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REPUTATION CYCLES

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ABSTRACT

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Abstract

This paper shows that endogenous cycles can arise when contracts between firms and their customers are incomplete and when products are experience goods. Then firms invest in the quality of their output in order to establish a good reputation. Cycles arise because investment in reputation causes self-fulfilling changes in the discount factor. Cycles are more likely to occur when information diffuses slowly and consumers exhibit high risk aversion. A rise in idiosyncratic uncertainty is of two kinds that work in opposite ways: Noise in observing effort is contractionary as it generally is in agency models. But a rise in the variance of the distribution of abilities is expansionary. A calibrated version produces realistic fluctuations in terms of peak-to-trough movements in consumption and the spacing of time between recessions.

Keywords: Endogenous Fluctuations, Reputation, Intangible Capital.

1 Introduction

A seller's reputation is often the only guarantee of quality of its products or services. By the same token, reputation building is often the seller's only incentive to deliver on

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quality. This is why consumers are willing to pay a premium for goods and services from established brands. Underprovision of quality today will be punished by lower prices in the future, and a positive surprise will be rewarded. Because of this, the seller's market value depends partly on his reputation or "brand value", an *intangible* component of his capital stock.[1]

Since reputational investment today pays off in the future, a seller's incentive to maintain or improve his reputation depends on his discount factor. And when the seller is risk averse, his discount factor depends negatively on his consumption growth. We show that if the seller is unable to smooth his consumption by other means, this force can give rise to cycles.

For a two-period cycle the intuition is simple. In a recession, current consumption is low relative to future consumption which means that the discount factor is also low. This reduces the incentive to create a good reputation and the recession is self fulfilling. Conversely, current consumption in a boom is higher than future consumption, which means that the discount factor is also high and so the boom too is self fulfilling. In other words, the discount factor and reputational concern are pro-cyclical because consumption growth is counter-cyclical.

We embed this mechanism into a general equilibrium model and study how investment in reputation affects aggregate outcomes. The model has no direct externalities and no aggregate shocks, but output oscillates. Cycles are sustainable because investment is always below its output-maximizing level, a wedge that is driven by the delay with which reputation reacts to investment. This market failure implies that output and investment in quality are positively correlated.

Consumption and investment are therefore both pro-cyclical, a basic feature of business cycles which is difficult to reconcile with canonical models of deterministic cycles based on multi-sectoral economies, such as Benhabib and Nishimura (1985), or on replacement echoes, such as Boucekkine, Germain and Licandro (1997). Besides these raw correlations, our model is also consistent with a series of stylized facts about business cycles. First, it predicts that output is more volatile than consumption but less than investment. Second, product quality is pro-cyclical as recently documented by Broda and Weinstein (2010) or Jaimovich, Rebelo and Wong (2015). Third, our model predicts that an inverted yield

[1] McGrattan and Prescott (2000) argue that as much as 40 percent of GNP is intangible capital and firm's reputation accounts for a large fraction of it.

curve precedes recessions (Ang, Piazzesi and Wei, 2006). Finally we show that asymmetric cycles with a string of positive growth rates followed by a highly negative growth rate – i.e., a disaster – can also arise repeatedly.

To go beyond qualitative insights, we use micro-data on firm dynamic to parametrize the model. We explain how variance parameters can be recovered from estimates about the evolution of TFP at the plant level. The speed at which consumers update their beliefs determines the length of the period. Setting it equal to the average frequency of 56 months separating booms and busts, we find that the calibrated parameters fall well within the region where cycles arise endogenously. Hence our model does not require extreme parameter values in order to generate deterministic cycles. In particular, cycles with a frequency of 56 months are sustainable whenever the annual discount factor is below 0.977, a number well above the upper-bounds of the competitive equilibrium models surveyed in Boldrin and Woodford (1990). This finding is in line with recent research pointing out that market imperfections considerably widen the range of discount factors compatible with endogenous cycles. For example, Beaudry, Galizia and Portier (2015) find that adding strategic complementarities to a standard DSGE model allows them to produce limit cycles that match US business fluctuations in employment and output. Our analysis shows that pecuniary externalities can lead to a similar conclusion, so that complementarities do not have to be directly embedded in the structure of the economy.

We find that two kinds of uncertainty affect output in opposite directions. A rise in the variance of the distribution of productivity strengthens reputational concerns and is expansionary. By contrast, a rise in the variance of output at the firm level makes it harder for consumers to infer investment and is therefore contractionary. Since it also dampens reputation cycles, an increase in the noisiness of the technology yields a negative correlation between micro and macro volatility that is qualitatively similar to the one observed during the great moderation.

Although ours is a general equilibrium paper targeting aggregate business cycles, we begin our analysis by showing that cyclical equilibria also arise in Holmström's (1999) partial equilibrium setting if the agent is risk averse and cannot borrow or lend. So, while significant extensions have been made on the learning process (e.g., Board and Meyer-ter-Vehn, 2013), our analysis suggests that similarly important insights can be gathered studying the specification of preferences in reputation models.

Section 2 lays out the model. Section 3 deals with partial equilibrium where the mechanism is more transparent while Section 4 deals with general equilibrium. Section 5 explains how the model can be parametrized, and shows that cycles arise for realistic parameters values. Then we review major hypotheses for endogenous cycles in Section 6 so as to highlight the novelty of reputation cycles.

2 A model of firm reputation

We build on Holmström (1999) where a risk-neutral agent faces a spot market with risk-neutral buyers. The unique equilibrium then entails a monotonic time path of effort. We will now show that if one assumes that the agent is risk averse, multiple equilibria arise and in some of them effort follows a 2-period cycle. In *partial equilibrium* our mechanism requires the following assumptions:

A1. Incomplete agency contracts with non-contractible effort and output;

A2. Risk averse agent facing risk-neutral principals;

A3. Agent's preferences have no wealth effect (CARA utility);

A4. Agent cannot borrow or lend.

In *general equilibrium* A4 is replaced by the fact that the economy is closed, and because of insurance possibilities among agents we can drop A3 and work with CRRA utility.

2.1 Set-up

Each period an agent produces output y_t by exerting hidden effort a_t:

$$y_t = \theta_t + a_t + \varepsilon_t. \tag{1}$$

Here $\varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right)$ is an i.i.d. shock. The variable θ is the agent's efficiency. θ is unknown, even to the the agent himself, and the common prior is $\mathcal{N}\left(0, \sigma_\theta^2\right)$.[2]

Contracts in which the period-t payment is contingent on y_t are not feasible. Instead, buyers pay the agent up front and the payment reflects the market belief at the beginning of the period. Thus the agent exerts effort only so as to raise his future income, and so we refer to a as "investment". Output has a persistent effect on prices because it affects the market's belief about the agent's efficiency which fluctuates over time as

$$\theta_t = \theta_{t-1} + \nu_t, \text{ with } \nu \sim \mathcal{N}\left(0, \sigma_\nu^2\right). \tag{2}$$

[2] We shall let the prior mean differ from zero in Section 5.

Each period the agent chooses an a from the feasible set $\mathcal{A} \subseteq \mathbb{R}_+$ and everyone observes y. The agent can also observe $\theta + \varepsilon$ but not its components. The market's participants observe y only, but since they will infer a from equilibrium, they too will be able to infer $\theta + \varepsilon$.

Within a period, events unfold as follows:

1. There are many identical and risk neutral potential buyers, and they get zero rents; the up-front payment the agent gets is $E\left[y_t \mid y^t\right]$, where $y^t \equiv \{y_s\}_{s=0}^{t-1}$ is his public history.

2. The agent chooses a_t, privately.

3. Output y_t is realized, and it becomes part of the agent's history y^{t+1}.

At date t, everyone knows the history y^t. Let a_t^* denote the agent's equilibrium action. We will derive an equilibrium in which a^* depends on t, but not on y^t. A sufficient statistic for the information revealed about θ_t is the sequence $x^t \equiv (x_0, ..., x_{t-1})$, where $x_t \equiv y_t - a_t^* = \theta_t + \varepsilon_t$. The market treats x as the signal, which is normally distributed. Because ν and ε are normal, the posterior is also normal: $\theta_t \sim \mathcal{N}\left(m_t, \sigma_{\theta,t}^2\right)$.

Evolution of market beliefs.—The posterior variance evolves deterministically as

$$\sigma_{\theta,t}^2 = \frac{1}{\sigma_{\theta,t-1}^{-2} + \sigma_\varepsilon^{-2}} + \sigma_\nu^2, \tag{3}$$

and in the long-run it converges to $\bar{\sigma}_\theta^2$, whose value is obtained setting $\sigma_{\theta,t} = \sigma_{\theta,t-1}$ in (3) so that

$$\bar{\sigma}_\theta^{-2} = \frac{1}{2}\left(\sqrt{\frac{1}{\sigma_\varepsilon^4} + \frac{4}{\sigma_\varepsilon^2 \sigma_\nu^2}} - \frac{1}{\sigma_\varepsilon^2}\right). \tag{4}$$

Assume that the agent's initial θ is drawn from $\mathcal{N}\left(0, \bar{\sigma}_\theta^2\right)$, so that $\sigma_{\theta,t} = \bar{\sigma}_\theta$ does not change over time, and age is not a state.[3] By contrast, the posterior mean m of market belief is a state and it follows the process

$$m_{t+1} = E\left[\theta_{t+1} \middle| m_t, x_t\right] = \lambda m_t + (1-\lambda)\, x_t, \text{ where } \lambda \equiv \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \bar{\sigma}_\theta^2}. \tag{5}$$

By deviating from a_t^*, the agent can manipulate m_{t+1} and drive a wedge between his own belief and that of the market; a_t^* is an equilibrium action if he rejects that option.

[3] Observe that $\partial \bar{\sigma}_\theta^{-2} / \partial \sigma_\varepsilon < 0$ and $\partial \bar{\sigma}_\theta^{-2} / \partial \sigma_\nu < 0$ as fluctuations in output and efficiency lower stationary precision.

2.2 Incentives

The agent is risk averse and infinitely lived. We focus on deterministic solutions of his optimization program[4]

$$\max_{\{a_t\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t U(c_t)\right], \text{ s.t. } c_t = m_t + a_t^* - g(a_t), \tag{6}$$

where m_t obeys the law of motion (5) and $a_t \in \mathcal{A}$. The utility function $U(\cdot)$ is twice differentiable and concave, while $g(\cdot)$ is an increasing, twice differentiable, cost function measured in terms of goods. Consumption in each period is equal to revenues $E[y_t \mid y^t] = m_t + a_t^*$ net of the investment cost $g(a_t)$. Hence we have implicitly assumed that:

(i) The agent cannot smooth consumption by borrowing and lending.

(ii) The action a is hidden.

(iii) Deterministic policies are optimal for the problem in (6), which is true if $U(c)$ is in the CARA class and if shocks to the agent's income are permanent. Income shocks, which reflect the history y^t, will in turn be permanent because buyers are risk neutral and because their beliefs have the martingale property.

Taking $\{a_s^*\}_{s=t}^{\infty}$ as given, Bayes rule and repeated substitution for c yields the following incentive constraint:[5]

$$g'(a_t^*) = \frac{1-\lambda}{\lambda} \sum_{s=t+1}^{\infty} (\beta\lambda)^{s-t} E_t\left[\frac{U'(c_s)}{U'(c_t)}\right]. \tag{7}$$

The left-hand side is the marginal cost, and the right-hand side is the discounted benefit because a deviation from a_t^* to $a_t^* + 1$ would raise the posterior mean at date s by

[4]We will show that optimal actions on and off the equilibrium path are a function of time only (See lemmata 9 and 10 in the technical Appendix A.4). Hence (i) given the law of motion of beliefs derived in the previous subsection, equilibrium strategies maximize the agent's utility; (ii) given the equilibrium actions, beliefs are updated via Bayes rule. Bayesian updating is always well defined since all output levels occur with positive probability. Solutions to problem (6) are therefore optimal.

[5]Condition (7) rules out one-shot deviations but does not guarantee that multiple deviations are not profitable. Whenever the agent deviates, he drives a persistent wedge between his belief and that of the market. Thus a deviation that is not attractive on the equilibrium path might nonetheless be profitable off path. In other words, the FOC (7) is necessary but not always sufficient. We show in the technical Appendix A.4 that sufficiency is not a concern for the models studied in this paper because optimal strategies *on and off* the equilibrium path are deterministic. A recent paper by Cisternas (2016) provides bounds guaranteeing sufficiency in more general settings in which optimal actions may depend on m.

$\partial m_s / \partial a_t = (1-\lambda)\,\lambda^{s-t-1}$, for all $s > t$.[6] Since c is increasing in m and m is a martingale, expected consumption depends on the agent's current reputation. Hence the optimality condition (7) varies with m_t and in general so would a_t^*, contrary to our assumption that a^* depends on t only. To ensure that $E_t\left[U'(c_s)/U'(c_t)\right]$ is not affected by m_t, we have to neutralize the wealth effect.

The wealth effect is easier to eliminate in general equilibrium where we assume that a representative family insures its members against wealth shocks. There we shall work with CRRA preferences. In partial equilibrium, however, we deal with one agent who cannot insure his income, and we shall assume that the agent's utility function is CARA. With either assumption, the agent's discount factor no longer depends on m.

3 Cycles in partial equilibrium

Following the discussion above, in this section we assume that utility is CARA:

$$U(c) = -\exp(-\gamma c), \text{ with } \gamma > 0. \tag{8}$$

Inserting (8) and (5) in (7), the term m_t cancels out from the expression of $E_t\left[U'(c_s)/U'(c_t)\right]$ because it is the forecastable component of m_s. We obtain an incentive constraint that is consistent with deterministic actions since (7) becomes equivalent to

$$g'(a_t^*) = \frac{1-\lambda}{\lambda} \sum_{s=t+1}^{\infty} (r\lambda)^{s-t} \exp\left(\gamma\left[a_t^* - g(a_t^*) - (a_s^* - g(a_s^*))\right]\right), \tag{9}$$

with

$$r \equiv \beta \exp\left(\frac{\gamma^2(1-\lambda)^2(\sigma_\varepsilon^2 + \bar{\sigma}_\theta^2)}{2}\right).$$

Note that $r > \beta$ because consumption is stochastic and because $U'''(\cdot) > 0$. Consumption volatility thus raises expected marginal utility in future periods, especially when γ is large. To ensure that expected returns are bounded, we focus on cases where $r < 1/\lambda$.

3.1 Linear costs

Steady-state and 2-period cycles.—Suppose that $g(a) = \kappa a$, with $\kappa \in (0,1)$ to ensure that investment raises output. We rule out infinite output levels by imposing an upper-bound $\bar{a}$ on the feasible set $\mathcal{A} = [0, \bar{a}]$. With linear costs, optimality can only be restored

[6] Eq. (7) is the counterpart of Holmström's (1999), eq. (22), with the essential addition of the term $E_t\left[U'(c_s)/U'(c_t)\right]$.

through changes in the discount factor because both marginal costs and marginal returns are constant. To see this, assume that both a_t^* and a_{t+1}^* belong to the interior of $\mathcal{A}$ and substitute in the FOC for $g'\left(a_{t+1}^*\right)$ on the right-hand side of (7) to obtain[7]

$$\frac{U'(a_{t+1}^*(1-\kappa))}{U'(a_t^*(1-\kappa))} = \frac{\kappa}{r(1+\lambda(\kappa-1))}. \tag{10}$$

Taking expectations about a_{t+1}^* as given, a_t^* adjusts until the equality above is satisfied. The equilibrium path is fully determined by the ratio of marginal costs to discounted marginal returns on the right-hand side of (10). Since this ratio is constant, all paths converge towards $\bar{a}$ when it is lower than one or, conversely, converge to zero when it is higher than one. To illustrate the dynamics with a phase portrait, we use (8) to rewrite (10) as

$$a_{t+1}^* = a_t^* - \frac{1}{\gamma(1-\kappa)} \log\left(\frac{\kappa}{r(1+\lambda(\kappa-1))}\right). \tag{11}$$

The law of motion is given by a line parallel to the 45 degree line with an intercept equal to the constant on the right-hand side of (11). As shown in Figure 1, in the knife-edge case where the intercept is zero, i.e., when $\kappa = (1-\lambda)/(r^{-1}-\lambda)$, the dynamic map and 45 degree line coincide so that any action in $\mathcal{A}$ is a potential rest point. Besides this singular case, investment converges to $\bar{a}$ when the intercept is positive and converges to zero when it is negative. Hence investment can only be constant at the boundary of the action set.

