

NBER WORKING PAPER SERIES

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Working Paper 21773  
<http://www.nber.org/papers/w21773>

NATIONAL BUREAU OF ECONOMIC RESEARCH  
1050 Massachusetts Avenue  
Cambridge, MA 02138  
December 2015

Prepared for the Fed St. Louis-JEDC-SCG-SNB-UniBern Conference, Gerzensee, October 2015. We thank Jarda Borovicka, Riccardo Colacito, and John Stachurski for helpful advice. We also thank the conference participants, including especially Giancarlo Corsetti, Robert Kollmann, Ayhan Kose, and Ravi Ravikumar. James Bullard suggested the analogy with wedges. We will post the code shortly at <https://github.com/NYUEcon/BCFL2016>. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

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NBER Working Paper No. 21773  
December 2015  
JEL No. F31,F41

**ABSTRACT**

In models with recursive preferences, endogenous variation in Pareto weights would be interpreted as wedges from the perspective of a frictionless model with additive preferences. We describe the behavior of the (relative) Pareto weight in a two-country world and explore its interaction with consumption and the real exchange rate.

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Supplementary materials available at <http://www.nber.org/papers/w21773>:  
- data appendix

# 1 Introduction

We explore the effects of recursive preferences and risk in an otherwise standard two-country exchange economy. We focus on the behavior of the (relative) Pareto weight, which characterizes consumption allocations across countries. When preferences are additive over time and across states, as they typically are, the Pareto weight is constant in frictionless environments. But when preferences are recursive and agents consume different goods, the Pareto weight can fluctuate even in frictionless environments. This variation in the Pareto weight acts like a wedge from the perspective of an additive model. Among the potential byproducts are changes in the behavior of consumption and the exchange rate.

The natural comparison is with models that use capital market frictions to resolve some of the anomalous features of the standard model. Baxter and Crucini (1995), Corsetti, Dedola, and Leduc (2008), Heathcote and Perri (2002), Kehoe and Perri (2002), and Kose and Yi (2006) are well-known examples. The frictions in these papers might be viewed as devices to produce variation in the Pareto weight, which is then reflected in prices and quantities. In the language of Chari, Kehoe, and McGrattan (2007), variations in Pareto weights would look like wedges in the frictionless model. The question for both approaches is whether these wedges are similar to those we observe when we confront frictionless models with evidence. Ultimately we would like to understand how the two approaches compare, but for now we're simply trying to understand how the Pareto weight works in models with recursive preferences.

We build in an obvious way on earlier work with multi-good economies by Colacito and Croce (2013, 2014), Colacito, Croce, Ho, and Howard (2014), Kollmann (2015), and Tretvoll (2011, 2013, 2015). We show how their models work and introduce some modest extensions. We also build on the fundamental work on recursive risk-sharing by Anderson (2005), Borovicka (2015), and Collin-Dufresne, Johannes, and Lochstoer (2015). These papers study one-good worlds, and in that respect are simpler than work with multi-good international models, but they lay out the structure of recursive risk-sharing problems. The last paper in the list also describes an effective computational method which we adapt to our environment.

One byproduct is a clearer picture of what drives the dynamics of the Pareto weights. In many one-good economies, Pareto weights aren't stable. Eventually one agent consumes everything. One of the insights of Colacito and Croce (2013) is that home bias and imperfect substitutability between goods can produce stable processes for Pareto weights and consumption shares. That's true here, as well, but we also show how changes in risk aversion and intertemporal substitutability affect the dynamics of the Pareto weight. Relative to the additive case, increasing risk aversion or decreasing intertemporal substitution generates more persistence in the real exchange rate. Since real exchange rates are very persistent in the data, this novel source of persistence is a welcome one.

Risk is a particularly interesting object in this context. Random changes in the relative supply of foreign and domestic goods also affect demand — with recursive preferences — through their impact on future utility. As in many dynamic models, it's not clear here how (if?) we might separate the concepts of supply and demand. A change in the conditional variance of future endowments, however, works only through the second channel; supplies (endowments) do not change. Risk affects allocations through its impact on the Pareto weight without any direct impact on supply.

All of these results are based on global numerical solutions to the Pareto problem. These solutions take much more computer time than the perturbation methods used in most related work, but they come with greater assurance that the solution is accurate even in states far from the mean of the distribution.

Notation and terminology. We use Latin letters for variables and Greek letters for parameters. Variables without time subscripts are means of logs. We use the term steady state to refer to the mean of the log of a variable. Thus steady state  $x$  refers to the mean value of  $\log x_t$ .

## 2 A recursive two-country economy

We study an exchange version of the Backus, Kehoe, and Kydland (1994) two-country business cycle model. Like their model, ours has two agents (one for each country), two intermediate goods (“apples” and “bananas”), and two final goods (one for each

country). Unlike theirs, ours has (i) exogenous output of intermediate goods, (ii) a unit root in productivity, (iii) recursive preferences, and (iv) stochastic volatility in productivity growth in one country. The first is for convenience. The second allows us to produce realistic asset prices. We focus on the last two, specifically their role in the fluctuations in consumption and exchange rates.

*Preferences.* We use the recursive preferences developed by Epstein and Zin (1989), Kreps and Porteus (1978), and Weil (1989). Utility from date  $t$  on in country  $j$  is denoted  $U_{jt}$ . We define utility recursively with the time aggregator  $V$ :

$$U_{jt} = V[c_{jt}, \mu_t(U_{jt+1})] = [(1 - \beta)c_{jt}^\rho + \beta\mu_t(U_{jt+1})^\rho]^{1/\rho}, \quad (1)$$

where  $c_{jt}$  is consumption in country  $j$  and  $\mu_t$  is a certainty equivalent function. The parameters are  $0 < \beta < 1$  and  $\rho \leq 1$ . We use the power utility certainty equivalent function,

$$\mu_t(U_{jt+1}) = [E_t(U_{jt+1}^\alpha)]^{1/\alpha}, \quad (2)$$

where  $E_t$  is the expectation conditional on the state at date  $t$  and  $\alpha \leq 1$ . Preferences reduce to the traditional additive case when  $\alpha = \rho$ .

Both  $V$  and  $\mu$  are homogeneous of degree one (hd1). The two functions together have the property that if consumption is constant at  $c$  from date  $t$  on, then  $U_{jt} = c$ .

In standard terminology,  $1/(1 - \rho)$  is the intertemporal elasticity of substitution (IES) (between current consumption and the certainty equivalent of future utility) and  $1 - \alpha$  is risk aversion (RA) (over future utility). The terminology is somewhat misleading, because changes in  $\rho$  affect future utility, the thing over which we are risk averse. As in other multi-good settings, there's no clean separation between risk aversion and substitutability.

*Technology.* Each country specializes in the production of its own intermediate good, “apples” in country 1 and “bananas” in country 2. In the exchange case we study, production in country  $j$  equals its exogenous productivity:

$$y_{jt} = z_{jt}. \quad (3)$$

Intermediate goods can be used in either country. The resource constraints are

$$a_{1t} + a_{2t} = y_{1t} \quad (4)$$

$$b_{1t} + b_{2t} = y_{2t}, \quad (5)$$

where  $b_{1t}$  is the quantity of country 2's good imported by country 1 and  $a_{2t}$  is the quantity of country 1's good imported by country 2.

Agents consume final 'goods, composites of the intermediate goods defined by the Armington aggregator  $h$ :

$$c_{1t} = h(a_{1t}, b_{1t}) = [(1 - \omega)a_{1t}^\sigma + \omega b_{1t}^\sigma]^{1/\sigma} \quad (6)$$

$$c_{2t} = h(b_{2t}, a_{2t}) \quad (7)$$

with  $0 \leq \omega \leq 1$  and  $\sigma \leq 1$ . The elasticity of substitution between the two intermediate goods is  $1/(1 - \sigma)$ . The function  $h$  is also hd1.

