as needed to satisfy the corresponding equation. In particular, classical solutions to the first-order HJB and KF equations (12) and (13) would need to be once differentiable. Similarly, classical solutions to second-order equations that arise for example if y follows a diffusion process as in Section 4.1 would need to be twice differentiable. In general, we do not expect to find such classical solutions to either HJB or KF equations. For instance, the value function v_j may have kinks and the distribution g_j may feature Dirac point masses. Instead, we generally look for certain *weak solutions* of these equations, that is solutions that may not be continuously differentiable or even continuous but still satisfy these equations in some

sense. As we explain in more detail in Online Appendix C, the correct notion for a weak solution of the HJB equation is a *viscosity solution* and that of the KF equation is a *measure-valued solution*.²⁵ See Evans (2010, Section 1.3) and Tao (2008) for illuminating discussions on the role of weak solutions in the study of partial differential equations more generally.

That being said, most of our paper employs classical methods. And, with the preceding paragraph in hand, the reader is well-equipped for those parts that do not.

2 Theoretical Results: Consumption, Saving and Inequality

This Section presents theoretical results about our continuous-time version of the Aiyagari-Bewley-Huggett model, including the four new results emphasized in the introduction. Sections 2.1 to 2.5 analyze the HJB and KF equations (7) and (8) in partial equilibrium, i.e. taking as given a fixed interest rate r (assumed to be less than ρ which will be the equilibrium outcome). Section 2.6 then imposes market clearing (11) and considers the stationary equilibrium, particularly its existence and uniqueness. Section 2.7 considers “soft” borrowing constraints.

2.1 An Euler Equation

Our first few theoretical results concern the consumption and saving behavior of individuals. Our characterization of individual behavior uses the following Lemma.

Lemma 1 *The consumption and saving policy functions $c_j(a)$ and $s_j(a)$ for $j = 1, 2$ corresponding to the HJB equation (7) satisfy*

$$\begin{aligned} (\rho - r)u'(c_j(a)) &= u''(c_j(a))c'_j(a)s_j(a) + \lambda_j(u'(c_{-j}(a)) - u'(c_j(a))), \\ s_j(a) &= y_j + ra - c_j(a). \end{aligned} \tag{18}$$

Proof: Differentiate the HJB equation (7) with respect to a (envelope condition) and use that $v'_j(a) = u'(c_j(a))$ and hence $v''_j(a) = u''(c_j(a))c'_j(a)$. \square

The differential equation (18) is an Euler equation. The right-hand side is simply the expected change of individual marginal utility of consumption $\mathbb{E}_t[du'(c_j(a_t))]/dt$.²⁶ Therefore

²⁵The standard notion of a measure-valued solution is only defined on the interior of the state space and therefore cannot be used to deal with a Dirac point mass at the boundary, a feature that arises in our application. We show in Appendix C.2 how to extend the standard notion to take into account this possibility.

²⁶This uses the extension of Ito’s formula to Poisson processes: $\mathbb{E}_t[du'(c_j(a_t))] = [u''(c_j(a_t))c'_j(a_t)s_j(a_t) + \lambda_j(u'(c_{-j}(a_t)) - u'(c_j(a_t)))]dt$.

(18) can be written in the more standard form

$$\frac{\mathbb{E}_t[du'(c_j(a_t))]}{u'(c_j(a_t))} = (\rho - r)dt.$$

2.2 Consumption and Saving Behavior of the Poor

Our first main result is obtained by analyzing the Euler equation (18) close to the borrowing constraint. The interesting case is when the behavior at the constraint differs qualitatively from that of rich individuals. Whether this is the case depends crucially on two factors: the tightness of the borrowing constraint \underline{a} , and the properties of the utility function at low levels of consumption. To focus on the interesting case, we make the following assumption.

Assumption 1 *The coefficient of absolute risk aversion $R(c) := -u''(c)/u'(c)$ when wealth a approaches the borrowing limit \underline{a} is finite, that is*

$$\underline{R} := -\lim_{a \rightarrow \underline{a}} \frac{u''(y_1 + ra)}{u'(y_1 + ra)} < \infty.$$

For any utility function, a sufficient condition for Assumption 1 is that the borrowing constraint is tighter than the “natural borrowing constraint” $\underline{a} > -y_1/r$. For example, with CRRA utility (5) we have $\underline{R} = \gamma/(y_1 + r\underline{a})$ which is finite whenever $\underline{a} > -y_1/r$. However, Assumption 1 is considerably weaker than this. In particular, it is also satisfied if the borrowing constraint equals the natural borrowing constraint $\underline{a} = -y_1/r$ and the coefficient of absolute risk aversion $-u''(c)/u'(c)$ is bounded as consumption goes to zero. This is for example the case with exponential utility (6) in which case $\underline{R} = \theta < \infty$ regardless of the tightness of the borrowing constraint. The only case in which Assumption 1 is not satisfied is if *both* (i) the borrowing constraint equals the natural borrowing constraint *and* (ii) risk aversion becomes unbounded as consumption goes to zero. For completeness, this case is covered in Proposition 1’ in the Appendix.

In what follows as well as elsewhere in the paper, we use the following asymptotic notation: for any two functions f and g , “ $f(a) \sim g(a)$ as $a \rightarrow \underline{a}$ ” is short-hand notation for $\lim_{a \rightarrow \underline{a}} f(a)/g(a) = 1$, i.e. f “behaves like” g for a close to \underline{a} .

Proposition 1 (MPCs and Saving at Borrowing Constraint) *Assume that $r < \rho$, $y_1 < y_2$ and that Assumption 1 holds. Then the solution to the HJB equation (7) and the corresponding policy functions have the following properties:*

1. $s_1(\underline{a}) = 0$ but $s_1(a) < 0$ all $a > \underline{a}$. That is, only individuals exactly at the borrowing constraint are constrained, whereas those with wealth $a > \underline{a}$ are unconstrained and decumulate assets.

2. as $a \rightarrow \underline{a}$, the saving and consumption policy function of the low income type and the corresponding instantaneous marginal propensity to consume satisfy

$$s_1(a) \sim -\sqrt{2\nu_1}\sqrt{a - \underline{a}}, \quad (19)$$

$$c_1(a) \sim y_1 + ra + \sqrt{2\nu_1}\sqrt{a - \underline{a}},$$

$$c'_1(a) \sim r + \sqrt{\frac{\nu_1}{2(a - \underline{a})}}, \quad (20)$$

$$\nu_1 := \frac{(\rho - r)u'(\underline{c}_1) + \lambda_1(u'(\underline{c}_1) - u'(\underline{c}_2))}{-u''(\underline{c}_1)} \quad (21)$$

$$\approx (\rho - r)\text{IES}(\underline{c}_1)\underline{c}_1 + \lambda_1(\underline{c}_2 - \underline{c}_1),$$

where $\underline{c}_j = c_j(\underline{a})$, $j = 1, 2$ is consumption of the two types at the borrowing constraint and $\text{IES}(c) := -u'(c)/(u''(c)c)$ is the intertemporal elasticity of substitution (IES).²⁷ This implies that the derivatives of c_1 and s_1 are unbounded at the borrowing constraint, $c'_1(a) \rightarrow \infty$ and $s'_1(a) \rightarrow -\infty$ as $a \rightarrow \underline{a}$.

The proof of the Proposition, like that of all others, is in the Appendix. The proof of the first part follows straight from the state constraint boundary condition (10). The second part of the proof follows from characterizing the limiting behavior of the squared saving policy function $(s_1(a))^2$ as wealth a approaches the borrowing constraint \underline{a} – hence the square root.

The consumption and saving behavior in the Proposition is illustrated in Figure 1. Im-

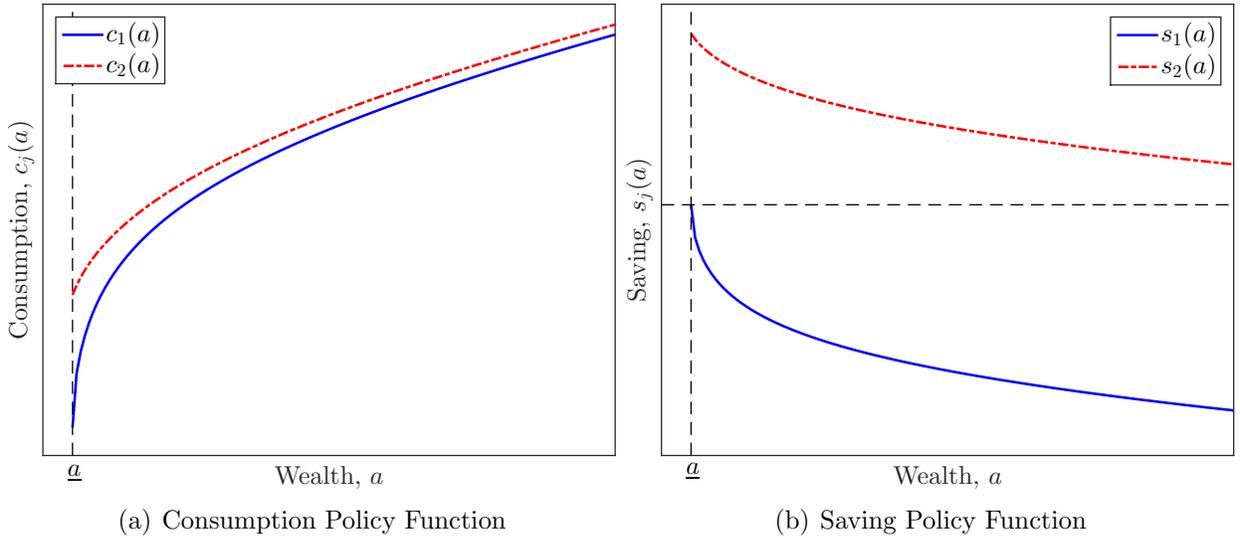


Figure 1: Consumption and Saving Behavior with $r < \rho$

portantly, the derivatives of type 1's consumption and saving policy functions become un-

²⁷Type 1's consumption at the borrowing constraint is given by $\underline{c}_1 = y_1 + r\underline{a}$ and type 2's consumption $\underline{c}_2 > \underline{c}_1$ is a more complicated object determined by the HJB equation (7) for $j = 2$.

bounded at the borrowing constraint. This unbounded derivative has an important implication, namely that individuals hit the borrowing constraint in finite time.

Corollary 1 (Hit Constraint in Finite Time) *Assume that $r < \rho$, $y_1 < y_2$ and that Assumption 1 holds. Then the wealth of an individual with initial wealth a_0 and successive low income draws y_1 converges to the borrowing constraint in finite time at speed governed by ν_1 :*

$$a(t) - \underline{a} \sim \frac{\nu_1}{2} (T - t)^2, \quad T := \sqrt{\frac{2(a_0 - \underline{a})}{\nu_1}}, \quad 0 \leq t \leq T, \quad (22)$$

where T is the “hitting time.”

The result that the borrowing constraint is reached in finite time bears some similarity to optimal stopping time problems (see e.g. Stokey, 2009). Just like in stopping time problems, continuous time avoids a type of integer problem arising in discrete time: the borrowing constraint would be reached after a non-integer time period, but discrete time forces this to occur after an integer number of periods.

Proposition 1 features an intuitive formula (21) for the speed at which individuals hit the borrowing constraint, ν_1 . In Section 2.4 we show that ν_1 is also the key quantity determining the marginal propensity to consume (MPC) out of a windfall income gain. We therefore postpone the discussion of formula (21) until that Section.

Intuition for Proposition 1 and Corollary 1: Two Useful Special Cases. To understand the intuition for the square root in Proposition 1, the implied saving behavior in Corollary 1 and the role of Assumption 1 it is useful to consider two special cases for which analytic solutions are available. Both abstract from income uncertainty which is not essential to the main point.²⁸

In the first special case, an individual has exponential utility (6), receives a deterministic income stream $y > 0$, faces a strict no borrowing constraint $a \geq 0$ and starts with some initial wealth $a_0 > 0$. The corresponding Euler equation and budget constraint are

$$\dot{c} = \frac{1}{\theta}(r - \rho), \quad (23)$$

$$\dot{a} = y + ra - c. \quad (24)$$

Conjecture that at some time $T > 0$, individuals hit the borrowing constraint, i.e. $a(T) = 0$

²⁸We are indebted to Xavier Gabaix for coming up with the first special case. Also see Holm (2017) who provides an elegant analytical characterization of consumption behavior with deterministic income and a borrowing constraint when utility is in the HARA class.

and hence $c(T) = y$. From the Euler equation (23) consumption for $t \leq T$ is

$$c(t) = y + \nu(T - t), \quad \nu := \frac{\rho - r}{\theta} > 0 \quad (25)$$

Substituting into the budget constraint (24), we have $\dot{a}(t) = ra(t) - \nu(T - t)$. Consider first the case $r = 0$ which contains all the intuition. In this case, we have $\dot{a}(t) = -\nu(T - t)$ with solution

$$a(t) = \frac{\nu}{2}(T - t)^2 \quad (26)$$

for $t \leq T$ and where the constant of integration is zero because $a(T) = 0$. Since $a(0) = a_0$, the hitting time is given by $T = \sqrt{2a_0/\nu}$. Note that (26) is the same expression as (22) in Corollary 1. Figure 2 plots the time paths of consumption, saving and wealth for this special case. Panels (a) and (b) show that consumption $c(t)$ decreases linearly toward $c(T) = y$ and savings $\dot{a}(t)$ increase linearly toward $\dot{a}(T) = 0$. Panel (c) plots the resulting wealth

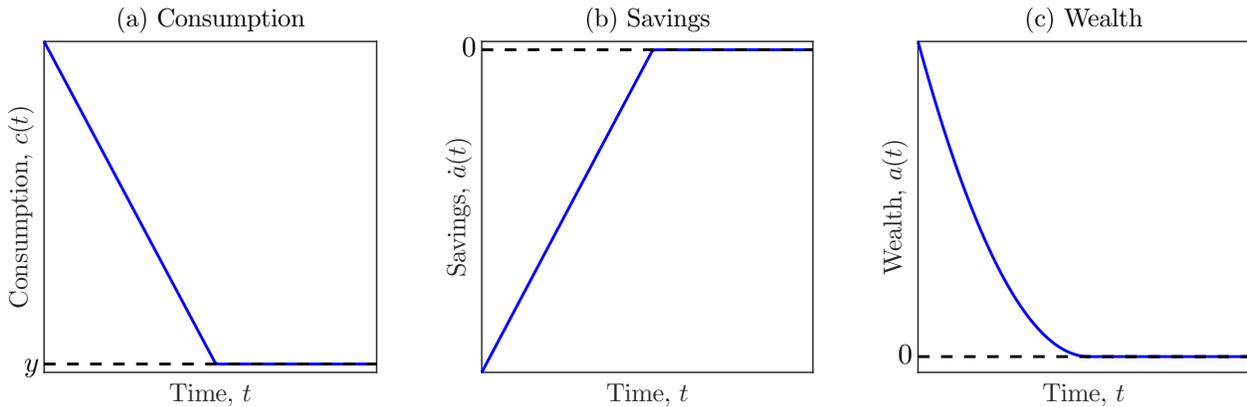


Figure 2: First special case in which borrowing constraint binds in finite time

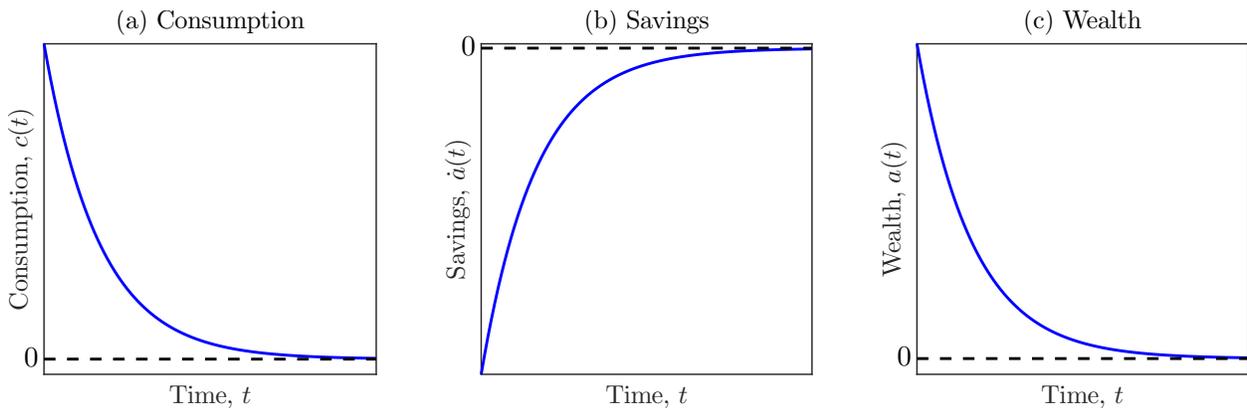


Figure 3: Second special case in which borrowing constraint never binds

dynamics and shows the quadratic shape of equation (26).

To understand the square root in the saving and consumption policy functions in Proposition 1, consider an individual at $t = 0$ with some initial wealth a_0 . From (25) and (26), we have $c(0) = y + \nu T$ and $\dot{a}(0) = -\nu T$ with $T = \sqrt{2a_0/\nu}$. Writing consumption and saving in terms of the state variable a rather than calendar time, we have $c(a) = y + \sqrt{2\nu a}$ and $s(a) = -\sqrt{2\nu a}$ which are the square-root expressions from Proposition 1. This simple derivation also explains why the consumption policy function is concave in wealth a (Figure 1). As the individual approaches the borrowing constraint, both her consumption and wealth decline. If both consumption and wealth declined at the same speed, then consumption would be a linear function of wealth. But this is not the case: instead, wealth declines more rapidly than consumption – quadratically rather than linearly – see Figures 2(a) and (c). Therefore, consumption is a strictly concave function of wealth. Given this logic, it is then also clear that the consumption policy function must be more concave the higher is the speed at which individuals hit the borrowing constraint ν .