If marginal costs are low, so that $\kappa < (1-\lambda)/(r^{-1}-\lambda)$, agents are tempted to raise their investment above that of the rest point $\hat{a}$. Hence $\hat{a}$ is sustainable only if such deviations are not feasible, that is if $\hat{a} = \bar{a}$.[8] This solution is also efficient because the cost parameter $\kappa < 1$ while marginal productivity is one.

Conversely, when marginal costs are high ($\kappa > (1-\lambda)/(r^{-1}-\lambda)$), reputational concerns are too weak and $\hat{a} = 0$ is the only incentive-compatible steady-state. This market failure arises even though consumers update their beliefs in each period. It is therefore of a different nature than the one in Holmström's (1999) model which was caused by the absence of learning in the long-run. Another important difference is that risk aversion

[7] Eq. (10) is a particular case of the recursive incentive constraint (13) where $g(a) = \kappa a$. The posterior m does not affect the ratio on the LHS of (10) because utility is CARA.

[8] Formally, the incentive constraint (9) does not have to hold as an equality at the bounds of the feasibility set. When $\hat{a} = \bar{a}$ and $\kappa < (1-\lambda)/(r^{-1}-\lambda)$, the RHS exceeds the LHS of (9) and the agent is constrained by the requirement that $a \leq \bar{a}$.

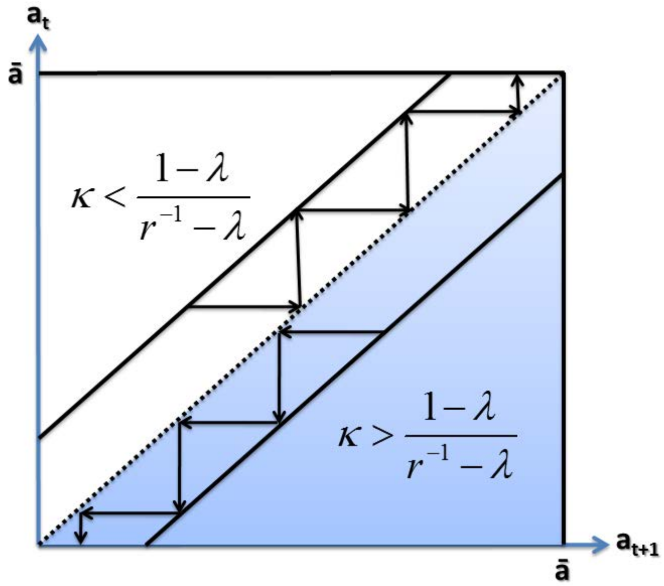

Figure 1: PHASE PORTRAITS FOR TWO DIFFERENT VALUES OF κ WHEN $g(a) = \kappa a$.

can give rise to deterministic solutions where a_t^* varies over time. In particular, welfare enhancing cycles of period 2 can be sustained.

Proposition 1 *Consider cases where costs are linear, so that* $g(a) = \kappa a$ *with* $a \in [0, \bar{a}]$. *Assume that* $\kappa \in \left(\frac{1-\lambda}{r^{-1}-\lambda}, 1\right)$ *and* $\bar{a} \geq \log\left(\frac{\kappa\left(1-(r\lambda)^2\right)}{(1-\lambda)r} - r\lambda\right) / [\gamma(1-\kappa)]$. *Then the steady-state* $\hat{a} = 0$ *is Pareto dominated by 2-period cycles in which investment oscillates between* 0 *and* $\bar{a}$.

Cycles arise because the discount factor fluctuates procyclically. High $a_t = \bar{a}$ entails above-normal output and consumption, leading to a low marginal utility $U'(c_t)$. The opposite is true next period as $a_{t+1} = 0$ delivers low consumption and relatively high marginal utility $U'(c_{t+1})$. Thus $E_t[U'(c_{t+1})/U'(c_t)]$ is large today, and this justifies the higher investment in reputation building. A similar but opposite mechanism operates in the next period, making low investment optimal. The oscillations of the discount factor capture the willingness of the agent to smooth consumption. Since the benefits of his action only accrue in the following period, he finds it optimal to transfer resources from good to bad times by overinvesting.

It is insightful to rewrite the incentive constraint at the upper-bound $\bar{a}$ as[9]

$$\kappa \leq \frac{1-\lambda}{\lambda}\left(\sum_{s\in\{1,3,5,\ldots\}}^{\infty}(r\lambda)^{s}\frac{U'(0)}{U'((1-\kappa)\,\bar{a})}+\sum_{s\in\{2,4,6,\ldots\}}^{\infty}(r\lambda)^{s}\right)$$
$$=\frac{r(1-\lambda)}{1-(r\lambda)^{2}}\left[\exp\left(\gamma(1-\kappa)\,\bar{a}\right)+r\lambda\right]. \tag{12}$$

Reputational returns are on the right-hand side of (12), and they are increasing in investment $\bar{a}$ because it raises the ratio of marginal utilities between bad and good times. Quite intuitively, smaller oscillations become sustainable when the power of incentives is strengthened. Take for example an increase in the discount factor β. It raises r as patient agents tend to be more concerned by their reputation. Hence the required wedge between marginal utilities is decreasing in β which allows the model to generate cycles for lower values of $\bar{a}$. A similar mechanism is triggered by changes in the degree of risk aversion because the effective rate of time preference r is increasing in γ. This effect is reinforced by the positive impact that γ has on the curvature of the utility function. When agents are more risk averse, similar oscillations in consumption generate greater swings in the discount factor, which lowers the value of $\bar{a}$ that restores incentive compatibility.

Asymmetric cycles.—The model can generate cycles with more than two states. As an illustration, Fig. 2 describes incentive-compatible cycles of period 3 where investment grows for two successive periods and then drops back to its initial level. This is the simplest instance of asymmetric cycles featuring protracted booms and sudden busts. A tendency for a time series to show large negative growth rates followed by several smaller positive growth rates is known as "steep asymmetry." The empirical counterpart of Fig. 2 is Fig. 8 which shows that steep asymmetry appears to be present in the U.S. consumption series. Moreover, the long-run frequency distribution of a_t places equal weight on the points 0, 0.71, and 1, and thus has longer left tail. A tendency for a detrended time series to have negative skew is known as "deep" asymmetry and it too is present in U.S. data.[10]

The longer period of growth is sustainable because investment costs κ are low, i.e., less than $(1-\lambda)/(r^{-1}-\lambda)$.[11] As explained before, when this inequality holds, the steady-state

[9] See proof of Proposition 1 for a derivation of (12).

[10] See Fig. 4 in Jovanovic (2006) for a graphical distinction between the two types of asymmetry, and Fig 1 for evidence of deep asymmetry in the GDP and industrial production series.

[11] By contrast, asymmetric cycles with protracted slumps and sudden booms can be constructed when the steady-state is inefficient, so that $\kappa > (1-\lambda)/(r^{-1}-\lambda)$.

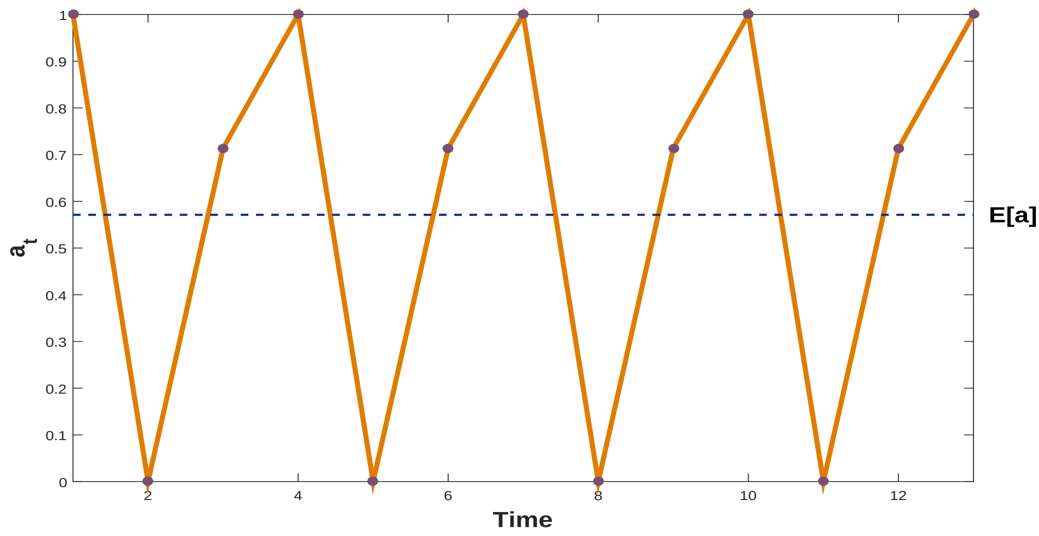

Figure 2: ASYMMETRIC CYCLES WITH CARA UTILITY AND LINEAR COSTS, $g(a) = \kappa a$. PARAMETERS: σ_ε=.1, σ_ν=.2, β=.8, γ=4, κ=.75, $\mathcal{A}$=[0,1].

is efficient because firms always find it profitable to raise their investment. But now they are discouraged from doing so by the fact that the marginal utility of consumption will be lower tomorrow than it is today. We show in the technical Appendix A.5 that this feature gives rise to asymmetric cycles similar to the one depicted in Fig. 2 whenever: (i) $\bar{a}$ is high enough, and (ii) $\kappa \in (\underline{\kappa}, (1-\lambda)/(r^{-1}-\lambda))$ for some $\underline{\kappa} > 0$. The logic of the proof can be extended to cycles with more than 3 states. Instead of investigating such variations, we turn our attention to cost functions that are not linear.

3.2 Convex costs

Assume now that $g(a)$ is strictly convex with $g'(0) = 0$. This ensures that, when $\mathcal{A} = \mathbb{R}_+$, the incentive constraint (7) admits an interior solution for a^*. Thus we can substitute in the expression of $g'\left(a^*_{t+1}\right)$ on the right-hand side of (7) so as to obtain a recursive FOC

$$g'\left(a^*_t\right) = \beta E_t \left[\frac{U'\left(c_{t+1}\right)}{U'\left(c_t\right)}\right] \left(1 - \lambda + \lambda g'\left(a^*_{t+1}\right)\right). \tag{13}$$

The recursive incentive constraint highlights that adding a unit of investment to a^*_t has two benefits:

(i) It raises $t+1$ earnings by $(1-\lambda)$, and (*ii*) it enables the firm to reduce its investment in period $t+1$ by λ in order to undo the deviation, thereby restoring the reputation associated to the equilibrium path in $t+2$ and beyond.

Both benefits are converted into today's utils through multiplication by the stochastic discount factor $\beta E_t \left[U'(c_{t+1})/U'(c_t)\right]$. When the utility function is CARA, (13) is equivalent to

$$g'(a_t^*) = r \exp\left(\gamma \left[a_t^* - g(a_t^*) - \left(a_{t+1}^* - g(a_{t+1}^*)\right)\right]\right)\left(1 - \lambda + \lambda g'\left(a_{t+1}^*\right)\right). \tag{14}$$

Setting a_{t+1}^* equal to a_t^* yields the rest point solution

$$\hat{a} = g'^{-1}\left(\frac{1-\lambda}{r^{-1}-\lambda}\right).$$

Efficient investment would require instead that $a = g'^{-1}(1)$. Thus, when r is smaller than one, $\hat{a} < g'^{-1}(1)$ and investment at the steady-state is suboptimal because costs are paid up-front, whereas reputational benefits accrue slowly over time. This is why the gap between optimal and actual investment is larger when agents are more impatient.

Stability of the steady-state.—Under risk neutrality, the steady-state is always unstable and any other action than $\hat{a}$ generates diverging trajectories.[12] In other words, $\hat{a}$ is the unique rational expectation solution. By contrast, when the agent is risk averse, the steady-state can be locally stable. Then, for any initial action a_0 in the neighborhood of $\hat{a}$, the solution path a_t converges back to $\hat{a}$ so that the equilibrium is not unique. The condition under which equilibrium multiplicity arises is laid-out in Proposition 2.

Proposition 2 *Assume that $g(a)$ is strictly convex with $g'(0) = 0$. Then the steady-state $\hat{a}$ is locally stable, and the model's equilibrium is indeterminate, if and only if*

$$\left(\frac{1-\lambda+2\lambda g'(\hat{a})}{1-\lambda+\lambda g'(\hat{a})}\right)\frac{g''(\hat{a})}{g'(\hat{a})} < 2\gamma(1-g'(\hat{a})). \tag{15}$$

When the coefficient of absolute risk aversion γ goes to zero, condition (15) is violated and the steady-state is unstable.[13] The impact of γ, as measured by the term on the

[12] See Fig. 3 for an illustration of the model's dynamics under risk neutrality.

[13] This is not obvious from (15) because γ affects $\hat{a}$ through its impact on r. However, since r converges to β when γ goes to zero, $\lim_{\gamma \to 0} \hat{a}(\gamma) = g'^{-1}\left((1-\lambda)/\left(\beta^{-1}-\lambda\right)\right)$. Hence the LHS of (15) converges to a positive value, whereas the RHS goes to zero, showing that (15) never holds in the limit.

right hand side of (15), is proportional to $1 - g'(\hat{a})$. Thus risk aversion is irrelevant when $1 = g'(\hat{a})$, that is when the steady state and the first best coincide. Intuitively, the curvature of the utility function matters only to the extent that changes in investment affect consumption. At the first best, benefits and costs are set equal and a marginal increase in a leaves consumption unchanged. But we have seen that $\hat{a}$ is suboptimal whenever agents discount future consumption. Then raising investment is beneficial as it increases output and reduces the agent's marginal utility. This makes it more attractive to invest today in order to raise tomorrow's consumption, which explains why an increase in investment is followed by a drop and a return to the steady state when the utility function has enough curvature.

Deterministic cycles.—Besides the continuum of converging paths in the neighborhood of the steady-state, the model also features global cycles of period 2.

Proposition 3 *When the costs function is quadratic, i.e., $g(a) = a^2/2$, deterministic 2-period cycles are sustainable whenever the steady-state is locally stable.*

The intuition why cycles are sustainable is the same as in the model with linear costs: Procyclical cycles in the discount factor make it optimal to invest during booms and to reduce expenses during busts. This mechanism is closely related to the one rendering the steady-state locally stable. Proposition 3 shows that the two phenomena are indeed driven by the same forces. Combining Propositions 2 and 3, we see that cycles arise whenever the agent is sufficiently risk averse.

Looking at the phase portrait generated by the incentive constraint (13) helps one understand how cycles arise. We highlight the impact of risk aversion by also illustrating the model's dynamics when agents are risk neutral. Then, as shown in the left panel of Fig. 3, the mapping between a_t^* and a_{t+1}^* is linear with a slope greater than one. Hence the rest point $\hat{a}$ is the only rational expectation solution that does not generate diverging action paths.