We will typically use  $\omega < 1/2$ , which puts more weight on the home good in the production of final goods. This "home bias" in final goods production is essential. If  $\omega = 1/2$ , the two final goods are the same. With identical hd1 utility functions, as we have here, any optimal allocation involves a constant Pareto weight and proportional consumption paths.

*Shocks.* Fluctuations in this economy reflect variation in the productivities cum endowments  $z_{jt}$  and the conditional variance of the first one. Logs of productivities have unit roots and are cointegrated:

$$\begin{bmatrix} \log z_{1t+1} \\ \log z_{2t+1} \end{bmatrix} = \begin{bmatrix} \log g \\ \log g \end{bmatrix} + \begin{bmatrix} 1 - \gamma & \gamma \\ \gamma & 1 - \gamma \end{bmatrix} \begin{bmatrix} \log z_{1t} \\ \log z_{2t} \end{bmatrix} + \begin{bmatrix} v_t^{1/2} w_{1t+1} \\ v_t^{1/2} w_{2t+1} \end{bmatrix}, \quad (8)$$

with  $0 < \gamma < 1/2$ . The only asymmetry in the model is the conditional variance of productivity. The conditional variance  $v$  of  $\log z_{2t+1}$  is constant. The conditional variance  $v_t$  of  $\log z_{1t+1}$  is AR(1):

$$v_{t+1} = (1 - \varphi_v)v + \varphi_v v_t + \tau w_{3t+1}. \quad (9)$$

The innovations  $\{w_{1t}, w_{2t}, w_{3t}\}$  are standard normals and are independent of each other and over time.

### 3 Features and parameter values

We describe some of the salient features of the model and the benchmark parameter values we use later on.

#### Features

*Consumption frontier.* The resource constraints define a possibilities frontier for consumption. We picture several examples in Figure 1,  $z_{1t} = z_{2t} = 1$ . In each case, we compute the maximum quantity  $c_{1t}$  consistent with a given quantity  $c_{2t}$ , the resource constraints (4,5), and the Armington aggregators (6,7). The shape depends on the aggregator. When  $\omega = 1/2$ , the two final goods are the same and the tradeoff is linear. When  $\omega \neq 1/2$ , the frontier is convex. The degree of convexity depends on the elasticity of substitution  $1/(1 - \sigma)$  in the Armington aggregator.

In a competitive equilibrium, the slope of the consumption frontier is (minus) the relative price of consumption in the two countries,  $p_{2t}/p_{1t} = e_t$ , the real exchange rate. From the figure, we can imagine variation in this price produced either by moving along the frontier or by changing the frontier through movements in the quantities of intermediate goods.

*Marginal rates of substitution.* Competitive equilibria and Pareto optima equate agents' marginal rates of substitution (mrs for short). With recursive preferences, the intertemporal mrs of agent  $j$  is

$$m_{jt+1} = \beta \left( \frac{c_{jt+1}}{c_{jt}} \right)^{\rho-1} \left( \frac{U_{jt+1}}{\mu_t(U_{jt+1})} \right)^{\alpha-\rho}. \quad (10)$$

The last term summarizes the impact of recursive preferences. If  $\alpha = \rho$ , preferences are additive and the term disappears. Otherwise anything that affects future utility can play a role in the marginal rate of substitution, hence in allocations. A change in risk, for example, affects future utility, and through it consumption quantities and prices. Persistence is critical here, because more persistent shocks have a larger impact on future utility. The discount factor  $\beta$  is similar: the larger it is, the greater the weight on future utility and the greater the impact on the mrs.

Note, too, the dynamics built into the recursive term. Its log is a risk adjustment plus white noise. The log of the numerator is

$$\log U_{jt+1} = E_t(\log U_{jt+1}) + [\log U_{jt+1} - E_t(\log U_{jt+1})],$$

the mean plus a white noise innovation. The log of the denominator is

$$\begin{aligned} \log \mu_t(U_{jt+1}) &= \alpha^{-1} \log E_t(e^{\alpha \log U_{jt+1}}) \\ &= E_t(\log U_{jt+1}) + \alpha^{-1} [\log E_t(e^{\alpha \log U_{jt+1}}) - E_t(\alpha \log U_{jt+1})]. \end{aligned}$$

The term in square brackets is the entropy of  $U_{jt+1}^\alpha$  and is positive. We multiply by  $\alpha$  to get what we term the risk adjustment, which is negative if  $\alpha$  is. The difference then is the innovation minus the risk adjustment. The innovation increases the volatility of the mrs, which is the primary source of success in asset pricing applications.

The mrs (10) is measured in units of agent  $j$ 's consumption good, whose price is  $p_{jt}$ . We refer to the relative price of the two consumption goods as the real exchange rate:  $e_t = p_{2t}/p_{1t}$ . The two mrs's are then connected by  $m_{2t+1} = (e_{t+1}/e_t)m_{1t+1}$ . We'll derive this in the next section, but it should be evident here the dynamics of the exchange rate reflect the mrs's. If the two mrs's are close to white noise, as we suggested, then the depreciation rate  $e_{t+1}/e_t$  has the same property.

*Productivity dynamics.* One way to think about our log productivity process (8) is that their average is a martingale and their difference is stable. Denote half the sum and half the difference by

$$\begin{aligned} \log \bar{z} &= (\log z_1 + \log z_2)/2 \\ \log \hat{z} &= (\log z_1 - \log z_2)/2. \end{aligned}$$

Then we can express the underlying productivities by  $\log z_1 = \log \bar{z} + \log \hat{z}$  and  $\log z_2 = \log \bar{z} - \log \hat{z}$ . Equation (8) implies that half the sum,

$$\log \bar{z}_{t+1} = \log g + \log \bar{z}_t + (v_t^{1/2} w_{1t+1} + v_t^{1/2} w_{2t+1})/2,$$

is a martingale with drift. The difference,

$$\log \hat{z}_{t+1} = (1 - 2\gamma) \log \hat{z}_t + (v_t^{1/2} w_{1t+1} - v_t^{1/2} w_{2t+1})/2, \quad (11)$$



is stable, which tells us  $\log z_{1t}$  and  $\log z_{2t}$  are cointegrated. Given the linear homogeneity of the model, changes in  $\bar{z}$  affect consumption quantities proportionately, with no effect on their relative price, the exchange rate. Changes in relative productivity  $\hat{z}$ , however, affect consumption quantities differentially and therefore affect the real exchange rate as well.

*Pareto problems.* We compute competitive equilibria in this environment by finding Pareto optimal allocations and their supporting prices. In a two-agent Pareto problem, we maximize one agent's utility subject to (i) the other agent getting at least some promised level of utility (the promise-keeping constraint) and (ii) the productive capacity of the economy (the resource constraints and shocks). The Lagrangian for this problem is

$$\mathcal{L} = U_{1t} + \lambda_t(U_{2t} - \bar{U}) + \text{resource constraints and shocks},$$

with  $\lambda_t$  the multiplier on the promise-keeping constraint. If utility functions are strictly concave, this is equivalent to traditional Mantel-Negishi maximization of their weighted average,

$$\theta_{1t}U_{1t} + \theta_{2t}U_{2t},$$

with positive Pareto weights  $(\theta_{1t}, \theta_{2t})$ . Evidently  $\lambda_t$  in the previous problem plays the same role as  $\theta_{2t}/\theta_{1t}$ . We refer to  $\lambda_t$  as the Pareto weight, although in terms of the latter version we might call it the relative Pareto weight.

*Transforming utility.* We find it convenient to use an hd1 time aggregator, but with additive preferences (the special case  $\rho = \alpha$ ) it's more common to transform utility to  $U_{jt}^* = U_{jt}^\rho/\rho$ . With this transformation, equation (1) becomes

$$U_{jt}^* = (1 - \beta)c_{jt}^\rho/\rho + \beta[E_t(U_{jt+1}^{*\alpha/\rho})]^\rho/\alpha. \quad (12)$$

When  $\rho = \alpha$  this takes the familiar additive form.

