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Working Paper 22016
<http://www.nber.org/papers/w22016>

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
February 2016, Revised April 2019

This paper has circulated under the title “The Demand for Diversification in Incomplete Markets”. We thank Kerry Back, Jennifer Conrad, George Constantinides, Kevin Crotty, John Donaldson, Alex Horenstein, Pedram Jahangiry, Robert Merton, Kalle Rinne, Manuel Santos, Yuhang Xing, and the seminar participants at Aalto University, Miami University, Rice University, Vienna University and the WU Gutmann Center for helpful comments. The usual caveat applies. Wahal is a consultant to Dimensional Fund Advisors. DFA provided no funding or data for this research. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

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NBER Working Paper No. 22016
February 2016, Revised April 2019
JEL No. G11,G12

ABSTRACT

In Merton (1987), idiosyncratic risk is priced in equilibrium as a consequence of incomplete diversification. We modify his model to allow the degree of diversification to vary with average idiosyncratic volatility. This simple recognition results in a state-dependent idiosyncratic risk premium that is higher when average idiosyncratic volatility is low, and vice versa. The data appear to be consistent with a positive state-dependent premium for idiosyncratic risk both in the US and in other developed markets.

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1. Introduction

A major research initiative in financial economics focuses on the determinants of the cross-sectional and time series properties of asset returns. There are two prominent classes of asset pricing models with microeconomic foundations that address this issue: the CAPM and the consumption CAPM (along with their numerous extensions). These models, while theoretically elegant, prove inadequate when confronted with data.¹ This has led to a third class of largely ad hoc empirical factor models that attempt to connect the expected returns on the assets with their ‘betas’.² While these models invoke some version of the arbitrage pricing theory (APT) and the ICAPM (Merton (1973)) for theoretical justification, it is not clear that they succeed as asset pricing models, in the sense of connecting returns to ‘risk premia’.³ A key abstraction common to these models is that information is costless and investors hold fully diversified portfolios. A fourth model class, where asset prices are determined by individual preferences and beliefs but where investors have incomplete information (and hence hold under diversified portfolios), has received less attention. This paradigm has its genesis in the early work of Levy (1978) and Mayshar (1979, 1981). It forms the basis of the asset pricing model proposed by Robert Merton in his 1986 presidential address to the American Finance Association. The

¹ See, for example, Sharpe (1964), Lintner (1965), Mossin (1966), Black (1972), Rubinstein (1976), Lucas (1978), Breeden (1979), and Ross (1976). The list of empirical studies that reject these models is long and catalogued in numerous review papers.

² Harvey et al. (2015) catalogue 316 anomalies proposed as potential factors in asset-pricing models and they note that there are others that do not make their list.

³ The APT was introduced by Ross(1976) and extended by Huberman (1982), Chamberlain (1983), Chamberlain and Rothschild (1983), Connor (1984), Reisman (1988) and Gilles and LeRoy (1991), amongst others.

central insight of Merton's model is that when investors hold under diversified portfolios, idiosyncratic risk should be priced, leading to a positive premium for bearing idiosyncratic risk. This is in sharp contrast to the implications of the CAPM and the CCAPM, which are predicated on frictionless markets with no role for idiosyncratic risk.

In this paper, we modify Merton's model. Our point of departure is the empirical observation that average idiosyncratic volatility varies considerably over time. This is illustrated in Figure 1, which shows average idiosyncratic volatility from July 1931 to December 2014. The figure shows that the variation in average idiosyncratic volatility over the entire time series is large. Even within decades, average idiosyncratic volatility can vary significantly.⁴ Our economic intuition is that since the marginal benefit from diversification is likely to be higher in states of the world characterized by high average idiosyncratic volatility, the idiosyncratic risk premium should be lower in such periods (and vice versa). This is perhaps most obvious in periods like the financial crisis of 2008 when diversification was especially valuable. The implication is that time series variation in average idiosyncratic volatility should lead to a state-dependent risk premium. In contrast to Merton's original model where the risk premium is positive and constant, in our modification, the risk premium varies inversely with the degree of diversification, which in turn, varies with average idiosyncratic volatility. This leads to an asset pricing

⁴ Campbell, Lettau, Malkiel and Xu (2001) argue that average idiosyncratic volatility has increased over time, although that conclusion is controversial since much of the attributed increase occurred in the 1990s. For our purpose, what matters is not an increase in average idiosyncratic volatility but time series variation over an investor's investment horizon.

model where the time series variation in the idiosyncratic risk premium is linked to average idiosyncratic volatility. We illustrate this in Figure 2, highlighting the difference between Merton's (1987) formulation and our modification.

Our aim in the empirical section is to test the implications of the model developed in this paper; an additional outcome is that our results shed light on the (mostly empirical) idiosyncratic volatility literature. All prior investigations of the Merton model, whether re-examining the theory, or its empirical content, ignore time series variation in average idiosyncratic volatility. For instance, Wu et al. (1996) allow for heterogeneous expectations and short-sale restrictions, which generate offsetting effects, but remain unconditional. Empirical investigations are far more voluminous. Most prominent and puzzling is Ang et al. (2006, 2009) who find that contrary to Merton's (1987) prediction, there is a negative relation between expected returns and lagged idiosyncratic volatility. Fu (2009) questions this result, claiming that it is expected, rather than lagged, idiosyncratic volatility that should matter. He finds a positive relation between contemporaneous returns and expected idiosyncratic volatility. Guo, Kassa and Ferguson (2014) point out that Fu's (2009) findings are driven by a look-ahead bias in his tests, and that there is in fact no statistically discernible relation between average returns and expected idiosyncratic volatility.

All of these tests, as well as numerous others that seek to understand this connection, are unconditional.⁵ Our formal model says that the premium should be positive and for the model to be meaningful, that pricing should be conditional. If the relevant state variable in conditional pricing was persistent, or deviated by small amounts in the time series, the economic impact of conditional versus unconditional pricing would be empirically unimportant. Indeed, this is precisely the point that Lewellen and Nagel (2006) make with respect to tests of the conditional CAPM – that CAPM betas move so slowly that conditional tests are not very different from unconditional tests. That is clearly not the case for idiosyncratic volatility; Figure 1 shows that average idiosyncratic volatility varies substantially over time. Since most existing empirical evidence is based on unconditional tests, it cannot be used to draw inferences about our conditional version of Merton’s model.

The most direct way to determine if the model has any traction in the data is to ask whether in the cross-section, the risk premium on idiosyncratic risk is positive and depends on average idiosyncratic volatility. We do so by estimating Fama-MacBeth regressions of monthly returns on contemporaneous expected idiosyncratic volatility, scaled by expected average idiosyncratic volatility. Scaling by expected average idiosyncratic volatility is important from both a theoretical and empirical standpoint.