In the case $r \neq 0$, (26) generalizes to $a(t) = \frac{\nu}{r^2} (r(T-t) - 1 + e^{-r(T-t)})$ which converges to (26) as $r \rightarrow 0$; individuals still hit the borrowing constraint in finite time and, in fact, $a(t) \sim \frac{\nu}{2}(T-t)^2$ as $t \rightarrow T$.²⁹ Also all other properties of consumption and saving behavior are unchanged for t close to T or, equivalently, for wealth close to the borrowing constraint.

In contrast, consider a second special which is identical except that $y = 0$ and that the individual has CRRA utility (5). In this case, the Euler equation and budget constraint change from (23) and (24) to

$$\frac{\dot{c}}{c} = \frac{1}{\gamma}(r - \rho), \quad \dot{a} = ra - c.$$

It is easy to show that savings and consumption are $\dot{a}(t) = -\eta a(t)$, $c(t) = (r + \eta)a(t)$ where $\eta := \frac{\rho - r}{\gamma}$. Therefore wealth is

$$a(t) = a_0 e^{-\eta t}, \quad t \geq 0. \quad (27)$$

The situation is depicted in Figure 3. As wealth decumulates towards the borrowing constraint, the rate of decumulation slows down more and more and individuals never actually hit the borrowing constraint in finite time. This is an immediate consequence of a linear saving policy function $s(a) = -\eta a$. Turning this logic around, the consumption policy function is linear in wealth because both consumption and wealth decline toward the borrowing constraint at the same speed – see Figures 3 (a) and (c) – rather than wealth declining faster than consumption as in the exponential case.

Our first special case with exponential utility also illustrates the role of Assumption 1

²⁹To see that the expression converges to (26) as $r \rightarrow 0$ use l'Hôpital's rule twice: $\lim_{r \rightarrow 0} a(t; r) = \lim_{r \rightarrow 0} \frac{\nu}{r^2} (r(T-t) - 1 + e^{-r(T-t)}) = \lim_{r \rightarrow 0} \frac{\nu}{2r} (1 - e^{-r(T-t)}) (T-t) = \frac{\nu}{2}(T-t)^2$. Similarly, we have $\lim_{t \rightarrow T} \frac{a(t)}{(T-t)^2} = \lim_{t \rightarrow T} \frac{\nu}{r^2(T-t)^2} (r(T-t) - 1 + e^{-r(T-t)}) = \frac{\nu}{2}$ where the last equality again uses l'Hôpital's rule twice; equivalently $a(t) \sim \frac{\nu}{2}(T-t)^2$ as $t \rightarrow T$.

and how it differs from the assumption that the borrowing constraint is tighter than the natural borrowing constraint. To this end, assume that $y = 0$ so that $a \geq -y/r = 0$ is also the natural borrowing constraint. Nevertheless, if utility is exponential, this does *not* change the individual's saving behavior and she still hits the borrowing constraint in finite time. This is because with exponential utility Assumption 1 is satisfied regardless of the borrowing constraint's tightness.

2.3 Consumption and Saving Behavior of the Wealthy

Proposition 1 characterizes the consumption and saving behavior close to the borrowing constraint. The following Proposition 2 characterizes consumption and saving behavior for large wealth levels. This will be useful below, when we characterize the upper tail of the wealth distribution.

Proposition 2 (Consumption and Saving Behavior of the Wealthy) *Assume that $r < \rho, y_1 < y_2$ and that relative risk aversion $-cu''(c)/u'(c)$ is bounded above for all c .*

1. *Then there exists $a_{\max} < \infty$ such that $s_j(a) < 0$ for all $a \geq a_{\max}, j = 1, 2$, and $s_2(a) \sim \zeta_2(a_{\max} - a)$ as $a \rightarrow a_{\max}$ for some constant ζ_2 . The wealth of an individual with initial wealth a_0 and successive high income draws y_2 converges to a_{\max} asymptotically (i.e. not in finite time): $a(t) - a_{\max} \sim e^{-\zeta_2 t}(a_0 - a_{\max})$.*
2. *In the special case of CRRA utility (5) individual policy functions are asymptotically linear in a . As $a \rightarrow \infty$, they satisfy*

$$s_j(a) \sim \frac{r - \rho}{\gamma} a, \quad c_j(a) \sim \frac{\rho - (1 - \gamma)r}{\gamma} a. \quad (28)$$

The first part of the Proposition is the analogue of Proposition 4 in Aiyagari (1993). The condition that $-cu''(c)/u'(c)$ is bounded above for all c for example rules out exponential utility (6) in which case $\gamma(c) = \theta c$.

The second part of the Proposition extends to continuous time a result by Benhabib, Bisin, and Zhu (2015) who have shown that, with CRRA utility, consumption and saving policy functions are asymptotically linear for large wealth. The proof makes use of a simple homogeneity property: it shows that, for all $\xi > 0$, the value function v expressed as a function of wealth a and income y satisfies $v(\xi a, y) = \xi^{1-\gamma} v(a, y/\xi)$. That is, doubling wealth a , besides scaling everything by a factor $2^{1-\gamma}$, effectively halves income y . Therefore, as wealth becomes large, it is as if the individual had no labor income. And it is well-known that the consumption-saving problem with CRRA and without labor income has an analytic solution with linear policy functions given by (28).

Another way of appreciating the asymptotic linearity is again with a simple special case without income risk. Assume that individuals have CRRA utility, (5) a deterministic stream of labor income $y > 0$, and the borrowing constraint equals the natural borrowing constraint $\underline{a} = -y/r$. Consumption and saving policy functions then have a closed-form solution given by

$$s(a) = \frac{r - \rho}{\gamma} \left(a + \frac{y}{r} \right), \quad c(a) = \frac{\rho - (1 - \gamma)r}{\gamma} \left(a + \frac{y}{r} \right). \quad (29)$$

As $a \rightarrow \infty$, y/r becomes irrelevant relative to a and the policy functions indeed satisfy (28).

The asymptotic linearity of consumption and saving policy functions with CRRA utility has played a key role in the literature. For instance, [Krusell and Smith \(1998\)](#) argue that this linearity explains their finding that the business cycle properties of their baseline heterogeneous agent model are virtually indistinguishable from its representative agent counterpart (see Figure 2 and surrounding discussion in their paper). Future studies may want to gauge the robustness of this result by relaxing the assumption of CRRA utility.

2.4 Marginal Propensities to Consume and Save

We now characterize further the consumption and saving policy functions and in particular the corresponding marginal propensities to consume and save, defined as the changes in consumption and saving in response to a windfall increase in available funds a . Propositions 1 and 2 characterize the slope of the consumption function $c'_j(a)$ or, equivalently, the *instantaneous MPC* which captures the consumption gain (per time unit) after such a windfall over an infinitesimally small time interval. This is an interesting object but it does not correspond to what is measured in the data, namely the fraction of income consumed out of a windfall income gain over a discrete time interval. We here show how to characterize this more empirically relevant object.

Definition 1 *The Marginal Propensity to Consume over a period τ is given by*

$$\text{MPC}_{j,\tau}(a) = C'_{j,\tau}(a), \quad \text{where} \quad (30)$$

$$C_{j,\tau}(a) = \mathbb{E} \left[\int_0^\tau c_j(a_t) dt \mid a_0 = a, y_0 = y_j \right]. \quad (31)$$

Similarly, the Marginal Propensity to Save over a period τ is given by

$$\text{MPS}_{j,\tau}(a) = S'_{j,\tau}(a), \quad \text{where} \quad (32)$$

$$S_{j,\tau}(a) = \mathbb{E} \left[a_\tau \mid a_0 = a, y_0 = y_j \right]. \quad (33)$$

To get a feel for the behavior of these objects and to see how they differ from their instantaneous counterparts $c'_j(a)$ and $s'_j(a)$, it is instructive to consider a time interval τ

that is sufficiently small so that individuals in the low income state do not switch income state. In this case, expected saving over a period τ for the low income type, $S_{1,\tau}(a)$ defined in (33), is simply given by a_τ starting from $a_0 = a$. But from Corollary 1 we know that $a_\tau - \underline{a} \sim \frac{\nu_1}{2} ((T - \tau)^+)^2$ with $T = \sqrt{2(a_0 - \underline{a})/\nu_1}$, i.e.

$$S_{1,\tau}(a) \sim \frac{\nu_1}{2} \left(\left(\sqrt{\frac{2(a - \underline{a})}{\nu_1}} - \tau \right)^+ \right)^2 + \underline{a}. \quad (34)$$

Differentiating this expression and using the budget constraint, we get the following result.

Corollary 2 *Assume τ is sufficiently small that individuals with current income draw y_1 do not receive the high-income draw y_2 ,³⁰ that $r < \rho$, and that Assumption 1 holds. Then*

$$\text{MPC}_{1,\tau}(a) \sim \min \left\{ \tau \sqrt{\frac{\nu_1}{2(a - \underline{a})}}, 1 \right\} + \tau r = \min \{ \tau c'_1(a), 1 + \tau r \}, \quad \text{as } a \rightarrow \underline{a}, \quad (35)$$

where $c'_1(a)$ is the instantaneous MPC characterized in Proposition 1. Similarly $\text{MPS}_{1,\tau}(a) \sim 1 + \tau r - \text{MPC}_{1,\tau}(a)$. Alternatively, (35) holds with equality in the special case in Section 2.2 with exponential utility, deterministic income and $\underline{a} = r = 0$.

This result yields a number of useful observations. First, in contrast to the instantaneous MPC $c'_1(a)$ which becomes unbounded as $a \rightarrow \underline{a}$, the MPC over a time period τ in (35) is bounded between zero and $1 + \tau r$. Second, for $a > \underline{a}$ and τ small enough, the marginal propensity to consume (35) is strictly decreasing in wealth a ; that is, consumption over a period τ , $C_{1,\tau}(a)$, is strictly concave in wealth a . Third, the key quantity determining the size of the MPC is ν_1 , the speed at which individuals hit the borrowing constraint. The intuition for the last two properties is the same as that discussed in Section 2.2: because wealth declines toward the borrowing constraint faster than consumption, the mapping from wealth to consumption is concave; and the faster wealth declines, the more concave this mapping.

In contrast, consider the second special case of Section 2.2 in which individuals never hit the borrowing constraint in finite time. From the expression for wealth dynamics (27) we have that savings over a period τ are given by $S_\tau(a) = ae^{-\eta\tau}$ where $\eta =: (\rho - r)/\gamma$. That is, both consumption and saving over a period τ are linear in wealth a and therefore, the marginal propensities to save and consume are independent of wealth:³¹

$$\text{MPS}_\tau(a) = e^{-\eta\tau} \approx 1 - \eta\tau, \quad \text{MPC}_\tau(a) = 1 - e^{-\eta\tau} + \tau r \approx \tau(\eta + r), \quad \eta := \frac{\rho - r}{\gamma}.$$

³⁰Alternatively we can take $\lambda_1 \rightarrow 0$ so that the low income state is close to being absorbing.

³¹Similar expressions can be obtained in the special case from Section 2.3 in which saving and consumption policy functions are linear throughout the wealth distribution and given by (29).

Summarizing, when people hit the borrowing constraint in finite time, MPCs depend on wealth and, in particular, are higher for poorer people.

When individuals experience new income draws within the time interval τ , it is no longer possible to characterize the MPC and MPS as tightly as in Corollary 2 because we lack a characterization of $c_2(a)$ in the vicinity of the borrowing constraint. However, the following Lemma shows how to easily compute the MPC numerically. The key idea is that $C_{j,\tau}(a)$ defined in (31) is a conditional expectation that can be easily computed using the *Feynman-Kac formula* which establishes a link between conditional expectations of stochastic processes and solutions to partial differential equations. Given knowledge of $C_{j,\tau}(a)$, we can then immediately compute $\text{MPC}_{j,\tau}(a) = C'_{j,\tau}(a)$.

Lemma 2 (Computation of MPCs using Feynman-Kac formula) *The conditional expectation $C_{j,\tau}(a)$ defined in (31) and therefore the MPC defined in (30) can be computed as $C_{j,\tau}(a) = \Gamma_j(a, 0)$ where $\Gamma_j(a, t)$ satisfies the system of two PDEs*

$$0 = c_j(a) + \partial_a \Gamma_j(a, t) s_j(a) + \lambda_j (\Gamma_{-j}(a, t) - \Gamma_j(a, t)) + \partial_t \Gamma_j(a, t), \quad j = 1, 2$$

on $(\underline{a}, \infty) \times (0, \tau)$, with terminal condition $\Gamma_j(a, \tau) = 0$ for all a .

Proof: This follows from a direct application of the Feynman-Kac formula for computing conditional expectations as solutions to partial differential equations. \square

Figure 4(a) plots the MPC computed according to this numerical strategy for the two income types and assuming that individuals have CRRA utility (5). For comparison, Figure 4(b) plots the “instantaneous MPC”, i.e. the slope of the consumption function. As expected,

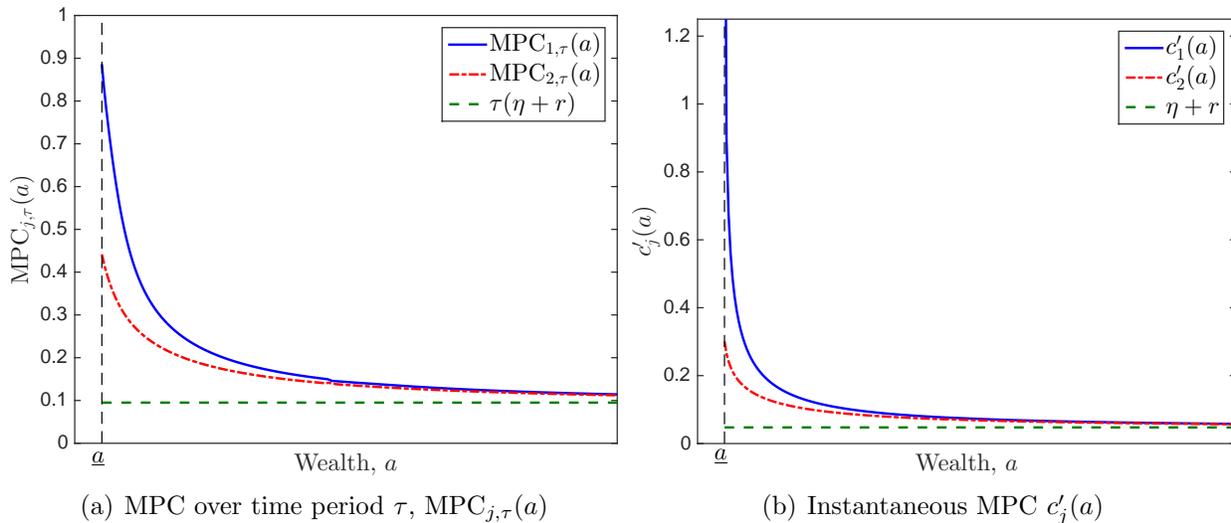


Figure 4: MPCs across the Wealth Distribution

the former is a smoother version of the latter and, in contrast to the latter, does not exceed

$1 + \tau r$. As wealth $a \rightarrow \infty$ the borrowing constraint becomes irrelevant and the slope of the consumption function converges to $\eta + r$ with $\eta = (\rho - r)/\gamma$ and therefore the MPC to $\tau(\eta + r)$. Finally, the strategy for computing MPCs using the Feynman-Kac formula in Lemma 2 is extremely general. For instance, [Kaplan, Moll, and Violante \(2016\)](#) apply it in a considerably more complicated setting with two assets and kinked adjustment costs.

As an aside, in some applications a slightly altered version of the MPCs in Definition 1 may be easier to map to the data. Empirical studies do not typically observe marginal propensities to consume out of an infinitesimal increase in resources. Instead, they observe the increase in consumption in response to a discrete increase in resources, say by \$500. To this end, define $\text{MPC}_{j,\tau}^x(a) := (C_{j,\tau}(a+x) - C_{j,\tau}(a))/x$. This is the MPC out of x dollars over a period τ , i.e. a discrete counterpart to the MPC in (30). [Kaplan, Moll, and Violante \(2016\)](#) compute such discrete MPCs and compare them to various empirical studies such as [Broda and Parker \(2014\)](#), [Misra and Surico \(2014\)](#), [Blundell, Pistaferri, and Saporta-Eksten \(2016\)](#), and [Fagereng, Holm, and Natvik \(2016\)](#).