The transformation also changes the look of derivatives. When we represent preferences with  $U_{jt}$ , marginal utility is

$$\partial U_{jt}/\partial c_{jt} = U_{jt}^{1-\rho}(1 - \beta)c_{jt}^{\rho-1}.$$

When we use  $U_{jt}^*$ , marginal utility takes the simpler form

$$\partial U_{jt}^* / \partial c_{jt} = (1 - \beta) c_{jt}^{\rho-1}.$$

We'll use this insight later on to simplify some of the expressions we get using the hd1 form of the time aggregator. This includes the Pareto weight, which is defined for a specific utility function.

## Parameter values

We make only a modest effort to use realistic parameter values. The goal instead is to highlight the effects of recursive preferences and stochastic volatility with parameter values in the ballpark of those used elsewhere in the literature. We summarize these choices in Table 1. The time interval is one quarter.

*Preferences.* We use  $\rho = -1$  (implying an IES of one-half) and  $\alpha = -9$  (implying risk aversion of 10). The former is a common value in business cycle modeling; Kydland and Prescott (1982), for example. The latter is widely used in asset pricing; Bansal and Yaron (2004) is the standard reference. The key feature of this configuration is that  $\alpha - \rho < 0$ . We set  $\beta = 0.98$ .

*Technology.* The Armington aggregator plays a central role here, specifically the elasticity of substitution  $1/(1 - \sigma)$  between foreign and domestic intermediate goods. A wide range of elasticities have been used in the literature. Some earlier work used elasticities greater than one. Colacito and Croce (2013, 2014) and Kollman (2015) use an elasticity of one. Heathcote and Perri (2002, Section 3.2) and Tretvoll (2013, Table 1; 2015, Table 3) suggest smaller values. We start with an elasticity of one ( $\sigma = 0$ ), but consider other values, particularly when we explore the interaction of the elasticity and the dynamics of the Pareto weight.

Given a choice of  $\sigma$ , we set the share parameter  $\omega$  like this. First-order conditions equate prices to marginal products:

$$\begin{aligned} p_{1t} &= c_{1t}^{1-\sigma} (1 - \omega) a_{1t}^{\sigma-1} \\ p_{2t} &= c_{1t}^{1-\sigma} \omega b_{1t}^{\sigma-1} \end{aligned}$$

In a symmetric steady state with import share  $s_m = b_1/(a_1 + b_1)$  and relative price  $p_{2t}/p_{1t} = 1$ , the ratio of these two equations implies

$$\left(\frac{1 - \omega}{\omega}\right) = \left(\frac{1 - s_m}{s_m}\right)^{1-\sigma}. \quad (13)$$

We set  $s_m = 0.1$ . Given a value for  $\sigma$ , the import share nails down  $\omega$ . One consequence of this calculation is that the parameter  $\omega$  approaches one-half as  $\sigma$  approaches one (and the elasticity of substitution approaches infinity).

*Shocks.* The mean growth rate is  $\log g = 0.004$ : 0.4% per quarter. The number comes from Tallarini (2000, Table 4) and applies to the US. We set the persistence parameter  $\gamma$  that governs productivity dynamics equal to 0.1, which implies an autocorrelation of  $1 - 2\gamma = 0.8$  for  $\log \hat{z}$ . Rabanal, Rubio-Ramirez, and Tuesta (2011, Table 5) estimate  $\gamma$  to be less than 0.01, which implies significantly greater persistence. The stochastic volatility process (9) is based on Jurado, Ludvigson, and Ng (2014) as described in Backus, Ferriere, and Zin (2014, Section 5.3).

## 4 Solving the recursive Pareto problem

We compute a competitive equilibrium indirectly as a Pareto optimum, a standard approach in this literature. The competitive prices then show up as Lagrange multipliers on the resource constraints. Similar models have been studied by Colacito and Croce (2013), Kollman (2015), and Tretvoll (2011).

### Pareto problem

We solve the Pareto problem recursively. We make a slight change in notation, and represent utility of agent 1 by  $J$ , the value function, and the utility of agent 2 by  $U$ , without the “2” subscript. The state at date  $t$  is then the exogenous variables  $s_t = (z_{1t}, z_{2t}, v_t)$  plus the utility promise  $U_t$  made to agent 2. The Bellman equation is

$$\begin{aligned} J(U_t, s_t) &= \max_{\{c_{1t}, U_{t+1}\}} V\{c_{1t}, \mu_t[J(U_{t+1}, s_{t+1})]\} \\ \text{s.t.} \quad &V\{c_{2t}, \mu_t(U_{t+1})\} \geq U_t \\ &\text{plus resource constraints and shocks.} \end{aligned}$$

The resource constraints include (4,5) for intermediate goods and (6,7) for final goods. The shocks follow (8,9).

We use Lagrange multipliers  $\lambda_t$  on promised utility and  $(q_{1t}, q_{2t}, p_{1t}, p_{2t})$  on the resource constraints. The first-order conditions are then

$$c_{1t} : \quad p_{1t} = J_t^{1-\rho}(1-\beta)c_{1t}^{\rho-1} \quad (14)$$

$$c_{2t} : \quad p_{2t} = \lambda_t U_t^{1-\rho}(1-\beta)c_{2t}^{\rho-1} \quad (15)$$

$$a_{1t} : \quad q_{1t} = p_{1t}c_{1t}^{1-\sigma}(1-\omega)a_{1t}^{\sigma-1} \quad (16)$$

$$b_{1t} : \quad q_{2t} = p_{1t}c_{1t}^{1-\sigma}\omega b_{1t}^{\sigma-1} \quad (17)$$

$$a_{2t} : \quad q_{1t} = p_{2t}c_{2t}^{1-\sigma}\omega a_{2t}^{\sigma-1} \quad (18)$$

$$b_{2t} : \quad q_{2t} = p_{2t}c_{2t}^{1-\sigma}(1-\omega)b_{2t}^{\sigma-1} \quad (19)$$

$$U_{t+1} : \quad J_t^{1-\rho}\beta\mu_t(J_{t+1})^{\rho-\alpha}J_{t+1}^{\alpha-1}J_{U_{t+1}} = -\lambda_t U_t^{1-\rho}\beta\mu_t(U_{t+1})^{\rho-\alpha}U_{t+1}^{\alpha-1}. \quad (20)$$

Note, in particular, that equation (20) applies to promises  $U_{t+1}$  in every state at  $t+1$ . It's many equations, not just one.

The envelope condition for  $U_t$  is

$$J_{U_t} = -\lambda_t,$$

which we use to replace  $J_{U_{t+1}}$  with  $\lambda_{t+1}$  in (20).

*Transforming the Pareto weight.* We can simplify the solution by transforming the Pareto weight  $\lambda$ . A Pareto weight is defined for specific a utility function; if we transform the utility function, we transform the Pareto weight along with it. The natural benchmark for the Pareto weight is the additive case, in which it's constant. Additive preferences are traditionally expressed using utility  $U_t^* = U_t^\rho/\rho$  and  $J_t^* = J_t^\rho/\rho$ . The associated Pareto weight is

$$\lambda_t^* = \lambda_t U_t^{1-\rho}/J_t^{1-\rho}.$$

We refer to  $\lambda_t^*$  as the additive Pareto weight — or simply, when it's clear, the Pareto weight.

With this substitution, we can clearly see the impact of recursive preferences. The first-order condition (20) becomes

$$\beta \left( \frac{J_{t+1}}{\mu_t(J_{t+1})} \right)^{\alpha-\rho} \lambda_{t+1}^* = \lambda_t^* \beta \left( \frac{U_{t+1}}{\mu_t(U_{t+1})} \right)^{\alpha-\rho}. \quad (21)$$

In the additive case,  $\alpha = \rho$ , this reduces to  $\lambda_{t+1}^* = \lambda_t^*$ : the Pareto weight is constant. This is, of course, well known, but it's nice to know we're on the right track.