⁵ The results in Ang et al. (2006) spawned a large literature attempting to explain this idiosyncratic risk “puzzle”. A partial list of papers includes Bali and Cakici (2008), Chen and Petkova (2012), Duarte et al. (2014), Han and Lesmond (2011), Herskovic et al. (2015), Hou and Lo (2016), and Spiegel and Wang (2006). Idiosyncratic risk is also often invoked as an impediment to arbitrage. See Pontiff (2006) for a detailed discussion.

Conceptually, the scaling variable is not ad hoc and follows directly from our theory – it lies at the very heart of the model which says that the marginal benefit of diversification is high when expected average idiosyncratic volatility is high. The fact that the model identifies the relevant state variable is a significant advantage, particularly in light of Cochrane’s (2001) assertion that the conditional CAPM is technically not testable because the econometrician cannot know the “right” state variable. Empirically, the scaling allows us to test the conditional model in a unified framework without resorting to subsample tests with limited power.

In US data from 1931 to 2014, controlling for conditional market betas, the slope on expected idiosyncratic volatility scaled by expected average idiosyncratic volatility is positive. This is in stark contrast to the unconditional idiosyncratic volatility literature which finds a negative slope or no relation between idiosyncratic volatility and expected returns. We also estimate similar regressions in markets outside the US. Since the tests require an adequate cross-section and a time series of returns, we restrict our attention to Canada, France, Japan and the UK. In these markets too, the slopes on scaled expected idiosyncratic volatility are positive and statistically significant. Overall, the data appear to be consistent with a conditional version of Merton (1987) in which the positive premium for idiosyncratic risk varies over time with average idiosyncratic risk.

A natural question that arises is whether the slopes on scaled expected idiosyncratic volatility are sensitive to the inclusion of size, momentum, and other empirically motivated variables. Our purpose is to evaluate the theoretical model developed in this

paper. The null hypothesis against which our (and Merton's) model should be judged is the CAPM, not an empirically motivated factor model. This is because the CAPM becomes a special case of our model when information costs go to zero. One could potentially generate any number of variables from the so-called factor zoo that drive out a theoretically motivated construction. Our position is that there is something to be learned from the conditional model, even if one can find factors that dominate it empirically. Our results suggest that perhaps it is premature to reject the ideas in Merton (1987).

The paper is organized as follows: in section 2, we present the model, both in summary and detail form. Section 3 describes our sample, measurement approach, and results. Section 4 concludes.

2. The Model

We provide a summary of the model and its intuition below, followed by a formal derivation. In the formal derivation, we describe Merton's (1987) original formulation as well as our critical modifications.

2.1 Model Summary

Merton (1987) presents a model where investors are under-diversified, the market portfolio is not mean – variance efficient, the CAPM does not hold and idiosyncratic risk is priced in equilibrium. In this paper we extend the Merton model by making two modifications:

1. We assume that the fraction of all investors who know about a security is proportional to its market value, relative to the value of the market portfolio. An intuitively appealing implication of this is that the idiosyncratic risk premium varies inversely with the average number of securities in an investor's portfolio.
2. In addition to the conditions in Merton (1987), we require that, in equilibrium, there is no incentive for investors to further diversify. We achieve this by imposing the condition that the marginal increase in utility due to increased diversification is offset by the marginal disutility due to the (implicit) costs, I , of information acquisition. As a result, the degree of diversification varies inversely with the costs of diversification and directly with average idiosyncratic volatility.

The rest of our model follows Merton (1987): investors are risk averse, have identical preferences, are price-takers, have the same initial wealth, are mean variance optimizers, and have conditional homogenous beliefs. Investors are less than fully diversified as they only invest in a security if they 'know' about that security in the sense that they know the mean and variance of its return distribution. This leads to an asset-pricing model with clear, testable implications.

The equilibrium expected return on security i in this model is:

$$\bar{R}_i = R_f + b_i \bar{b} \delta + \frac{\sigma_i^2 \delta}{Q^*} \quad (1)$$

where δ is the coefficient of risk aversion, R_f is the risk free rate, σ_i^2 is the idiosyncratic volatility of security i , b_i is its beta, \bar{Q}^* is the average number of stocks held by an investor in equilibrium, and \bar{b} is the average beta of the investor's portfolio. As in Merton (1987), there is a positive premium for idiosyncratic volatility. The key deviation from his model is that the parameter \bar{Q}^* , representing portfolio diversification, is determined in equilibrium as follows:

$$\bar{Q}^* = \sqrt{\frac{\delta \bar{\sigma}^2}{2I}} \quad (2)$$

\bar{Q}^* is determined by risk aversion, average idiosyncratic volatility ($\bar{\sigma}^2$) and the cost of information acquisition (I), which accords with our intuition. Combining equations (1) and (2), we can express (1) as:

$$\bar{R}_i = R_f + b_i \bar{b} \delta + \pi \sigma_i^2 \quad (3)$$

where

$$\pi = \sqrt{\frac{2I\delta}{\bar{\sigma}^2}}$$

is the state dependent idiosyncratic risk premium. Equation (3) highlights the role of both average idiosyncratic volatility ($\bar{\sigma}^2$) and the costs of information (I) in portfolio diversification. In the limiting case, with perfect information ($I = 0$), investors are fully diversified and the idiosyncratic risk premium π disappears. When I is not zero, changes

in average idiosyncratic volatility influence the disutility of under-diversification and therefore the idiosyncratic risk premium.⁶

2.2 The Formal Model

The economy has N firms, $N \gg 1$. The return \tilde{R}_i from investing in firm i has a factor structure:

$$\tilde{R}_i = \bar{R}_i + b_i \tilde{Y} + \sigma_i \tilde{\varepsilon}_i, \quad i = 1, \dots, N \quad (4)$$

where \tilde{Y} is a common factor with $E(\tilde{Y}) = 0$, $E(\tilde{Y}^2) = 1$, b_i is the factor loading of security i , $\tilde{\varepsilon}_i$ is a firm-specific random variable with

$$E(\tilde{\varepsilon}_i) = E(\tilde{\varepsilon}_i | \tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_{i-1}, \tilde{\varepsilon}_{i+1}, \dots, \tilde{\varepsilon}_N, Y) = 0, \quad i = 1, \dots, N \quad (5)$$

$E(\tilde{\varepsilon}_i^2) = 1$, σ_i^2 is the idiosyncratic volatility of security i , and $\bar{\sigma}^2$ is the value weighted average idiosyncratic volatility across the N securities. \bar{R}_M denotes the value weighted expected return of the N securities.