We introduce risk aversion in the right panel of Fig. 3. The dynamic map solves the recursive equation (14) with a quadratic cost function $g(a) = a^2/2$, and is therefore equivalent to

$$\exp\left(-\gamma(a_t^* - a_t^*/2)\right) a_t^* = r \exp\left(-\gamma\left(a_{t+1}^* - a_{t+1}^*/2\right)\right)\left[1 - \lambda + \lambda a_{t+1}^*\right]. \qquad (16)$$

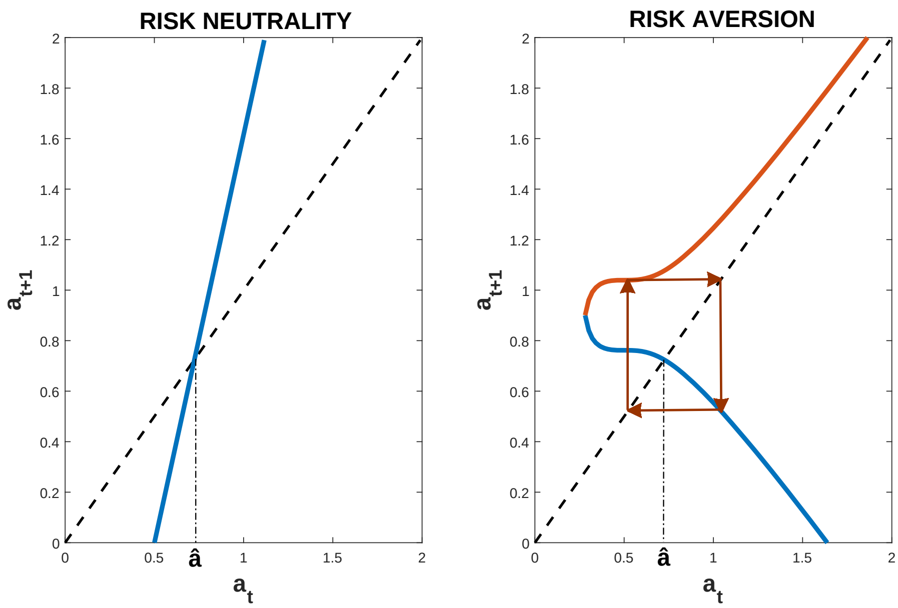

Figure 3: Phase portraits with linear and CARA utility. Parameters: $\lambda = .4,\ \beta = .9,\ \gamma = 4,\ g(a) = a^2/2$.

Let us first explain why small values for a_t^* can never be incentive compatible. Letting a_t^* go to zero, we see that the expression on the left-hand side of (16) also converges to zero. By contrast, the expression on the right-hand side of (16) has a positive minimum. By continuity, there always exists an $a_{min}^* > 0$ such that (16) cannot hold whenever $a_t^* \in [0, a_{min}^*]$. In economic terms, there is no expectation about a_{t+1}^* that sustains an investment level smaller than a_{min}^*. Intuitively, today's marginal loss converges to 0 whereas tomorrow's marginal returns are bounded below by $1-\lambda$, which explains why the dynamic map in Fig. 3 is empty close to the origin.

Besides this empty interval, we see that, instead of the one-to-one mapping prevailing under risk neutrality, the dynamic map is a correspondence that associates a pair of incentive compatible a_{t+1}^* to any a_t^*. The incentive constraint is satisfied by two different a_{t+1}^* because tomorrow's investment shifts the discount factor and marginal costs in opposite directions. Increasing a_{t+1}^* raises tomorrow's consumption, which lowers the discount factor and counteracts the increase in marginal costs $g'(a_{t+1}^*)$. When the elasticity of the discount factor with respect to a_{t+1}^* is higher than that of the marginal costs, it is possible to perturb a pair of sustainable actions and restore incentive compatibility by adjusting

a^*_{t+1} until its effect on the discount factor offsets the change in marginal costs.

For the economy to settle down at the rest point $\hat{a}$, the expectation-formation mechanism has to select the lower branch of the phase portrait. By contrast, 2-period cycles are *regime switching* equilibria since expectations oscillate between the upper and lower branch.

Other equilibria.—We have focused on deterministic cycles of period 2, but our model has a continuum of equilibria, i.e., a continuum of sequences $\{a_t\}$ solving (14). Instead of alternating selection of branches, one could assume other patterns of regime switching including sunspots.[14] We leave a comprehensive investigation of all equilibrium outcomes to further research and focus instead on the simplest form of regime switching cycles. We evaluate their ability to match aggregate business cycles in the next section where we embed our model into a general equilibrium setting.

4 Cycles in general equilibrium

We now move to general equilibrium. We keep the production technology (1) and the assumptions about (θ, ε) summarized by (2)-(5). We also keep assumptions A1 and A2 but we drop A3 which is not needed in GE, and replace A4 with the requirement that the economy is closed. Cycles are now in *aggregate* consumption and since there is no access to outside finance, agents cannot smooth them.[15]

4.1 Set-up

Firms' aggregate distribution.—We continue to assume that firms draw their θs from $\mathcal{N}(\bar{m}, \bar{\sigma}_\theta^2)$, with $\bar{\sigma}_\theta^2$ being equal to the stationary precision given in (4). Then posterior precision remains constant over time so that a firm's individual state is just m.[16]

[14]There are of course models in which an extrinsic or even intrinsic shock acts so as to shift an economy from one regime to another (e.g., Hamilton, 1990). See also Christiano and Harrison (1999) for a model with stochastic regime switching where consumption satisfies the Euler equation. Here we assume that such switches of regime alternate in a deterministic manner.

[15]This assumption fails for a small open economy facing an exogenous interest rate.

[16]Restricting our attention to stationary priors ensures that precision, and thus investment, do not vary across firms of different vintages.

To obtain a stationary distribution for m, we also assume that firms are randomly hit by a death shock with probability δ per period. As $\sigma_{\theta,t} = \bar{\sigma}_\theta$ is a constant, (5) implies that a firm's state m follows a random walk with constant incremental variance. Since a firm's lifetime is a geometrically distributed random variable and an additional period of life adds $(1-\lambda)^2(\sigma_\varepsilon^2 + \bar{\sigma}_\theta^2)$ to the variance of m, the *time-invariant* distribution of types $\Upsilon(m)$ is a mixture of normal distributions with mean $\bar{m}$ and stationary variance

$$\mathrm{Var}(m) = (1-\lambda)^2\left(\sigma_\varepsilon^2 + \bar{\sigma}_\theta^2\right)\sum_{t=1}^{\infty} t\delta(1-\delta)^t = (1-\lambda)^2\left(\sigma_\varepsilon^2 + \bar{\sigma}_\theta^2\right)\frac{1-\delta}{\delta}.$$

Product markets.—The risk-averse household fully diversifies its purchases of goods over firms so as to eliminate the risk in the random variable $\theta - m + \varepsilon$. Any firm's product is marginal to the family, and it pays up front the expected value of output for it. Since all market participants observe the history of outputs, and the sequence of equilibrium actions a^* is common knowledge, firm m has revenue $m + a^*$ and profit

$$\pi^m(a) = m + a^* - g(a), \tag{17}$$

where $g(\cdot)$ is measured in output units. The firm pays its profits to the manager's family.

Preferences and assets.—The only store of value are one period bonds in zero net supply. The price of the bonds then is

$$\frac{1}{1+r_t} \equiv \beta\frac{U'(c_{t+1})}{U'(c_t)},$$

which is today's value of a unit of income next period. Today's value of a unit of income 2 periods from now is $\beta^2\frac{U'(c_{t+2})}{U'(c_t)}$, and so on. Thus in units of current consumption, the value of any income stream $\{w_s\}_{s=t+1}^{\infty}$ is $\sum_{t+1}^{\infty}\beta^{s-t}\frac{U'(c_{t+s})}{U'(c_t)}w_s$.

The representative family has the option of borrowing and lending at the risk-free rate, but rejects that option because every family is identical. Then the consumption of the family is simply the average profit of its members

$$c_t = \int_{\mathbb{R}} (m + a - g(a))d\Upsilon(m) \tag{18}$$

The stationary distribution Υ exists because of the death rate δ among the managers. A manager's reputation dies with him and he is replaced by a newborn drawn from the stationary distribution $\mathcal{N}(\bar{m}, \bar{\sigma}_\theta^2)$.

A firm's decision problem.—A manager maximizes the discounted value of his family's consumption. Since the family owns many firms, idiosyncratic variations in efficiency and reputation wash out. This is why we no longer need CARA preferences to derive deterministic policies.[17]

Definition of equilibrium.—An equilibrium consists of functions $(c_t, a_t^*, r_t, \pi_t^m)$, where a^* solves (7) and (c, r, π) clear the asset and goods markets. That is, c, r and a^* are deterministic sequences, whereas π depends on t and m. A simpler definition of equilibrium uses only the firms' first-order condition and the bond-price equation:

Definition 4 *An equilibrium path is a pair of sequences $\{a_t^*, r_t\}_{t=0}^{\infty}$ that solves the Incentive-Constraint and bond-pricing equations*

$$(IC) \ : \ g'(a_t^*) = \frac{1-\lambda}{\lambda} \sum_{s=t+1}^{\infty} (\rho\lambda)^{s-t} \frac{U'(c_s)}{U'(c_t)}, \tag{19}$$

$$(AP) \ : \ \frac{1}{1+r_t} = \beta \frac{U'(c_{t+1})}{U'(c_t)}, \tag{20}$$

where $c_t = \bar{m} + a_t^* - g(a_t^*)$ and $\rho \equiv \beta(1-\delta)$. The economy always has a rest-point solution $(\hat{a}, \hat{c}, \hat{r})$ where

$$\hat{a} = g'^{-1}\left(\frac{1-\lambda}{\rho^{-1}-\lambda}\right), \ \hat{c} = \bar{m} + \hat{a} - g(\hat{a}), \ \text{and} \ \frac{1}{1+\hat{r}} = \frac{\rho}{1-\rho}\hat{c}. \tag{21}$$

If $U(\cdot)$ is linear then $(a_t, c_t, P_t) = (\hat{a}, \hat{c}, \hat{r})$ for all t.

The model has a unique equilibrium when consumers are risk neutral. By contrast, when $U''(\cdot) < 0$, the model has other equilibria, some of which are cyclical.

4.2 Stability and cycles

The structure of incentives is not fundamentally affected by whether the problem is formulated in partial or general equilibrium. Hence the results on stability and cycles presented in Section 3 continue to hold in our macro setting. If anything, they hold more generally

[17] The exogeneity of aggregate consumption implies that the manager's payoff is linear in investment. Since this is also true in Holmström (1999), both models share important features. In particular, optimal investment is deterministic and, as shown in the technical Appendix A.4, the necessary condition (19) is also sufficient.

because the representative family runs many firms and is insured against firm-level shocks. The lack of idiosyncratic risk implies that: (i) utility does not have to be CARA for optimal investment to be deterministic; (ii) steady-state consumption does not depend on the degree of risk aversion.

Stability.—Using (i) we can generalize Proposition 2 to arbitrary utility functions. Replacing the coefficient of absolute risk aversion, γ, with a potentially consumption-dependent measure, $A_U(c) \equiv -U''(c)/U'(c)$, we find that the steady-state $\hat{a}$ is locally stable[18] if and only if

$$\left(\frac{1-\lambda+2\lambda g'(\hat{a})}{1-\lambda+\lambda g'(\hat{a})}\right)\frac{g''(\hat{a})}{g'(\hat{a})} < 2A_U(\hat{a}-g(\hat{a}))(1-g'(\hat{a})). \tag{22}$$

Condition (22), aside from being more general than its partial equilibrium counterpart (15), is also more informative because $\hat{a}$ does not depend anymore on risk aversion. This is why (22) allows us to derive parametric restrictions under which the steady-state is stable, as done in Proposition 5 for the most common classes of utility functions.[19]

Proposition 5 *Assume that the utility function is either CARA, i.e.,* $U(c) = -\exp(-\gamma c)$, *or CRRA, i.e.,* $U(c) = c^{1-\gamma}/(1-\gamma)$. *Then there exists a function* $\tilde{\gamma}(\rho, \lambda)$ *such that the steady-state is locally stable whenever* $\gamma > \tilde{\gamma}(\rho, \lambda)$.

Cycles.—As in partial equilibrium, the model features global cycles of period 2. Finally, as in partial equilibrium, local stability and global cycles are concomitant phenomena. Propositions 5 and 6 confirm the intuition that cycles become sustainable when risk aversion is strong enough.

Proposition 6 *Assume that: (i) the equilibrium conditions in Definition 4 with* $\bar{m} = 0$ *are satisfied; (ii) the utility function is either CARA or CRRA; (iii) and the cost function is quadratic, i.e.,* $g(a) = a^2/2$. *Then deterministic 2-period cycles are sustainable whenever the steady-state is locally stable.*

Cyclical properties of key variables—Among other properties of cycles, the following have some empirical support.

[18] See proof of Proposition 2.

[19] See technical Appendix A.6 for an analysis of the effect of ρ on equilibrium stability under both CARA and CRRA preferences.

1. **Output is more volatile than consumption but less than investment:** Aggregate output a is by definition equal to the sum of consumption $c = \bar{m} + a - g(a)$ and investment $g(a)$. Thus it is certainly smoother than investment since the latter is a convex function of a. Consumption being given by the difference between the two, it has to fluctuate less than output.

2. **Product quality is pro-cyclical:** Product quality is just a plus the average value of m and, since the latter is fixed, quality is procyclical as recently documented by Broda and Weinstein (2010) and Jaimovich, Rebelo and Wong (2015).

3. **Interest rates and consumption growth are positively related at low frequencies:** The model implies that the rate of interest at t should be positively correlated with consumption growth between t and $t+1$. Related to this, Brainard, Nelson, and Shapiro (1991) and Parker and Julliard (2005, Fig. 2) found that the Consumption-based Asset-Pricing Model performs better at a horizon of 2 or 3 years. Our own calculations and figures, shown in Appendix A.3, provide more detailed empirical support for correlation at business-cycle frequencies and at the 5-year frequency.

5 Calibration to micro data

5.1 Parametrization

We now show that the model produces realistic cycles when parameters are chosen to fit micro data. The most important of these is period length, which depends on how fast a firm's history becomes public. We discuss this parameter last after all other dimensions of the model have been parametrized.

Choosing $(\sigma_\varepsilon, \sigma_\nu)$.—These parameters determine the volatility of firms' sales. Since the same amount a_t^* of input is used by all firms, they actually correspond to production units and changes in sales are observationally equivalent to changes in revenues based Total Factor Productivity (TFP).