With recursive preferences, the Pareto weight need not be constant, although it's an open question how important this is quantitatively.

*Consumption and the exchange rate.* The same transformation of the Pareto weight changes equations (14,15) to

$$(1 - \beta)c_{1t}^{\rho-1}/p_{1t} = \lambda_t^*(1 - \beta)c_{2t}^{\rho-1}/p_{2t}$$

or

$$p_{2t}/p_{1t} = \lambda_t^*(c_{2t}/c_{1t})^{\rho-1}. \quad (22)$$

If the prices of final goods are equal, as they are in a one-good world, the first equation tells us to equate weighted marginal utilities across agents.

In the second equation, the left side is the real exchange rate  $e_t$ . The right is the product of the Pareto weight and the consumption ratio. In the additive case, the Pareto weight is constant and there's a linear relation between the logs of the real exchange rate and the consumption ratio. In the data, there is little sign of such a relation. See, among many others, Backus and Smith (1993), Chari, Kehoe, and McGrattan (2002), Corsetti, Dedola, and Leduc (2008), Colacito and Croce (2013), Kollmann (1995), and Tretvoll (2011). We might say that the data suggests a wedge between the price and consumption ratios.

If we further simplify the model to have a single good, then the real exchange rate is one in all states. In the additive case,  $\lambda_t^*$  is constant and equation (22) tells us that the ratio of consumptions is also constant. Any variation in consumption by one agent is exactly mirrored by the other. This is, of course, counterfactual, and one of the standard anomalies of international business cycle models.

*Marginal rates of substitution.* The Pareto problem equates marginal rates of substitution, but since the agents consume different goods this involves the relative price  $e_t = p_{2t}/p_{1t}$ . The ratio of equation (22) at dates  $t$  and  $t + 1$  is

$$\beta \left( \frac{c_{1t+1}}{c_{1t}} \right)^{\rho-1} \left( \frac{e_{t+1}}{e_t} \right) = \left( \frac{\lambda_{t+1}^*}{\lambda_t^*} \right) \beta \left( \frac{c_{2t+1}}{c_{2t}} \right)^{\rho-1}.$$

Combining this with (21) gives us

$$\beta \left( \frac{c_{1t+1}}{c_{1t}} \right)^{\rho-1} \left( \frac{J_{t+1}}{\mu_t(J_{t+1})} \right)^{\alpha-\rho} \left( \frac{e_{t+1}}{e_t} \right) = \beta \left( \frac{c_{2t+1}}{c_{2t}} \right)^{\rho-1} \left( \frac{U_{t+1}}{\mu_t(U_{t+1})} \right)^{\alpha-\rho}$$

as noted earlier. Note the role of  $\alpha - \rho$ . If the difference is zero, the recursive term drops out. Otherwise the sign of the impact of future utility depends on the sign of  $\alpha - \rho$ .

## Computation

*Scaling.* One challenge with computing solutions to this model is that the productivity process (8) has a unit root. We deal with that by scaling: we divide everything by  $\bar{z}_t$  and label it with a tilde:  $\tilde{c}_{1t} = c_{1t}/\bar{z}_t$ ,  $\tilde{c}_{2t} = c_{2t}/\bar{z}_t$ ,  $\tilde{U}_t = U_t/\bar{z}_t$ , and so on. We define the relevant state by  $\tilde{s}_t = (\hat{z}_t, v_t)$  and the growth rate in the scaling variable  $\bar{z}_t$  by  $g_{t+1} = \bar{z}_{t+1}/\bar{z}_t$ .

Since the functions are all hd1, we can divide the whole Bellman equation by  $\bar{z}_t$ :

$$\begin{aligned} \tilde{J}(\tilde{U}_t, \tilde{s}_t) &= \max_{\{\tilde{c}_{1t}, \tilde{U}_{t+1}\}} V\{\tilde{c}_{1t}, \mu_t[g_{t+1}\tilde{J}(\tilde{U}_{t+1}, \tilde{s}_{t+1})]\} \\ \text{s.t.} \quad &V\{\tilde{c}_{2t}, \mu_t(g_{t+1}\tilde{U}_{t+1})\} \geq \tilde{U}_t \\ &\text{plus resource constraints and shocks.} \end{aligned}$$

The scaled resource constraints become

$$\begin{aligned} \tilde{a}_{1t} + \tilde{a}_{2t} &= \tilde{y}_{1t} = \hat{z}_t \\ \tilde{b}_{1t} + \tilde{b}_{2t} &= \tilde{y}_{2t} = 1/\hat{z}_t \\ \tilde{c}_{1t} &= h(\tilde{a}_{1t}, \tilde{b}_{1t}) \\ \tilde{c}_{2t} &= h(\tilde{a}_{2t}, \tilde{b}_{2t}). \end{aligned}$$

The laws of motion for the shocks are (11) for  $\widehat{z}_t$  and (9) for  $v_t$  (no scaling required).  $\bar{z}_t$  drops out, leaving us with a lower-dimensional state.

Given a solution to the scaled problem, we can multiply by  $\bar{z}_t$  to produce a solution to the original problem. The notation is horrendous. The point is simply that we can convert our problem to one with stable shocks. And we do.

*Algorithm.* Another challenge in computing a solution is that at each date  $t$ , we need to choose utility promises  $U_{t+1}$  for every state the following period. We adapt the algorithm of Collin-Dufresne, Johannes, and Lochstoer (2015) to a multi-good setting. Their algorithm has three essential features. First, they make a clever choice of state variable. Second, they use backward recursion, starting at a terminal date and computing the value function recursively at earlier dates. Third, they use the first-order conditions to solve for the optimal policies, rather than simply choosing the best from a finite set of possibilities.

Consider the state variable. We characterized the problem using promised utility  $U_t$  as the state. Collin-Dufresne, Johannes, and Lochstoer replace promised utility with the consumption share. Given the consumption share, the conditions of the solution give us promised utility as a byproduct. We use the (additive) Pareto weight the same way, with consumption shares and promised utility as byproducts. The choice of variable does not change the solution, but in our experience it can have a large impact on the behavior of the algorithm.

Now consider backward recursion. We approximate an infinite-horizon dynamic programming problem with a long finite-horizon problem. At the terminal date  $T$ , we set utility equal to current consumption. This corresponds to a steady state in which consumptions are constant at these values at all future dates. If we know the Pareto weight  $\lambda_T^*$  in all states at  $T$ , we can compute allocations of the two intermediate goods and consumptions of the two agents in the same states. The consumptions then give us value functions for each state at  $T$  that we can use the previous period.

In any previous period  $t$ , we need to choose both current consumptions and continuation values for the Pareto weight in every succeeding state at  $t + 1$ . We choose the Pareto weights  $\lambda_{t+1}^*$  and current consumptions  $c_{jt}$  by solving the planner's first-order

conditions. We speed this up by pre-solving the allocation problem on a fine grid for the state variables. Given a solution for the Pareto weight, the time aggregators then give us current utility and the value function.

We repeat this process until the value function and decision rules converge. We use a sup norm criterion over both value functions and decision rules.

We use a discrete grid for the Pareto weight and other state variables. Our numerical implementation has 451 points for the Pareto weight, 13 points each for the exogenous state variables  $\widehat{z}_t$  and  $v_t$ , and 5 quadrature nodes for each of the three future innovations — a total of 9.5 million points. We then use Hermite quadrature to compute the expectations in the certainty equivalent functions. If the number of points seems high, we have found that it's essential to have a fine grid over the Pareto weight to describe its dynamics adequately. We could probably work with a less fine grid than we did, but this gives us some confidence that the solutions are accurate.