In addition to the N securities issued by firms, the economy has two “inside” securities with zero net supply:

- (a) a ‘*factor mimicking*’ security with return, $\tilde{R}_{N+1} = \bar{R}_{N+1} + \tilde{Y}$
- (b) a riskless security with return R_f

⁶ Bekaert, Hodrick and Zhang (2012), Brown and Kapadia (2007) and others offer explanations for the source of time series variation in average idiosyncratic volatility, but for our purpose, it is exogenous and outside the model.

The economy has K investors, $K \gg N$. Investors are risk averse, with identical mean-variance preferences:

$$U_k = E(\tilde{R}^k) - \frac{\delta}{2} \text{Var}(\tilde{R}^k), \quad k = 1, \dots, K \quad (6)$$

\tilde{R}^k denotes the portfolio return, and δ is the coefficient of risk aversion. Investors are price takers and assumed to have identical initial wealth W_o , which we normalize to 1.

An investor only includes security i in his portfolio if he is “informed” in the sense that he knows $(\bar{R}_i, b_i, \sigma_i^2)$. Information is costly and as a consequence investor k selects only a subset of the N available securities to include in his portfolio.⁷ We assume that the securities he selects, Q_k are much smaller than N ($Q_k \ll N$) and that the probability of selecting a firm is proportional to its value relative to the market portfolio. Θ_k is the set of integers that index the Q_k firms selected by investor k .⁸

In addition to firm-specific knowledge, each investor’s information set contains common knowledge: $(R_f, \bar{R}_{N+1}, \bar{R}_M, \bar{\sigma}^2)$.

Equilibrium in capital markets is characterized as follows:

- (a) Given the set of securities selected, each investor chooses an optimal portfolio.
- (b) Markets clear.

The optimal portfolio holdings, for any investor k is determined as follows:

⁷ These subsets will in general differ across the K investors.

⁸ They are a subset of the first N natural numbers.

From (4) and (6), an investor's portfolio return can be specified as:

$$\tilde{R}^k = \bar{R}^k + b^k \tilde{Y} + \sigma^k \tilde{\varepsilon}^k \quad (7)$$

where:

$$b^k = \sum_{i \in \Theta_k} w_i^k b_i + w_{N+1}^k \quad (8)$$

$$(\sigma^k)^2 = \sum_{i \in \Theta_k} (w_i^k)^2 \sigma_i^2 \quad (9)$$

w_i^k and w_{N+1}^k denote the fraction of investor k 's wealth allocated to security i and $N+1$

. The expected portfolio return and variance are:

$$E(\tilde{R}^k) = R_f + b^k (\bar{R}_{N+1} - R_f) + \sum_{i \in \Theta_k} w_i^k \Delta_i \quad (10)$$

$$Var(\tilde{R}^k) = (b^k)^2 + \sum_{i \in \Theta_k} (w_i^k)^2 \sigma_i^2 \quad (11)$$

where:

$$\Delta_i = (\bar{R}_i - R_f) - b_i (\bar{R}_{N+1} - R_f), \quad i \in \Theta_k \quad (12)$$

The investor's optimal portfolio choice is the solution to the following problem:

$$\max_{\{b^k, w_i^k\}} \left[E(\tilde{R}^k) - \frac{\delta}{2} Var(\tilde{R}^k) \right], \quad i \in \Theta_k \quad (13)$$

$$\text{Subject to } \sum_{i \in \Theta_k} w_i^k + w_{N+1}^k + w_f^k = 1$$

From (10) and (11), the first-order conditions for (13) are:

$$\bar{R}_{N+1} - R_f - b^k \delta = 0 \quad (14)$$

$$\Delta_i - w_i^k \sigma_i^2 \delta = 0, \quad i \in \Theta_k \quad (15)$$

From (8), (14), and (15), the investor's optimal portfolio solution is:

$$b^k = \frac{(\bar{R}_{N+1} - R_f)}{\delta} \quad (16)$$

$$w_i^k = \frac{\Delta_i}{\sigma_i^2 \delta}, \quad i \in \Theta_k \quad (17)$$

$$w_{N+1}^k = b^k - \sum_{i \in \Theta_k} w_i^k b_i \quad (18)$$

$$w_f^k = 1 - b^k + \sum_{i \in \Theta_k} w_i^k (b_i - 1) \quad (19)$$

We aggregate to determine equilibrium expected returns. From (16), all investors choose the same b^k . Let $b^k = B$, $k = 1, \dots, K$. Thus, from (16), we have:

$$\bar{R}_{N+1} = R_f + B\delta \quad (20)$$

From (17), the aggregate demand for security i is:

$$D_i = \sum_{k=1}^{K_i} W_o w_i^k = \sum_{k=1}^{K_i} W_o \frac{\Delta_i}{\sigma_i^2 \delta} \quad (21)$$

In the equation above, K_i is the number of investors who know about the firm i .

From (18) and (19), the aggregate demand for "inside" securities is:

$$D_{N+1} = \sum_{k=1}^K W_o w_{N+1}^k = \sum_{k=1}^K W_o B - \sum_{i=1}^N b_i D_i \quad (22)$$

$$D_f = \sum_{k=1}^K W_o w_f^k = \sum_{k=1}^K W_o - \sum_{i=1}^{N+1} D_i \quad (23)$$

In equilibrium the demand for these securities is zero: $D_{N+1} = D_f = 0$. As a consequence

$$B = \frac{\sum_{i=1}^N b_i D_i}{\sum_{k=1}^K W_o} = \sum_{i=1}^N x_i b_i = \bar{b} \quad (24)$$

where x_i is the fraction of investors' total wealth allocated to security i . Using (24), we

can rewrite (20) as:

$$\bar{R}_{N+1} = R_f + \bar{b} \delta \quad (25)$$

If V_i denotes the equilibrium value of firm i , then

$$x_i = \frac{V_i}{\sum_{k=1}^K W_o} \quad (26)$$

is the fraction of investors' total wealth invested in firm i . Market clearing implies that

$V_i = D_i$, and hence:

$$x_i = \frac{V_i}{\sum_{k=1}^K W_o} = \frac{D_i}{\sum_{k=1}^K W_o} = q_i \frac{\Delta_i}{\sigma_i^2 \delta} \quad (27)$$

and

$$q_i = \frac{\sum_{k=1}^{K_i} W_o}{\sum_{k=1}^K W_o} = K_i / K \quad (28)$$

where q_i is the fraction of investors who invest in firm i . Equation (27) corresponds to equation 15 in Merton (1987).