Estimates of the volatility of TFP are readily available in the empirical literature on industry dynamics. Castro, Clementi and Lee (2015) use the Annual Survey and Census of Manufactures, for the years 1972 through 1997, to estimate the following equation

$$z_{i,t+1} = \nu z_{i,t} + \mu_i + \boldsymbol{\gamma_t}\mathbf{X}_{i,t} + \omega_{i,t}, \tag{23}$$

where $z_{i,t}$ is the log-TFP for plant i at time t as estimated from a first stage regression of real sales on capital, labor and materials. $\mathbf{X}_{i,t}$ is a vector of observables that are systematically related to innovations in TFP.[20] Equation (23) is the empirical counterpart[21] of

$$m_{t+1}^i = \lambda m_t^i + (1-\lambda)\,\theta_{\tau^i}^i + (1-\lambda)\left[\sum_{s=\tau^i+1}^{t} \nu_s^i + \varepsilon_t^i\right], \tag{24}$$

where τ^i is the vintage of firm i.[22] Since the survival probability of firms follows a geometric distribution, the cross-sectional variance of $\omega_{i,t}$ is given by

$$\sigma_\omega^2 = (1-\lambda)^2\left[\sigma_\nu^2 \sum_{t=1}^{\infty} t\delta\,(1-\delta)^t + \sigma_\varepsilon^2\right] = (1-\lambda)^2\left[\sigma_\nu^2 \frac{1-\delta}{\delta} + \sigma_\varepsilon^2\right] .$$

As λ only depends on σ_ε and σ_ν, one can use the autocorrelation coefficient ρ and the conditional standard deviation of TFP growth to identify both volatility coefficients. More precisely, we estimate $(\sigma_\varepsilon, \sigma_v)$ by solving two equations involving TFP and its autocorrelation as follows:

$$\begin{aligned} \text{Cond. Std.}\,(TFP) &= \frac{\sigma_\varepsilon^{-2}}{\sigma_\varepsilon^{-2} + \bar{\sigma}_\theta^{-2}} \sqrt{\sigma_\nu^2 \frac{1-\delta}{\delta} + \sigma_\varepsilon^2}, \\ \text{Autocorr.}\,(TFP) &= \frac{\bar{\sigma}_\theta^{-2}}{\sigma_\varepsilon^{-2} + \bar{\sigma}_\theta^{-2}} = \lambda, \end{aligned} \tag{25}$$

where $\bar{\sigma}_\theta^{-2}$ is given in (4). Castro, Clementi and Lee (2011) find that the conditional standard deviation of TFP across all manufacturing plants is equal to 20.53% while the autocorrelation coefficients of TFP are centered around 0.5. Combining these two moments with the average exit rate of 5% for US firms in the 1990's (see Bartelsman, Haltiwanger and Scarpetta, 2004), we find that $\sigma_\varepsilon = .243$ and $\sigma_\nu = .172$.

[20] Castro, Clementi and Lee (2015) control for the industry in which firms operate as well as their size and age.

[21] The dependent variable $z_{i,t+1}$ in Eq. (23) can be interpreted as a log-linear approximation of its theoretical counterpart m_{t+1}^i. The approximation being taken around the average value $\bar{m}$, it will be accurate when $\bar{m}$ is close to one, a requirement that is satisfied by our calibration since our preferred value for $\bar{m}$ is 1.06.

[22] Equation (24) follows reinserting $x_t^i = \theta_t^i + \varepsilon_t^i$ and $\theta_t^i = \theta_{t-1}^i + \nu_t^i$ into the law of motions of beliefs (5).

Table 1: Baseline Parameters

Parameter	Interpretation†	Moment/Source
$\gamma = 4$	Relative risk aversion	Standard
$\delta = .05$	Exit rate of firms	Bartelsman *et al.* (04), J&P (85)
$\beta = .97$	Annual discount factor	Jarrell & Peltzman (1985)
$\bar{m} = 1.06$	Average firm efficiency	GDP loss from recessions = 8%
$\sigma_\varepsilon = .243^*$	Volatility of firms' output	Std. dev. plant lnTFP=20%
$\sigma_\nu = .172^*$	Volatility of firms' efficiency	Autocorr. plant lnTFP=0.5

Notes. †When applicable, parameter values are for yearly frequency. *Inferred jointly as described above.

Choosing $\bar{m}$.—Having solved for $(\sigma_\varepsilon, \sigma_v)$, we relax the normalization to zero of the average firm efficiency $\bar{m}$. Adding this extra degree of freedom allows the model to generate recessions of similar magnitude and frequency to the ones observed in the data. The NBER measure shows that, from 1854 to 2009, the duration between US recessions averaged 56 months with a loss in GDP from peak to through of around 8%.[23] For a quadratic cost function, $g(a) = a^2/2$, these two targets are perfectly matched when the average efficiency of firms $\bar{m} = 1.06$.[24] For this average efficiency, the normality of output is not a concern anymore because the share of firms with negative valuation is negligible.[25]

Choosing (ρ, β).—In today's consumption, the value of a firm with market posterior m is given recursively by

$$p_t^{m_t} = \rho E_t \left[\frac{U'(c_{t+1})}{U'(c_t)} \left(m_{t+1} + a_{t+1}^* - g\left(a_{t+1}^*\right) + p_{t+1}^{m_{t+1}} \right) \mid m_t \right]. \tag{26}$$

This would be the stock price of such a firm if it were traded. Jarrell and Peltzman (1985) estimate the stock-price impact of a product recall, and they estimate the "direct costs "of a product recall by assuming that all of the defective units become worthless on recall. We interpret a recall as a negative surprise amounting to an output drop equal to the direct cost. The direct cost is then a subtraction from today's dividend, and eq. (26) along with the martingale property of m_t imply that p_t^m should fall by an amount

[23] Source material available at http://www.nber.org/cycles.html.
[24] See discussion of the bifurcation point below.
[25] In our benchmark calibration, only one out of ten thousand firms has negative value.

$\rho / (1-\rho)$ where $\rho = \beta(1-\delta)$. Jarrell and Peltzman, p. 521, estimate this ratio to be 12 which is matched by our values for β and δ. Our choice of parameters is summarized in Table 1. Since we refer to TFP in annual data, the parameters are also reported at yearly frequencies.

5.2 Period length

The speed of learning determines the length of the period and, hence, time between booms and recessions. For the set of parameters summarized in Table 1, the bifurcation point $\tilde{\rho}$ has a value of 0.708. Whether or not this is a reasonable discount factor depends on the frequency at which consumers update their beliefs, i.e., the speed with which information about a firm's performance leaks out to the general public.

Let n denote the duration of the model's period in years, so that $y_t = \theta_t + a_t + \sum_{j=1}^{n} \varepsilon_{t+j/n}$ and $\theta_t = \theta_{t-1} + \sum_{j=1}^{n} \nu_{t-1+j/n}$. The standard deviations of output and productivity are both linearly increasing in the period's duration and so is the dispersion of posteriors $\bar{\sigma}_\theta$, since its expression in (4) is homogeneous of degree 1 in $(\sigma_\nu^2, \sigma_\varepsilon^2)$. Hence, raising n leaves the gain parameter λ unchanged. In other words, the speed of information acquisition affects the model's solution solely through its impact on the discount factor ρ. It follows that $\tilde{n} = log(\tilde{\rho})/log\left(\beta(1-\delta)\right) = 4.27$ is the period length above which endogenous cycles can occur.

Fig. 4 illustrates the effect that the speed of learning – i.e., period length – has on cycles. Investment in the low and high phases of the cycle, as well as in the steady-state, is reported in the left side panel. As the period duration rises the amplitude of the oscillation increases and, as can be seen in the right side panel, generates larger cycles in consumption.

Using our yearly estimates for both β and δ reported in Table 1, one can directly infer the yearly updating frequency. The time distance between peak and trough is 56 months or $56/12 = 4.7$ years. Since this exceeds the threshold value of $\tilde{n} = 4.27$, the model is capable of generating cycles at this frequency. Then the 4.7-year discount factor is $\tilde{\beta} = .97^{56/12} = 0.87$, and the 4.7-year exit rate is $\tilde{\delta} = 1 - (1-0.05)^{\frac{56}{12}} = 0.21$. At this updating frequency, cycles raise average reputational investment by 4.1%. The output gain nearly compensates the cost of income volatility as welfare decreases by only 0.7%, a loss that is equivalent to a compensating variation of 0.25% in consumption.

Micro evidence shows reputation-diffusion lags can be on either side of the 4.7 years

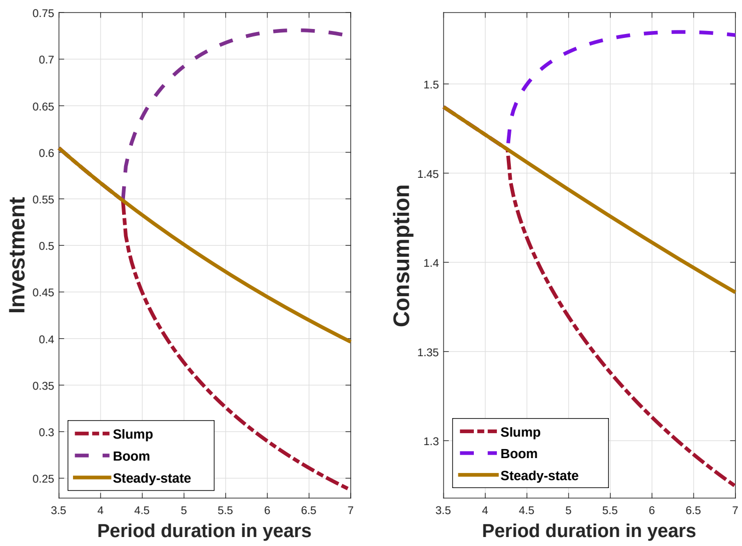

Figure 4: Investment and Consumption as a function of period length Lag. Parameters reported in Table 1.

threshold, depending on what type of product or service is involved. In other words, the 4.7 diffusion lag that the macro data call for is well within the range of what the micro data say. We now discuss these pieces of evidence and explain how they can be gathered from micro data.

5.2.1 Product age at recall as a measure of period length

Using data on automobile and drug recalls, Jarrell and Peltzman (1985) find that a firm's stock price plunges when it recalls one of its products, roughly at the same time. Thus information about the firm's quality (implied by the recall) spreads no faster and no slower than the product recall.[26]

We updated the auto recall data from the Department of Transportation, obtaining 48,000 cases. We measure the difference between the "start of manufacture" of the product and the product's recall date to be 4.14 years. This number is within $4.14/4.7 = 88\%$ of the peak-to-trough business cycle distance. Details are in Appendix A.2 with Fig. 7 showing the frequency distribution of the ages of the products – there is considerable

[26]By contrast, the price of a takeover target rises several months ahead of the takeover announcement.

heterogeneity in the products' ages at recall, suggesting differences in the speed at which information spreads.

Cyclical behavior of recalls.—If low quality products are more likely to be produced when a is low, i.e., in a recession, then product recalls should be higher in the boom, one period later. That is what our model predicts: Recalls and consumption should be positively correlated, and they are. We construct a time series of recalls and correlate it with aggregate consumption – both series logged and detrended. Over the period 1978-2007 the correlation is 0.30, and the series are shown in Fig. 6 in Appendix A.2.

5.2.2 Diffusion of innovations

Product-adoption lags are estimated in the marketing literature and a popular formulation is the Bass model which can be summarized by is market-penetration function F, with $F = 1$ denoting full penetration

$$\frac{dF}{dt} = (p + qF)(1 - F).$$

Sultan, Farley and Lehmann (1990) look at a broad range of product innovations and estimate that $p = 0.03$, and $q = 0.38$, leading to a solution to the equation $F(T) = 1/2$ of $T = 6.5$. Thus for the average product innovation, 50% of potential customers have adopted it after 6.5 years.

Process-diffusion lags are slightly longer. Consistent with the evidence on technology adoption (Comin and Hobijn, 2010; Cox and Alm, 1996), Anzoategui *et al.* (2015) calibrate the mean technology diffusion lag to 7 years for the U.S. These delays would be caused by a combination of awareness lags and switching costs. Such estimates overstate the lags in customer response at the intensive margin – a customer can just buy less without switching to a new supplier.

5.2.3 Sluggish growth of new plants

Foster, Haltiwanger and Syverson (2016) find that new plants have much lower demand than incumbents in their industries, in spite of being at least as efficient as incumbents. They attribute it to sluggishness in new firms' customer base, and argue that it is unlikely that capital adjustment costs could explain the bulk of the 15+ years that it takes for plants in their sample to close their measured idiosyncratic demand gaps. Entrants (businesses 0-4 years old) have demand that is about 60% lower than that of "old" incumbents (i.e., businesses over 15 years old). By the time plants are "medium" aged (10-14 years old), their demand is only about 30% lower than incumbents. So it appears that it takes 5-10 years for half of the demand gap to be closed.

In our model entrants come in as a representative draw of θ from the stationary distribution. Eq. (24) then gives the law of morion of beliefs conditional on the initial draw of θ, but since θ is a random walk this too is the expected long-run value of the firm's quality. Then our estimate of $\lambda = 0.5$ implies that the half-life between starting belief and long-run belief is just one period, which for us is close to 5 years.[27]

5.3 Micro and macro uncertainty

It is well documented (e.g., Comin and Philippon, 2005) that macro and micro volatility moved in opposite directions during the great moderation. According to our model, this negative correlation can be explained by an increase in the noisiness of the technology. Although aggregate cycles are driven by the learning process, and thus depend on the degree of idiosyncratic uncertainty, higher volatility at the micro level does not necessary translate into greater cycles at the macro level. The sign of the relationship depends on whether higher firm volatility is due to more innovations in fundamental productivity, as measured by σ_ν, or to more noisiness in output, as measured by σ_ε.

These two sources of idiosyncratic uncertainty have opposite effects on λ, i.e., on the inertia of the updating process. More weight is put on recent observations when σ_ν is higher because types are more volatile. Then reputation is more responsive to investment which strengthens the power of incentives. On the other hand, σ_ε lowers the power of incentives because recent observations are less informative when output quality is noisier. This explains why the two parameters affect steady-state investment in opposite directions.

Proposition 7 *The steady-state level of investment $\hat{a}$ is increasing in the volatility of productivity σ_ν but decreasing in the volatility of output σ_ε.*

[27] In eq. (5) a deviation x raises beliefs next period by the factor $1 - \lambda$. The value was obtained not from the peak-trough interval lengths, but from the estimate in eq. (25).

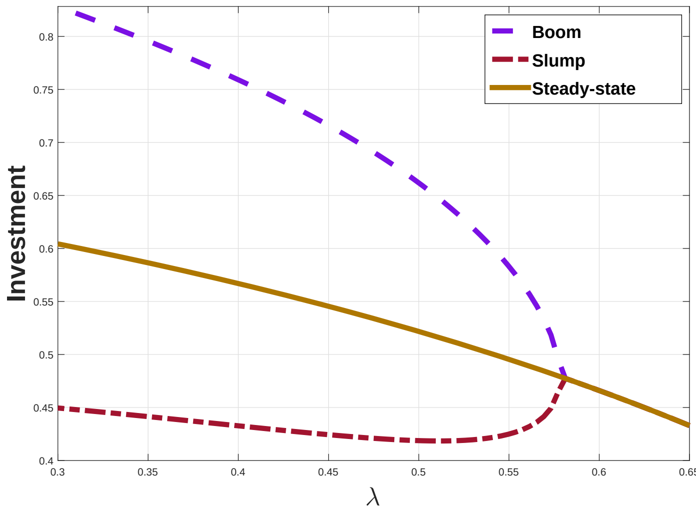

Figure 5: INVESTMENT AS A FUNCTION OF $\lambda = 1 - 2/\left(\sqrt{1 + 4\frac{\sigma_\varepsilon^2}{\sigma_\nu^2}} + 1\right)$. PARAMETERS REPORTED IN TABLE 1.

The proposition is illustrated in Fig. 5. Firms' reputations become more sluggish as λ increases which lowers their reputational concerns, resulting in less investment, and thus consumption, at the steady-state. Fig. 5 also shows that a similar conclusion holds for cycles: their amplitude declines when λ goes up. And since λ is increasing in σ_ε, more noisiness in output quality lowers macro-volatility but increases micro-volatility.

6 Related models

In contrasting to other work, it helps to recall why reputation, an intangible, differs from other types of capital, human or physical. Our model offers two reasons:

(i) investment in reputation raises *current* consumption, and

(ii) investment today does *not* raise future output or aggregate consumption.

We now discuss related models, with special reference to (i) and (ii).