We do all of these calculations in Julia, a programming language that combines the convenience of dynamic vector-based languages like Matlab with the speed of compiled languages like C or Fortran.

## 5 Properties of the exchange economy

We compute an accurate global solution to the scaled Pareto problem and describe its properties. We start with the Pareto weight, then go on to explore the dynamics of the Pareto weight, the connection between consumption and the real exchange rate, and the responses of consumption and other variables to changes in various state variables.

*The Pareto frontier.* One of the outputs of the numerical solution is the value function  $J$ , a function of promised utility  $U$  and the exogenous state variables. Given values for the exogenous state variables, this gives us the Pareto frontier: the highest utility of agent 1 ( $J$ ) consistent with a given level of utility for agent 2 ( $U$ ) and the productive capacity of the economy.

We describe the Pareto frontier in Figure 2 with state variables  $z_{1t} = z_{2t} = \widehat{z}_t = 1$  and  $v_t = v$ . The outer curve in the figure is the consumption frontier, which echoes



Figure 1. The inner curve is the Pareto frontier. We see that it has much the same shape. It's inside the consumption frontier largely because of risk: utility is below consumption because risk reduces the level of utility. If we increase risk aversion to 50 ( $\alpha = -49$ , not shown), it shifts in further.

Changes in the state variables lead to changes in the frontiers. Movements in productivity and output shift the frontiers — both of them — in and out ( $\bar{z}_t$ , which affects the two intermediate goods proportionately) or twist them ( $\hat{z}_t$ , which affects the two goods differently). Changes in risk twist the Pareto frontier, since it affects the two goods differently, but not the consumption frontier, since it has no effect on quantities of intermediate goods.

*Dynamics of the Pareto weight.* We see the impact of recursive preferences in Figure 3, where we graph  $\log \lambda_t^*$  against time for a (very long) simulation of the model. The flat horizontal line refers to the additive case ( $\alpha = \rho = -1$ ). As we know, the Pareto weight doesn't change in this case. The other line refers to the recursive case, and we see clear variation in the Pareto weight. We also see that the variation is both large and very persistent. Persistence is important, as we'll see, in producing persistent movements in the real exchange rate.

This touches on a question that's been discussed extensively: Is the Pareto weight stable, or does one agent eventually consume everything? Anderson (2005) and Borovicka (2015) document some of the difficulties of establishing stability in similar one-good settings. Colacito and Croce (2014) prove stability in a two-good world with elasticities of substitution between goods and over time equal to one. Colacito and Croce (2013), Kollmann (2015), and Tretvoll (2011, 2013, 2015) solve similar models numerically and report that the solutions are stable. We also find that they're stable, but extremely persistent.

We get a sense of how stability works in Figure 4, where we plot the expected change in the log Pareto weight against its level. The exogenous state variables here have been set equal to their means. In the additive case, the change is zero. The log Pareto weight is a martingale with no variance. With greater risk aversion, mean reversion becomes evident. If the Pareto weight is below its steady state value of one ( $\log \lambda^* = 0$ ), it's expected to increase. If above, it's expected to decrease. The effect

is stronger when we increase risk aversion to 50. There is also an evident nonlinearity in the solution, as there is in Colacito and Croce (2014, Figure 5), but most of it occurs in regions of the state space we rarely reach.

The elasticity of substitution between foreign and domestic intermediate goods also plays a role in persistence. See Figure 5. With smaller values, mean reversion is slower. And with larger values, it's faster. As the elasticity increases, the line gets flatter and we approach the one-good world with a constant Pareto weight.

The intertemporal elasticity of substitution also has an effect, but with the numbers we've chosen the effect is smaller. See Figure 6. Evidently smaller values of  $\rho$ , and larger values of the IES  $[1/(1 - \rho)]$ , lead to flatter lines.

*Consumption and exchange rate.* We noted earlier that the relation between the log consumption ratio  $[\log(c_{2t}/c_{1t})]$  and the log of the real exchange rate  $[\log e_t = \log(p_{2t}/p_{1t})]$  is mediated by the log Pareto weight  $(\log \lambda_t^*)$ . See equation (22). In the additive case, the Pareto weight is constant and we have a perfect linear relationship between the two variables. We see exactly this in the line in Figure 7.

The scatter of points in the same figure represents the recursive case, where the Pareto weight acts like a wedge from the perspective of the additive model. With our numbers, the variation in the Pareto weight is enough to change a negative correlation of minus one between the consumption ratio and exchange rate to a slight positive correlation. Colacito and Croce (2013), Kollmann (2015), and Tretvoll (2011) show the same. If we increase risk aversion  $1 - \alpha$  to 50, the correlation becomes strongly positive. In the recursive model, we can produce almost any correlation we like by varying the risk aversion parameter.

Recursive preferences also have an impact on exchange rate dynamics. Another well-known feature of international data is the extreme persistence of the real exchange rate. The mainstream view is that real exchange rates exhibit modest mean reversion, with a half-life in the neighborhood of five years. We see in Figure 8 that the additive model is much less persistent: The half-life (where the autocorrelation function equals one-half) is about a year. By five years, the autocorrelation is essentially zero.

Exchange rate dynamics reflect, in this case, the modest persistence of relative productivity  $\hat{z}_t$ . With recursive preferences, the exchange rate is much more persistent. In fact with these parameter values, it's virtually a martingale. We can reproduce any level of persistence we like by varying risk aversion between the two cases, but the larger point is that recursive preferences are a device that can deliver a high degree of persistence.

*Responses to productivity and volatility shocks.* We get another perspective on the model's dynamics from impulse responses. Starting at the steady state, we increase one of the exogenous state variables by one standard deviation at date one and compute the mean dynamics of the other variables in the model.

In Figure 9 we describe responses to an increase in (the log of) relative productivity  $\hat{z}_t$ . The effect of the impulse declines geometrically over time as described by equation (11). Volatility remains constant. Holding constant  $\bar{z}_t$ , this leads to an increase in  $\log z_{1t}$  and an equal decrease in  $\log z_{2t}$ . The quantity of apples goes up, and the quantity of bananas goes down. Because of home bias, consumption goes up in country 1 (the apple eaters) and down in country 2 (banana eaters). The exchange rate rises as scarce bananas become more expensive.

All of this would be true in the additive case as well. What's different is the response of the Pareto weight. It goes up as we compensate the agent in country 2 with promises of higher future consumption. This effect eventually wears off, but it does so very slowly.

In Figure 10 we describe the responses to an increase in volatility  $v_t$ . Here there's no change in the quantities of intermediate goods. In the additive case, there would be virtually no effect. In the recursive case, utility falls, but it falls more for country 1 because of its home bias in favor of the good whose supply is more risky. The social planner responds by decreasing the Pareto weight on agent 2. Consumption therefore rises in country 1 and falls in country 2. The real exchange rate falls. This is entirely a demand-side effect. By increasing the weight on agent 1, the demand for apples goes up and the demand for bananas goes down. The magnitudes are small, but it's an interesting effect that we would like to explore further in a production economy, where supply can respond to changes in market conditions.

## 6 Open questions

We have documented the behavior of the Pareto weight in a relatively simple environment. We showed, as others have, that recursive preferences can change the quantitative properties of the model in useful ways. The behavior of exchange rates, in particular, is much different from the additive case.

Beyond this, we are left with a number of open questions:

- What parameters govern the persistence of the Pareto weight? Can we be more precise about the impact of risk aversion and intertemporal substitution on the dynamics of the Pareto weight? Of the substitutability of foreign and domestic goods in the in the Armington aggregator?
- How would this change in a production economy? Production offers opportunities to respond to changes in exogenous variables, particularly changes in risk. If production in one country becomes more risky, do we shift production to the other country? Does capital flow to the less risky country? Are the magnitudes plausible?
- How do changes in the Pareto weight generated by frictions and recursive preferences compare? Are they similar or different? Are the two mechanisms complements or substitutes?