Our first modification is that we assume that probability of an investor knowing about a security is proportional to the security's market capitalization. This assumption

suggests that the fraction of all investors who know about a security is proportional to the weight of the security in the market portfolio.⁹ This implies that q_i is proportional to x_i

$$x_i = \phi q_i \quad (29)$$

Using (17), (27), and (29),

$$w_i^k = \frac{x_i}{q_i} = \phi, \quad i \in \Theta_k \quad (30)$$

Since

$$\sum_{k=1}^K \left[\sum_{i \in \Theta_k} w_i^k + w_{N+1}^k + w_f^k \right] = K \quad (31)$$

using (30) we get

$$K = \sum_{k=1}^K \sum_{i \in \Theta_k} \phi + \sum_{k=1}^K (w_{N+1}^k + w_f^k) = \sum_{k=1}^K \phi Q_k = \phi \sum_{k=1}^K Q_k \quad (32)$$

Here we have used the observation that the number of firms in Θ_k is Q_k and that the holdings of security $N+1$ and the risk-free asset sum to zero across all investors.

Hence

$$\phi = 1 / \frac{1}{K} \sum_{k=1}^K Q_k = 1 / \bar{Q} \quad (33)$$

⁹ We are implicitly assuming the law of large numbers, as investors $N \gg I$. The law of large numbers implies an equality between the actual fraction of investors who know about a security and the probability of knowing it.

where $\bar{Q} = \frac{1}{K} \sum_{k=1}^K Q_k$ is the average number of securities in a portfolio.

From (18), (19), (24), (30), (33), we have:

$$w_i^k = \frac{1}{\bar{Q}} \quad (34)$$

$$w_{N+1}^k = \bar{b} - \sum_{i \in \Theta_k} \frac{b_i}{\bar{Q}} \quad (35)$$

$$w_f^k = 1 - \bar{b} + \sum_{i \in \Theta_k} \frac{b_i - 1}{\bar{Q}} \quad (36)$$

As noted in (34), w_i^k is the same for each investor in firm i , while w_{N+1}^k, w_f^k can be different across investors. From (10), (25-29) and (33), we observe that the expected security returns

$$\bar{R}_i = R_f + b_i \bar{b} \delta + \frac{\sigma_i^2 \delta}{\bar{Q}}, \quad i = 1, \dots, N \quad (37)$$

are linear in idiosyncratic volatility and the idiosyncratic risk premium ($\frac{\delta}{\bar{Q}}$) varies inversely with the average number of securities:

Our second modification is that we assume that in equilibrium investors have no incentive to increase their holdings Q_k . We achieve this by imposing the condition that the marginal increase in utility due to increased diversification is offset by the disutility due to the (implicit) costs of information acquisition, I .

From (10-12) and (34-37), the expected portfolio return and portfolio variance are:

$$E(\tilde{R}^k) = R_f + \bar{b}^2 \delta + \frac{\delta}{\bar{Q}^2} \sum_{i \in \Theta_k} \sigma_i^2 \quad (38)$$

$$Var(\tilde{R}^k) = \bar{b}^2 + \frac{1}{\bar{Q}^2} \sum_{i \in \Theta_k} \sigma_i^2 \quad (39)$$

Thus, the utility of investor k is:

$$U_k = E(\tilde{R}^k) - \frac{\delta}{2} Var(\tilde{R}^k) = R_f + \frac{\bar{b}^2 \delta}{2} + \frac{\delta}{2\bar{Q}^2} \sum_{i \in \Theta_k} \sigma_i^2 \quad (40)$$

With access to an additional security a , where a is an element of $\{N\} \setminus \Theta_k$, the investor's new optimal portfolio allocation is the solution to the maximization problem:

$$\max_{\{b^k, w_i^k\}} \left[E(\tilde{R}^k) - \frac{\delta}{2} Var(\tilde{R}^k) \right], \quad i \in \Theta_k \cup \{a\} \quad (41)$$

The resulting expected portfolio return and variance *conditional* on selecting security a are:

$$E(\tilde{R}_k | a) = R_f + \bar{b}^2 \delta + \frac{\delta}{\bar{Q}^2} \left(\sum_{i \in \Theta_k} \sigma_i^2 + \sigma_a^2 \right) \quad (42)$$

$$Var(\tilde{R}^k | a) = \bar{b}^2 + \frac{1}{\bar{Q}^2} \left(\sum_{i \in \Theta_k} \sigma_i^2 + \sigma_a^2 \right) \quad (43)$$

Since the probability of selecting an additional security is proportional to its market capitalization, the expected idiosyncratic volatility of the additional security is the value weighted average idiosyncratic volatility across the securities he doesn't hold:

$$E[\sigma_a^2] = \sum_{i \in \{N\} \setminus \Theta_k} x_i \sigma_i^2 \quad (44)$$

where $\{N\}$ is the set of integers $1, \dots, N$. Since $N \gg Q_k$ (the investor only knows a small fraction of all securities) $E[\sigma_a^2]$ can be approximated as:

$$E[\sigma_a^2] \cong \sum_{i=1}^N x_i \sigma_i^2 = \overline{\sigma^2} \quad (45)$$

Using (45), the *unconditional* expected portfolio return and variance can be written as:

$$E(\tilde{R}^k) = R_f + \bar{b}^2 \delta + \frac{\delta}{\bar{Q}^2} \left(\sum_{i \in \Theta_k} \sigma_i^2 + \overline{\sigma^2} \right) \quad (46)$$

$$Var(\tilde{R}^k) = \bar{b}^2 + \frac{1}{\bar{Q}^2} \left(\sum_{i \in \Theta_k} \sigma_i^2 + \overline{\sigma^2} \right) \quad (47)$$

and the expected utility of investor k becomes:

$$U'_k = E(\tilde{R}^k) - \frac{\delta}{2} Var(\tilde{R}^k) = R_f + \frac{\bar{b}^2 \delta}{2} + \frac{\delta}{2\bar{Q}^2} \left(\sum_{i \in \Theta_k} \sigma_i^2 + \overline{\sigma^2} \right) \quad (48)$$

Comparing (40) with (48), we see that the expected increase in utility ΔU_k is:

$$\Delta U_k = U'_k - U_k = \frac{\delta}{2\bar{Q}^2} \overline{\sigma^2} \quad (49)$$

Note that as a result of the approximation in (45) ΔU_k is same for all investors

and we have

$$\Delta U = \Delta U_k = \frac{\delta}{2\bar{Q}^2} \overline{\sigma^2} \quad \forall k$$

For investors to have no incentive to learn about an additional security ΔU must be no greater than the disutility of the cost of information acquisition I .¹⁰

$$\Delta U - I \leq 0 \quad (50)$$

From (49) and (50) we have:

$$\frac{\delta}{2\bar{Q}^2} \sigma^2 = I \quad (51)$$

where \bar{Q}^* is the average number of stocks held by investor k in equilibrium.