Using the Analytic Expression for ν_1 to Better Understand MPCs. As part of Proposition 1 we obtained an analytic expression (21) for ν_1 , the speed of hitting the borrowing constraint. As just discussed, ν_1 is also the key quantity governing the size of MPCs. The formula (21) is therefore also useful to examine how MPCs depend on various model parameters. In particular, it can be used to shed some light on various numerical results that may seem counterintuitive at first. For instance, consider the dependence of the low-income type's $\text{MPC}_{1,\tau}(a)$ on the low income realization y_1 . This low income realization may, for example, represent the size of unemployment benefits. Figure 5(a) graphs this relationship

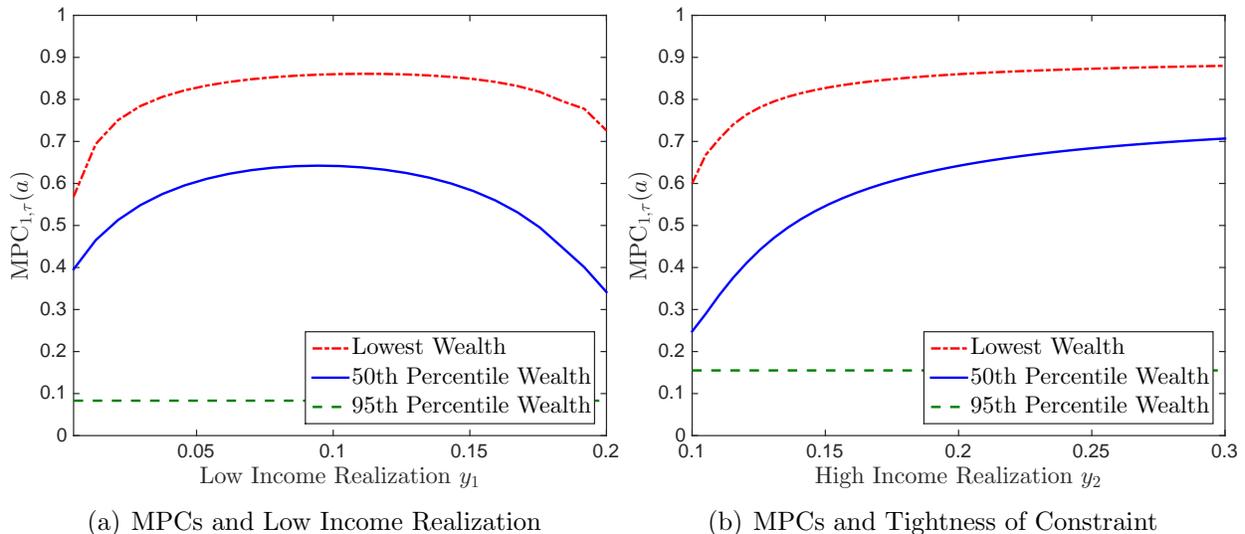


Figure 5: Dependence of MPCs on Parameters

for y_1 ranging between 0 and $y_2 = 0.2$ separately for various percentiles of the wealth distribution and assuming that individuals have CRRA utility (5). Perhaps surprisingly, the MPC is a hump-shaped function of the low income realization y_1 . But formula (21) easily resolves the apparent mystery. With CRRA utility (5) so that the IES is constant

$$\nu_1 \approx (\rho - r) \frac{\underline{c}_1}{\gamma} + \lambda_1(\underline{c}_2 - \underline{c}_1),$$

where $\underline{c}_1 = y_1 + r\underline{a}$. An increase in y_1 has two offsetting effects. The intuitive part is that as y_1 increases, individuals are better insured against idiosyncratic income risk and therefore have a low MPC (as in models without risk and borrowing constraints). In the formula, as y_1 increases toward y_2 , $\lambda_1(\underline{c}_2 - \underline{c}_1)$ converges to zero and this results in a lower ν_1 . But there is an offsetting effect captured by the term $(\rho - r)\underline{c}_1/\gamma$: if consumption conditional on hitting the constraint \underline{c}_1 is high, individuals do not mind hitting the constraint as much. Hence they converge to it faster or, equivalently, have a higher MPC.

Figure 5(b) instead graphs the dependence of the low income type's MPC on the realization of the *high* income y_2 . The MPC is increasing in y_2 and the intuition can again be seen from our formula for ν_1 which shows that the MPC is higher the larger is the consumption gain from getting a high income draw $\lambda_1(\underline{c}_2 - \underline{c}_1)$. Other comparative statics are as follows: individuals have higher MPCs, the lower is the interest rate r relative to the rate of time preference ρ , and the higher is the likelihood λ_1 of getting a high income draw (so that getting stuck at the constraint is less likely). Similarly, MPCs tend to be higher the higher is the IES and the tighter is the borrowing constraint, i.e. the closer to zero is \underline{a} .

2.5 The Stationary Wealth Distribution

We now present the paper's second main theoretical result: an analytic solution to the Kolmogorov Forward equation characterizing the stationary distribution with two income types (8) for any given individual saving policy functions. This analytic solution yields a number of insights about properties of the stationary wealth distribution, particularly at the borrowing constraint and in the right tail.

The derivation of this analytic solution is constructive and straightforward and we therefore present it in the main text. Summing the KF equation (8) for the two income types, we have $\frac{d}{da}[s_1(a)g_1(a) + s_2(a)g_2(a)] = 0$ for all a , which implies that $s_1(a)g_1(a) + s_2(a)g_2(a)$ equals a constant. Because any stationary distribution must be bounded, we must then have $s_1(a)g_1(a) + s_2(a)g_2(a) = 0$ for all a . Substituting into (8) and rearranging, we have

$$g'_j(a) = - \left(\frac{s'_j(a)}{s_j(a)} + \frac{\lambda_j}{s_j(a)} + \frac{\lambda_{-j}}{s_{-j}(a)} \right) g_j(a), \quad j = 1, 2. \quad (36)$$

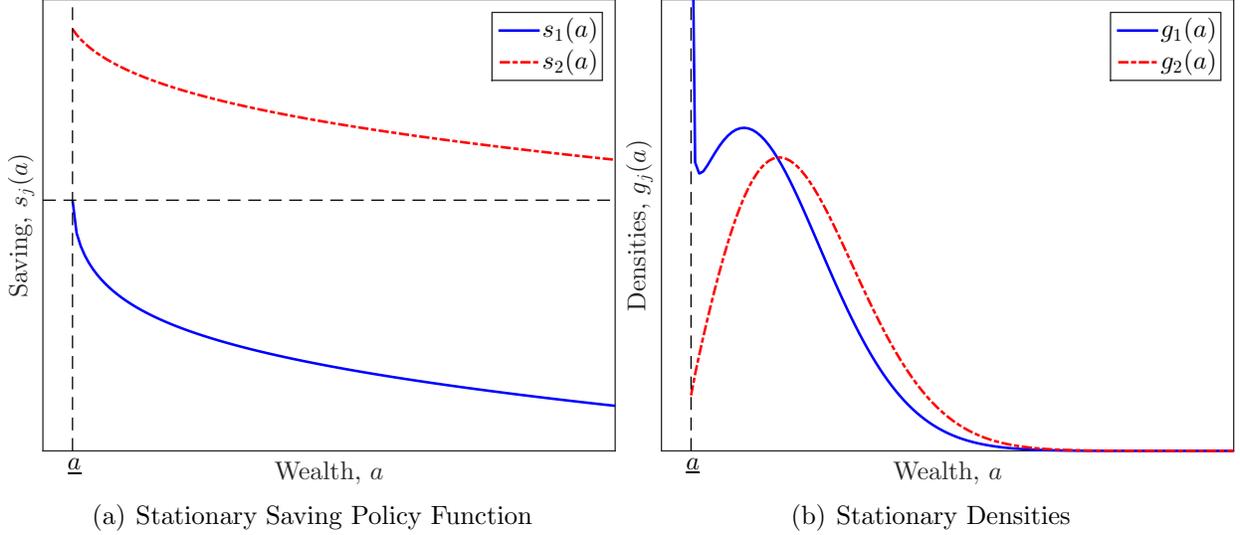


Figure 6: Saving Behavior and Stationary Wealth Distribution with $r < \rho$

Importantly, (36) are two *independent* ODEs for g_1 and g_2 rather than the coupled system of two ODEs (8) we started out with. Together with two boundary conditions they can be solved separately. To obtain these boundary conditions, we simply impose that the densities integrate to the mass of agents with the respective income types:

$$\int_{\underline{a}}^{\infty} dG_1(a) = \frac{\lambda_2}{\lambda_1 + \lambda_2}, \quad \int_{\underline{a}}^{\infty} dG_2(a) = \frac{\lambda_1}{\lambda_1 + \lambda_2}, \quad (37)$$

where G_1, G_2 are the CDFs corresponding to g_1, g_2 . We here allow for the possibility of Dirac point masses in the distributions which will be relevant momentarily. Using the fact that the two ODEs (36) can be solved analytically and our characterization of the optimal saving policy functions from Section 2.2 we obtain our second main theoretical result.

Proposition 3 (Stationary Wealth Distribution with Two Income Types) *Assume that $r < \rho, y_1 < y_2$, that relative risk aversion $-cu''(c)/u'(c)$ is bounded above for all c , and that Assumption 1 holds. Then there exists a unique stationary distribution given by*

$$g_j(a) = \frac{\kappa_j}{s_j(a)} \exp\left(-\int_{\underline{a}}^a \left(\frac{\lambda_1}{s_1(x)} + \frac{\lambda_2}{s_2(x)}\right) dx\right), \quad j = 1, 2 \quad (38)$$

for some constants of integration $\kappa_1 < 0$ and $\kappa_2 > 0$ which satisfy $\kappa_1 + \kappa_2 = 0$ and are uniquely pinned down by (37). The stationary wealth distribution has the following properties:

1. (Close to the borrowing constraint) The stationary distribution of low income types $g_1(a)$ has a Dirac point mass m_1 at the borrowing constraint \underline{a} . More precisely, the

corresponding cumulative distribution function satisfies

$$G_1(a) \sim m_1 \exp\left(\lambda_1 \sqrt{2(a - \underline{a})/\nu_1}\right) \quad \text{as } a \downarrow \underline{a}, \quad \text{with} \quad (39)$$

$$m_1 = \frac{\lambda_2}{\lambda_1 + \lambda_2} \tilde{m}_1, \quad \frac{1}{\tilde{m}_1} = \lambda_2 \int_{\underline{a}}^{a_{\max}} \left\{ \frac{1}{s_2(a)} \exp\left(-\int_{\underline{a}}^a \left(\frac{\lambda_1}{s_1(x)} + \frac{\lambda_2}{s_2(x)}\right) dx\right) \right\} da.$$

The stationary density of high income types at the borrowing constraint is bounded, $g_2(\underline{a}) < \infty$.

2. (In the right tail) The stationary wealth distribution is bounded above at some $a_{\max} < \infty$ defined in Proposition 2. The wealth distribution $g(a) := g_1(a) + g_2(a)$ satisfies

$$g(a) \sim \xi(a_{\max} - a)^{\lambda_2/\zeta_2 - 1} \quad \text{as } a \rightarrow a_{\max} \quad (40)$$

where $\zeta_2 = |s_2'(a_{\max})|$ and ξ is a constant. Therefore $g(a_{\max}) = 0$ for large λ_2 (so that $\lambda_2 > \zeta_2$). In contrast, $g_2(a) \rightarrow \infty$ as $a \rightarrow a_{\max}$ for small λ_2 . In neither case is there a Dirac mass.

3. (Smoothness) In contrast to the analogous discrete-time economy, the density of wealth is smooth everywhere except exactly at the borrowing constraint, i.e. for all $a > \underline{a}$.
4. (Shape of the wealth distribution) The exact shape of g_1 and g_2 is ambiguous. However, both g_1 and g_2 are ratios of well-understood functions, in particular $g_j(a) = \kappa_j f(a)/s_j(a)$, $j = 1, 2$ where $f(a) := \exp\left(-\int_{\underline{a}}^a \left(\frac{\lambda_1}{s_1(x)} + \frac{\lambda_2}{s_2(x)}\right) dx\right)$ and $\kappa_1 < 0$, $\kappa_2 > 0$. The function f is strictly log-concave and single-peaked with $f'(a)/f(a) \rightarrow \infty$ as $a \downarrow \underline{a}$ and $f'(a)/f(a) \rightarrow -\infty$ as $a \uparrow a_{\max}$.
5. (Joint distribution of labor income and wealth) For any given wealth level a , the fraction of individuals that have the high income y_2 , $\Pr(y_2|a) := \frac{g_2(a)}{g_1(a) + g_2(a)}$ satisfies $\Pr(y_2|a) = \frac{1}{1 - s_2(a)/s_1(a)}$ and similarly $\Pr(y_1|a) = 1 - \Pr(y_2|a) = \frac{1}{1 - s_1(a)/s_2(a)}$.

The Dirac property in part 1 of the Proposition follows immediately from Proposition 1 and that individuals hit the borrowing constraint in finite time. If Assumption 1 is satisfied, then (i) g_1 in (38) explodes as $a \rightarrow \underline{a}$ and (ii) there is a Dirac mass at \underline{a} , $G_1(\underline{a}) = m_1 > 0$. This is illustrated in panel (b) of Figure 6. In particular, note the spike in the density $g_1(a)$ at $a = \underline{a}$. In contrast, if Assumption 1 is violated, then there is no Dirac mass. The derivation of (39) is simple and instructive so we state it here. First, (8) implies that G_1 satisfies $0 = -s_1(a)G_1'(a) - \lambda_1 G_1(a) + \lambda_2 G_2(a)$. As $a \downarrow \underline{a}$, $G_2(a) \rightarrow 0$ and hence G_1 satisfies $G_1'(a)/G_1(a) \sim -\lambda_1/s_1(a)$ with solution

$$G_1(a) \sim m_1 \exp\left(-\int_{\underline{a}}^a \frac{\lambda_1}{s_1(x)} dx\right),$$

for a constant of integration $m_1 > 0$. From Proposition 1, $s_1(a) \sim -\sqrt{2\nu_1}\sqrt{a-\underline{a}}$. Substituting in and integrating yields (39). At the end of this subsection, we discuss in more detail the role of Assumption 1 by making use of our two simple special cases from Section 2.2.

Part 2 of the Proposition states that the stationary wealth distribution in our economy is bounded above. Like discrete-time versions of Aiyagari-Bewley-Huggett economies with idiosyncratic labor income risk only, our model therefore has difficulties explaining the high observed wealth concentration in developed economies like the United States (e.g. that the top one percent of the population own roughly thirty-five percent of total wealth). In particular, the wealth distribution in the data appears to feature a fat Pareto tail. Motivated by this observation, we show in Section 4.4 how to extend the model to feature a stationary distribution with a Pareto tail by introducing a second, risky asset. This caveat aside, Proposition 3 provides a complete characterization of the wealth distribution’s tail in the vicinity of its upper bound. From (40) top wealth inequality is high (g declines towards zero at a_{\max} only slowly) if individuals face a high likelihood of dropping out of the high income state (λ_2 is high) and if high-income types accumulate wealth only slowly (ζ_2 is low). Intuitively, wealth accumulation requires both time and luck (consecutive high income draws). And under the circumstances just mentioned, only a few individuals obtain sufficiently long enough high income spells to accumulate large riches. Hence, wealth inequality is high.

Part 3, which can also be seen in Figure 1, highlights an important difference between our continuous-time formulation and the traditional discrete-time one: except for the Dirac mass exactly at the borrowing constraint \underline{a} , the wealth distribution is smooth for all $a > \underline{a}$. This is true even though income follows a process with discrete states (a two-state Poisson process). In contrast, the wealth distribution in discrete-time versions of Aiyagari-Bewley-Huggett models with discrete-state income processes, tends to feature “spikes” on the interior of the state space. See for example Figure 17.7.1 in Ljungqvist and Sargent (2004).³²

Part 4 characterizes the wealth distribution for intermediate wealth levels. It shows that the shapes of g_1 and g_2 in Figure 6 are not simply due to a particular numerical example. Instead both density functions are simple ratios of well-understood functions $g_j(a) = \kappa_j f(a)/s_j(a)$ where f is defined in the Proposition and hump-shaped. For instance consider g_1 as a increases: as in Figure 6, g_1 tends to be first decreasing, then increasing again and finally decreasing. Similarly, consider g_2 as a increases: it tends to be first increasing and then decreasing (hump-shaped), again as in the Figure.

³²To see why this must happen in discrete time, consider a discrete-time Huggett economy with two income states. All individuals with wealth $a = \underline{a}$ who get the high income draw choose the same wealth level $a' > \underline{a}$. So if there is a Dirac mass at \underline{a} , there must also be a Dirac mass at a' . But all individuals with wealth a' who get the high income draw, also choose the same wealth level a'' . So there must also be a Dirac mass at $a'' > a'$. And so on. Through this mechanism the Dirac mass at the borrowing constraint “spreads into the rest of the state space.” In continuous time this does not happen because individuals leave the borrowing constraint in a smooth fashion (here according to a Poisson process, i.e. a process with a random and continuously distributed arrival time).

Part 5 characterizes the joint distribution of income and wealth. The fraction of high income types conditional on wealth $\Pr(y_2|a)$ depends only on the saving rates s_1 and s_2 but, perhaps surprisingly and in contrast to the fraction of high income types in the population $\Pr(y_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$, it does not depend directly on the intensities λ_1 and λ_2 .

Our Two Special Cases Again. We now briefly return to part 1 of Proposition 3 and illustrate in more detail when and, if so, why the wealth distribution features a Dirac mass at the borrowing constraint. To this end, consider again the two special cases without income risk from Section 2.2. In the first special case with exponential utility $s(a) = -\sqrt{2\nu a}$. in the second special case with CRRA utility $s(a) = -\eta a$. To obtain a stationary wealth distribution in the absence of income risk, assume that individuals die at rate λ . When an individual dies, she is replaced by a newborn with starting wealth a_{\max} .³³ Because $r < \rho$ so that everyone decumulates wealth, a_{\max} is also the upper bound of the wealth distribution (hence the notation).

It turns out to be convenient to work with the cumulative distribution function $G(a)$ which satisfies³⁴

$$0 = -s(a)G'(a) - \lambda G(a), \quad 0 < a < a_{\max} \quad (41)$$

with boundary condition $G(a_{\max}) = 1$. This equation can be solved easily: integrating $G'(a)/G(a) = -\lambda/s(a)$ with $G(a_{\max}) = 1$ we have

$$G(a) = \exp\left(\int_a^{a_{\max}} \frac{\lambda}{s(x)} dx\right). \quad (42)$$

In the first special case with $s(a) = -\sqrt{2\nu a}$, the CDF in (42) becomes

$$G(a) = \exp\left(\lambda\sqrt{2a/\nu} - \lambda\sqrt{2a_{\max}/\nu}\right). \quad (43)$$

Note in particular that $m := G(0) = \exp\left(-\lambda\sqrt{2a_{\max}/\nu}\right) > 0$, i.e. there is a Dirac mass at the borrowing constraint $a = 0$. In contrast, in the second special case $s(a) = \eta a$, (42) becomes

$$G(a) = \left(\frac{a}{a_{\max}}\right)^{\lambda/\eta}. \quad (44)$$

Therefore $G(0) = 0$ i.e. there are no individuals at the borrowing constraint. As can be seen clearly in their derivations, the difference between (43) and (44) is solely due to the saving behavior (linearity versus unbounded derivative at $a = 0$) which determines whether

³³Alternatively, we could assume that newborns draw their starting wealth from some distribution Ψ with support $[0, a_{\max}]$. In this case, (41) below is identical but with an additional term $+\lambda\Psi(a)$.