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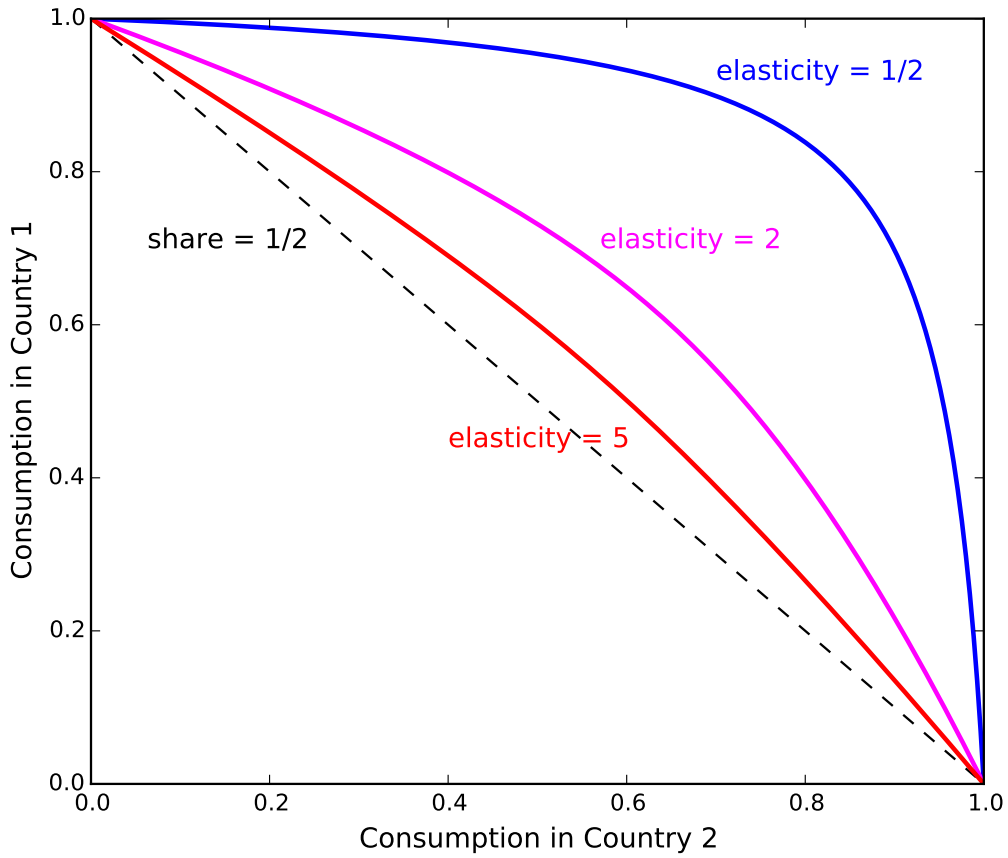
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**Table 1.** Benchmark parameter values.

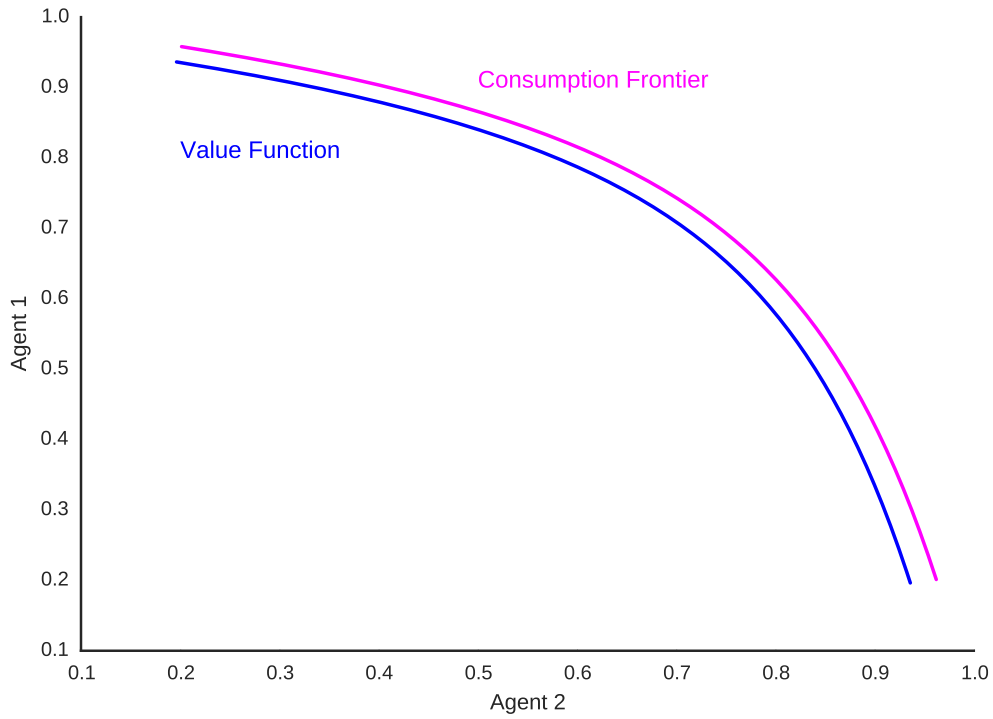
Parameter	Value	Comment
<i>Preferences</i>		
$\rho$	-1	IES = $1/(1 - \rho) = 1/2$
$\alpha$	-9	RA = $1 - \alpha = 10$
$\beta$	0.98	
<i>Armington aggregator</i>		
$\sigma$	0	Cobb-Douglas
$\omega$	0.1	chosen to hit import share of 0.1
<i>Productivity growth</i>		
$\log g$	0.004	Tallarini (2000, Table 4)
$v^{1/2}$	0.015	Tallarini (2000, Table 4), rounded off
$\varphi_v$	0.95	Backus, Ferriere, and Zin (2015, Table 1)
$\tau$	$0.74 \times 10^{-5}$	makes $v$ three standard deviations from zero
$\gamma$	0.1	persistence of productivity difference



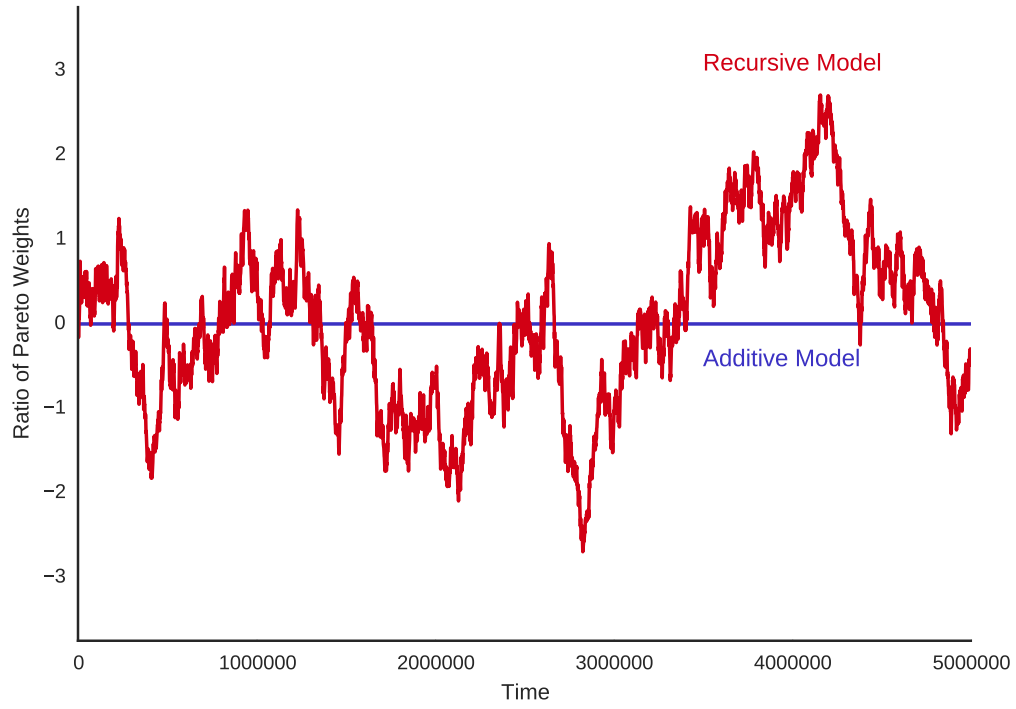
**Figure 1.** Consumption frontiers. Lines represent the frontier quantities of consumption given unit quantities of the intermediate goods. The dashed black line has  $\omega = 1/2$ , making the two final goods the same. For the others, we choose an import share of 0.1 and use (13) to adjust  $\omega$  as we vary  $\sigma$ . The elasticities of substitution noted in the figure are  $1/(1 - \sigma)$ .