Hence:

$$\bar{Q}^* = \sqrt{\frac{\delta \sigma^2}{2I}} \quad (52)$$

Using (37) and (52), the expected return on asset i can be written as:

$$\bar{R}_i = R_f + b_i \bar{b} \delta + \frac{\sigma_i^2 \delta}{\bar{Q}^*} \quad (53)$$

After substituting for \bar{Q}^* , we can rewrite equation 53 as

$$\bar{R}_i = R_f + b_i \bar{b} \delta + \pi \sigma_i^2 \quad (54)$$

where

$$\pi = \sqrt{\frac{2I\delta}{\sigma^2}}$$

is the state dependent idiosyncratic risk premium. This corresponds to equation (3) in the model summary above.

¹⁰ In this framework $U(I) = \text{a constant} \times I$. We have normalized the constant to be 1 as it does not affect the subsequent analysis.

3. Empirical tests

3.1 Testing Strategy

To test the model, we transform the beta coefficients in equation (53) to their counterparts with respect to the market, resulting in the following equation:

$$\bar{R}_i = R_f + \beta_i \sigma_m^2 \delta + \frac{\sigma_i^2 \delta}{\bar{Q}^*} (1 - q_i) \quad (55)$$

where $\beta_i = Cov(\tilde{R}_i, \tilde{R}_m) / \sigma_m^2$ and $\sigma_m^2 = Var(\tilde{R}_m)$.

We make the approximation that $1 - q_i \approx 1$ where $1 - q_i$ is the fraction of investors who *do not* hold asset i .¹¹ This results in the following equation that we test:

$$\bar{R}_i = R_f + \beta_i \sigma_m^2 \delta + \frac{\sigma_i^2 \delta}{\bar{Q}^*} \quad (56)$$

Our interest is in the risk premium in the last term of equation 55. The most straightforward way to test the model is to recognize that $\bar{Q}^* = \sqrt{\frac{\delta \sigma^2}{2I}}$. Substituting

this expression into the last term of equation 55 we get

¹¹Alternatively we could assume that $\beta_i \sigma_m^2 - x_i \sigma_i^2 \approx \beta_i \sigma_m^2$ to get to equation 56. This is similar to the approximation in equation 9 in Dybvig and Ross (1985). Quantitatively the approximation is innocuous, the fraction of the market portfolio invested in asset i , x_i , is of the order of 10^{-2} or less. Using typical values for the other parameters, ($\beta_i \approx 1$, $\sigma_m^2 \approx 0.04$, $\sigma_i^2 \approx 0.16$) we see that while $\beta_i \sigma_m^2$ is of the order 10^{-2} , the neglected term $x_i \sigma_i^2$ is of order 10^{-3} . For the indicated parameters the approximation involves using 0.04 instead of the exact value 0.0384.

$$\bar{R}_i = R_f + \beta_i \sigma_m^2 \delta + \gamma \frac{\sigma_i^2}{\sqrt{\sigma^2}} \quad (57)$$

where $\gamma = \sqrt{2I\delta}$

Equations 56 and 57 say that controlling for conditional betas, the average slope of regressions of stock returns on the ratio of expected idiosyncratic volatility to the (square root of) expected average idiosyncratic volatility should be positive. More precisely, the regression coefficient is identified as γ .

3.2 Sample Construction

Our sample of US stocks is derived from the CRSP-Compustat universe with CRSP share codes 10 or 11 that restrict the universe to common stocks, and with exchange codes 1, 2 and 3 corresponding to NYSE, Amex and Nasdaq listed securities. We eliminate stocks with a share price below \$1 at the beginning of the month. The tests are based on a sample period from 1931 to 2014 because we need at least 5 years of data to calculate expected idiosyncratic volatility.

For the international sample, we obtain a time series of market information from Datastream. We start with an unconstrained universe of all firms in the following developed markets between 1990 and 2014: Canada, France, Japan, and the United Kingdom. We restrict our attention to these countries because the tests require an adequate cross-section of securities as well as a reasonable time series. The universe of stocks includes live as well as dead stocks. We apply the sequence of filters described in Goyal and Wahal (2015), retaining only equity issues from the primary exchange of the

country, and ensuring that we only sample local (not cross-listed) stocks. US dollar returns are computed by converting local currency returns using the conversion function built into Datastream, which uses spot rates. Market values are similarly converted to US dollar equivalents.

3.3 Measurement

For each security-month, we estimate daily time series market model regressions of excess stock returns on the excess market return. We use the market to generate residuals because the CAPM serves as the natural theoretical counterpart to Merton's (1987) model and our modification. The idiosyncratic component ε_i from these regressions is assumed to be normally distributed. The model says that that conditional expected returns should be positively related to *expected* (not lagged) idiosyncratic volatility. We model expected idiosyncratic volatility for stock i in month t using the EGARCH process used by Guo, Kassa and Ferguson (2014) as follows.

$$\ln \sigma_{i,t}^2 = a_{i,t-1} + \sum_{l=1}^p b_{i,l,t-1} \ln \sigma_{i,t-l}^2 + \sum_{k=1}^q c_{i,k,t-1} \left\{ \theta_{i,t-1} \left(\frac{\varepsilon_{i,t-k}}{\sigma_{i,t-k}} \right) + \gamma_{i,t-1} \left[\left| \frac{\varepsilon_{i,t-k}}{\sigma_{i,t-k}} \right| - \left(\frac{2}{\pi} \right)^{1/2} \right] \right\} \quad (58)$$

In estimating the above, we ensure that the sample used stops in month $t-1$ so that there is no look-ahead bias in the estimates. As in Guo, Kassa, and Ferguson (2014), we require at least 60 monthly observations to estimate month t idiosyncratic volatility. We consider nine EGARCH specifications, corresponding to values of p and q from $\{1,2,3\}$ and choose the one that converges with the lowest Akaike Information Criterion (AIC).

Estimates of expected idiosyncratic volatility are trimmed at the 95th percentile to prevent outliers from influencing the tests.

We calculate the empirical counterpart of the market-wide average expected idiosyncratic volatility ($\overline{\sigma^2}$) using a two-step process as follows.

$$\sqrt{\sigma_{SL}^2} = \sqrt{\frac{1}{2} \left(\sum_{s=1}^S w_s \sigma_s^2 + \sum_{l=1}^L w_l \sigma_l^2 \right)} \quad (59)$$

where the subscripts s and l refer to small and large stocks respectively, the weights w_s and w_l are market capitalization weights within small and large stocks, and the expected idiosyncratic volatility estimates (σ_i^2) are derived from equation 58 above. We use the NYSE median market capitalization in the prior month to designate each security into small and large stock groups. This process of value-weighting expected idiosyncratic volatility for small and large stocks separately, and then taking a simple average of the two, ensures a balance between small and large stocks. As a robustness check, we also compute average expected idiosyncratic volatility using market-wide equal- and value-weights as below.

$$\sqrt{\sigma_{EW}^2} = \sqrt{\left(\frac{1}{N} \sum_{i=1}^N \sigma_i^2 \right)} \quad (60)$$

$$\sqrt{\sigma_{VW}^2} = \sqrt{\left(\sum_{i=1}^N w_i \sigma_i^2 \right)} \quad (61)$$

where w_i is a market capitalization weight across all stocks. We caution, however, that equal-weighted average idiosyncratic volatility is heavily influenced by the large number

of more volatile small stocks. In the value-weighted average, a small number of large less volatile stocks dominate the calculation.