³⁴The KF equation is $0 = -(s(a)g(a))' - \lambda g(a)$ for $0 < a < a_{\max}$. Integrating and using $G(a) = \int_0^a g(x)dx$ yields (41). Working with the CDF is only more convenient in this special case. In the case with two income types above, it is instead more convenient to work with the densities.

individuals hit the borrowing constraint in finite time (see Section 2.2).

The special case with exponential utility and no income risk also yields some instructive comparative statics that carry over to numerical solutions of the more general case. Death risk at rate λ results in a higher effective discount rate $\rho + \lambda$ and hence the natural formula for the parameter governing the speed at which individuals hit the constraint is $\nu = (\rho - r + \lambda)/\theta$. Using this, the number of individuals at the borrowing constraint is $m = G(0) = \exp\left(-\lambda\sqrt{\frac{2\theta a_{\max}}{\rho - r + \lambda}}\right)$. This quantity is decreasing in the coefficient of absolute risk aversion θ , increasing in the gap $\rho - r$ and decreasing in the Poisson rate λ .³⁵ Numerical experiments in the model with a two-state Poisson process for income show that the same comparative static holds with respect to λ_1 , the Poisson rate of leaving the low income state.

2.6 Stationary Equilibrium: Existence and Uniqueness

We construct stationary equilibria along the same lines as in Aiyagari (1994). That is, we fix an interest rate $r < \rho$, solve the individual optimization problem (7), find the corresponding stationary distribution from (8), and then find the interest rate r that satisfies the market clearing condition (11), i.e. $S(r) = B$. While we continue to focus on the case of two income types for the sake of continuity, all results in this section generalize to any stationary Markovian process for income y , e.g. continuous diffusion or jump-diffusion processes.

Figure 7 illustrates the typical effect of an increase in r on the solutions to the HJB equation (7) and the KF equation (8). An increase in r from r_L to $r_H > r_L$ leads to an increase in individual saving at most wealth levels and the stationary distribution shifts to the right. Aggregate saving $S(r)$ as a function of the interest rate r typically looks like in Figure 8, i.e. it is increasing. A stationary equilibrium is then an interest rate r such that $S(r) = B$. But we have so far not proven that $S(r)$ is increasing or that it intersects B and hence there may, in principle, be no or multiple equilibria. Existence of a stationary equilibrium can be proved with a graphical argument due to Aiyagari (1994) that is also the foundation for a number of existence results in the literature (e.g. Acikgoz, 2016).

Proposition 4 (Existence of Stationary Equilibrium) *Assume that relative risk aversion $-cu''(c)/u'(c)$ is bounded above for all c and that Assumption 1 holds. Then there exists a stationary equilibrium in our continuous-time version of Huggett's economy.*

The logic behind the proof is simple. One can show that the function $S(r)$ defined in (11) is continuous. To guarantee that there is at least one r such that $S(r) = B$, it then suffices

³⁵An increase in λ has two offsetting effects: on one hand, individuals approach the borrowing constraint faster (ν), thereby increasing m ; on the other hand, individuals are more likely to die before they reach the constraint, thereby decreasing m . Differentiation of the expression for m shows that the latter effect always dominates.

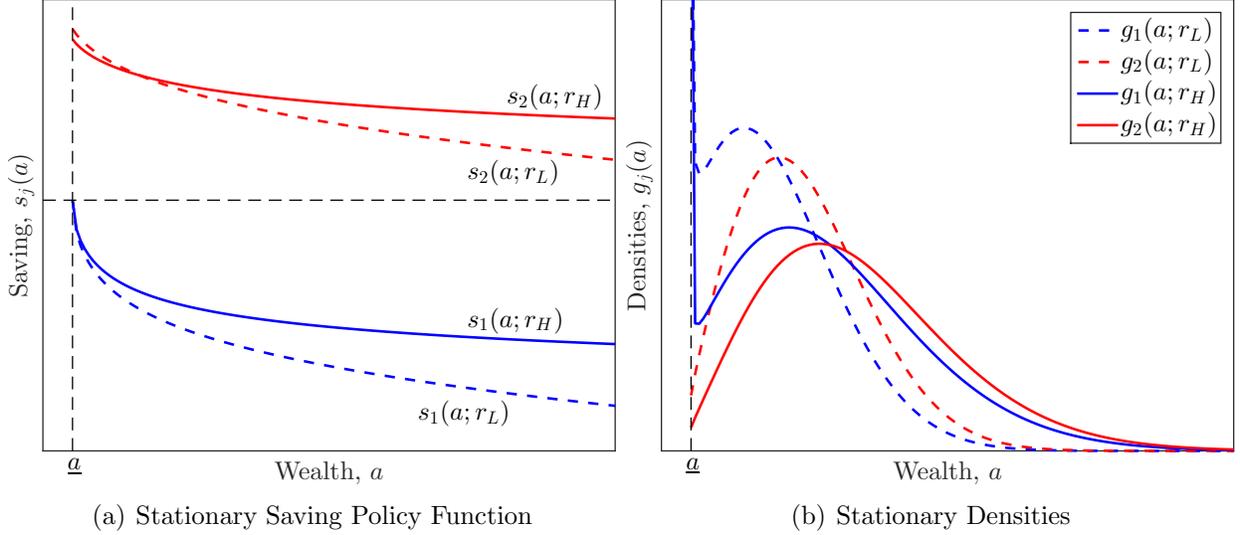


Figure 7: Effect of an Increase in r on Saving Behavior and Stationary Distribution

to show that

$$\lim_{r \uparrow \rho} S(r) = \infty, \quad \lim_{r \downarrow -\infty} S(r) = \underline{a}.$$

We next turn to our third main theoretical result, namely a proof of uniqueness of a stationary equilibrium.

Proposition 5 (Uniqueness of Stationary Equilibrium) *Assume that the intertemporal elasticity of substitution is weakly greater than one for all consumption levels*

$$\text{IES}(c) := -\frac{u'(c)}{u''(c)c} \geq 1 \quad \text{for all } c \geq 0, \quad (45)$$

and that the borrowing constraint takes the form of a strict no-borrowing limit $a \geq 0$. Then:

1. Individual consumption $c_j(a; r)$ is strictly decreasing in r for all $a > 0$ and $j = 1, 2$.
2. Individual saving $s_j(a; r)$ is strictly increasing in r for all $a > 0$ and $j = 1, 2$.
3. An increase in the interest rate leads to a rightward shift in the stationary distribution in the sense of first-order stochastic dominance: $G_j(a; r)$, $j = 1, 2$ is strictly decreasing in r for all a in its support.
4. Aggregate saving $S(r)$ is strictly increasing and hence our continuous-time version of Huggett's economy has at most one stationary equilibrium.

We briefly sketch key steps in the proof. Part 1 makes use of an important result by [Olivi \(2017\)](#) who analyzes the continuous-time income fluctuation problem put forth in the

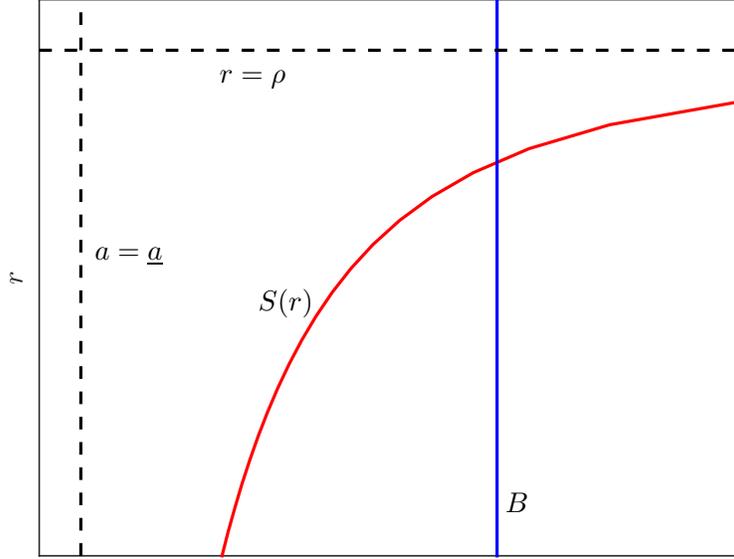


Figure 8: Equilibrium in the Bond Market

current paper and shows that the consumption response to a change in the interest rate can be decomposed into substitution and income effects as:

$$\frac{\partial c_j(a)}{\partial r} = \underbrace{\frac{1}{u''(c_0)} \mathbb{E}_0 \int_0^\tau e^{-\int_0^t \xi_s ds} u'(c_t) dt}_{\text{substitution effect} < 0} + \underbrace{\frac{1}{u''(c_0)} \mathbb{E}_0 \int_0^\tau e^{-\int_0^t \xi_s ds} u''(c_t) (\partial_a c_t) a_t dt}_{\text{income effect} > 0}$$

with $\xi_t := \rho - r + \partial_a c_t > 0$ and where $\tau := \inf\{t \geq 0 | a_t = 0\}$ is the stopping time at which wealth reaches the borrowing constraint. Here the expectations are over sample paths of (a_t, y_t) starting from $(a_0, y_0) = (a, y_j)$ and $\partial_a c_t$ is short-hand notation for the instantaneous MPC $\partial_a c_t = c'_j(a_t)$. [Olivi \(2017\)](#) further simplifies these substitution and income effects and expresses them in terms of potentially observable sufficient statistics like the MPC. We instead pursue a different avenue: we show that a sufficient condition for the substitution effect dominating the income effect and hence $\partial c_j(a)/\partial r < 0$ for all $a > 0$ is that the IES is weakly greater than one.

Part 2 uses the budget constraint $s_j(a) = y_j + ra - c_j(a)$. If consumption is decreasing in r then this immediately implies that saving is increasing in r for $a \geq 0$. Because there is a positive mechanical effect of r on saving through interest income ra , the assumption that the IES is greater than one is likely overly strong and saving may also be increasing in r if the IES is less than one. Consistent with this idea, consider the simple deterministic example with CRRA utility in [\(29\)](#): whether consumption is increasing in r depends on the IES $1/\gamma$; but saving $s(a)$ is increasing in a independently of $1/\gamma$.³⁶ Future work should try to prove uniqueness under weaker assumptions than the IES being greater than one. Either way,

³⁶Differentiating $s(a)$ in [\(29\)](#) yields $\frac{\partial s(a)}{\partial r} = \frac{1}{\gamma} \left(a + \frac{y}{r} \right)$ which is positive for all $a \geq -y/r$ as long as $r < \rho$.

$\text{IES}(c) \geq 1$ is an intuitive assumption and it includes the commonly used case of logarithmic utility $\text{IES}(c) = 1$.

Part 3, first-order stochastic dominance, uses a method for characterizing the dependence of solutions to partial differential equations on parameters, i.e. doing “comparative statics,” *without solving them*. Since this method may be useful in other applications, we explain it briefly in the context of the simple example without income uncertainty (41). The key idea is to derive an equation for $f(a) := \frac{\partial G(a)}{\partial r}$ and to prove that $f(a) < 0$ for all $a \in (0, a_{\max})$. Differentiating (41) with respect to r we have $0 = -\frac{\partial s(a)}{\partial r}g(a) - s(a)f'(a) - \lambda f(a)$ with $f(a_{\max}) = 0$. Next and to obtain a contradiction, suppose that $f(a) \geq 0$ for some a . Since $f(a_{\max}) = 0$ and if $f(0) = 0$,³⁷ f has to attain a maximum at some interior a^* at which $f(a^*) \geq 0$ and $f'(a^*) = 0$. Therefore $\lambda f(a^*) = -\frac{\partial s(a^*)}{\partial r}g(a^*) < 0$ but this contradicts the assumption that $f(a^*) \geq 0$ and hence proves the result. Again, the method is considerably more general.

Finally, first-order stochastic dominance of the wealth distribution immediately implies that aggregate asset supply is increasing in the interest rate and hence uniqueness of the stationary equilibrium (Part 4).³⁸

2.7 Soft Borrowing Constraints and Non-Participation

Empirical wealth distributions typically have the following properties: there are individuals with both positive and negative net worth but there is a spike at close to zero net worth. This empirical observation does not square well with the Aiyagari-Bewley-Huggett model we have considered thus far. If we set the borrowing constraint to $\underline{a} = 0$, we get the spike at zero but there are no individuals with negative net worth; if we set $\underline{a} < 0$, we get a spike at a strictly negative wealth level. Both are counterfactual. A simple way of generating the empirical observation just mentioned is to model a “soft” borrowing constraint as opposed to the “hard” constraint (3), that is, a wedge between borrowing and saving rates. This form of constraint is used in a number of recent papers including [Alonso \(2016\)](#) and [Kaplan, Moll, and Violante \(2016\)](#). In this section, we provide the first theoretical characterization of such soft borrowing constraints.

Consider the Huggett model from Section 1 with one modification: there is a wedge between borrowing and lending rates. That is, we replace the budget constraint (2) by

$$\dot{a}_t = y_t + r(a_t)a_t - c_t, \quad r(a) = \begin{cases} r_+, & a \geq 0 \\ r_-, & a < 0 \end{cases}, \quad r_- > r_+.$$

³⁷Since $G(0) > 0$, it is typically not true that $f(0) = \frac{\partial G(0)}{\partial r} = 0$ but this assumption can be easily relaxed.

³⁸[Acikgoz \(2016\)](#) provides a numerical example of multiple steady states in an Aiyagari-Bewley-Huggett model with an IES of 1/6.5, i.e. considerably below one, as well as a somewhat special income process.

We show below that, in order to obtain a stationary wealth distribution with a spike at zero and positive mass on both sides of zero, it is necessary to introduce more than two income types. In particular, the simplest extension of the model that yields the desired result is to have three income types i.e. $y_t \in \{y_0, y_1, y_2\}$ with $y_0 < y_1 < y_2$.

The next Proposition characterizes the saving behavior with a soft borrowing constraint. To avoid the somewhat cluttered notation resulting from considering three income types, it only considers the deterministic case $y_t = y$ for all t . This case has all the intuition and the extension to stochastic income is straightforward.

Proposition 6 (Saving Behavior with Soft Borrowing Constraint) *Assume that $r_+ < \rho < r_-$, that $y_t = y$ for all t and that $y > 0$ (so that $-u''(y)/u'(y) < \infty$, the analogue of Assumption 1). Then the solution to the HJB equation (7) and the corresponding saving policy function (9) have the following properties:*

1. $s(0) = 0$ but $s(a) < 0$ all $a > 0$ and $s(a) > 0$ all $a < 0$.
2. close to $a = 0$, the saving and consumption policy functions satisfy

$$s(a) \sim -\sqrt{2\nu_+ a}, \quad c'(a) \sim r + \frac{1}{2}\sqrt{\frac{2\nu_+}{a}}, \quad \nu_+ := \frac{(\rho - r_+)u'(y)}{-u''(y)} > 0 \quad \text{as } a \downarrow 0,$$

$$s(a) \sim \sqrt{2\nu_- a}, \quad c'(a) \sim r + \frac{1}{2}\sqrt{\frac{2\nu_-}{a}}, \quad \nu_- := \frac{(\rho - r_-)u'(y)}{-u''(y)} < 0 \quad \text{as } a \uparrow 0.$$

This implies that the derivatives of s and c are unbounded at zero, with $s'(a) \rightarrow -\infty$ and $c'(a) \rightarrow \infty$ both as $a \uparrow 0$ and $a \downarrow 0$.

3. Individuals with $a > 0$ decumulate wealth and hit $a = 0$ in finite time. Individuals with $a < 0$ instead accumulate wealth and also hit $a = 0$ in finite time.

The main takeaway from the Proposition is that a soft borrowing constraint results in an interesting symmetry in the saving policy function around zero net worth. To understand this property consider the blue solid line labelled $s_1(a)$ in Figure 9(a) (we will return to the other two lines below). The behavior for $a > 0$ with a soft borrowing constraint is identical to that with a hard borrowing constraint but at $a = 0$: as $a \downarrow 0$ it behaves like $-\sqrt{a}$. See for example Figure 1(b). The main takeaway from the Proposition is that the behavior of the saving policy function for $a < 0$ is simply a mirror image around the forty-five degree line of the behavior for $a > 0$: as $a \uparrow 0$ it behaves like $\sqrt{-a}$.³⁹ A simple extension of Corollary 1 then implies that individuals with $a > 0$ decumulate wealth and hit $a = 0$ in finite time; individuals with $a < 0$ instead accumulate wealth and also hit $a = 0$ in finite time.

³⁹Interestingly, for $a < 0$ the consumption function is *convex*, that is, the instantaneous MPC $c'(a)$ is *increasing* in wealth a . See the Proposition which, ignoring constants, shows that $c'(a) \sim \sqrt{1/(-a)}$ as $a \uparrow 0$.