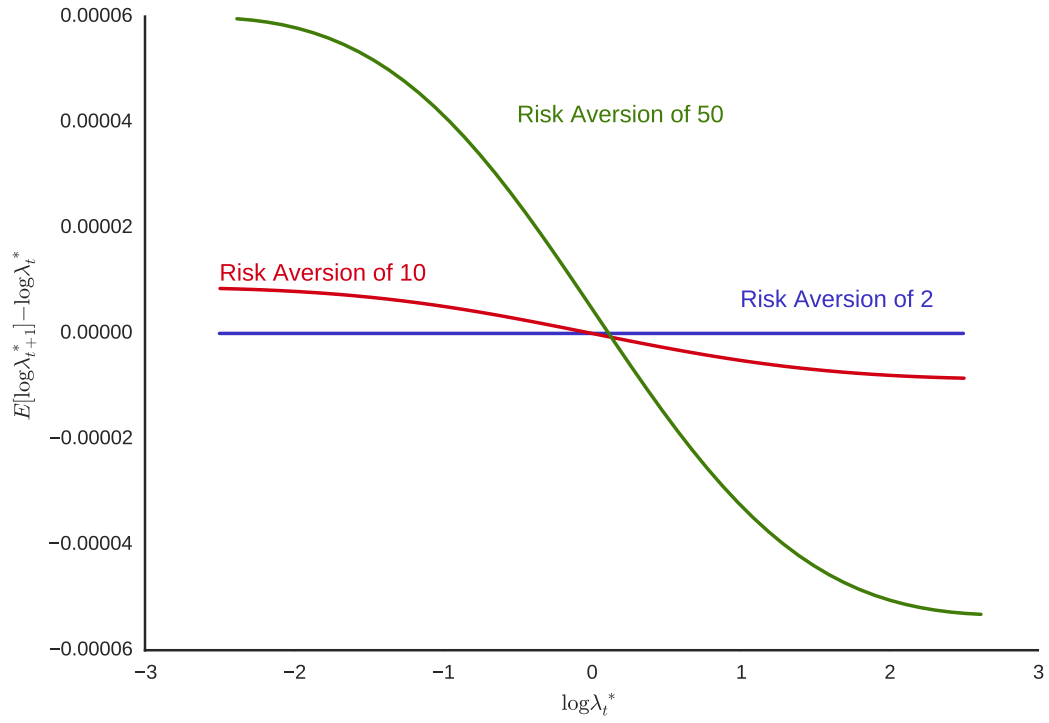
**Figure 2.** Pareto and consumption frontiers. The outer line is the consumption frontier with benchmark parameter values. The inner line is the Pareto frontier: the utility  $J$  of agent 1 given promised utility  $U$  to agent 2. In each case, the other state variables are  $z_{1t} = z_{2t} = \hat{z}_t = 1$  and  $v_t = v$ .



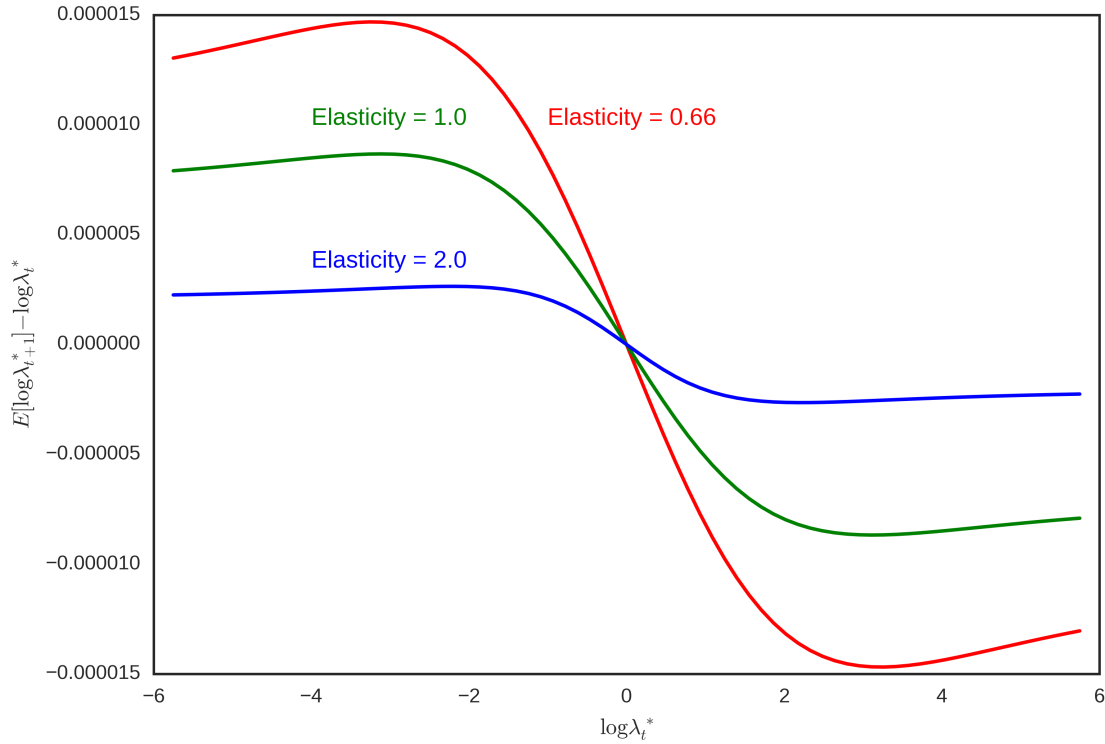
**Figure 3.** Dynamics of the additive and recursive Pareto weight. The two lines represent simulations of models with additive ( $\alpha = \rho = -1$ ) and recursive ( $\alpha = -9, \rho = -1$ ) preferences. The simulations use the same paths for exogenous state variables. In each case, we plot  $\log \lambda_t^*$  against time.