1. *Learning by doing.*—LBD is the closest-related line of research because it shares with our model the feature that higher effort today raises an agent's income in the future. Here is a version of LBD that compares closely to our model.[28] Let k denote the stock

[28] In-depth discussions of LBD are in Chang, Gomes and Schorfheide (2002), Qureshi (2008), and Gunn

of firm-specific human capital and a the investment in human capital, again measured in terms of goods. Assume that

$$y = k + a - g(a) \quad \text{and,} \tag{27}$$

$$k' = (1-\delta)k + a, \quad \text{with} \quad \delta \in (0,1). \tag{28}$$

Property (i) does not hold because firms set a beyond the point where y is maximized[29] and therefore increasing a *reduces* current consumption instead of raising it. Property (ii) does not hold either because in (28), higher investment raises k', and so increases output and consumption tomorrow. Thus higher values of a lower the discount factor and LBD exerts a stabilizing force which prevents the emergence of deterministic cycles.

2. *Discount-factor shocks.*—Our model generates discount factor movements. One can also shock the discount factor exogenously as do Werning (2012), or Albertini and Poirier (2014), and obtain cycles with procyclical incentives to invest. But unless the shocked model features property (i), cycles would be mitigated by a rise in current $U'(c)$ that would arise as c drops.

3. *Other pecuniary externalities.*—Our model features a strategic complementarity in a induced by a pecuniary external effect. Cycles in Shleifer (1986) also originate in a pecuniary external effect, but one that is due to simultaneous product introductions by monopolists. One needs risk aversion to be low ($\gamma \leq 1$) for his model to yield cycles. In the same vein, Judd (1985) and Matsuyama (1999) feature cycles with alternating periods of competition and monopoly.

4. *Models with direct externalities.*—Christiano and Harrison (1999) analyze regime switching in a model whose geometric structure is related to ours and where, as in Benhabib and Farmer (1994), multiplicity of equilibria is driven by direct externalities in production.

5. *Echo effects and intertemporal substitution of consumption.*—When the age distribution of capital has spikes and when intertemporal substitution in consumption is high

and Johri (2011). In contrast to our paper, these models feature exogenous aggregate shocks.

[29] Current output is maximized when $a = g^{-1}(1)$. Thus the observation that investment exceeds its output-maximizing level directly follows from the firm's FOC

$$g'(a_t) = 1 + \sum_{s=t+1}^{\infty} \beta^{s-t} \frac{U'(c_s)}{U'(c_t)} \frac{\partial k_s}{\partial a_t} > 1. \tag{29}$$

enough, investment can have echo effects that take a long time to die out (See Boucekkine, Germain and Licandro, 1997; Mitra, Ray and Roy, 1990). Permanent investment cycles arise only if utility is linear. In a two sector model in which production of the consumption good is capital intensive compared to that of the capital good, an abundance of capital today raises current consumption and lowers investment and capital tomorrow, at which point the process is reversed. Benhabib and Nishimura (1985) show that this mechanism works if factor intensities differ sufficiently, and if the utility function is not too concave. These models all entail a counterfactual negative correlation between consumption and investment over time.

Finally, the literature on reputation concerns has seemingly not addressed cycles. Kondo and Papanikolaou (2013) study how the value of future business acts as a (productive) discipline device – a firm that expropriates the knowledge of a partner acquires a bad reputation and it is precluded from partnering with others in the future. One can imagine time-dependent punishments that would generate movements in aggregate activity.

7 Conclusion

We have shown that deterministic cycles may arise when contracts are incomplete – one cannot condition payment on output – and when products are experience goods. Calibrated to fit some micro facts, the model produces realistic cycles in terms of peak-to-trough movements in consumption and the spacing of time between recessions. The frequency of booms and recessions depends on the speed with which reputations spread – the slower the diffusion, the longer is the inter-arrival time of recessions.

For the mechanism to work the economy must be closed or, at the micro level, borrowing and lending must not be possible. A promising avenue for future research would be to relax these assumptions. Adding a storable commodity would allow agents to smooth consumption and thus partially dampen fluctuations. Then risk aversion should have an ambiguous effect on the sustainability of cycles since the consumption-smoothing motive is increasing in the curvature of the utility function. Oscillatory behavior will probably continue to arise for intermediate degree of risk aversion, especially if regime switches are allowed to be stochastic. Another interesting extension would be to let firms adjust the size of their observable output. Enriching the technology of production so that both quantity and hidden quality are optimally set would make it possible to embed our mechanism within otherwise standard macro models, thereby providing a new microfoundation for intangible capital as well as more elaborate tests for the relevance of reputation cycles.

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A Appendix

A.1 Proofs

Proof. Proposition 1: We are interested in 2-period cycles where $a_t^* = \bar{a}$ when t is even, and $a_t^* = 0$ when t is odd. Using the notation

$$S(a) \equiv \exp\left(\gamma\left(1-\kappa\right)a\right), \tag{30}$$

and taking into account the feasibility constraint $a \in [0, \bar{a}]$, we see that the Kuhn-Tucker conditions resulting from the incentive compatibility constraint (7) are

$$\kappa \leq \frac{1-\lambda}{\lambda}\left(\sum_{i\in\{1,3,5,\ldots\}}^{\infty}(r\lambda)^i S(\bar{a}) + \sum_{i\in\{2,4,6,\ldots\}}^{\infty}(r\lambda)^i\right) \quad \text{if } t \text{ is even so that } a_t^* = \bar{a},$$

and

$$\kappa \geq \frac{1-\lambda}{\lambda}\left(\sum_{i\in\{1,3,5,\ldots\}}^{\infty}(r\lambda)^i S(\bar{a})^{-1} + \sum_{i\in\{2,4,6,\ldots\}}^{\infty}(r\lambda)^i\right) \quad \text{if } t \text{ is odd so that } a_t^* = 0.$$

These two conditions can be re-written as

$$\frac{1}{S(\bar{a})} \leq f(\kappa, r, \lambda) \leq S(\bar{a}), \;\; \text{where } f(\kappa, r, \lambda) \equiv \kappa\left[\frac{1-(r\lambda)^2}{(1-\lambda)r}\right] - r\lambda. \tag{31}$$

The following cases have to be distinguished:

1. $\kappa \leq \frac{r^2\lambda-(r\lambda)^2}{1-(r\lambda)^2}$: Then $f(\kappa, r, \lambda) \leq 0$ and, since $S(\bar{a}) > 0$, the first inequality in (31) cannot be satisfied. Intuitively, the costs κ are so low that it is never optimal to set $a^* = 0$.

2. $\kappa \in \left(\frac{r^2\lambda-(r\lambda)^2}{1-(r\lambda)^2}, \frac{r(1-\lambda)}{1-r\lambda}\right]$: Then $f(\kappa, r, \lambda) \in (0, 1]$ and, using the definition of $S(a)$ in (30), we find that the first inequality in (31), $1/S(\bar{a}) \leq f(\kappa, r, \lambda)$, is satisfied whenever $\bar{a} \geq -\log(f(\kappa, r, \lambda))/[\gamma\left(1-\kappa\right)]$. As for the second inequality, it follows from $1/S(\bar{a}) \leq 1$ that $S(\bar{a}) \geq 1 \geq f(\kappa, r, \lambda)$.

3. $\kappa \in \left(\frac{r(1-\lambda)}{1-r\lambda}, 1\right)$: Then $f(\kappa, r, \lambda) > 1$ and the second inequality in (31), $f(\kappa, r, \lambda) \leq S(\bar{a})$, is satisfied whenever $\bar{a} \geq \log(f(\kappa, r, \lambda))/[\gamma\left(1-\kappa\right)]$. The first inequality immediately follows from $S(\bar{a}) \geq f(\kappa, r, \lambda) > 1 > 1/S(\bar{a})$.

Proposition 1 focuses on case 3. It also compares it to the steady-state solution. To show that $\hat{a} = 0$ is the unique steady-state when $\kappa \in ((1-\lambda)/(r^{-1}-\lambda), 1)$, set $a_s^* = \hat{a}$ for all $s \geq t$ in (9). The resulting incentive constraint reads $\kappa = r(1-\lambda+\lambda\kappa)$, a requirement that cannot be satisfied since we are focusing on cases where $\kappa > (1-\lambda)/(r^{-1}-\lambda)$. Hence $\hat{a} > 0$ cannot be incentive compatible as the agent would like to deviate by investing less than $\hat{a}$. However, such deviations are not feasible when $a_t^* = 0$ for all t, and so $\hat{a} = 0$ is indeed the only rest point. ■

Proof. Proposition 2: We want to characterize deterministic dynamics near the steady-state $\hat{a}$. Let $\varphi(a_t) = a_{t+1}$ denote the implicit map so that $l(a) = h(\varphi(a))$, where $l(a) \equiv g'(a)\, U'(a - g(a))$ and $h(a) \equiv rU'(a - g(a))\,[1-\lambda+\lambda g'(a)]$. Differentiating the incentive constraint (13) at the steady-state, we find that

$$\varphi'(\hat{a}) \equiv \left.\frac{da_{t+1}}{da_t}\right|_{a_t=\hat{a}} = \frac{A_g(\hat{a}) - A_U(\hat{a} - g(\hat{a}))\,[1-g'(a)]}{\frac{\lambda g'(\hat{a})}{1-\lambda+\lambda g'(\hat{a})} A_g(\hat{a}) - A_U(\hat{a} - g(\hat{a}))\,[1-g'(a)]}, \tag{32}$$

where $A_U(c) = -U''(c)/U'(c)$ and $A_g(a) = g''(a)/g'(a)$.

The steady-state $\hat{a}$ is locally stable if $|\varphi'(\hat{a})| \in [0,1)$. Since the numerator in (32) is always higher than the denominator, the stability condition can be satisfied solely if the denominator is negative. Let us focus first on cases where the numerator is positive while the denominator is negative. Then it is easy to verify that $\varphi'(\hat{a}) \in (-1, 0]$ whenever (15) is satisfied. The other possibility is that both numerator and denominator are negative, then

$$\varphi'(\hat{a}) \in [0,1) \Leftrightarrow A_g(\hat{a}) < A_U(\hat{a} - g(\hat{a}))(1-g'(\hat{a})), \tag{33}$$

a condition that is actually more stringent than (15). ■

Proposition 3 can be proved in a similar way to Proposition 1. Thus we first use a direct approach by changing variable and defining a new fixed point problem.

Lemma 8 *Let*

$$a(s) \equiv \frac{r(1-\lambda)}{1-(r\lambda)^2}(r\lambda + s),$$

2-period cycles are sustainable when the fixed point problem

$$s = \psi(s) \equiv \frac{U'(c(a(1/s)))}{U'(c(a(s)))} \tag{34}$$

admits a solution s^*.

Proof. Lemma 8: We focus on 2-period cycles and denote this period action by a, next period by a', the period after by a, and so on. In other words, we have the discount factors from today til tomorrow and from tomorrow til the day after, respectively,

$$S \equiv S\left(a,a'\right) = U'(c(a'))/U'(c(a)), \tag{35}$$

$$S' \equiv S\left(a',a\right) = 1/S\left(a,a'\right). \tag{36}$$

Therefore, if we start at $t=0$, so that a is the action at $t=0,2,4,6...$ and a' the action for $t=1,3,5,7,...$, then the incentive constraint (7) is satisfied when (a,a') solve the following two equations

$$a = \frac{1-\lambda}{\lambda}\left(\sum_{\text{t=1,3,5,7,...}}^{\infty}(r\lambda)^t S + \sum_{\text{t=2,4,6,8,...}}^{\infty}(r\lambda)^t\right) = \frac{1-\lambda}{\lambda}\left(\frac{r\lambda S}{1-(r\lambda)^2}+\frac{(r\lambda)^2}{1-(r\lambda)^2}\right),$$

$$a' = \frac{1-\lambda}{\lambda}\left(\sum_{\text{t=1,3,5,7,...}}^{\infty}(r\lambda)^t S' + \sum_{\text{t=2,4,6,8,...}}^{\infty}(r\lambda)^t\right) = \frac{1-\lambda}{\lambda}\left(\frac{r\lambda S'}{1-(r\lambda)^2}+\frac{(r\lambda)^2}{1-(r\lambda)^2}\right).$$

These simplify to

$$a = \frac{1-\lambda}{1-(r\lambda)^2}r\left(r\lambda+S\right), \tag{37}$$

$$a' = \frac{1-\lambda}{1-(r\lambda)^2}r\left(r\lambda+S'\right). \tag{38}$$

Thus there are 4 equations (35), (36), (37), and (38), and 4 unknowns, (a,a',S,S'). One solution is $(\hat{a},\hat{a},1,1)$ where $\hat{a}=(1-\lambda)/(r^{-1}-\lambda)$, which is a version of Holmström's (1999) Proposition 1. Now let's treat S as a parameter to begin with. Investment as a function of $s=S(a,a')$ is given by

$$a\left(s\right) = \frac{r(1-\lambda)}{1-(r\lambda)^2}\left(r\lambda+s\right), \text{ and } a'\left(s\right) = \frac{r(1-\lambda)}{1-(r\lambda)^2}\left(r\lambda+s^{-1}\right).$$

Thus our problem is equivalent to looking for a fixed point in s of the function $\psi\left(s\right)$ defined in (34). ■

Proof. Proposition 3: Since $U(c)=-\exp(-\gamma c)$, it follows from the definition in (34) of $\psi\left(\cdot\right)$ that $\psi\left(1\right)=1$. Let $\bar{s}\equiv 2\frac{1-(r\lambda)^2}{r(1-\lambda)}-r\lambda$, since $c\left(\bar{s}\right)=0$ and $c\left(1/\bar{s}\right)\in(0,1)$, we have $\psi\left(\bar{s}\right)<1$. By continuity of the mapping $\psi\left(\cdot\right)$, there will be a fixed point $s\in(1,\bar{s})$ if $\psi'\left(1\right)>1$. Differentiating $\psi\left(\cdot\right)$, we obtain

$$\psi'\left(s\right) = \gamma\left(1-a(s)+\frac{1-a(1/s)}{s^2}\right)\frac{r\left(1-\lambda\right)}{1-(r\lambda)^2}\exp\left(\gamma\left[a(s)-\frac{a(s)^2}{2}-\left(a(1/s)-\frac{a(1/s)^2}{2}\right)\right]\right),$$

and so

$$\psi'(1) > 1 \Leftrightarrow 2\gamma\left[1 - a(1)\right]\frac{r(1-\lambda)}{1-(r\lambda)^2} > 1.$$

This expression can be simplified as $a(1) = \hat{a} = (1-\lambda)/(r^{-1}-\lambda)$ so that

$$\frac{r(1-\lambda)}{1-(r\lambda)^2} = \frac{\hat{a}}{1+r\lambda} = \frac{a(1)}{1+r\lambda}\ . \tag{39}$$

Reinserting this equality into the previous equation we get

$$\psi'(1) > 1 \Leftrightarrow 2\gamma\left[1-a(1)\right]a(1) > 1 + r\lambda,$$

which implies in turn that

$$\frac{\frac{1}{a(1)} - \gamma(1-a(1))}{\frac{r\lambda}{a(1)} - \gamma(1-a(1))} \in (-1,1)\ .$$

This allows us to conclude that the steady-state is stable since equation (32) with quadratic costs and CARA utility reads

$$\varphi'(a(1)) = \left.\frac{da_{t+1}}{da_t}\right|_{a_t=a(1)} = \frac{\frac{1}{a(1)} - \gamma(1-a(1))}{\frac{1}{\frac{1-\lambda}{\lambda}+a(1)} - \gamma(1-a(1))} = \frac{\frac{1}{a(1)} - \gamma(1-a(1))}{\frac{r\lambda}{a(1)} - \gamma(1-a(1))},$$

where the last equality follows from expression of the rest-point $a(1) = (1-\lambda)/(r^{-1}-\lambda)$. ■