3.4 Results

Panel A of Table 1 contains average slopes and t-statistics from monthly Fama-MacBeth regressions of returns on conditional market betas measured over the prior 3 months using daily returns (Lewellen and Nagel (2006)), and expected idiosyncratic volatility scaled by the square root of average expected idiosyncratic volatility ($\frac{\sigma_i^2}{\sqrt{\sigma_{SL}^2}}$).

The slopes are multiplied by 100 for expositional convenience. Conditional betas are statistically indistinguishable from zero. This is inconsistent with the model as specified in equation 55. It is, however, consistent with existing evidence that the conditional CAPM does not perform much better than the unconditional CAPM. More importantly, from our perspective, the slopes on expected idiosyncratic volatility scaled by average expected idiosyncratic volatility are positive. In equal-weighted regressions, the slope is 0.91 with a t-statistic of 2.04. In value-weighted regressions, which are less subject to the presence of outliers and to the large number of stocks in the sample, the slope on expected volatility rises to 2.22 with a t-statistic of 4.00.¹²

¹² Since the slopes are equal to $\sqrt{2I\delta}$, it is tempting to make assumptions about either the cost of information (I) or risk reversion (δ), and infer the other. We resist this temptation because the cost of information and risk aversion *jointly* determine the slope.

Panels B and C show estimates when average expected idiosyncratic volatility is measured using equal- or value-weighted averages ($\sqrt{\sigma_{EW}^2}$ and $\sqrt{\sigma_{VW}^2}$ respectively). These approaches do not appear to make a difference to inferences. The coefficients on conditional betas do not move and the slopes on scaled expected idiosyncratic volatility are quite similar.

It is also interesting to examine the variation in the regression slopes over time. Individual coefficients from monthly Fama-MacBeth regressions are quite noisy so we compute 10-year rolling averages. These, along with a 10-year rolling average of average expected idiosyncratic volatility, are plotted in Figure 3. Visual inspection, which is only suggestive, is indicative of an inverse relation between the risk premium and average idiosyncratic volatility.

Prima facie, these results suggest that the data are consistent with a conditional version of Merton's model. Models are parsimonious descriptions of the behavior of homo economicus and are agnostic to countries. It is therefore useful to test them in other countries as a crude out of sample test. Since power is an important consideration, we can only do so in markets that have a sufficiently large cross-section of securities and a long enough time series. Four countries for which we have data meet that criteria: Canada, France, Japan, and the UK. Table 2 contains similar regressions for these countries. As in the US data, the slopes on conditional betas are statistically insignificant. In value-weighted regressions, the slopes on scaled expected idiosyncratic volatility are reliably positive in Canada, Japan and the UK, with t-statistics above 2.00. In France,

the slope is positive but with a t-statistic of only 1.67. In equal-weighted regressions, the slopes on scaled idiosyncratic volatility are positive for Canada and France with t-statistics of 2.29 and 1.65 respectively. In Japan and the UK, the slopes are insignificantly different from zero, suggesting that the relation is weaker in small stocks.¹³

While unconditional tests of Merton (1987) provide little empirical support for his model, our regressions suggest that a conditional version of Merton's model has a footprint in the data. One could complain that these regressions ignore the existing evidence on size, value, profitability, investment, accruals, and other such variables that have explanatory power for returns. This omission is willful. Empirically motivated variables may have explanatory power but do not constitute tests of asset pricing models and are subject to the factor zoo problem. We avoid the inclusion of ad hoc variables to maintain the integrity of the test of the theory in section 2. We include conditional betas because the CAPM generates that natural equilibrium counter to the under-diversification that is at the heart of both Merton's original model and our modification.

4. Conclusion

The key insight in Merton's (1987) model of asset pricing under incomplete diversification is that there should be a positive premium for bearing idiosyncratic risk. We propose a simple, yet important modification to his model – the premium for bearing

¹³ It is tempting to ask whether the returns on portfolios sorted by scaled idiosyncratic volatility are monotonic across the sorting variable. This is not feasible. Since average idiosyncratic volatility does not vary across securities, the scaling variable generates no dispersion; the sorts are effectively just sorts on (unscaled) idiosyncratic volatility, which are uninformative about the empirical content of the model.

idiosyncratic risk should vary with average idiosyncratic risk. When average idiosyncratic risk is high, the marginal benefit from diversification is also high, implying a lower risk premium (and vice versa). This simple intuition delivers a conditional asset pricing model in the spirit of Merton (1987), where the relevant state variable, average idiosyncratic risk, is identified by the theory.

We test the model in the spirit of classical tests of the CAPM (Fama and MacBeth (1973)): regressions of returns on expected idiosyncratic volatility scaled by average expected idiosyncratic risk. The coefficient on this scaled variable is positive in the US between 1931 and 2014. This variable also has a positive slope Canada, France, Japan and the UK between 1990 and 2014. These results are in stark contrast to the negative relation between lagged idiosyncratic volatility and expected returns documented by Ang et al. (2006) and explored by numerous others. The key difference is that the theory demands conditional tests because the state variable (average idiosyncratic risk) is economically meaningful and readily identifiable. Both the model and results suggest that a rejection of the ideas in Merton (1987) might be hasty.