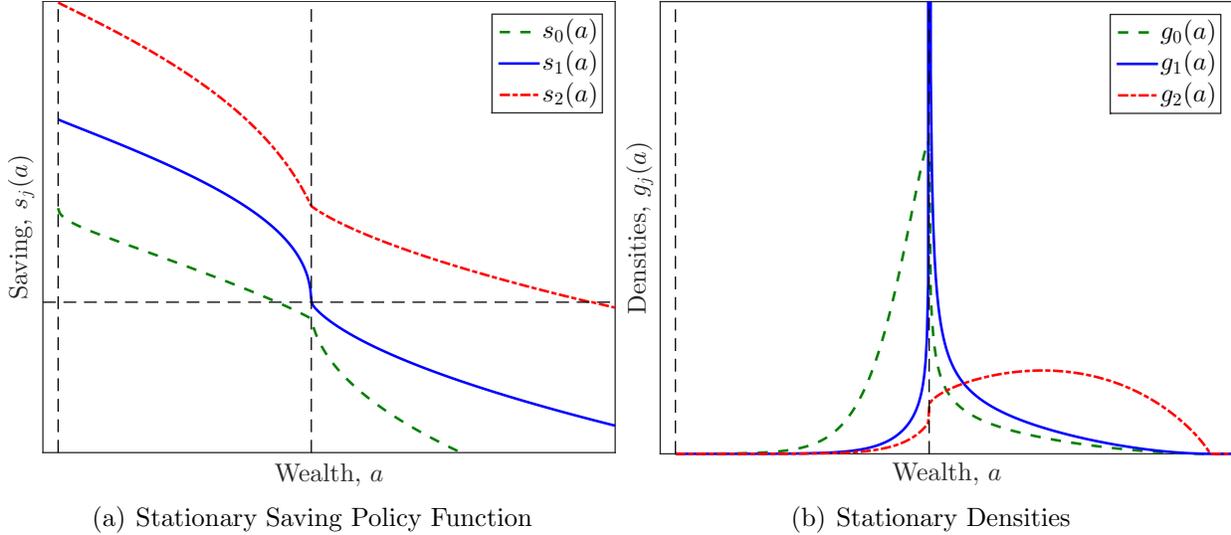


Figure 9: Saving Behavior and Wealth Distribution with Soft Borrowing Constraint

Of course with only one income type, the stationary wealth distribution will only be a Dirac point mass at $a = 0$. With two income types, it will be a Dirac mass at $a = 0$ combined with some mass either to the left ($a < 0$) or to the right ($a > 0$) *but not both*. Therefore to speak to the empirical observation of a spike at zero combined with mass both to the left and the right of zero it is necessary to introduce (at least) another income type. Figure 9(a) plots the saving policy functions in such a version with three income types $y_0 < y_1 < y_2$. Figure 9(b) plots the resulting wealth densities g_0, g_1 and g_2 . The unconditional wealth distribution is the sum of these three densities. As expected, it has a spike at zero and mass both to the left and the right.

3 Computation

We now describe our algorithm for numerically computing equilibria of continuous-time heterogeneous agent models. We use a finite difference method based on work by [Achdou and Capuzzo-Dolcetta \(2010\)](#) and [Achdou \(2013\)](#) which is simple, efficient and easily extended to other environments. We explain our method in the context of the baseline heterogeneous agent model of Section 1. But the algorithm is, in fact, considerably more general and applies to *any* heterogeneous agent model with a continuum of atomistic agents (and without aggregate shocks). It is particularly well-suited for computing transition dynamics and solving problems with non-convexities, a fact we illustrate in Section 4 by computing equilibria of such economies. Codes for these applications (and many more) are available from <http://www.princeton.edu/~moll/HACTproject.htm> in Matlab as well as Python, Julia and C++.

3.1 Computational Advantages relative to Discrete Time

Before explaining our algorithm, we provide a brief overview of some of its computational advantages relative to traditional discrete-time methods. We here list four computational advantages that we consider crucial and that contribute notably to the efficiency gains over traditional methods. The first of these advantages is special to the solution of problems with borrowing constraints. The second to fourth advantages concern the solution of heterogeneous agent models more broadly (e.g. models with heterogeneous firms).

To appreciate the first two advantages, contrast the first-order condition of the continuous-time income fluctuation problem (7) with that in the analogous discrete-time problem. For concreteness also assume CRRA utility (5) so that the two conditions are

$$c^{-\gamma} = v'_j(a), \quad j = 1, 2 \quad \text{in continuous time and} \quad (46)$$

$$c^{-\gamma} \geq \beta \sum_{k=1}^2 \pi_{jk} v'_k(a'), \quad a' = y_j + (1+r)a - c, \quad j = 1, 2 \quad (47)$$

in discrete time, where $0 < \beta < 1$ is a discount factor and $\pi_{jk} = \Pr(y' = y_k | y = y_j)$ are the entries of the Markov transition matrix for the analogous discrete-time income process. The first advantage of our continuous-time approach is that, as explained in Section 1.2, the borrowing constraint (3) only shows up in the boundary condition (10) and therefore the first-order condition (46) holds *with equality* everywhere in the interior of the state space. In contrast, the discrete-time first-order condition (47) holds with complementary slackness and therefore is an *inequality*. This is because the borrowing constraint may bind one time period ahead. Continuous time allows us to completely sidestep any technical difficulties arising due to such occasionally binding constraints.

Second and related, the first-order condition in (46) is “static” in the sense that it only involves contemporaneous variables. Given (a guess for) the value function $v_j(a)$ it can be solved by hand: $c_j(a) = (v'_j(a))^{-1/\gamma}$, $j = 1, 2$. In contrast, the discrete-time condition (47) defines the optimal choice *only implicitly*. Typical solution methods therefore employ costly root-finding operations. Our continuous-time approach again sidesteps this difficulty.⁴⁰ Intuitively, discrete time distinguishes between “today” and “tomorrow” but in continuous time, “tomorrow” *is* “today.”⁴¹

The third advantage of continuous time is a form of “sparsity.” To solve the HJB and KF equations (7) and (8), we discretize these so that their solution boils down to solving

⁴⁰In this regard, it shares some similarities with the “endogenous grid method” of Carroll (2006). The difference is that in continuous time this also works with “exogenous grids.”

⁴¹Related, (46) also does not involve an expectation operator as in (47) that makes it necessary to calculate a summation over future income states or a costly numerical integral (in the context of more general income processes). Instead the HJB equation (7) captures the evolution of the stochastic process for y_t with an additive terms $\lambda_j(v_{-j}(a) - v_j(a))$ that does not affect the first-order condition.

systems of linear equations. The resulting matrices are typically extremely sparse, namely “tridiagonal” or at least “block-tridiagonal.” This sparsity generates considerable efficiency gains because there are well-developed routines for solving sparse linear systems, either implemented as part of commercial software packages like Matlab or open-source libraries like SuiteSparse. The reason that tridiagonal matrices arise is that a discretized continuous-time process either stays at the current grid point, takes one step to the left or one step to the right. But it never jumps.⁴²

Fourth, in all heterogeneous agent models, there is a tight link between solving the HJB and KF equations. One can typically “kill two birds with one stone” in the sense that, having computed the solution to the HJB equation one gets the solution to the KF equation “for free”: the matrix in the discretized version of the latter is the transpose of the matrix in that of the former. The underlying mathematical reason is that the KF equation is the “transpose problem” of the HJB equation or, more precisely, that the differential operator in the KF equation is the adjoint of the operator in the HJB equation.⁴³

3.2 Bird’s Eye View of Algorithm for Stationary Equilibria

Our aim is to calculate stationary equilibria – functions v_1, v_2 and g_1, g_2 and a scalar r satisfying (7), (8) and (11) – given a specified function u , and values for the parameters $\rho, \lambda_1, \lambda_2$ and \underline{a} . Transition dynamics are the subject of Section 3.5. Before we describe the algorithm in detail, we provide a bird’s eye view of the algorithm’s general structure. We focus on two distinct challenges. First, the HJB and KF equations describing a stationary equilibrium are coupled and one therefore has to iterate on them somehow. Second, solving these differential equations requires approximating the value function and distribution.

Iterating on the Equilibrium System From a bird’s eye perspective our algorithm for solving the stationary equilibrium shares many similarities with algorithms typically used to solve discrete-time heterogeneous agent models. In the context of our Huggett economy, we use a bisection algorithm on the stationary interest rate. We begin an iteration with an initial guess r^0 . Then for $\ell = 0, 1, 2, \dots$ we follow

1. Given r^ℓ , solve the HJB equation (7) using a finite difference method. Calculate the saving policy function $s_j^\ell(a)$.

⁴²Except of course if the process is a Poisson process, i.e. if jumps are “built in.” That being said, the sparsity property survives as long as there is at least one continuously moving state variable (like wealth), i.e. not all individual state variables follow discrete-state Poisson processes.

⁴³In principle, one can use an analogous approach in discrete time: form the endogenous Markov transition matrix between states and use it both to iterate backward over value functions (or Euler equations) and to iterate forward over distributions. This method does not appear to be very popular and researchers typically use Monte-Carlo simulation to solve for the distribution. A possible reason is that transition matrices are not (or less) sparse resulting in a less dramatic efficiency gain.

2. Given $s_j^\ell(a)$, solve the KF equation (8) for $g_j^\ell(a)$ using a finite difference method.
3. Given $g_j^\ell(a)$, compute the net supply of bonds $S(r^\ell) = \int_a^\infty a(g_1^\ell(a) + g_2^\ell(a))da$ and update the interest rate: if $S(r^\ell) > B$, decrease it to $r^{\ell+1} < r^\ell$ and vice versa.

When $r^{\ell+1}$ is close enough to r^ℓ , we call $(r^\ell, v_1^\ell, v_2^\ell, g_1^\ell, g_2^\ell)$ a stationary equilibrium. As already noted, this algorithm is extremely close to typical algorithms used to solve a discrete-time Huggett economy. The difference of our continuous-time approach – and the resulting efficiency gains – instead lie in the solutions of the dynamic programming equation and the equation describing the distribution.

Discretization of the Equilibrium System In order to solve the differential equations (7) and (8), the value function and distribution need to be approximated in some fashion. We explain our approach – a finite difference method – in more detail in the next two subsections. But a brief sketch is as follows. In a nutshell, the key idea is that this finite difference method transforms our system of differential equations into a system of sparse matrix equations. With this goal in mind, we approximate both v_1, v_2 and g_1, g_2 at I discrete points in the space dimension, $a_i, i = 1, \dots, I$. Denote the value function and distribution along this discrete grid using the vectors $\mathbf{v} = (v_1(a_1), \dots, v_1(a_I), v_2(a_1), \dots, v_2(a_I))^T$ and $\mathbf{g} = (g_1(a_1), \dots, g_1(a_I), g_2(a_1), \dots, g_2(a_I))^T$; both \mathbf{v} and \mathbf{g} are of dimension $2I \times 1$, the total number of grid points in the individual state space. The end product of our discretization method will then be the following system of matrix equations:

$$\rho \mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v}; r) \mathbf{v}, \quad (48)$$

$$\mathbf{0} = \mathbf{A}(\mathbf{v}; r)^T \mathbf{g}, \quad (49)$$

$$B = S(\mathbf{g}; r). \quad (50)$$

The first equation is the discretized HJB equation (7), the second equation is the discretized KF equation (8) and the third equation is the discretized market clearing condition (11). The $2I \times 2I$ matrix $\mathbf{A}(\mathbf{v}; r)$ will have the interpretation of a transition matrix that captures the evolution of the idiosyncratic state variables in the discretized state space. It will turn out to be extremely sparse. $\mathbf{A}(\mathbf{v}; r)^T$ in the second equation denotes the transpose of that same matrix, i.e. the discretized KF equation is the “transpose problem” of the discretized HJB equation. As already noted, (48) to (50) is simply a system of sparse matrix equations that can be easily solved on a computer by following (the analogues of) Steps 1 to 3 described in the previous paragraph.

3.3 Step 1: Solving the HJB Equation

For step 1, we solve the HJB equation (7) using a finite difference method. We now explain this approach. The Online Appendix contains a more detailed explanation.

Theory for Numerical Solution of HJB Equations (Barles-Souganidis) Before we explain our approach we should note that there is a well-developed theory concerning the numerical solution of HJB equations using finite difference schemes in the same way as there is a well-developed theory concerning the numerical solution of discrete-time Bellman equations. The key result is due to Barles and Souganidis (1991) who have proven that, under certain conditions, the solution to a finite difference scheme converges to the (unique viscosity) solution of the HJB equation.⁴⁴ The interested reader should consult Barles and Souganidis’ original (and relatively accessible) paper or the introduction by Tourin (2013). In short, for their result to hold, the finite difference scheme needs to satisfy three conditions: (i) “monotonicity”, (ii) “stability” and (iii) “consistency.” These are spelled out in the Online Appendix. Here it suffices to note that (ii) and (iii) are typically easy to satisfy and, in practice, the main difficulty for the numerical solution of HJB equations is to design a finite difference scheme that satisfies (iii), the monotonicity condition.

Finite Difference Method We here explain the finite difference method for solving the stationary HJB equation for a special case we have already examined earlier in the paper, namely the one with no income uncertainty $y_1 = y_2 = y$. The generalization to income risk is straightforward. The HJB equation in this special case is

$$\rho v(a) = \max_c u(c) + v'(a)(y + ra - c). \quad (51)$$

As already mentioned, the finite difference method approximates the function v at I discrete points in the space dimension, $a_i, i = 1, \dots, I$. We use equispaced grids, denote by Δa the distance between grid points, and use the short-hand notation $v_i := v(a_i)$. The derivative $v'_i = v'(a_i)$ is approximated with either a forward or a backward difference approximation

$$v'(a_i) \approx \frac{v_{i+1} - v_i}{\Delta a} =: v'_{i,F} \quad \text{or} \quad v'(a_i) \approx \frac{v_i - v_{i-1}}{\Delta a} =: v'_{i,B}.$$

The finite difference approximation to (51) is then

$$\rho v_i = u(c_i) + v'_i s_i, \quad s_i := y + ra_i - c_i, \quad c_i = (u')^{-1}(v'_i), \quad i = 1, \dots, I \quad (52)$$

⁴⁴See Appendix C.1 for an “economist-friendly” introduction to the theory of viscosity solutions.

where v'_i is either the forward or backward difference approximation. Which of the two approximations is used and where in the state space is extremely important. The reason is that this choice determines whether Barles and Souganidis' monotonicity condition is satisfied.

Upwinding As just mentioned, it is important whether and when a forward or a backward difference approximation is used. The ideal solution to this problem is to use a so-called “upwind scheme.” The rough idea is to use a forward difference approximation whenever the drift of the state variable (here, saving $s_i = y + ra_i - c_i$) is positive and to use a backward difference whenever it is negative. This is intuitive: if saving is positive, what matters is how the value function changes when wealth increases by a small amount; and vice versa when saving is negative. The right thing to do is therefore to approximate the derivative in the direction of the movement of the state. To this end use the notation $s_i^+ = \max\{s_i, 0\}$, i.e. s_i^+ is “the positive part of s_i ” and analogously $s_i^- = \min\{s_i, 0\}$. The upwind version of (52) is then

$$\rho v_i = u(c_i) + \frac{v_{i+1} - v_i}{\Delta a} s_i^+ + \frac{v_i - v_{i-1}}{\Delta a} s_i^-, \quad i = 1, \dots, I, \quad (53)$$

where also the finite difference approximation v'_i used to compute $c_i = (u')^{-1}(v'_i)$ depends on the sign of s_i in the same way. This simplified exposition ignores two important and related issues. First, that the HJB equation (51) is highly non-linear due to the presence of the max operator, and therefore so is its finite difference approximation (53). It therefore has to be solved using an iterative scheme and one faces a choice between using so-called “explicit” and “implicit” schemes. Related, from the first-order condition $c_i = (u')^{-1}(v'_i)$, saving s_i and consumption c_i themselves depend on whether the forward or backward approximation is used so (53) has a circular element to it. The solution to both these issues is described in detail in the Online Appendix.

The upwind finite difference scheme for the HJB equation (53) can be conveniently written in matrix notation. Denoting by $\mathbf{v} = (v_1, \dots, v_I)^T$ the vector collecting the value function at different grid points, we have the matrix equation (48). The matrix $\mathbf{A}(\mathbf{v}; r)$ has a special structure: first, it is sparse; more precisely, it is tridiagonal: all entries are zero except for those on the main diagonal, the first diagonal below this, and the first diagonal above the main diagonal. Second, all diagonal entries are negative and given by $\frac{s_i^-}{\Delta a} - \frac{s_i^+}{\Delta a} \leq 0$ and all off-diagonal entries are positive and given by $-\frac{s_i^-}{\Delta a} \geq 0$ and $\frac{s_i^+}{\Delta a} \geq 0$. Third, all rows of $\mathbf{A}(\mathbf{v}; r)$ sum to zero. All these properties are extremely intuitive. In effect, the finite difference method approximates the law of motion for the continuous state variable a with a discrete-state Poisson process on the grid $a_i, i = 1, \dots, I$ and the matrix $\mathbf{A}(\mathbf{v}; r)$ summarizes the corresponding Poisson intensities. The properties noted above are precisely the properties that a Poisson transition matrix needs to satisfy. For these reasons we will sometimes refer to $\mathbf{A}(\mathbf{v}; r)$ as “Poisson transition matrix” or “intensity matrix.” All this will

be useful in Section 3.4 below when we solve the KF equation (8).

Boundary Conditions and Handling the Borrowing Constraint Besides guaranteeing that the Barles-Souganidis monotonicity condition holds, an upwind scheme like (53) has an additional great advantage: the handling of boundary conditions. First, consider the upper end of the state space a_I . If this upper end is large enough, saving will be negative $s_I < 0$ so that $s_I^+ = 0$.⁴⁵ It can then be seen from (53) that the forward difference is never used at the upper end of the state space. As a result no boundary condition needs to be imposed. Next, consider the lower end of the state space and in particular the question how to impose the state constraint boundary condition (10) which holds with equality only when the constraint binds. To impose this, we can exploit the special structure of the upwind scheme: set the boundary condition

$$v'_{1,B} = u'(y + ra_1)$$

but only for the backward difference approximation and not for the forward difference approximation $v'_{1,F}$ which is instead computed as $(v_2 - v_1)/\Delta a$. Then let the upwind scheme itself select whether this boundary condition is used. From (53) we can see that the boundary condition is only imposed if it would be the case that $s_1 < 0$; but it is not used if $s_1 > 0$. This ensures that the borrowing constraint is never violated.

3.4 Step 2: Solving the Kolmogorov Forward Equation

For step 2, consider the stationary Kolmogorov Forward equation (8). We again discretize the equation using a finite difference scheme. In contrast to the HJB equation which is non-linear in v , the KF equation is linear in g . Its discretized counterpart can therefore be solved in one iteration. There are a number of potentially admissible finite difference schemes, but one of these is particularly convenient and deeply rooted in mathematical theory: the discretization (49) which involves the *transpose* of $\mathbf{A}(\mathbf{v}, r)$, the transition matrix of the discretized stochastic process in (a, y_j) -space.