Proof. Corollary 5: The corollary immediately follows from the fact that investment at the steady-state $\hat{a} = (1-\lambda)/(\rho^{-1}-\lambda)$ does not depend on the degree of risk aversion. Since $A_U(\hat{a} - g(\hat{a})) = \gamma/(\hat{a} - g(\hat{a}))$ when the function is CRRA, and $A_U(c) = \gamma$ when the function is CARA, it is easily seen that, in both cases, (22) is equivalent to imposing a lower bound on γ. ■

Proof. Proposition 6: When the utility function is CARA, the proof is similar to the one of Proposition 3 with ρ replacing r. When the utility is CRRA, so that $U(c) = c^{1-\gamma}/(1-\gamma)$, it follows from the definition (34) of $\psi(\cdot)$ that $\psi(1) = 1$. Let $\bar{s} \equiv 2\frac{1-(\rho\lambda)^2}{\rho(1-\lambda)} - \rho\lambda$, since $c(\bar{s}) = 0$ and $c(\bar{s}^{-1}) > 0$, we have $\psi(\bar{s}) = 0$. By continuity of the mapping $\psi(\cdot)$, there will be a fixed point $s \in (1, \bar{s})$ if $\psi'(1) > 1$. Differentiating $\psi(\cdot)$, we obtain

$$\psi'(s) = \gamma\left(\frac{c(s)}{c(s^{-1})}\right)^{\gamma-1}\frac{\left[(1-a(s))\,c(s^{-1}) - c(s)\,(1-a(s^{-1}))\,(-1/s^2)\right]}{c(s^{-1})^2}a'(s).$$

Since $a'(s) = \rho(1-\lambda) / \left[1-(\rho\lambda)^2\right]$, we have

$$\psi'(1) > 1 \Leftrightarrow \frac{2\gamma\left[1-a(1)\right]}{c(1)} \frac{\rho(1-\lambda)}{1-(\rho\lambda)^2} > 1 \; .$$

Reinserting (39), we get

$$\psi'(1) > 1 \Leftrightarrow \frac{2\gamma\left[1-a(1)\right]}{c(1)} > \frac{1+\rho\lambda}{a(1)} \; ,$$

which implies in turn that the steady-state is stable since equation (32) with quadratic costs and CRRA utility reads

$$\varphi'(a(1)) = \left.\frac{da_{t+1}}{da_t}\right|_{a_t=a(1)} = \frac{\frac{1}{a(1)} - \frac{\gamma}{c(1)}(1-a(1))}{\frac{\rho\lambda}{a(1)} - \frac{\gamma}{c(1)}(1-a(1))}.$$

■

Proof. Proposition 7: Since $\rho < 1$, the ratio $(1-\lambda)/(\rho^{-1}-\lambda)$ is decreasing in λ. Then, given that $g'(\hat{a})$ is increasing in $\hat{a}$, the definition of $\hat{a} = g'^{-1}((1-\lambda)/(\rho^{-1}-\lambda))$ implies that the rest-point is decreasing in λ. To see how this gain parameter varies with the variance coefficients σ_ν^2 and σ_ε^2, note that

$$\lambda = 1 - \frac{\sigma_\varepsilon^{-2}}{\bar{\sigma}_\theta^{-2} + \sigma_\varepsilon^{-2}} = 1 - \frac{2\sigma_\varepsilon^{-2}}{\sqrt{\frac{1}{\sigma_\varepsilon^4} + \frac{4}{\sigma_\varepsilon^2\sigma_\nu^2}} + \frac{1}{\sigma_\varepsilon^2}} = 1 - \frac{2}{\sqrt{1+4\frac{\sigma_\varepsilon^2}{\sigma_\nu^2}}+1},$$

so that $\partial\lambda/\partial\sigma_\varepsilon > 0$ while $\partial\lambda/\partial\sigma_\nu < 0$. Hence the variance coefficients have opposite effect on λ, and thus $\hat{a}$. ■

A.2 Product recall data and calculations

The product recall data are taken from the National Highway Traffic Safety Administration of the Department of Transportation (NHTSA). The data contain all NHTSA safety-related information on defects and compliance from the late 1960s. This includes report-received date, record-creation date, model of the car, and the date of manufacture. We construct the quarterly recall data as follows: We

1. removed the observations with missing report-received date, and/or date of the start of manufacture and/or date of the end of manufacture. We ended up with 48014 product-recall cases in total.

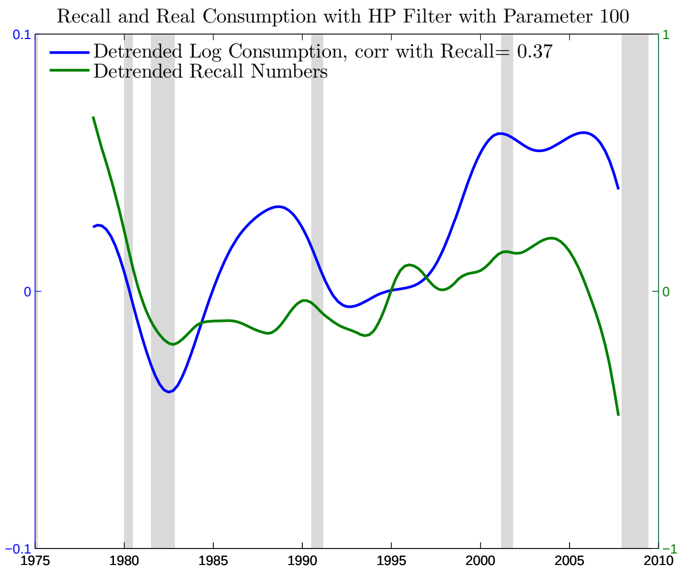

Figure 6: DETRENDED CONSUMPTION AND PRODUCT-RECALLS SERIES 1978:1-2007:3

2. sorted the cases by the report-received date, and created *quarterly* bins from 1966Q4 to 2012Q3.

3. calculated the number of total recalls in each bin.

4. further removed the bins with consecutive zero observations and ended up with the final sample spanning from 1978Q1 to 2007Q3.

5. took logs and time-detrended the observations in each remaining bin.

The time series plot of resulting recalls is in Fig. 6. The two series are positively correlated as the 2-period cycle equilibrium implies; faulty products are made in recessions and recalled the next period, i.e., in the boom.

Fig. 7 shows the frequency distribution of the ages of the products at the time of recall.

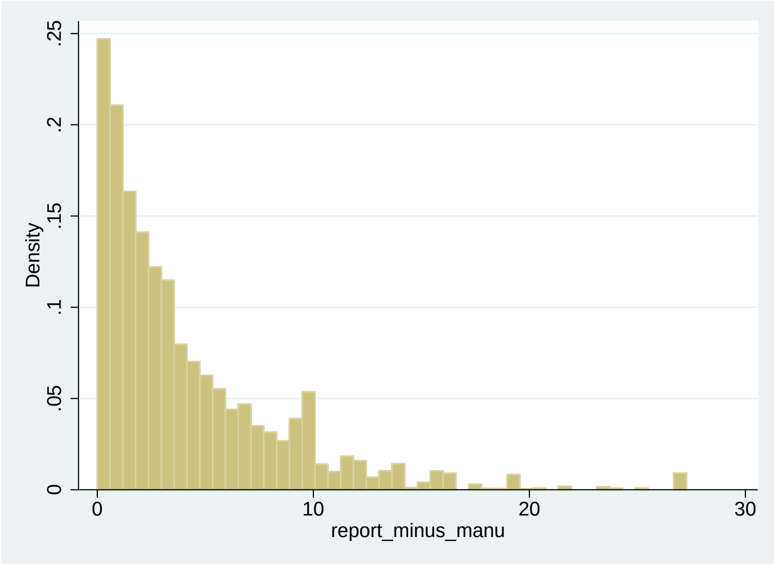

Figure 7: Ages of recalled products in years since start of manufacture

A.3 Consumption growth and the real rate

Fig. 8 reports real consumption growth and real interest rates over the 14 peak-trough or through-peak episodes that our data cover. The correlation between the two series (both are annualized) is 0.11.[30] For example, the two data points for 2001 mean the following:

(i) Nominal personal consumption expenditure growth 1991-2001 (annualized),

(ii) the annualized nominal 10-year bond rate realized in 1991,

(iii) the annualized inflation rate from 1991-2001 was substracted from both.

The model prediction is given by the formula

$$1 + r_i = \beta^{-T_i} g_i^{\gamma}$$

where T_i is the time interval between the peak and trough or trough and peak as the case may be, and where $\beta = 0.97$ and $\gamma = 4$ as in the calibration. Interest rates are smoother

[30]For real consumption we use the monthly total personal consumption expenditure (PCE), deflated by the PCE deflator. The sample interest rates consists of 1-year, 3-year, 5-year and 10-year treasury constant maturity rate. By matching – as closely as possible – the gap between each recession date with the maturity of the bond, the sample includes nine 1-year rates, one 3-year rate, two 5-year rates and three 10-year rates. The real interest rate is obtained by deflating the nominal rate by the PCE deflator. Neither real consumption growth nor real interest rates are annualized. The sample includes 8 NBER recession dates from 1960:03 to 2009:06.

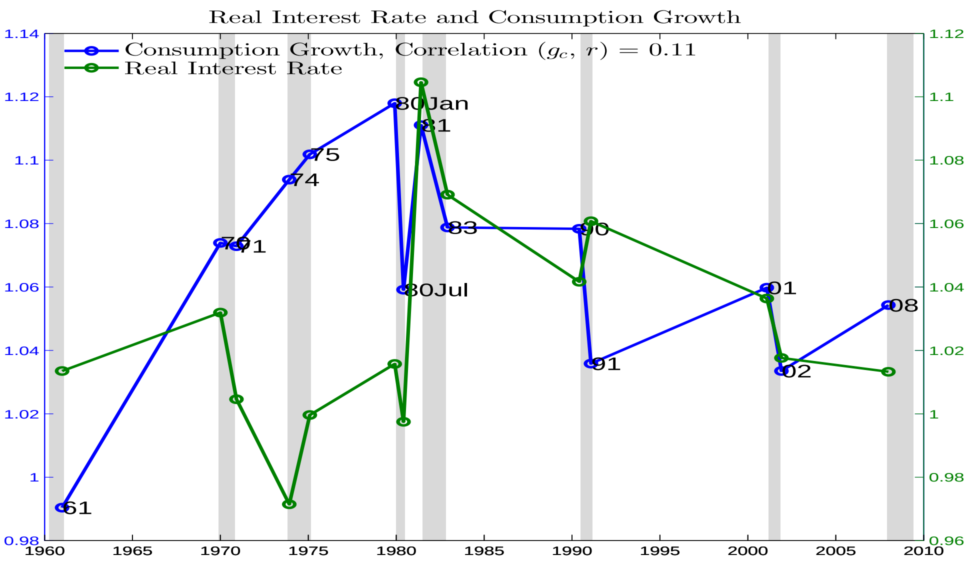

Figure 8: r AND g_c OVER PEAKS AND TROUGHS

than what our model implies, but the correlation is positive.

More generally, and as discussed below, the empirically relevant frequency for aggregate cycles is 4.7 years. Thus we are referring to 4.7-year bond rates predicting 4.7-year consumption growth. Table 2 reports some supporting evidence that uses the 5-year frequency for both consumption growth and bond returns.[31]

Forecasting Regression

We regress 5-year nominal consumption growth $g_{t,t+5}$ on the 5-year bond nominal return $r_{t,t+5}$ and on five lagged inflation rates $\pi_{t-1}, \pi_{t-2},..., \pi_{t-5}$. Regression results are reported with and without a constant in column (1) and (2) of Table 2 respectively.

Data Sources.— All series are annual. Consumption is Personal Consumption Expenditures in Billions of Dollars (PCE on FRED). The 5-year bond yield is the 5-Year Treasury Constant Maturity Rate (GS5 on FRED). Past inflation is calculated from the Consumer Price Index for All Urban Consumers (CPIAUCSL_PCH on FRED).

[31] Fig. 8 is the empirical counterpart of Fig. 2 in the text.

	(1)		(2)	
	$g_{t,t+5}$		$g_{t,t+5}$	
$r_{t,t+5}$	0.0085	(0.83)	**0.0518**	(5.11)
π_{t-1}	0.0188	(1.73)	0.0039	(0.27)
π_{t-2}	-0.0084	(-0.52)	0.0006	(0.03)
π_{t-3}	0.0067	(0.35)	-0.0012	(-0.05)
π_{t-4}	0.0036	(0.21)	0.0029	(0.12)
π_{t-5}	-0.0077	(-0.70)	-0.0133	(-0.90)
Constant	**0.2300**	(6.25)		
N	52		52	

Table 2: Note: t statistics in parentheses. Bold numbers indicate statistical significance at 5 percent level.

TECHNICAL APPENDIX

A.4 Sufficiency of necessary conditions.

The problem analyzed in Holmström (1999) is extremely tractable because returns are linear in effort. In order to allow for concave returns, we first recursively derive the agent's problem both on and off the equilibrium path. Hence we consider arbitrary strategies and let $\delta_t \equiv a_t - a_t^*$ denote the deviation from equilibrium effort at each date. We wish to relate the agent's and market's posteriors about θ, which we denote m^δ and m respectively. Bayes rule implies that

$$m_{t+1}^\delta - m_{t+1} = \frac{1-\lambda}{\lambda}\sum_{s=0}^{t}\lambda^{t-s}\left(x_s^\delta - x_s\right) = -\frac{1-\lambda}{\lambda}\sum_{s=0}^{t}\lambda^{t-s}\delta_s \ . \tag{40}$$

Let Δ_t denote the following weighted mean of past deviations

$$\Delta_t = \lambda\Delta_{t-1} + (1-\lambda)\delta_{t-1} \text{ with } \Delta_0 = 0 \ . \tag{41}$$

It follows from (40) that

$$m_t^\delta = m_t - \Delta_t \ .$$

We can use this relationship to specify the agent's expectations about the law of motion of beliefs. First notice that $y_t - m_t^\delta = \theta_t - m_t^\delta + a_t + \varepsilon_t$. Since $\theta_t - m_t^\delta$ is normally distributed with mean 0 and variance $\bar{\sigma}_\theta^2$, the innovation process from the agent's standpoint is a normally distributed variable, $u \sim \mathcal{N}(0, \bar{\sigma}_\theta^2 + \sigma_\epsilon^2)$, such that $y_t = m_t^\delta + a_t + u_t$. Reinserting this decomposition into the law of motion of market's beliefs, we find that

$$\begin{aligned} m_{t+1} &= \lambda m_t + (1-\lambda)x_t = m_t + (1-\lambda)[y_t - a_t^* - m_t] = m_t + (1-\lambda)[a_t - a_t^* + m_t^\delta - m_t + u_t] \\ &= m_t + (1-\lambda)[\delta_t - \Delta_t + u_t]. \end{aligned} \tag{42}$$

The normality of u_t implies that the distribution of the posterior is

$$P\left(m_{t+1} \middle| m_t, \Delta_t, \delta_t\right) = \Phi\left(\frac{m_{t+1} - [m_t + (1-\lambda)(\delta_t - \Delta_t)]}{(1-\lambda)(\bar{\sigma}_\theta + \sigma_\epsilon)}\right),$$

where $\Phi(\cdot)$ is the standard normal CDF.