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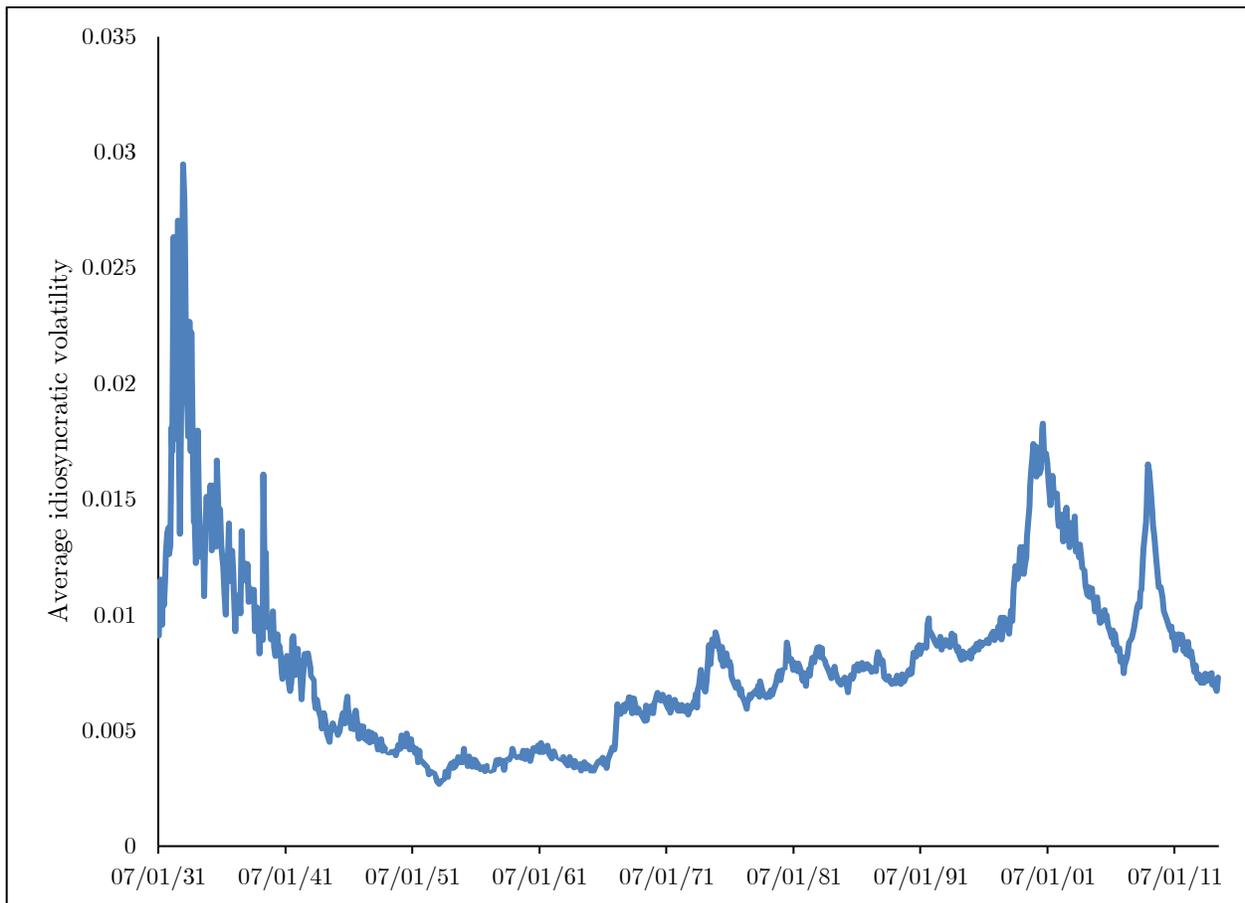


Figure 1. We compute the value-weighted average idiosyncratic volatility for small and large capitalization stocks, and then calculate a simple average of the two to obtain average idiosyncratic volatility for each month. We use NYSE median breaks to separate small and large cap stocks. Each month is classified as a low or high average idiosyncratic risk month if the month's average idiosyncratic volatility is above or below the trailing 10 year average.

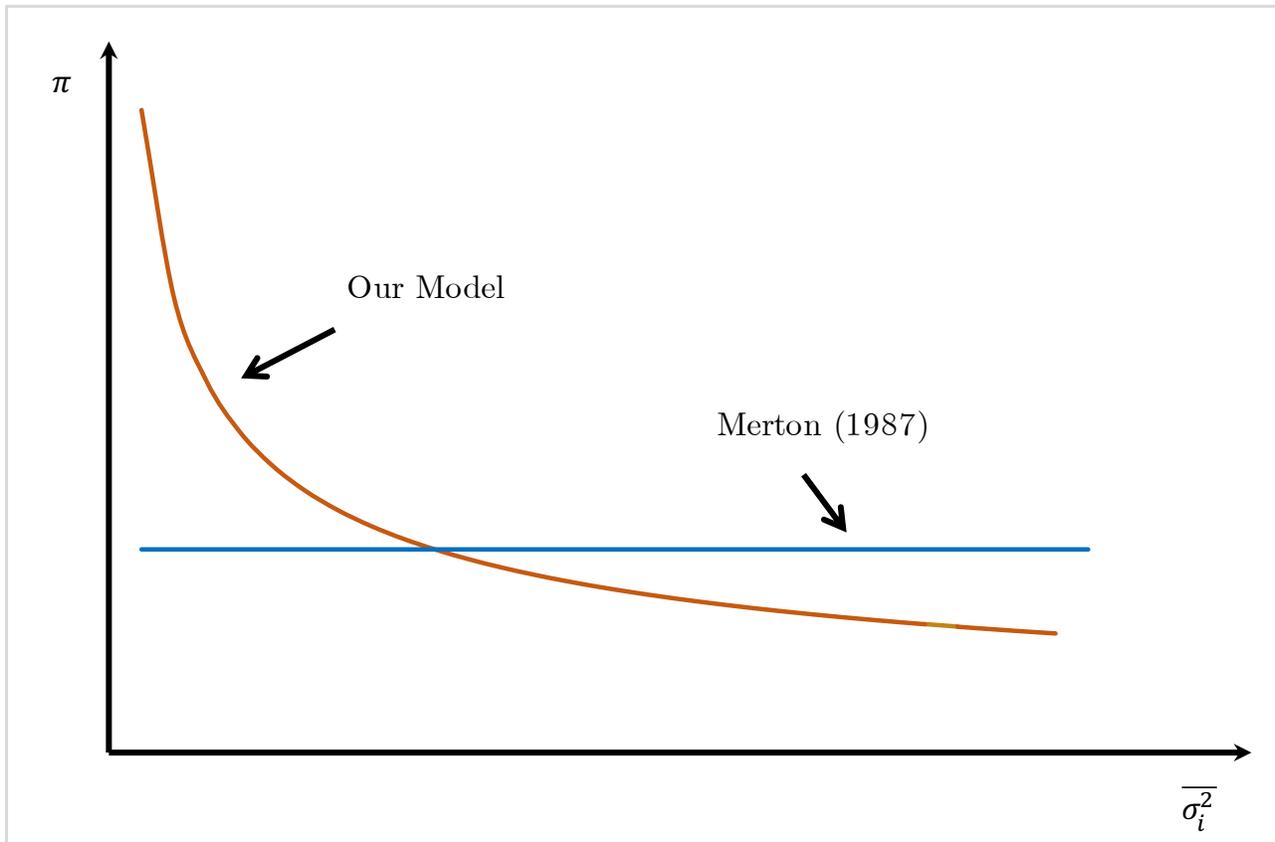


Figure 2: The x-axis shows average idiosyncratic volatility. The y-axis shows the premium associated with idiosyncratic volatility.