The deep underlying reason for this choice of discretization is that the KF equation actually *is* the “transpose” problem of the HJB equation. More precisely, the differential operator in the KF equation (8) is the adjoint of the operator in the HJB equation (7), the “infinitesimal generator.”⁴⁶ Our transpose discretization of the KF equation (49) is not

⁴⁵In fact, in the special case without uncertainty analyzed in the present section and under the assumption $r < \rho$, saving is negative everywhere in the state space. The condition that a_I needs to be large enough really only matters for the case with uncertainty. That the case without uncertainty is special also applies to the subsequent discussion: without uncertainty the state constraint (10) always holds with equality.

⁴⁶The “infinitesimal generator” is the continuous-time analogue of a discrete-time transition matrix, and the adjoint of an operator is the infinite-dimensional analogue of a matrix transpose. In our context, the

only well-founded in mathematics; it is also extremely convenient: having solved the HJB equation, the solution of the Kolmogorov Forward equation is essentially “for free.”

The same numerical method – building the matrix \mathbf{A} and then working with its transpose – can also be used when solving problems that involve *only* the KF equation, e.g. because the optimal decision rules can be solved for analytically. This approach is, for example, pursued in Jones and Kim (2014) and Gabaix, Lasry, Lions, and Moll (2016).⁴⁷

Finally, some readers may be concerned that the presence of a Dirac point mass in the stationary wealth distribution g_1 may cause problems for our finite difference method. We show in Appendix D that this is not the case. First, we show theoretically that the only implication of the Dirac mass is that some care is required when *interpreting* the output of the numerical algorithm, in particular the first element of the vector \mathbf{g} (corresponding to the density of income type y_1 at the point $a = \underline{a}$). Second, we use the analytic solution for the wealth distribution in Proposition 3 as a test case for our numerical algorithm and show that it performs extremely well in practice unless the wealth grid is very coarse.

3.5 Computing Transition Dynamics

The algorithm we use to calculate time-varying equilibria – functions v_1, v_2, g_1, g_2 and r satisfying (4), (12) and (13) given an initial condition (16) and a terminal condition (17) – is the natural generalization of that used to compute stationary equilibria. We again use a bisection method, this time on the entire function $r(t)$. We begin an iteration with an initial guess $r^0(t), t \in (0, T)$. Then for $\ell = 0, 1, 2, \dots$ we follow

1. Given $r^\ell(t)$ and the terminal condition (17), solve the HJB equation (12), marching backward in time. Calculate the saving policy function $s_j^\ell(a, t)$.
2. Given $s_j^\ell(a, t)$ and the initial condition (16), solve the KF equation, marching forward in time, for $g_j^\ell(a, t)$.
3. Given $g_j^\ell(a, t)$, compute the net supply of bonds $S^\ell(t) = \int_{\underline{a}}^\infty a(g_1^\ell(a, t) + g_2^\ell(a, t))da$ and update the interest rate as $r^{\ell+1}(t) = r^\ell(t) - \xi \frac{dS^\ell(t)}{dt}$ where $\xi > 0$.

infinitesimal generator captures the evolution of the process in (a, y_j) -space. This operator – let us denote it by \mathcal{A} – is defined as follows: for any vector of functions $[f_1(a), f_2(a)]^T$

$$\mathcal{A} \begin{bmatrix} f_1(a) \\ f_2(a) \end{bmatrix} = \begin{bmatrix} f_1'(a)s_1(a) + \lambda_1(f_2(a) - f_1(a)) \\ f_2'(a)s_2(a) + \lambda_2(f_1(a) - f_2(a)) \end{bmatrix}.$$

Next, one can show that the operator in the KF equation (8) is the adjoint of this operator: denoting by \mathcal{A}^* the adjoint of \mathcal{A} , (8) is $0 = \mathcal{A}^* \begin{bmatrix} g_1(a) \\ g_2(a) \end{bmatrix}$. Equation (49) is the discretized version of this problem.

⁴⁷Festa, Gomes, and Velho (2017) develop the idea of exploiting the adjoint property of the KF equation more systematically, also in the context of alternative numerical schemes (e.g. a semi-Lagrangian scheme).

When $r^{\ell+1}(t)$ is close enough to $r^\ell(t)$ for all t , we call $(r^\ell, v_1^\ell, v_2^\ell, g_1^\ell, g_2^\ell)$ an equilibrium.

The finite difference method used for computing the time-dependent HJB and KF equations for a given time path $r^\ell(t)$ is similar to those used to compute their stationary counterparts. In addition to discretizing wealth a , we now also discretize time t on a grid $t^n, n = 1, \dots, N$, for instance with equal-sized time steps of length Δt . Denoting by \mathbf{v}^n and \mathbf{g}^n the stacked, discretized value function and distribution at time t^n , the time-dependent version of the discretized equilibrium system (48) to (50) becomes

$$\begin{aligned}\rho \mathbf{v}^n &= \mathbf{u}(\mathbf{v}^{n+1}) + \mathbf{A}(\mathbf{v}^{n+1}; r^n) \mathbf{v}^n + \frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t}, \\ \frac{\mathbf{g}^{n+1} - \mathbf{g}^n}{\Delta t} &= \mathbf{A}(\mathbf{v}^n; r^n)^\top \mathbf{g}^{n+1}, \\ B &= S(\mathbf{g}^n; r^n),\end{aligned}$$

for time steps $n = 1, \dots, N$, with terminal condition $\mathbf{v}^N = \mathbf{v}$ where \mathbf{v} is the steady state solution to (48) and with initial condition $\mathbf{g}^1 = \mathbf{g}_0$. All this is explained in more detail in the Online Appendix.

3.6 The Method's Performance: Some Illustrative Results

All of Figures 1, 6 and 8 for the Huggett economy in Section 1 earlier in the paper were computed using a Matlab implementation of the algorithm just laid out. Even though we work with a fine wealth grid with $I = 1000$ grid points, solving for a stationary equilibrium takes about 0.25 seconds on a MacBook Pro laptop computer. Next, consider the corresponding transition dynamics. With $I = 1000$ wealth grid points, $N = 400$ time steps and the same hardware, computing (12) and (13) for a fixed time path $r(t)$ takes about 2 seconds. Iterating on $r(t)$ until an equilibrium transition is found takes about 4 minutes (even though market clearing conditions like (4) that implicitly define prices are notoriously difficult to impose during transitions).⁴⁸

⁴⁸In contrast, computing transitions for the Aiyagari model in Section 4.2, where prices are explicit functions of the aggregate capital stock as in (61), takes only 1 minute and 40 seconds. The code for the stationary equilibrium and transition dynamics of the Huggett model are available at http://www.princeton.edu/~moll/HACTproject/huggett_equilibrium_iterate.m and http://www.princeton.edu/~moll/HACTproject/huggett_transition.m. The code for the Aiyagari model is at http://www.princeton.edu/~moll/HACTproject/ayagari_poisson_MITshock.m.

3.7 Relation to Candler (1999) and Alternatives to Finite Difference Method

Candler (1999) has previously used a finite difference method to solve HJB equations arising in economics (with the neoclassical growth model as his prime example) and also discusses upwinding. Our numerical method adds to his in three dimensions. First and most obviously, we consider coupled HJB and KF equations rather than just the HJB equation in isolation: the system (7), (8) and (11) rather than just (7). We show that there is a tight mapping between the KF equation and the HJB equation and that this mapping can be conveniently exploited in finite difference schemes (the adjoint/transpose property discussed in Section 3.4). Candler, in contrast, does not discuss Kolmogorov Forward equations. Second, even when considered in isolation, our HJB equation differs from Candler’s and this is reflected in the solution method. In particular, we show how to handle borrowing constraints by mathematically casting them as state constraints (a step that requires using the notion of viscosity solution) and designing an upwind method that respects these constraints. Borrowing constraints are, of course, a ubiquitous feature of heterogeneous agent models. Third, we show that our solution method has well-developed theoretical underpinnings by making the connection to the Barles and Souganidis (1991) convergence theory.

Besides the finite difference method, there are many alternative methods for solving partial differential equations in general and HJB and KF equations in particular. Examples include finite-element, finite-volume, semi-Lagrangian, and Markov-chain approximation methods as well as approximation via orthogonal (e.g. Chebyshev) polynomials.⁴⁹ In principle, these other methods can also be used to solve heterogeneous agent models of the type discussed here; in particular by following the same Steps 1 to 3 laid out in Section 3.2 but simply exchanging the solution method used *within* Steps 1 and 2.

There is no sense in which the finite difference method laid out here dominates these other methods. In fact, some of these other methods may even be more accurate for coarse discretizations of the value function and distribution. We nevertheless prefer it over other methods for two reasons. First, the finite difference method is transparent and easy to implement. In case the algorithm spits out junk, it is usually easy to track down the problem. Second, and as explained in Section 3.4, the finite difference method delivers a useful symmetry in the solution methods for the HJB and KF equations (that the matrix in the latter is the transpose of that in the former). Other methods typically do not have this property. Finally, because the finite difference method is fast, choosing fine grids is usually

⁴⁹Some of these methods have also been used in economics. For example, Golosov and Lucas (2007) and Barczyk and Kredler (2014) use the Markov Chain approximation method of Kushner and Dupuis (2013). Because some of our paper’s readers have wondered about this in the past, we should emphasize that a finite difference method is different from the Kushner-Dupuis method: their approach essentially transforms the continuous-time problem into a discrete-time problem and then solves it using value function iteration.

not costly and hence it is not a concern that other methods may be more accurate with coarse discretizations.

4 Generalizations and Other Applications

Sections 4.1 and 4.2 discuss generalizations of the baseline Huggett model we have analyzed thus far. Sections 4.3 to 4.6 present a number of other theories, mostly to showcase the portability of our computational algorithm.

4.1 More General Income Processes

Our baseline model assumed that income y_t takes one of two values, high and low. We now extend many of our results to an environment with a continuum of productivity types.⁵⁰ In particular, the computational algorithm laid out in Section 3 carries over without change. This is true even though the system of equations describing an equilibrium will be a system of PDEs rather than a system of ODEs.

As in Section 1.1, there is a continuum of individuals that are heterogeneous in their wealth a and income y . The state of the economy is the joint distribution of income and wealth $g(a, y, t)$. The simplest way of introducing a continuum of income types is to work with a continuous diffusion process. Individual income evolves stochastically over time on a bounded interval $[\underline{y}, \bar{y}]$ with $\bar{y} > \underline{y} \geq 0$, according to the stationary diffusion process⁵¹

$$dy_t = \mu(y_t)dt + \sigma(y_t)dW_t. \quad (54)$$

This is simply the continuous-time analogue of a Markov process (without jumps). W_t is a Wiener process or standard Brownian motion and the functions μ and σ are called the drift and the diffusion of the process. We normalize the process such that its stationary mean equals one. An individual's problem is now to maximize (1) subject to (2), (3) and (54), taking as given the evolution of the interest rate r_t for $t \geq 0$.⁵²

Similarly to Section 1, a *stationary* equilibrium can be written as a system of partial differential equations. The problem of individuals and the joint distribution of income and

⁵⁰Among the theoretical results, we extend Propositions 1, 2, 4 and 5. That is, all Propositions from Section 2 with the exception of Propositions 3 (the analytic solution for the stationary distribution with two income types) and 6 (soft borrowing constraint with deterministic income).

⁵¹The process (54) either stays in the interval $[\underline{y}, \bar{y}]$ by itself or is reflected at \underline{y} and \bar{y} . From a theoretical perspective there is no need for restricting the process to a bounded interval, and unbounded processes can be easily analyzed. Instead the motivation for this assumption is purely practical: we ultimately solve the problem numerically and any computations necessarily require income to lie in a bounded interval.

⁵²The corresponding “natural borrowing constraint” is now $a_t \geq -y \int_t^\infty \exp(-\int_t^s r_\tau d\tau) ds$. As before, the borrowing constraint \underline{a} only binds if it is tighter than this “natural” borrowing limit.

wealth satisfy stationary HJB and KF equations:

$$\rho v(a, y) = \max_c u(c) + \partial_a v(a, y)(y + ra - c) + \partial_y v(a, y)\mu(y) + \frac{1}{2}\partial_{yy}v(a, y)\sigma^2(y), \quad (55)$$

$$0 = -\partial_a(s(a, y)g(a, y)) - \partial_y(\mu(y)g(a, y)) + \frac{1}{2}\partial_{yy}(\sigma^2(y)g(a, y)). \quad (56)$$

on $(\underline{a}, \infty) \times (\underline{y}, \bar{y})$. The function s is the saving policy function

$$s(a, y) = y + ra - c(a, y), \quad \text{where } c(a, y) = (u')^{-1}(\partial_a v(a, y)). \quad (57)$$

The function v again satisfies a state constraint boundary condition at $a = \underline{a}$ which is now

$$\partial_a v(\underline{a}, y) \geq u'(y + r\underline{a}), \quad \text{all } y. \quad (58)$$

Because the diffusion is reflected at \underline{y} and \bar{y} , the value function also satisfies the boundary conditions

$$\partial_y v(a, \underline{y}) = 0, \quad \partial_y v(a, \bar{y}) = 0, \quad \text{all } a. \quad (59)$$

A stationary equilibrium is a scalar r and functions v and g satisfying the PDEs (55) and (56) with s given by (57), boundary conditions (58), (59), with an equilibrium condition analogous to (11), namely $\int_{\underline{y}}^{\bar{y}} \int_{\underline{a}}^{\infty} ag(a, y)dady = B$. Transition dynamics again satisfy a system of time-dependent PDEs analogous to that in Section 1.

Importantly, the computational algorithm laid out in Section 3 carries over without change: from a computational perspective it is immaterial whether we solve a system of ODEs like (7) and (8) or a system of PDEs like (55) and (56). This would not be true if we had relied on an pre-built ODE solver (say one that is part of Matlab) to solve the ODEs (7) and (8).

Other income processes are possible as well. For instance, [Kaplan, Moll, and Violante \(2016\)](#) consider a “jump-drift process” with transitory and permanent components. As in (54) there is a continuum of types for each component; but rather than moving continuously over time as in (54), each component is subject to Poisson jumps. Income could also follow a jump-diffusion process.

Figure 10 plots the stationary saving policy function and wealth distribution when income follows a diffusion. Both inherit all important properties of the saving policy function and wealth distribution from the baseline model with two income types from Sections 1 and 2. This is not just a numerical result. Instead Propositions 7 and 8 in Appendix E.1 generalize Propositions 1 and 2 from the case with a two-state Poisson process to other processes including the diffusion process (54). It shows, for example, that – as can be seen in panel (a) of the Figure – the saving policy function has an unbounded derivative at $a = \underline{a}$ for income y below some threshold, and that therefore individuals with persistent low income realizations

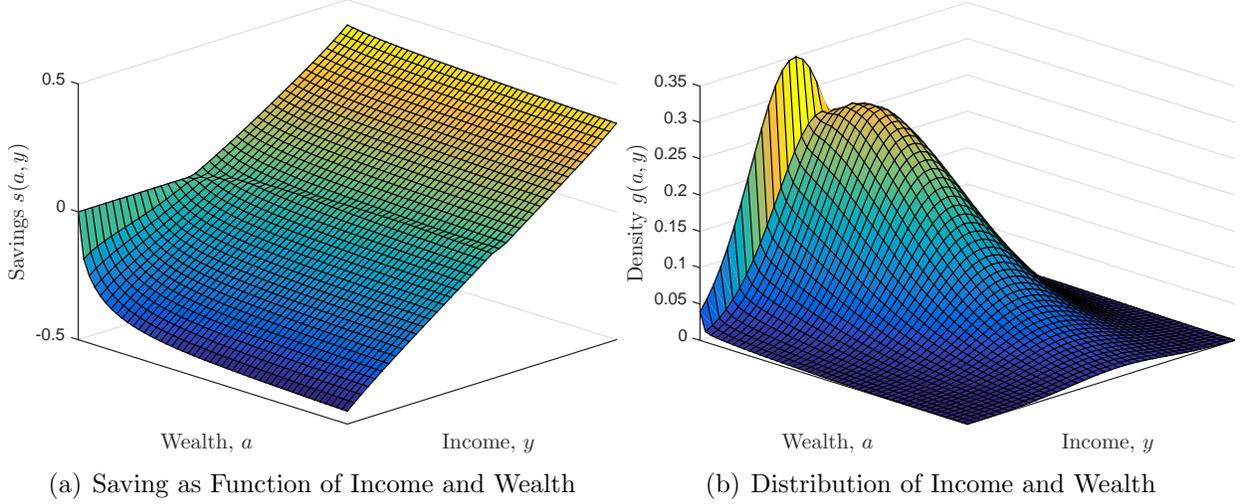


Figure 10: Stationary Equilibrium of Huggett Model with Diffusion Process

hit the borrowing constraint in finite time. This results in the spike in the wealth distribution at $a = \underline{a}$ in panel (b).⁵³ Finally, Proposition 7 also shows that formula (21) governing the MPC generalizes to

$$\nu(y) = (\rho - r)\text{IES}(\underline{c}(y))\underline{c}(y) + \left(\mu(y) - \frac{\sigma^2(y)}{2}\mathcal{P}(\underline{c}(y)) \right) \underline{c}'(y) + \frac{\sigma^2(y)}{2}\underline{c}''(y),$$

where $\underline{c}(y) := c(a, y)$ and $\mathcal{P}(c) := -u'''(c)/u''(c)$ is absolute prudence.