Given the per-period utility function $U(m, a^*, \delta)$, and the deterministic but potentially time-varying discount factor β_t, the agent's value function *on and off* the equilibrium path is given by the fixed point of the following functional equation

$$V_t(m,\Delta) = \max_{\delta} \left\{ U(m, a_t^*, \delta) + \beta_t \int V_{t+1}(m', \Delta')\, dP(m' | m, \Delta, \delta) \right\} \tag{43}$$
$$s.t.\ m' = m + (1-\lambda)[\delta - \Delta + u],$$
$$\Delta' = \lambda\Delta + (1-\lambda)\delta\ .$$

Partial equilibrium model.—In Section 3, the discount factor β is constant over time and the utility function reads $U(m, a^*, \delta) = -\exp(-\gamma[m + a^* - g(a^* + \delta)])$. We first show that the value function is log-linear in m.

Lemma 9 *The value function of the partial equilibrium model with CARA utility is of the form*

$$V_t(m,\Delta) = \exp(-\gamma m) V_t(0,\Delta)\,, \tag{44}$$

and the agent's policy function is deterministic.

Proof. To verify the conjecture, we change the variable of integration of the Bellman eq. (43)

$$\begin{aligned} V_t(m,\Delta) &= \max_{\delta} \left\{ \begin{array}{l} -\exp(-\gamma[m + a_t^* - g(a_t^* + \delta)]) \\ +\beta \int V_{t+1}(m + (1-\lambda)[\delta - \Delta + u], \Delta')\, d\Phi\left(\frac{u}{\bar{\sigma}_\theta + \sigma_\epsilon}\right) \end{array} \right\} \\ &= \exp(-\gamma m) \max_{\delta} \left\{ \begin{array}{l} -\exp(-\gamma[a_t^* - g(a_t^* + \delta)]) \\ +\beta \int V_{t+1}((1-\lambda)[\delta - \Delta + u], \Delta')\, d\Phi\left(\frac{u}{\bar{\sigma}_\theta + \sigma_\epsilon}\right) \end{array} \right\} \\ &= \exp(-\gamma m) V_t(0,\Delta)\,. \end{aligned}$$

The second equality follows reinserting our guess (44), while the third equality follows directly from the Bellman equation (43) defining V. Since the functional equation is a contraction mapping, the value function is unique and its policy function δ identical across all market's beliefs m. ■

Hence, the agent chooses a sequence of deterministic actions that only depend on the cumulative stock of past deviations Δ. In other words, optimal policies are deterministic *on and off* the equilibrium path. We can therefore focus, without loss of generality, on sequences of deterministic solutions, which allows us to directly establish the sufficiency of the FOC.

Let $a = (a_0, a_1, ..)$ denote a sequence of actions from date 0 onwards. Given that investment in each period belongs to the feasibility set $\mathcal{A} \subseteq \mathbb{R}_+$, sequences are element of the infinite Cartesian product $\mathcal{A}^\infty = \{(a_t)_{t=0}^\infty | a_t \in \mathcal{A} \text{ for all } t = 0, 1, ...\}$. The market's posterior can be computed using the Bayesian map, which is linear,

$$m_{t+1}(a, x^\delta | m_0) = \lambda^t m_0 + \frac{1-\lambda}{\lambda} \sum_{s=0}^{t} \lambda^{t-s} \left(x_s^\delta - a_s^* + a_s \right). \tag{45}$$

These notations allow us to redefine the agent's utility as a functional in the space of action sequences, i.e.,

$$V_0(0,0) = \max_{a \in \mathcal{A}^\infty} \mathcal{U}(a) = \max_{a \in \mathcal{A}^\infty} E \left[-\sum_{t=0}^{\infty} \beta^t \exp(-\gamma [m_t(a) + a_t^* - g(a_t)]) \right],$$

where $m_t(a)$ is a shorthand version of $m_t(a, x^\delta | m_0)$ introduced in (45).

It is easily seen that $\mathcal{U}$ is a concave functional. For any $\alpha \in (0,1)$ and all pair of sequences $a^1, a^2 \in \mathcal{A}^\infty$ with $a^1 \neq a^2$, we have

$$\begin{aligned}
&\mathcal{U}(\alpha a^1 + (1-\alpha) a^2) = E \left[-\sum_{t=0}^{\infty} \beta^t \exp\left(-\gamma \left[m_t(\alpha a^1 + (1-\alpha) a^2) + a_t^* - g\left(\alpha a_t^1 + (1-\alpha) a_t^2\right) \right]\right) \right] \\
&= E \left[-\sum_{t=0}^{\infty} \beta^t \exp\left(-\gamma \left[m_t(\alpha a^1) + m_t((1-\alpha) a^2) + a_t^* - g\left(\alpha a_t^1 + (1-\alpha) a_t^2\right) \right]\right) \right] \\
&< E \left[-\sum_{t=0}^{\infty} \beta^t \exp\left(-\gamma \left[m_t(\alpha a^1) + a_t^* - g\left(\alpha a_t^1\right) \right]\right) \right] + E \left[-\sum_{t=0}^{\infty} \beta^t \exp\left(-\gamma \left[m_t((1-\alpha) a^2) + a_t^* - g\left((1-\alpha) a_t^2\right) \right]\right) \right] \\
&= \mathcal{U}(\alpha a^1) + \mathcal{U}((1-\alpha) a^2).
\end{aligned} \tag{46}$$

The second equality holds true because the Bayesian map (45) is linear in a, while the inequality is a direct consequence of the concavity of the per-period utility function combined with the convexity of the cost function. We can conclude from (46) that $\mathcal{U}$ is a *strictly concave* functional in the space of action sequences. Thus any local maximum of $\mathcal{U}$ is also a global maximum. A necessary condition for $\mathcal{U}$ to have a maximum at a^* is that its Gateaux derivative $d\mathcal{U}(a^*; h) = 0$ for all $h \in \mathcal{A}^\infty$. This requirement yields the necessary FOC (9) which, by strict concavity of the objective, is also sufficient.

General equilibrium model.—In Section 4, the objective of the manager is linear as $U(m, a^*, \delta) = m + a^* - g(a^* + \delta)$, but the discount factor, $\beta_t = \rho U'(c_{t+1}) / U'(c_t)$, follows the changes in aggregate consumption.

Lemma 10 *The agent's value function in the general equilibrium model reads*

$$V_t(m, \Delta) = K_t^0 + K_t^1 m + K_t^2 \Delta, \tag{47}$$

where K_t^0, K_t^1 and K_t^2 are time-varying coefficients whose expressions are given in (49), (50) and (51), respectively. The deterministic equilibrium path a^* is incentive compatible if the necessary condition (19) is satisfied for all $t \geq 0$.

Proof. The Bellman equation reads

$$\begin{aligned} V_t(m, \Delta) &= \max_\delta \left\{ \begin{array}{l} m + a^* - g(a^* + \delta) \\ +\beta_t \int V_{t+1}(m + (1-\lambda)[\delta - \Delta + u], \Delta')\, d\Phi\left(\frac{u}{\bar{\sigma}_\theta + \sigma_\epsilon}\right) \end{array} \right\} \\ &= \max_\delta \left\{ \begin{array}{l} m + a^* - g(a^* + \delta) \\ +\beta_t \left[K_{t+1}^0 + K_{t+1}^1(m + (1-\lambda)(\delta - \Delta)) + K_{t+1}^2(\lambda\Delta + (1-\lambda)\delta)\right] \end{array} \right\}. \end{aligned} \tag{48}$$

The second equality follows using our guess (47) for the next period value function and replacing Δ' by its law of motion (41). Reinserting (47) on the left-hand side of the Bellman equation, we find that the conjecture is verified when $\delta = 0$ and

$$K_t^0 = a_t^* - g(a_t^*) + \sum_{s=t}^{\infty}\left(\prod_{\tau=t}^{s}\beta_\tau\right)[a_{s+1}^* - g(a_{s+1}^*)] = \sum_{s=t}^{\infty}\frac{U'(c_s)}{U'(c_t)}\rho^{s-t}[a_s^* - g(a_s^*)], \tag{49}$$

$$K_t^1 = 1 + \sum_{s=t}^{\infty}\left(\prod_{\tau=t}^{s}\beta_\tau\right) = \sum_{s=t}^{\infty}\frac{U'(c_s)}{U'(c_t)}\rho^{s-t}, \tag{50}$$

$$K_t^2 = -(1-\lambda)\left[\sum_{s=t}^{\infty}\left(\prod_{\tau=t}^{s}\beta_\tau\right)K_{s+1}^1\lambda^{s-t}\right] = -(1-\lambda)\left[\sum_{s=t+1}^{\infty}\frac{U'(c_s)}{U'(c_t)}\rho^{s-t}\left(\sum_{\tau=t+1}^{s}\lambda^{\tau-t-1}\right)\right]. \tag{51}$$

We still have to check that $\delta_t = 0$ when the necessary condition (19) is satisfied. Differentiating (48) with respect to δ_t yields the following FOC

$$\begin{aligned} g'(a_t^*) &= (1-\lambda)\beta_t\left[K_{t+1}^1 + K_{t+1}^2\right] = (1-\lambda)\left[\sum_{s=t+1}^{\infty}\frac{U'(c_s)}{U'(c_t)}\rho^{s-t}\left(1 - (1-\lambda)\left(\sum_{\tau=t+2}^{s}\lambda^{\tau-t-2}\right)\right)\right] \\ &= \frac{1-\lambda}{\lambda}\sum_{s=t+1}^{\infty}\frac{U'(c_s)}{U'(c_t)}(\rho\lambda)^{s-t}, \end{aligned} \tag{52}$$

which is indeed equivalent to the incentive constraint (19). ■

When c_t is constant over time, the discount factor β_t remains fixed and (52) is equivalent to the incentive constraint (22) in Holmström (1999), where β and μ stands for ρ and

λ in our notation. Lemma 10 shows that adding a discount factor that varies over time in a deterministic fashion does not modify the structure of the problem. An additional insight is that agents with private information about their types take the same action than those that are on the equilibrium path. Hence, multiple deviations are never optimal when returns are linear. By contrast, in the partial equilibrium model with CARA utility, optimal strategies on and off the equilibrium path did not coincide.

A.5 Asymmetric cycles.

Proposition 11 derives conditions under which cycles of period 3 can be constructed using the partial equilibrium model with linear costs described in Subsection 3.1. It focuses on asymmetric cycles with protracted booms and sudden busts. The logic of the proof can be applied to study reverse cases and to show that cycles with protracted slumps and sudden booms can be sustained when $\kappa > (1-\lambda)/(r^{-1}-\lambda)$. Furthermore, it is cumbersome but relatively straightforward to extend the proof so as to construct cycles with more than 3 states.

Proposition 11 *Consider the partial equilibrium model with CARA utility and linear costs, i.e., $g(a) = \kappa a$. Assume that κ satisfies the compatible conditions*

$$\frac{1-\lambda}{r^{-1}-\lambda} > \kappa > \frac{r(1-\lambda)}{1-(r\lambda)^3}\left[\frac{r^2\lambda(1-\lambda+\lambda\kappa)}{\kappa} + (r\lambda)^2\right]. \tag{53}$$

Then there exists a unique $\bar{a}_{min} > 0$ such that deterministic cycles of period three, where

$$a_t^* = \begin{cases} \bar{a} \text{ when } t = \{0,3,6,...\} \\ 0 \text{ when } t = \{1,4,7,...\} \\ \tilde{a} \in (0,\bar{a}) \text{ when } t = \{2,5,8,...\}, \end{cases}$$

are sustainable whenever $\bar{a} \geq \bar{a}_{min}$.

Proof. Proposition 11: We propose a constructive proof. We study each action in turn and prove their incentive compatibility

1. $a_t^* = \bar{a}$: Since we are focusing on cases where $\kappa < r(1-\lambda)/(1-r\lambda)$, we have

$$\kappa \leq \frac{1-\lambda}{\lambda}\sum_{i=\{1,2,...\}}^{\infty}(r\lambda)^i < \frac{1-\lambda}{\lambda}\sum_{i=\{1,2,...\}}(r\lambda)^i S(\bar{a}-a_{t+i}^*). \tag{54}$$

The last inequality follows from the definition of $S(\cdot)$ in (30) because $\bar{a} \geq a^*_{t+i}$ for all i, and so $S(\bar{a} - a^*_{t+i}) \geq 1$, with strict inequality for some i. Since the last expression in (54) measures the discounted returns from investment at date t, the feasibility constraint binds and $a^*_t = \bar{a}$ is indeed incentive compatible.

2. $a^*_t = 0$: At the lower-bound of the feasibility set, costs must exceed returns so that

$$\begin{aligned}
\kappa &\geq \frac{1-\lambda}{\lambda} \sum_{i=1,2,\ldots}^{\infty} (r\lambda)^i \, S\left(-a^*_{t+i}\right) \\
&= S\left(-\tilde{a}\right) r \left(1 - \lambda + \lambda \left[\frac{1-\lambda}{\lambda} \sum_{i=\{2,3,\ldots\}} (r\lambda)^{i-1} \, S(\tilde{a} - a^*_{t+i})\right]\right) \\
&= S\left(-\tilde{a}\right) r \left(1 - \lambda + \lambda\kappa\right).
\end{aligned}$$

The last equality follows from the fact that the incentive constraint must hold with equality in the next period because $a^*_{t+1} = \tilde{a} \in (0, \bar{a})$. The condition is satisfied whenever

$$\tilde{a} \geq \tilde{a}_{min} \equiv \frac{-\log\left(\frac{\kappa}{r(1-\lambda+\lambda\kappa)}\right)}{\gamma\left(1-\kappa\right)} > 0. \tag{55}$$

3. $a^*_t = \tilde{a} \in (0, \bar{a})$: First, we assume that condition (55) holds as an equality and we show that there exists a unique value of $\bar{a}$ which renders $\tilde{a}_{min}$ incentive compatible. Since $\tilde{a}$ is interior, the incentive constraint has to hold exactly, i.e.,

$$\kappa = \frac{1-\lambda}{\lambda} \sum_{i=1,2,\ldots}^{\infty} (r\lambda)^i \, S\left(\tilde{a} - a^*_{t+i}\right) \tag{56}$$

$$\begin{aligned}
&= S\left(\tilde{a} - \bar{a}\right) r \left(1 - \lambda + \lambda \left[\frac{1-\lambda}{\lambda} \sum_{i=\{2,3,\ldots\}} (r\lambda)^{i-1} \, S(\bar{a} - a^*_{t+i})\right]\right) \\
&> S\left(\tilde{a} - \bar{a}\right) r \left(1 - \lambda + \lambda\kappa\right) > S\left(\tilde{a} - \bar{a}\right) \kappa.
\end{aligned}$$

The first inequality follows from step 1 above, while the second one holds true because $\kappa < r(1-\lambda)/(1-r\lambda)$. It shows that $S\left(\tilde{a} - \bar{a}\right)$ has to be inferior to one, thus requiring that $\bar{a}$ be strictly larger than $\tilde{a}$. Keeping $\tilde{a}$ constant and differentiating (56) with respect to $\bar{a}$, we find that returns are strictly decreasing in $\bar{a}$. Furthermore, if

$$\kappa > \frac{1-\lambda}{\lambda} \left[\frac{(r\lambda)^2 r \left(1 - \lambda + \lambda\kappa\right)/\kappa + (r\lambda)^3}{1 - (r\lambda)^3}\right], \tag{57}$$

there exists a unique value, which we denote $\bar{a}_{min}$, such that $\tilde{a} = \tilde{a}_{min}$ and (56) is satisfied. To verify that (57) is consistent with $r(1-\lambda)/(1-r\lambda) > \kappa$, notice that

$$\frac{r(1-\lambda)}{1-r\lambda} > \frac{1-\lambda}{\lambda}\left[\frac{(r\lambda)^2 r\left(1-\lambda+\lambda\kappa\right)/\kappa+(r\lambda)^3}{1-(r\lambda)^3}\right] \Leftrightarrow 1 > r\lambda\left[\frac{r\left(1-\lambda+\lambda\kappa\right)}{\kappa}-1\right].$$

This inequality is equivalent to $\kappa > r^2\lambda(1-\lambda)/\left[1+r\lambda-(r\lambda)^2\right]$, which yields a lower-bound that is inferior to the term on the right-hand side of (53) as well as to $r(1-\lambda)/(1-r\lambda)$. Hence the two conditions in (53) are compatible since they define a non-empty interval. As with period-2 cycles, low levels of investment can never be incentive compatible when costs are too small.