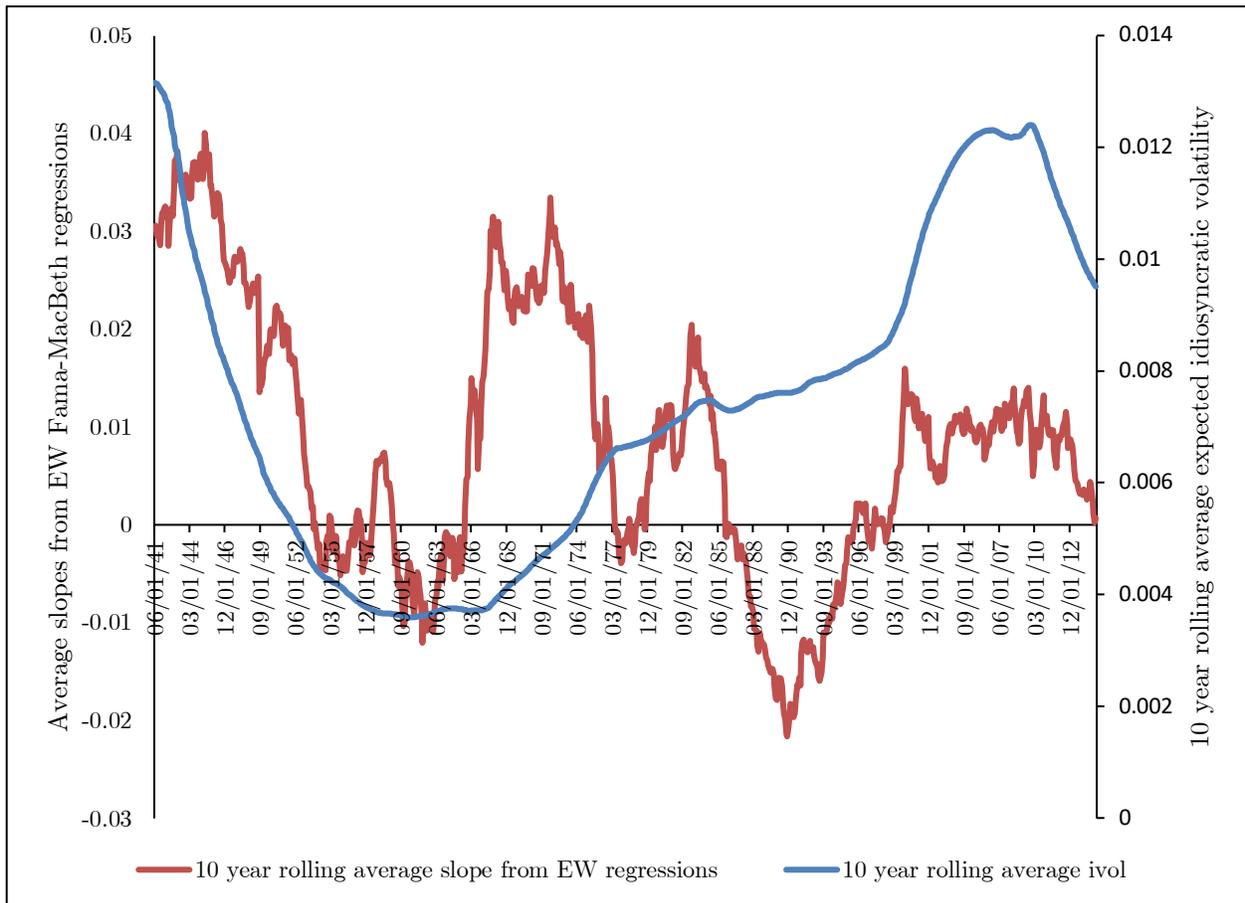


Figure 3: The figure reports 10-year rolling average slopes from the equal-weighted Fama-MacBeth regressions in Table 1, along with 10-year rolling average expected idiosyncratic volatility over the same period.

Table 1

Fama-MacBeth regressions of returns on conditional CAPM beta and scaled expected idiosyncratic volatility for US markets, 1931-2014

The table reports average slope estimates and t-statistics from monthly Fama and MacBeth (1973) regressions on stock returns on conditional market betas and expected idiosyncratic volatility scaled by average expected idiosyncratic volatility. The sample is from July 1931 through 2014. Conditional market betas are measured over the prior three months using daily returns. Expected idiosyncratic volatility is measured over the prior 60 months using an EGARCH(1,3) model but employing the lowest Akaike Information Criterion (AIC) to generate estimates. All coefficients are multiplied by 100. T-statistics are based on Newey-West standard errors with 4 lags.

	Equal-weighted Regressions	Value-weighted Regressions
Panel A		
β	0.02 (0.02)	-0.10 (-1.10)
$\frac{\sigma_i^2}{\sqrt{\sigma_{SL}^2}}$	0.91 (2.04)	2.22 (4.00)
Panel B		
β	0.02 (0.02)	-0.10 (-1.10)
$\frac{\sigma_i^2}{\sqrt{\sigma_{EW}^2}}$	1.00 (1.96)	2.67 (4.19)
Panel C		
β	0.02 (0.02)	-0.10 (-1.10)
$\frac{\sigma_i^2}{\sqrt{\sigma_{VW}^2}}$	0.70 (1.93)	1.68 (3.84)

Table 2

Fama-MacBeth regressions of returns on conditional CAPM beta and scaled expected idiosyncratic volatility for international markets, 1990-2014

The table reports average slope estimates and t-statistics from monthly value-weighted Fama and MacBeth (1973) regressions on stock returns on conditional market betas and expected idiosyncratic volatility scaled by average expected idiosyncratic volatility for four international markets. The sample period is from July 1990 through 2014. Conditional market betas are measured over the prior three months using daily returns. Expected idiosyncratic volatility is measured over the prior 60 months using an EGARCH(1,3) model but employing the lowest Akaike Information Criterion (AIC) to generate estimates. All coefficients are multiplied by 100. T-statistics are based on Newey-West standard errors with 4 lags.

	Canada	France	Japan	UK
Value-Weighted Regressions				
β	0.20 (0.28)	-0.31 (0.59)	0.31 (1.58)	0.32 (1.01)
$\frac{\sigma_i^2}{\sqrt{\sigma_{SL}^2}}$	4.28 (2.18)	3.77 (1.67)	2.61 (2.13)	5.33 (2.92)
Equal-Weighted Regressions				
β	-0.18 (1.35)	-0.18 (1.02)	0.11 (0.60)	0.12 (0.93)
$\frac{\sigma_i^2}{\sqrt{\sigma_{SL}^2}}$.33 (2.29)	1.10 (1.65)	0.85 (0.94)	-0.15 (-0.22)