4.2 An Alternative Way of Closing the Model: Aiyagari (1994)

Section 1 assumed that wealth takes the form of bonds that are in fixed supply. It is, of course, possible to make other assumptions. In particular, we can assume as in Aiyagari (1994) that wealth takes the form of productive capital that is used by a representative firm which also hires labor. Each individual's income is the product of an economy-wide wage w_t and her idiosyncratic labor productivity z_t and her wealth follows (2) with $y_t = w_t z_t$. The total amount of capital supplied in the economy equals the total amount of wealth. In a stationary equilibrium it is given by

$$K = \int_{\underline{z}}^{\bar{z}} \int_{\underline{a}}^{\infty} a g(a, z) da dz := S(r, w). \quad (60)$$

⁵³Given that the figure plots the density $g(a, y)$, some readers may wonder why the spike representing the Dirac mass at $a = \underline{a}$ is finite. The answer is that the figure plots the output of our numerical scheme, $g_{i,j}$ over grids $a_i, i = 0, \dots, I$ and $y_j, j = 1, \dots, J$. As explained in Appendix D the correct interpretation is that $g_{i,j} \approx g(a_i, y_j)$ for all grid points in the interior $i > 0$. But at the boundary $g_{0,j}\Delta a \approx m(y_j)$ where $m(y)$ is the Dirac mass. In the figure for example, $g_{0,j}$ equals about 0.35 at its highest point. The correct interpretation is: since the computation uses $\Delta a = 0.3$, the corresponding Dirac mass is $g_{0,j}\Delta a = 0.35 \times 0.3 = 0.105$.

Capital depreciates at rate δ . There is a representative firm with a constant returns to scale production function $Y = F(K, L)$. Since factor markets are competitive, the wage and the interest rate are given by

$$r = \partial_K F(K, 1) - \delta, \quad w = \partial_L F(K, 1), \quad (61)$$

where we use that the mean of the stationary distribution of productivities z equals one.

Because the income fluctuation problem at the heart of the Aiyagari model is the same as that in the Huggett model all of Propositions 1 to 3 apply without change. So does Proposition 4. Proposition 5 applies by exploiting a homogeneity property noted by Auclert and Rognlie (2016), namely that individual policy functions and therefore aggregate saving is homothetic in the wage rate, $S(r, w) = wS(r, 1)$ for all $w > 0$.⁵⁴ The computational algorithm is again unchanged except that, in Step 3, it imposes (60) and (61) rather than (11).

4.3 Non-Convexities: Indivisible Housing, Mortgages, Poverty Traps

An important class of economic theories involves non-convexities as in Skiba (1978). Another important class features prices in financial constraints as in Kiyotaki and Moore (1997). We here provide a parsimonious example of a theory that features both model elements: individuals can take out a mortgage to buy houses subject to a down-payment constraint – hence the price in the constraint – and housing is indivisible – hence the non-convexity. The purpose of this subsection is not to propose a quantitatively realistic model of housing; rather it is to showcase what kind of models can be solved with our computational algorithm. In particular, viscosity solutions and finite difference methods are *designed* to handle non-differentiable and non-convex problems like the one analyzed here.

Setup Individuals have preferences over consumption c_t and housing services h_t :

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t, h_t) dt.$$

They can borrow and save in a riskless bond b_t and buy housing at price p . The key restriction is that there are no houses below some threshold size $h_{\min} > 0$. That is, an individual can

⁵⁴Uniqueness requires one additional technical assumption about the production function F . To see this note that the homotheticity property implies that (60) becomes $S(r, 1) = k(r)$ where $k(r) := K(r)/w(r)$ is normalized capital demand. Since $S(r, 1)$ slopes upward by Proposition 5, the equilibrium is unique if $k(r)$ slopes downward. Auclert and Rognlie show that this is indeed the case if $\alpha < \varepsilon$ where α is the capital share and ε is the elasticity of substitution corresponding to F (both of which may depend on K/L).

either not own a house $h_t = 0$ or one that is larger than h_{\min} ; compactly:

$$h_t \in \{0, [h_{\min}, \infty)\}.$$

An individual's budget constraint is

$$\dot{b}_t + ph_t = y_t + rb_t - c_t.$$

As before $y_t \in \{y_1, y_2\}$ follows a two-state Poisson process. When buying a house, the individual can take out a mortgage and borrow up to a fraction $\theta \in [0, 1]$ of the value of the house:

$$-b_t \leq \theta ph_t.$$

Equivalently the down-payment needs to be at least a fraction $1 - \theta$ of the house's value. The interest rate r and house price p are determined in equilibrium. Housing is in fixed supply normalized to unity and bonds are in zero net supply.

Stationary Equilibrium It is convenient to work with net worth $a_t := b_t + ph_t$ which follows $\dot{a}_t = y_t + r(a_t - ph_t) - c_t$. Similarly, the borrowing constraint becomes $ph_t \leq \phi a_t$ where $\phi := \frac{1}{1-\theta}$. Denote by $\mathcal{H}(a)$ the set of admissible housing choices. We have

$$\mathcal{H}(a) = \{h : ph \leq \phi a\} \cap \{0, [h_{\min}, \infty)\}.$$

A stationary equilibrium is fully characterized by the following system of equations

$$\begin{aligned} \rho v_j(a) &= \max_{c, h \in \mathcal{H}(a)} u(c, h) + v'_j(a)(y_j + r(a - ph) - c) + \lambda_j(v_{-j}(a) - v_j(a)), & (62) \\ 0 &= -\frac{d}{da}[s_j(a)g_j(a)] - \lambda_j g_j(a) + \lambda_{-j} g_{-j}(a), \\ 1 &= \int_0^\infty (h_1(a)g_1(a) + h_2(a)g_2(a))da, \\ 0 &= \int_0^\infty (b_1(a)g_1(a) + b_2(a)g_2(a))da, \end{aligned}$$

where $c_j(a), h_j(a), s_j(a) = y_j + r(a - ph_j(a)) - c_j(a)$ and $b_j(a) = a - ph_j(a)$ are the optimal consumption, housing, saving and bond holding policy functions.

In what follows we solve this equilibrium system under the additional assumption that utility is quasi-linear $u(c, h) = \tilde{u}(c + f(h))$. This assumption is convenient because the optimal housing choice separates from the consumption-saving problem and allows for a simple connection to theories in the development literature with non-convex production technologies. That being said, the model can easily be solved numerically for general preferences

$u(c, h)$. Assuming quasi-linear utility and defining $x = c + f(h)$, (62) becomes

$$\begin{aligned}\rho v_j(a) &= \max_x \tilde{u}(x) + v'_j(a)(y_j + \tilde{f}(a) + ra - x) + \lambda_j(v_{-j}(a) - v_j(a)), \\ \tilde{f}(a) &= \max_{h \in \mathcal{H}(a)} \{f(h) - rph\}.\end{aligned}$$

The function \tilde{f} is the pecuniary equivalent of the utility benefit of housing (net of the opportunity cost of holding housing rather than interest-bearing bonds). Figures 11(a) and (b) plot the solution to the optimal housing choice problem as a function of wealth a : panel (a) plots the housing policy function $h(a)$ (the maximand of the second equation) and panel (b) plots the benefit from housing $\tilde{f}(a)$ (the corresponding maximum). The vertical line in the two graphs is at $a^* := ph_{\min}/\phi$ which is the down-payment necessary to buy the smallest available house h_{\min} . An individual with wealth a_t below this threshold simply cannot buy a house at time t , as reflected by the fact that both the housing policy function and the benefit from housing are zero, $h(a) = 0$ and $\tilde{f}(a) = 0$ for $a \leq a^*$. As her wealth increases above a^* , the individual is first up against the constraint $ph(a) = \phi a$ so that the size of her house increases linearly with wealth; when her wealth is large enough, she chooses the unconstrained house size given by the solution to the unconstrained first-order condition $f'(h) = rp$.

Importantly the function \tilde{f} in panel (b) is convex-concave as a function of wealth a . Note the similarity to theories of economic growth with convex-concave production functions (Skiba, 1978) and to theories of entrepreneurship with financial constraints and non-convexities in production, either due to fixed costs in production or to an occupational choice (see e.g. Buera, 2009; Buera, Kaboski, and Shin, 2011; Buera and Shin, 2013; Buera, Kaboski, and Shin, 2015). In fact, our computational method again carries over to the solution of such theories.

Figure 11(c) plots the resulting saving policy function. The black, dashed horizontal line is at zero, i.e. savings are positive above that line and negative below. Optimal saving has the typical feature of problems with non-convexities: for each income type, there is a threshold wealth level (the “Skiba point”) below which individuals decumulate assets and above which they accumulate assets. In panel (c) this point is where the saving policy functions intersect zero while sloping upward. As usual, the “Skiba point” is strictly below the point of the non-convexity a^* (the dashed vertical line). Figure 11(d) plots the corresponding value functions: importantly, they feature convex kinks both at the “Skiba point” and at the non-convexity a^* . Since derivatives of v_j do not exist at these kink points, it becomes necessary to explicitly invoke the theory of viscosity solutions. However, the Barles-Souganidis theory still applies in the presence of kinks and therefore our computational algorithm can be applied without change.

Since the theory features classic poverty trap dynamics, there can be multiple stationary

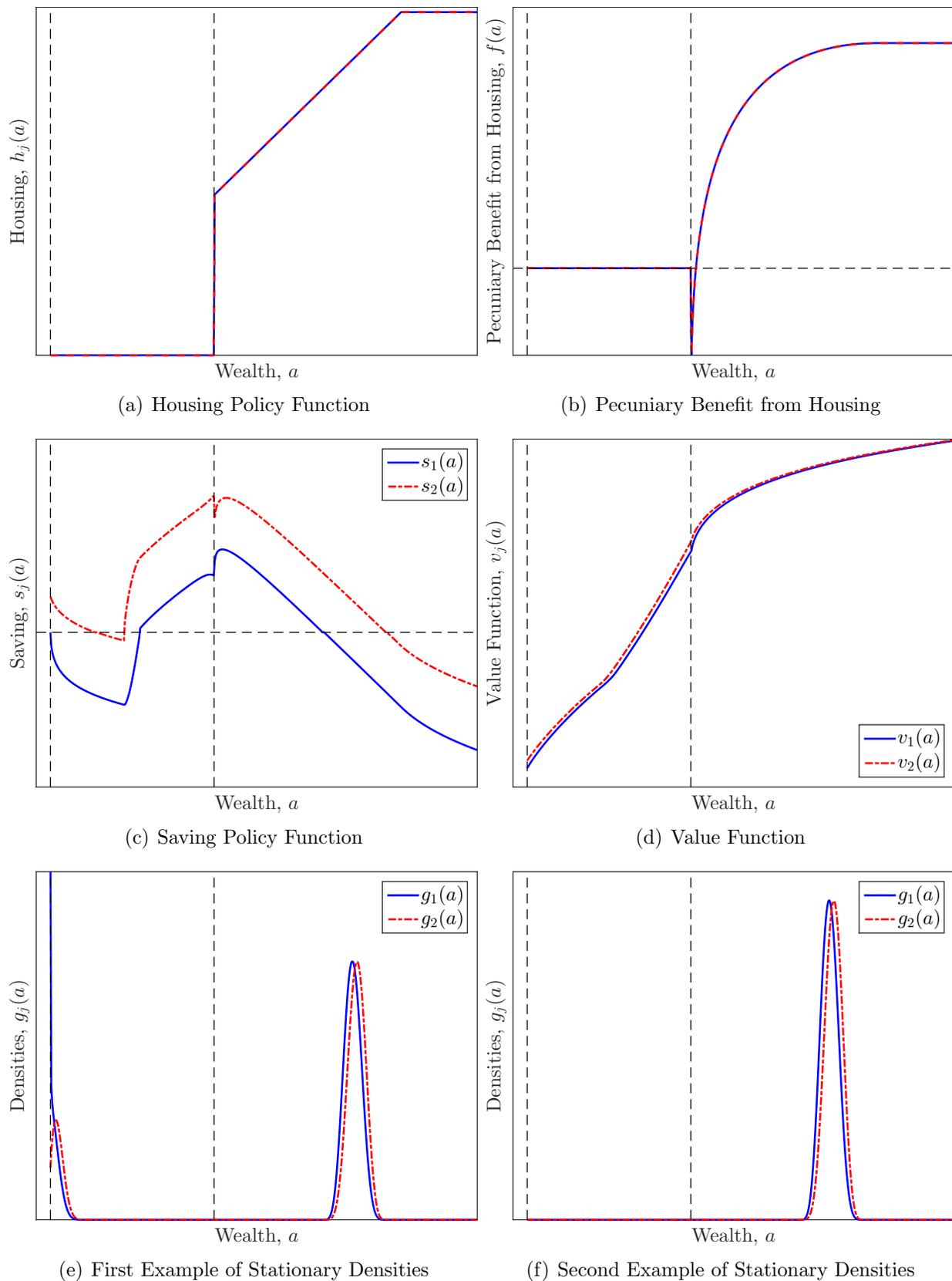


Figure 11: Model with Indivisible Housing: Policy and Value Functions and Multiple Stationary Distributions

wealth distributions. Figures 11 (e) and (f) confirm this possibility: they plot two possible stationary wealth distributions. In fact there is a continuum of stationary wealth distributions. In results not shown here due to space constraints, we have also computed the model’s transition dynamics. Not surprisingly given the discussion thus far, the economy features history dependence in the sense that initial conditions determine where the economy ends up in the long run. As already noted, the point of this subsection is not to argue quantitatively that the presence of indivisible housing and down-payment constraints creates history dependence. Rather it is to showcase the possibilities of our computational algorithm.

4.4 Fat Tails

As shown in Proposition 3, the stationary wealth distribution in the Huggett economy with a bounded income process is bounded above. More generally, any income process with a thin-tailed stationary distribution results in a thin-tailed wealth distribution.⁵⁵ This property of the model is, of course, problematic vis-à-vis wealth distributions observed in the data which are typically heavily skewed and feature fat upper tails. In Online Appendix E.2 we show how to extend the Huggett model of Section 1 to feature a fat-tailed stationary wealth distribution. We do this by introducing a risky asset in addition to the riskless bond. The insight that the introduction of “investment risk” into a Bewley model generates a Pareto tail for the wealth distribution is due to Benhabib, Bisin, and Zhu (2015) and our argument mimics several of their steps. Also see Jones (2015) and De Nardi and Fella (2017).

4.5 Multiple Assets with Adjustment Costs

The models discussed in Sections 4.3 and 4.4 featured two assets: bonds and housing in the former; bonds and a risky asset in the latter. But in both cases portfolio adjustment between the two assets was costless and we could therefore collapse the two assets into one state variable, net worth. Costless portfolio adjustment may be a bad assumption in many applications. For instance, buying a house may entail both fixed and variable transaction costs. The same is true for illiquid assets more generally. For example, an individual withdrawing funds from her retirement account typically incurs a penalty. Kaplan, Moll, and Violante (2016) show how to extend the computational algorithm developed here to handle multiple assets with kinked (but convex) adjustment costs and argue that the two-asset structure is important for understanding the monetary transmission mechanism.

⁵⁵As discussed by Benhabib and Bisin (2016) the only exception to this is if income itself has a stationary distribution with a fat (Pareto) tail. However, this assumption would generate the counterfactual prediction that the tail of the wealth distribution is equally fat as that of the income distribution. In the data, instead, the wealth distribution has a considerably fatter tail than the income distribution.

4.6 Stopping Time Problems

In models with multiple assets like the one discussed above, one may want to allow for non-convex adjustment costs, e.g. fixed costs. In work in progress we show how our algorithm also generalizes to solve such problems. Non-convex adjustment costs result in the individual’s problem becoming a stopping time problem rather than a standard dynamic optimization problem (Stokey, 2009). The value function then no longer solves an HJB equation; instead it solves a so-called “HJB Variational Inequality” (Øksendal, 1995; Tourin, 2013).⁵⁶ Nevertheless, the algorithm developed here can be generalized in a relatively straightforward manner to solve these type of stopping time problems.⁵⁷ The computational method for solving stopping time problems also promises to be useful in other applications, e.g. problems involving default by individuals (see e.g. Livshits, MacGee, and Tertilt, 2007) or by sovereign states (see e.g. Aguiar, Amador, Farhi, and Gopinath, 2013, for a continuous-time formulation).

5 Conclusion

This paper makes two types of contributions. First, we prove a number of new theoretical results about the Aiyagari-Bewley-Huggett model, the workhorse theory of income and wealth distribution in macroeconomics: (i) an analytic characterization of the consumption and saving behavior of the poor, particularly their marginal propensities to consume; (ii) a closed-form solution for the wealth distribution in a special case with two income types; (iii) a proof that there is a unique stationary equilibrium if the intertemporal elasticity of substitution is weakly greater than one; (iv) a characterization of “soft” borrowing constraints. Second, we develop a simple, efficient and portable algorithm for numerically solving both stationary equilibria and transition dynamics of a wide class of heterogeneous agent models, including – but not limited to – this model. Both types of contributions were made possible by recasting the Aiyagari-Bewley-Huggett model in continuous time, thereby transforming the model into a system of partial differential equations.

⁵⁶Economists typically tackle these types of problems by splitting the state space into an inaction region and an adjustment region, postulate that the value function satisfies an HJB equation in the adjustment region and a “smooth pasting” condition on the boundary. This approach is restrictive: in one dimension, the boundary of the inaction region is just a threshold; but as soon as there is more than one state variable, it becomes exceedingly hard to find the boundaries of the adjustment region. The approach is even somewhat misleading: using the HJB Variational Inequality formulation the smooth pasting condition is a result rather than an exogenously imposed axiom (Øksendal, 1995). Solving stopping time problems in multiple dimensions then poses no conceptual problem over solving one-dimensional ones.

⁵⁷See the applications labelled “Stopping Time Problem” here <http://www.princeton.edu/~moll/HACTproject.htm>, particularly a simple problem of exercising an option borrowed from Stokey (2009) http://www.princeton.edu/~moll/HACTproject/option_simple.pdf and a model with two assets and non-convex adjustment costs http://www.princeton.edu/~moll/HACTproject/liquid_liquid_numerical.pdf.

It is our hope that the methods developed in this paper, particularly the numerical algorithm, will also prove useful in other applications. One potential application is to spatial theories of trade and development as in [Rossi-Hansberg \(2005\)](#) and [Allen and Arkolakis \(2014\)](#). These theories typically feature a continuum of producers and households distributed over a continuum of locations. In dynamic versions, space would simply be an additional variable in the HJB and KF equations. A challenge would be how to solve for equilibrium prices which are typically functions of space rather than a small number of (potentially time-varying) scalars. Related, a second avenue for future research is to explore richer interactions between individuals. In the class of theories we have considered here, individuals interact only through prices. But for many questions of interest, richer interactions may be important: for instance there may be more “local” interactions in the form of knowledge spillovers or diffusion (see e.g. [Perla and Tonetti, 2014](#); [Lucas and Moll, 2014](#); [Burger, Lorz, and Wolfram, 2016](#); [Benhabib, Perla, and Tonetti, 2017](#)). In principle, the apparatus put forward in this paper – the Mean Field Game system of coupled HJB and KF equations – is general enough to encompass richer models such as these.