We still have to prove that our cycles are sustainable when $\bar{a} > \bar{a}_{min}$. Differentiating (56) with respect to both $\bar{a}$ and $\tilde{a}$, one finds that $\partial\tilde{a}/\partial\bar{a} \in (0,1)$. The derivative being positive, $\tilde{a} > \tilde{a}_{min}$ if $\bar{a} > \bar{a}_{min}$, and the condition (55) for incentive compatibility of $a_t^* = 0$ is satisfied. Furthermore, the derivative being smaller than one ensures that $\tilde{a}$ remains within the interior of the feasible set as $\bar{a}$ increases. ■

A.6 Effect of discount factor ρ on stability of the steady-state.

Proposition 12 *Assume that: (i) the equilibrium conditions in Definition 4 are satisfied; (ii) the utility function is CRRA; (iii) costs are quadratic, i.e., $g(a) = a^2$. There exists a unique bifurcation point $\tilde{\rho}(\gamma,\lambda) \in (0,1)$ such that the steady-state is locally stable if and only if $\rho < \tilde{\rho}(\gamma,\lambda)$. The bifurcation point $\tilde{\rho}$ is increasing in the coefficient of risk aversion γ. Moreover, productivity volatility, σ_ν, and output volatility, σ_ε, have opposite effects on $\tilde{\rho}$.*

Proof. Proposition 12: With quadratic cost and CRRA utility, the incentive constraint is satisfied if

$$c_t^{-\gamma}a_t = \rho\lambda c_{t+1}^{-\gamma}\left(\frac{1-\lambda}{\lambda}+a_{t+1}\right).$$

Taking logs on both side and differentiating with respect to a_t yields

$$-\frac{\gamma}{c_t}\left(1-a_t\right)+\frac{1}{a_t} = -\frac{\gamma}{c_{t+1}}\left(1-a_{t+1}\right)\frac{da_{t+1}}{da_t}+\frac{1}{\frac{1-\lambda}{\lambda}+a_{t+1}}\frac{da_{t+1}}{da_t}.$$

Evaluated at the rest point $(\hat{a},\hat{c})$, this condition reads

$$\left.\frac{da_{t+1}}{da_t}\right|_{\hat{a}} = \frac{\frac{1}{\hat{a}}-\frac{\gamma}{\hat{c}}\left(1-\hat{a}\right)}{\frac{1}{\frac{1-\lambda}{\lambda}+\hat{a}}-\frac{\gamma}{\hat{c}}\left(1-\hat{a}\right)}.$$

Reinserting the rest-point solution $\hat{a} = (1-\lambda)/(\rho^{-1}-\lambda)$, we find that

$$\left.\frac{da_{t+1}}{da_t}\right|_{\hat{a}} = \frac{\frac{1}{\hat{a}} - \frac{\gamma}{\hat{c}}(1-\hat{a})}{\frac{\rho\lambda}{\hat{a}} - \frac{\gamma}{\hat{c}}(1-\hat{a})}.$$

Hence $\tilde{\rho}\lambda$ solves

$$\left.\frac{da_{t+1}}{da_t}\right|_{\hat{a}} = -1 \Rightarrow \frac{1+\tilde{\rho}\lambda}{\hat{a}} = \frac{2\gamma(1-\hat{a})}{\hat{c}}.$$

Replacing the expression of $\hat{a}$ into this condition yields

$$\frac{\gamma}{1+\tilde{\rho}\lambda} = \frac{1}{4}\left[1 + \frac{1-\tilde{\rho}\lambda}{1-\tilde{\rho}}\right]. \tag{58}$$

The left hand side is decreasing in $\tilde{\rho}$ and goes from γ to $\gamma/(1+\lambda)$ as $\tilde{\rho}$ increases from 0 to 1. By contrast, the right hand side is increasing and goes from 1/2 to infinity. Thus there exists a unique $\tilde{\rho}$ solving (58) whenever $\gamma > 1/2$.
The impact of σ_ε and σ_ν on $\tilde{\rho}$ follows from the definition of λ

$$\lambda = 1 - \frac{\sigma_\varepsilon^{-2}}{\bar{\sigma}_\theta^{-2} + \sigma_\varepsilon^{-2}} = 1 - \frac{2\sigma_\varepsilon^{-2}}{\sqrt{\frac{1}{\sigma_\varepsilon^4} + \frac{4}{\sigma_\varepsilon^2\sigma_\nu^2}} + \frac{1}{\sigma_\varepsilon^2}} = 1 - \frac{2}{\sqrt{1+4\frac{\sigma_\varepsilon^2}{\sigma_\nu^2}}+1},$$

so that $\partial\lambda/\partial\sigma_\varepsilon > 0$ while $\partial\lambda/\partial\sigma_\nu < 0$. Hence the variance coefficients have opposite effect on the implicit equation (58) defining $\tilde{\rho}$ since it only depends on λ. As for the effect of γ, totally differentiating (58), we find that

$$\frac{\partial\tilde{\rho}}{\partial\gamma} = \frac{\frac{1}{1+\tilde{\rho}\lambda}}{\frac{\gamma\lambda}{(1+\tilde{\rho}\lambda)^2} + \frac{1}{4}\left[\frac{1-\lambda}{(1-\tilde{\rho})^2}\right]} > 0.$$

■

The model exhibits a flip bifurcation so that a period 2 cycle coexists with the steady state solution as γ gets close enough to the critical value $\tilde{\gamma}$. At the bifurcation point $\tilde{\gamma}$, the rest point becomes unstable and there are no other stable solutions in its neighborhood. Fig. 9 separates the (ρ, γ) plane into a stable and an unstable region. As stated in Proposition 5, there exists a value of γ above which the steady-state is always stable. The discount factor ρ has an opposite impact to γ since, as stated in Proposition 12, the equilibrium is stable when ρ is smaller than some threshold. This is intuitive because an increase in ρ lowers the degree of absolute risk aversion through its positive effect on output. Patient firms invest more in their reputation, which raises consumption and generates a wealth effect that renders agents less risk averse. This mechanism is not

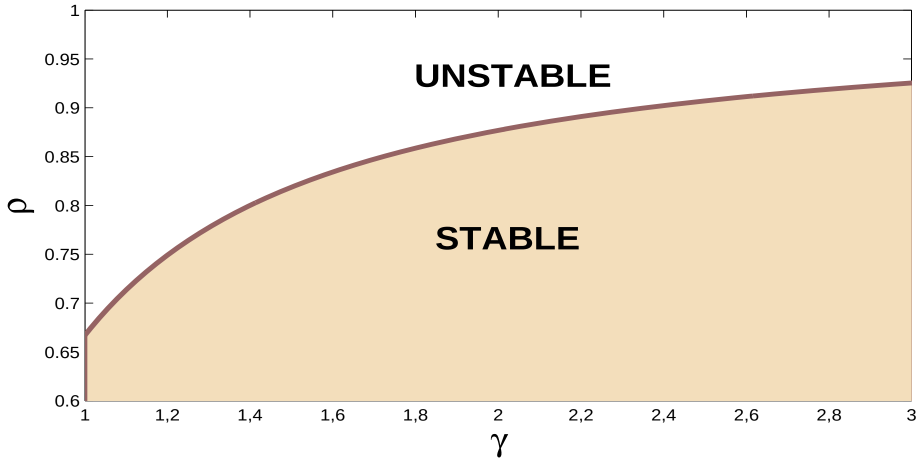

Figure 9: General Equilibrium: Local stability of the steady-state $\hat{a}$ as a function of γ and ρ when utility is CRRA. Parameters: λ=.5 $g(a) = a^2/2$.

operative when preferences are CARA and not CRRA. Then, the absolute degree of risk aversion remains constant and, as shown in Proposition 13, the relationship between the discount factor and the stability of the equilibrium becomes ambiguous.

Proposition 13 *Assume that: (i) the equilibrium conditions in Definition 4 are satisfied; (ii) the utility function is CARA; (iii) costs are quadratic, i.e., $g(a) = a^2$. The steady-state is always unstable when $\gamma < 4\lambda^2$. By contrast, when $\gamma > 4/(2-\lambda)$, there exists a non empty interval $(\rho_l(\gamma, \lambda), \rho_h(\gamma, \lambda))$ such that the steady-state is locally stable if and only if $\rho \in (\rho_l(\gamma, \lambda), \rho_h(\gamma, \lambda))$.*

Proof. **Proposition 13:** When preferences are CARA and costs are quadratic, i.e., $U(c) = -\exp(-\gamma c)$ and $g(a) = a^2/2$, equation (32) reads

$$\varphi'(\hat{a}) = \frac{N(\hat{a})}{D(\hat{a})} = \frac{\frac{1}{\hat{a}} - \gamma(1-\hat{a})}{\frac{1}{\frac{1-\lambda}{\lambda}+\hat{a}} - \gamma(1-\hat{a})} = \frac{\frac{1}{\hat{a}} - \gamma(1-\hat{a})}{\frac{\rho\lambda}{\hat{a}} - \gamma(1-\hat{a})}\ , \tag{59}$$

where the last equality follows from the expression of the rest-point $\hat{a} = \rho(1-\lambda)/(1-\rho\lambda)$. The steady-state $\hat{a}$ is locally stable if $|\varphi'(\hat{a})| \in [0, 1)$. We now study under which conditions this requirement is satisfied:

(i) If $\gamma < 4\lambda^2$, the denominator $D(\hat{a})$ in (59) is always positive. Since the numerator $N(\hat{a}) > D(\hat{a})$, we have $\varphi'(\hat{a}) = N(\hat{a})/D(\hat{a}) > 1$, and the steady-state is locally unstable.

(ii) If $\gamma \in (4\lambda^2, 4)$, the numerator $N(\hat{a})$ is always positive while the denominator $D(\hat{a})$ might be negative. Thus the steady-state is stable when $\varphi'(\hat{a}) \in (-1, 0)$, i.e., when

$$f(\rho) \equiv 2\gamma(1-\lambda)\frac{\rho(1-\rho)}{(1-\rho\lambda)^2} - (1+\rho\lambda) > 0. \tag{60}$$

As ρ goes from 0 to 1, the first term of (60) starts at 0 and converges again to 0, while the second term goes from -1 to $-(1+\lambda)$. Hence, the number of roots of $f(\cdot)$ has to be even. Differentiating $f(\cdot)$ twice, one finds that

$$f'(\rho) = 2\gamma(1-\lambda)\frac{1-2\rho+\rho_\gamma\lambda}{(1-\rho\lambda)^3} - \lambda,$$
$$f''(\rho) = \frac{4\gamma(1-\lambda)[\rho\lambda(\lambda-2)+2\lambda-1]}{(1-\rho\lambda)^4}.$$

We distinguish two cases:

1. $\lambda < 1/2$: Then $f''(\rho) < 0$ for all $\rho \in (0,1)$ and $f'(\cdot)$ can be positive solely if $f'(0) = 2\gamma(1-\lambda) - \lambda > 0$, i.e., if $\gamma > \lambda/[2(1-\lambda)]$. Since $f(0) = -1$, there is no root when $\lambda < 1/2$ and $\gamma < \lambda/[2(1-\lambda)]$. Conversely, $f'(\cdot)$ may have some roots when $\gamma > \lambda/[2(1-\lambda)]$. However, the strict concavity of $f(\cdot)$, along with the terminal conditions $f(0) = -1 > f(1) = -1-\lambda$, ensure that $f(\cdot)$ has either zero or two roots in $(0,1)$.

2. $\lambda > 1/2$: Then $f''(\rho) \gtrless 0 \Leftrightarrow (2\lambda-1)/(2\lambda-\lambda^2) \gtrless \rho$, and so the function $f(\cdot)$ has a unique inflection point in $(0,1)$. This implies in turn that the terminal conditions $f(0) = -1 > f(1) = -1-\lambda$ can be satisfied solely if $f(\cdot)$ has either zero or two roots in $(0,1)$.

Having shown that $f(\cdot)$ has at most two roots, we now establish a sufficient condition for their existence. The continuity of $f(\cdot)$, and the fact that it is negative at both ends of the unit interval, imply that it is sufficient to identify a parametric restriction under which $f(\cdot)$ reaches positive values within $(0,1)$. A tractable expression is obtained focusing on $\hat{a} = 1/2$, since

$$f\left(\frac{1}{2-\lambda}\right) = \frac{\gamma}{2} - \frac{2}{2-\lambda} > 0 \Leftrightarrow \gamma > 4/(2-\lambda).$$

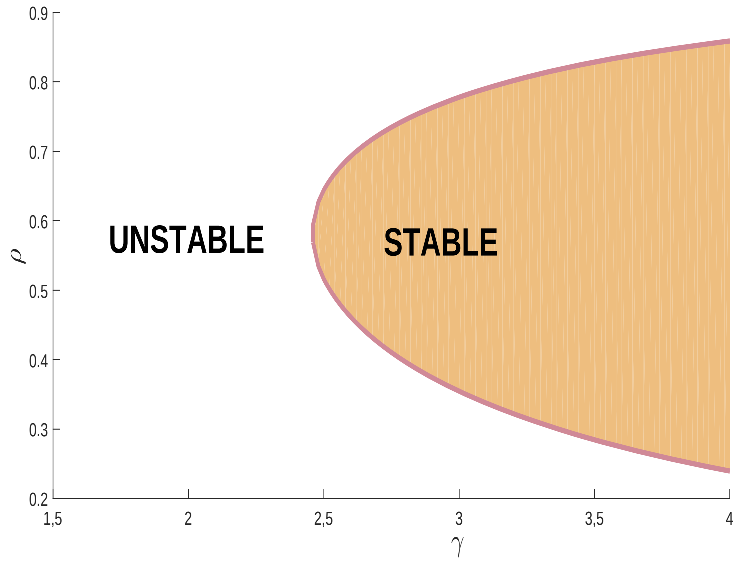

Figure 10: General Equilibrium: Local stability of the steady-state $\hat{a}$ as a function of γ and ρ when utility is CARA. Parameters: $\lambda = .39$, $g(a) = a^2/2$.

Thus, provided that $\gamma > 4/(2-\lambda)$, $f(\cdot)$ has two roots $(\rho_l(\gamma), \rho_h(\gamma)) \in (0,1)^2$. Furthermore, $f(\rho_\gamma) > 0$, and so the steady-state is stable for all $\rho \in (\rho_l(\gamma), \rho_h(\gamma))$.

(iii) If $\gamma > 4$, the numerator $N(\hat{a})$ and denominator $D(\hat{a})$ may become negative. However, the inequality $N(\hat{a}) > D(\hat{a})$ ensures that $\varphi'(\hat{a}) = N(\hat{a})/D(\hat{a}) \in (0,1)$ whenever $N(\hat{a}) < 0$, which implies in turn that ρ must belong to $(\rho_l(\gamma), \rho_h(\gamma))$. Since $\varphi'(\hat{a})$ is continuous over the interval $[\hat{a}(\rho_l(\gamma)), \hat{a}(\rho_h(\gamma))]$, we know that $|\varphi'(\hat{a})| \in (0,1)$, implying that the steady-state is stable for all $\rho \in (\rho_l(\gamma), \rho_h(\gamma))$. ■

Fig. 10 shows how the risk aversion coefficient γ widens the stable interval described in Proposition 13. Eventually, stability is always ensured for high enough values of γ.