References

- ACHDOU, Y. (2013): “Finite difference methods for mean field games,” in *Hamilton-Jacobi equations: approximations, numerical analysis and applications*, vol. 2074 of *Lecture Notes in Math.*, pp. 1–47. Springer, Heidelberg.
- ACHDOU, Y., AND I. CAPUZZO-DOLCETTA (2010): “Mean field games: numerical methods,” *SIAM J. Numer. Anal.*, 48(3), 1136–1162.
- ACIKGOZ, O. (2016): “On the Existence and Uniqueness of Stationary Equilibrium in Bewley Economies with Production,” Working paper, Yeshiva University.
- AGUIAR, M., M. AMADOR, E. FARHI, AND G. GOPINATH (2013): “Crisis and Commitment: Inflation Credibility and the Vulnerability to Sovereign Debt Crises,” NBER Working Papers 19516, National Bureau of Economic Research, Inc.
- AHN, S. (2017): “Sparse Grid Methods for Continuous-Time Heterogeneous Agent Models,” Working paper, Norges Bank.
- AHN, S., G. KAPLAN, B. MOLL, T. WINBERRY, AND C. WOLF (2017): “When Inequality Matters for Macro and Macro Matters for Inequality,” in *NBER Macroeconomics Annual 2017, volume 32*, NBER Chapters. National Bureau of Economic Research, Inc.
- AIYAGARI, S. R. (1993): “Uninsured idiosyncratic risk and aggregate saving,” Working Papers 502, Federal Reserve Bank of Minneapolis.
- AIYAGARI, S. R. (1994): “Uninsured Idiosyncratic Risk and Aggregate Saving,” *The Quarterly Journal of Economics*, 109(3), 659–84.
- ALLEN, T., AND C. ARKOLAKIS (2014): “Trade and the Topography of the Spatial Economy,” *The Quarterly Journal of Economics*, 129(3), 1085–1140.
- ALONSO, C. (2016): “Hard vs. Soft Financial Constraints: Implications for the Effects of a Credit Crunch,” Working paper, Princeton.
- ALVAREZ, F., AND R. SHIMER (2011): “Search and Rest Unemployment,” *Econometrica*, 79(1), 75–122.

- AUCLERT, A. (2017): “Monetary Policy and the Redistribution Channel,” NBER Working Papers 23451, National Bureau of Economic Research, Inc.
- AUCLERT, A., AND M. ROGNLIE (2016): “Inequality and Aggregate Demand,” Work in progress.
- (2017): “Aggregate Demand and the Top 1 Percent,” *American Economic Review*, 107(5), 588–592.
- BARCZYK, D., AND M. KREDLER (2014): “Altruistically Motivated Transfers under Uncertainty,” *Quantitative Economics*, 5(3), 705–749.
- BARLES, G., AND P. E. SOUGANIDIS (1991): “Convergence of approximation schemes for fully nonlinear second order equations,” *Asymptotic Analysis*, 4, 271–283.
- BAYER, C., AND K. WÄLDE (2010): “Matching and Saving in Continuous Time: Theory,” CESifo Working Paper Series 3026, CESifo Group Munich.
- BÉNABOU, R. (2002): “Tax and Education Policy in a Heterogeneous-Agent Economy: What Levels of Redistribution Maximize Growth and Efficiency?,” *Econometrica*, 70(2), 481–517.
- BENHABIB, J., AND A. BISIN (2016): “Skewed Wealth Distributions: Theory and Empirics,” Working paper.
- BENHABIB, J., A. BISIN, AND S. ZHU (2011): “The Distribution of Wealth and Fiscal Policy in Economies With Finitely Lived Agents,” *Econometrica*, 79(1), 123–157.
- (2015): “The Wealth Distribution in Bewley Economies with Capital Income Risk,” *Journal of Economic Theory*, 159(PA), 489–515.
- (2016): “The Distribution Of Wealth In The Blanchard?Yaari Model,” *Macroeconomic Dynamics*, 20(02), 466–481.
- BENHABIB, J., J. PERLA, AND C. TONETTI (2017): “Reconciling Models of Diffusion and Innovation: A Theory of the Productivity Distribution and Technology Frontier,” NBER Working Papers 23095, National Bureau of Economic Research, Inc.
- BERGER, D., V. GUERRIERI, G. LORENZONI, AND J. VAVRA (2015): “House Prices and Consumer Spending,” NBER Working Papers 21667, National Bureau of Economic Research, Inc.
- BEWLEY, T. (1986): “Stationary Monetary Equilibrium with a Continuum of Independently Fluctuating Consumers,” in *Contributions to Mathematical Economics in Honor of Gerard Debreu*, ed. by W. Hildenbrand, and A. Mas-Colell. North-Holland, Amsterdam.
- BLUNDELL, R., L. PISTAFERRI, AND I. SAPORTA-EKSTEN (2016): “Consumption Inequality and Family Labor Supply,” *American Economic Review*, 106(2), 387–435.
- BRODA, C., AND J. A. PARKER (2014): “The Economic Stimulus Payments of 2008 and the Aggregate Demand for Consumption,” *Journal of Monetary Economics*, 68, S20–S36.
- BUERA, F. (2009): “A dynamic model of entrepreneurship with borrowing constraints: theory and evidence,” *Annals of Finance*, 5(3), 443–464.
- BUERA, F. J., J. P. KABOSKI, AND Y. SHIN (2011): “Finance and Development: A Tale of Two Sectors,” *American Economic Review*, 101(5), 1964–2002.
- (2015): “Entrepreneurship and Financial Frictions: A Macroeconomic Perspective,” *Annual Review of Economics*, 7(1), 409–436.
- BUERA, F. J., AND Y. SHIN (2013): “Financial Frictions and the Persistence of History: A Quantitative Exploration,” *Journal of Political Economy*, 121(2), 221–272.
- BUNGARTZ, H.-J., AND M. GRIEBEL (2004): “Sparse grids,” *Acta Numerica*, 13, 147–269.
- BURGER, M., A. LORZ, AND M.-T. WOLFRAM (2016): “On a Boltzmann Mean Field Model for Knowledge Growth,” *SIAM Journal on Applied Mathematics*, 76(5), 1799–1818.
- CANDLER, G. V. (1999): “Finite-Difference Methods for Dynamic Programming Problems,” in *Computational Methods for the Study of Dynamic Economies*. Cambridge University Press, Cambridge, England.

- CAO, D., AND W. LUO (2017): “Persistent Heterogeneous Returns and Top End Wealth Inequality,” Working paper, Georgetown.
- CAPUZZO-DOLCETTA, I., AND P. L. LIONS (1990): “Hamilton-Jacobi Equations with State Constraints,” *Transactions of the American Mathematical Society*, 318(2), 643+.
- CARDIALAGUET, P. (2013): “Notes on mean-field games (from P.-L. Lions’ lectures at Collège de France),” <https://www.ceremade.dauphine.fr/~cardaliaguet/MFG20130420.pdf>.
- CARROLL, C. D. (2006): “The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems,” *Economics Letters*, 91(3), 312–320.
- CONZE, A., J. M. LASRY, AND J. SCHEINKMAN (1993): “Borrowing Constraints and International Comovements,” *Hitotsubashi Journal of Economics*, 34(Special I), 23–47.
- CRANDALL, M. G., AND P. L. LIONS (1983): “Viscosity Solutions of Hamilton-Jacobi Equations,” *Transactions of the American Mathematical Society*, 277(1), 1–42.
- DE NARDI, M., AND G. FELLA (2017): “Saving and Wealth Inequality,” CEPR Discussion Papers 11746, C.E.P.R. Discussion Papers.
- DEATON, A. (2016): “Measuring and Understanding Behavior, Welfare, and Poverty,” *American Economic Review*, 106(6), 1221–1243.
- DEN HAAN, W. J. (1997): “Solving Dynamic Models With Aggregate Shocks And Heterogeneous Agents,” *Macroeconomic Dynamics*, 1(02), 355–386.
- EVANS, L. (2010): *Partial Differential Equations*. American Mathematical Society.
- FAGERENG, A., M. B. HOLM, AND G. J. NATVIK (2016): “MPC Heterogeneity and Household Balance Sheets,” Discussion paper, Statistics Norway.
- FESTA, A., D. A. GOMES, AND R. M. VELHO (2017): “An Adjoint-based Numerical Method for a class of nonlinear Fokker-Planck Equations,” *ArXiv e-prints*.
- GABAIX, X., J.-M. LASRY, P.-L. LIONS, AND B. MOLL (2016): “The Dynamics of Inequality,” *Econometrica*, 84(6), 2071–2111.
- GERSTNER, T., AND M. GRIEBEL (2010): *Sparse Grids* John Wiley & Sons, Ltd.
- GOLOSOV, M., AND R. E. LUCAS (2007): “Menu Costs and Phillips Curves,” *Journal of Political Economy*, 115, 171–199.
- GUÉANT, O., J.-M. LASRY, AND P.-L. LIONS (2011): *Mean Field Games and Applications*. Springer Berlin Heidelberg, Berlin, Heidelberg. 205–266.
- GUERRIERI, V., AND G. LORENZONI (2017): “Credit Crises, Precautionary Savings, and the Liquidity Trap*,” *The Quarterly Journal of Economics*, 132(3), 1427–1467.
- GUVENEN, F. (2011): “Macroeconomics With Heterogeneity: A Practical Guide,” NBER Working Papers 17622, National Bureau of Economic Research, Inc.
- HAGEDORN, M., I. MANOVSKI, AND K. MITMAN (2017): “The Fiscal Multiplier,” Working paper.
- HEATHCOTE, J., K. STORESLETTEN, AND G. L. VIOLANTE (2009): “Quantitative Macroeconomics with Heterogeneous Households,” *Annual Review of Economics*, 1(1), 319–354.
- (2014): “Consumption and Labor Supply with Partial Insurance: An Analytical Framework,” *American Economic Review*, 104(7), 2075–2126.
- HOLM, M. B. (2017): “Consumption with Liquidity Constraints: An Analytical Characterization,” Discussion paper, Statistics Norway.
- HOPENHAYN, H. A. (1992): “Entry, Exit, and Firm Dynamics in Long Run Equilibrium,” *Econometrica*, 60(5), 1127–50.
- HUGGETT, M. (1993): “The risk-free rate in heterogeneous-agent incomplete-insurance economies,” *Journal of Economic Dynamics and Control*, 17(5-6), 953–969.
- JONES, C. I. (2015): “Pareto and Piketty: The Macroeconomics of Top Income and Wealth Inequality,” *Journal of Economic Perspectives*, 29(1), 29–46.

- JONES, C. I., AND J. KIM (2014): “A Schumpeterian Model of Top Income Inequality,” NBER Working Papers 20637, National Bureau of Economic Research, Inc.
- JOVANOVIĆ, B. (1979): “Job Matching and the Theory of Turnover,” *Journal of Political Economy*, 87(5), 972–90.
- KAPLAN, G., B. MOLL, AND G. L. VIOLANTE (2016): “Monetary Policy According to HANK,” Working Paper 21897, National Bureau of Economic Research.
- KAPLAN, G., AND G. VIOLANTE (2014): “A Model Of The Consumption Response To Fiscal Stimulus Payments,” *Econometrica*, forthcoming.
- KASA, K., AND X. LEI (2017): “Risk, Uncertainty, and the Dynamics of Inequality,” Discussion Papers dp17-06, Department of Economics, Simon Fraser University.
- KIYOTAKI, N., AND J. MOORE (1997): “Credit Cycles,” *Journal of Political Economy*, 105(2), 211–48.
- KREBS, T. (2003): “Human Capital Risk And Economic Growth,” *The Quarterly Journal of Economics*, 118(2), 709–744.
- KRUEGER, D., K. MITMAN, AND F. PERRI (2015): “Macroeconomics and Heterogeneity, Including Inequality,” Working paper.
- KRUSELL, P., AND A. A. SMITH (1998): “Income and Wealth Heterogeneity in the Macroeconomy,” *Journal of Political Economy*, 106(5), 867–896.
- KUSHNER, H. J., AND P. DUPUIS (2013): *Numerical Methods for Stochastic Control Problems in Continuous Time*. Springer.
- LASRY, J.-M., AND P.-L. LIONS (2007): “Mean field games,” *Japanese Journal of Mathematics*, 2, 229–260.
- LIGHT, B. (2017): “Uniqueness of Equilibrium in a Bewley-Aiyagari Model,” Discussion paper, Stanford.
- LIPPI, F., S. RAGNI, AND N. TRACHTER (2013): “State Dependent Monetary Policy,” Working Paper 13-17, Federal Reserve Bank of Richmond.
- LISE, J. (2013): “On-the-Job Search and Precautionary Savings,” *Review of Economic Studies*, 80(3), 1086–1113.
- LIVSHITS, I., J. MACGEE, AND M. TERTILT (2007): “Consumer Bankruptcy: A Fresh Start,” *American Economic Review*, 97(1), 402–418.
- LJUNGQVIST, L., AND T. J. SARGENT (2004): *Recursive Macroeconomic Theory*. The MIT Press.
- LUCAS, R. E., AND B. MOLL (2014): “Knowledge Growth and the Allocation of Time,” *Journal of Political Economy*, 122(1), 1–51.
- LUTTMER, E. G. J. (2007): “Selection, Growth, and the Size Distribution of Firms,” *The Quarterly Journal of Economics*, 122(3), 1103–1144.
- (2011): “On the Mechanics of Firm Growth,” *Review of Economic Studies*, 78(3), 1042–1068.
- (2015): “An Assignment Model of Knowledge Diffusion and Income Inequality,” Staff Report 509, Federal Reserve Bank of Minneapolis.
- MISRA, K., AND P. SURICO (2014): “Consumption, Income Changes, and Heterogeneity: Evidence from Two Fiscal Stimulus Programs,” *American Economic Journal: Macroeconomics*, 6(4), 84–106.
- MOLL, B. (2014): “Productivity Losses from Financial Frictions: Can Self-financing Undo Capital Misallocation?,” *American Economic Review*, forthcoming.
- MOSCARINI, G. (2005): “Job Matching and the Wage Distribution,” *Econometrica*, 73(2), 481–516.
- NUÑO, G., AND B. MOLL (2017): “Social Optima in Economies with Heterogeneous Agents,” Working paper, Bank of Spain.

- NUÑO, G., AND C. THOMAS (2017): “Optimal Monetary Policy with Heterogeneous Agents,” Working paper, Bank of Spain.
- ØKSENDAL, B. (1995): *Stochastic Differential Equations*. Springer Berlin Heidelberg, Berlin, Heidelberg.
- OLIVI, A. (2017): “Sufficient Statistics for Heterogeneous Agent Models,” Work in progress, MIT.
- PARRA-ALVAREZ, J., O. POSCH, AND M.-C. WANG (2017): “Identification and estimation of heterogeneous agent models: A likelihood approach,” Working paper, Aarhus University.
- PERLA, J., AND C. TONETTI (2014): “Equilibrium Imitation and Growth,” *Journal of Political Economy*, 122(1), 52–76.
- QUADRINI, V., AND J.-V. RÍOS-RULL (2015): “Inequality in Macroeconomics,” *Handbook of Income Distribution*, 2, 1229 – 1302, Handbook of Income Distribution.
- ROCHETEAU, G., P.-O. WEILL, AND T.-N. WONG (2015): “Working through the Distribution: Money in the Short and Long Run,” NBER Working Papers 21779, National Bureau of Economic Research, Inc.
- ROSSI-HANSBERG, E. (2005): “A Spatial Theory of Trade,” *American Economic Review*, 95(5), 1464–1491.
- SCHEINKMAN, J. A., AND L. WEISS (1986): “Borrowing Constraints and Aggregate Economic Activity,” *Econometrica*, 54(1), 23–45.
- SHAKER AKHTEKHANE, S. (2017): “Firm Entry and Exit in Continuous Time,” Working paper, Ohio State University.
- SKIBA, A. K. (1978): “Optimal Growth with a Convex-Concave Production Function,” *Econometrica*, 46(3), 527–39.
- SONER, H. M. (1986a): “Optimal Control with State-Space Constraint I,” *SIAM Journal on Control and Optimization*, 24(3), 552–561.
- (1986b): “Optimal Control with State-Space Constraint II,” *SIAM J. Control Optim.*, 24(6), 1110–1122.
- STOKEY, N. (2014): “The Race Between Technology and Human Capital,” Working paper, University of Chicago.
- STOKEY, N. L. (2009): *The Economics of Inaction*. Princeton University Press, Princeton, NJ.
- TAO, T. (2008): “Generalized Solutions,” in *The Princeton Companion to Mathematics*, ed. by T. Gowers, J. Barrow-Green, and I. Leader. Princeton University Press, http://www.math.ucla.edu/~tao/preprints/generalized_solutions.pdf.
- TODA, A. A., AND K. WALSH (2015): “The Double Power Law in Consumption and Implications for Testing Euler Equations,” *Journal of Political Economy*, 123(5), 1177–1200.
- TOURIN, A. (2013): “An Introduction to Finite Difference Methods for PDEs in Finance,” in *Optimal Stochastic Target problems, and Backward SDE*, ed. by N. Touzi, Fields Institute Monographs. Springer, available online at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2396142.
- VINDIGNI, A., S. SCOTTI, AND C. TEALDI (2014): “Uncertainty and the Politics of Employment Protection,” *Journal of Labor Economics*.
- WANG, N. (2007): “An Equilibrium Model of Wealth Distribution,” *Journal of Monetary Economics*, 54(7), 1882–1904.