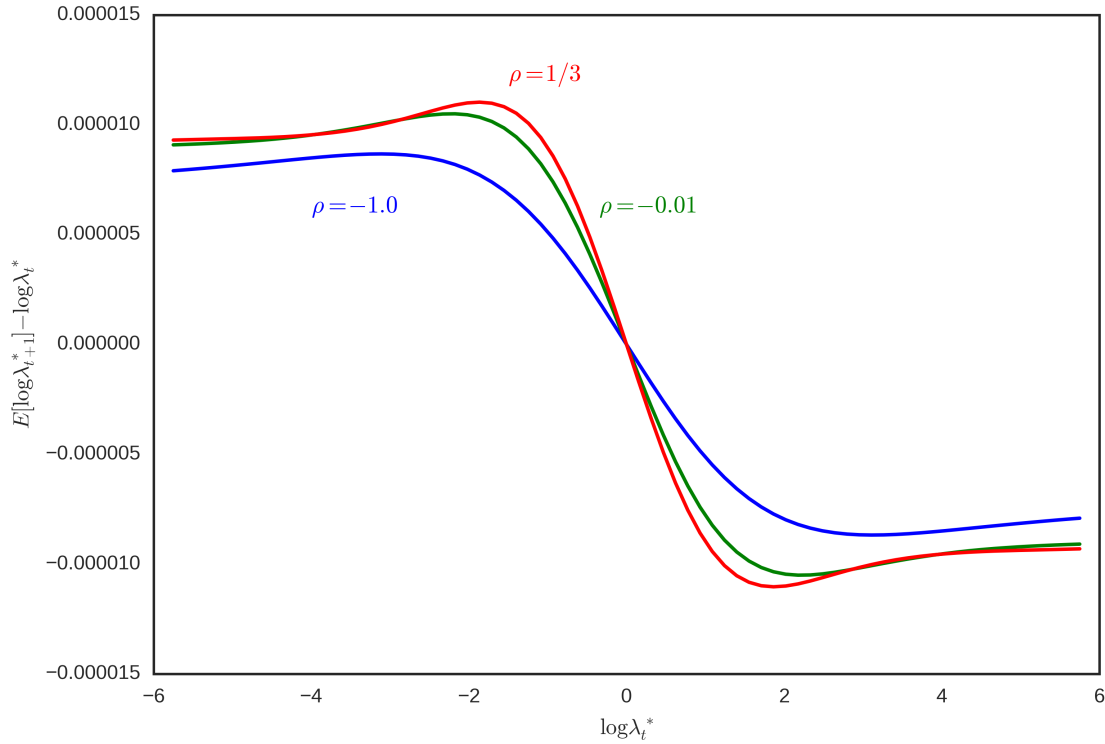
**Figure 4.** Risk aversion and expected changes in the Pareto weight. The lines represent the expected change in  $\log \lambda_t^*$  [ $E_t(\Delta \log \lambda_{t+1}^*)$ ] with three values of risk aversion  $1 - \alpha$ : 2 (additive), 10, and 50.



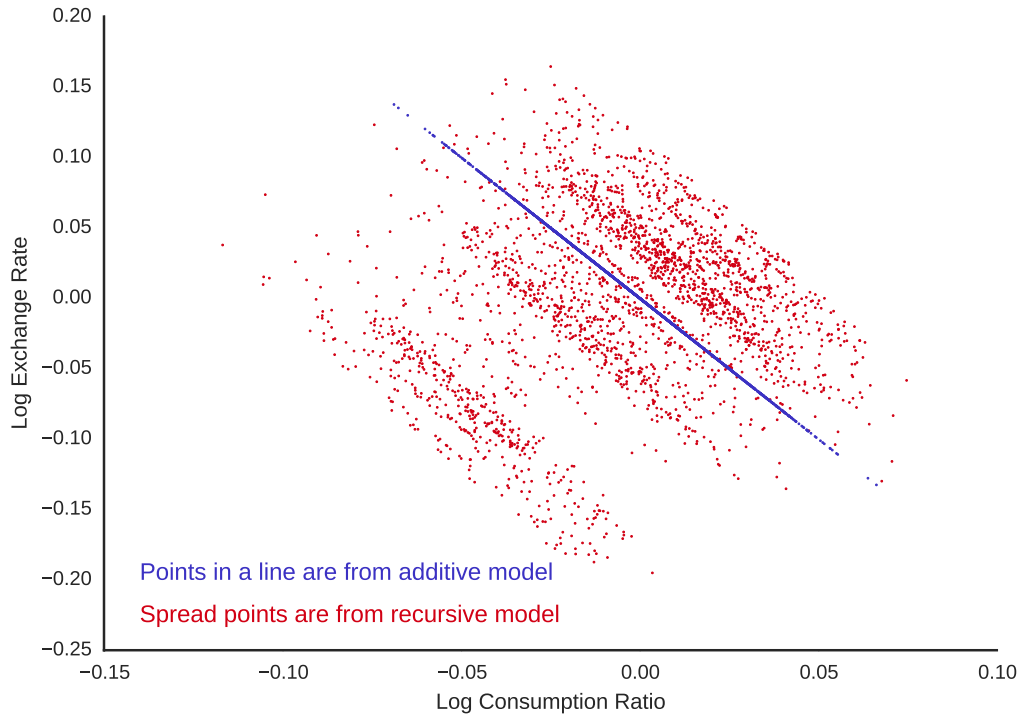
**Figure 5.** Armington substitutability and expected changes in the Pareto weight. The lines represent the expected change in  $\log \lambda_t^*$  [ $E_t(\Delta \log \lambda_{t+1}^*)$ ] with three values of the substitutability parameter  $\sigma$  in the Armington aggregator. The elasticities  $1/(1 - \sigma)$  are 2/3, 1, and 2.



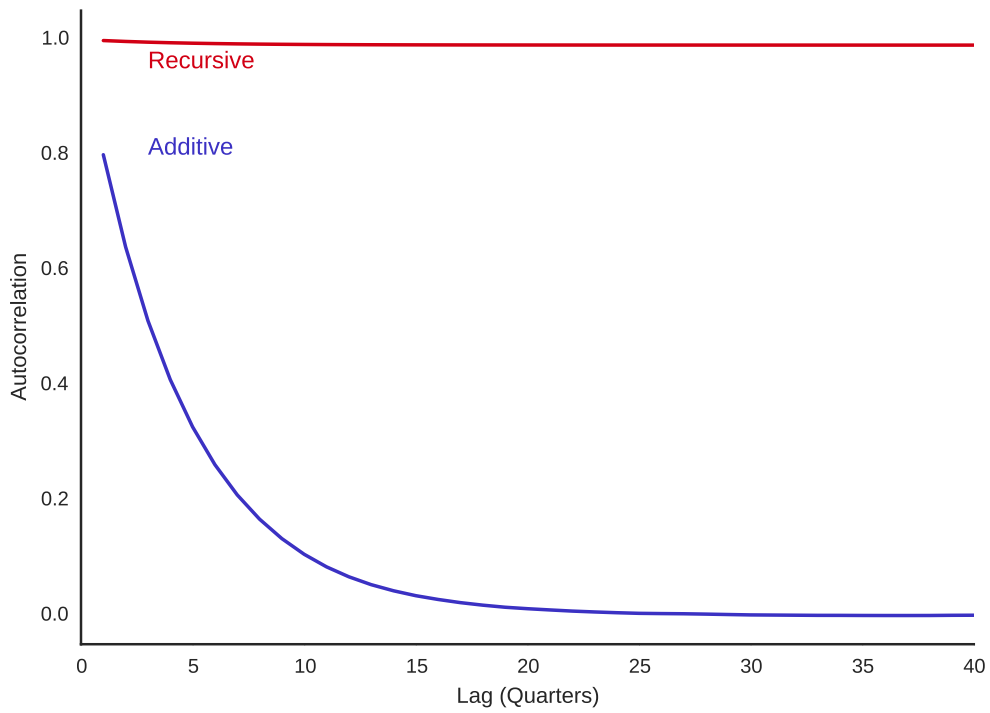
**Figure 6.** Intertemporal substitution and expected changes in the Pareto weight. The lines represent the expected change in  $\log \lambda_t^*$  [ $E_t(\Delta \log \lambda_{t+1}^*)$ ] with three values of the substitutability parameter  $\rho$  in the time aggregator:  $-1$ ,  $-0.01$ , and  $1/3$ . They correspond to intertemporal elasticities of substitution of  $1/2$ ,  $0.99$ , and  $3/2$ .



**Figure 7.** Consumption and the real exchange rate. The dots represent simulations of models with additive ( $\alpha = \rho = -1$ ) and recursive ( $\alpha = -9$ ) preferences. In each case, we plot  $\log e_t = \log(p_{2t}/p_{1t})$  against  $\log(c_{2t}/c_{1t})$  for a simulation of the model.

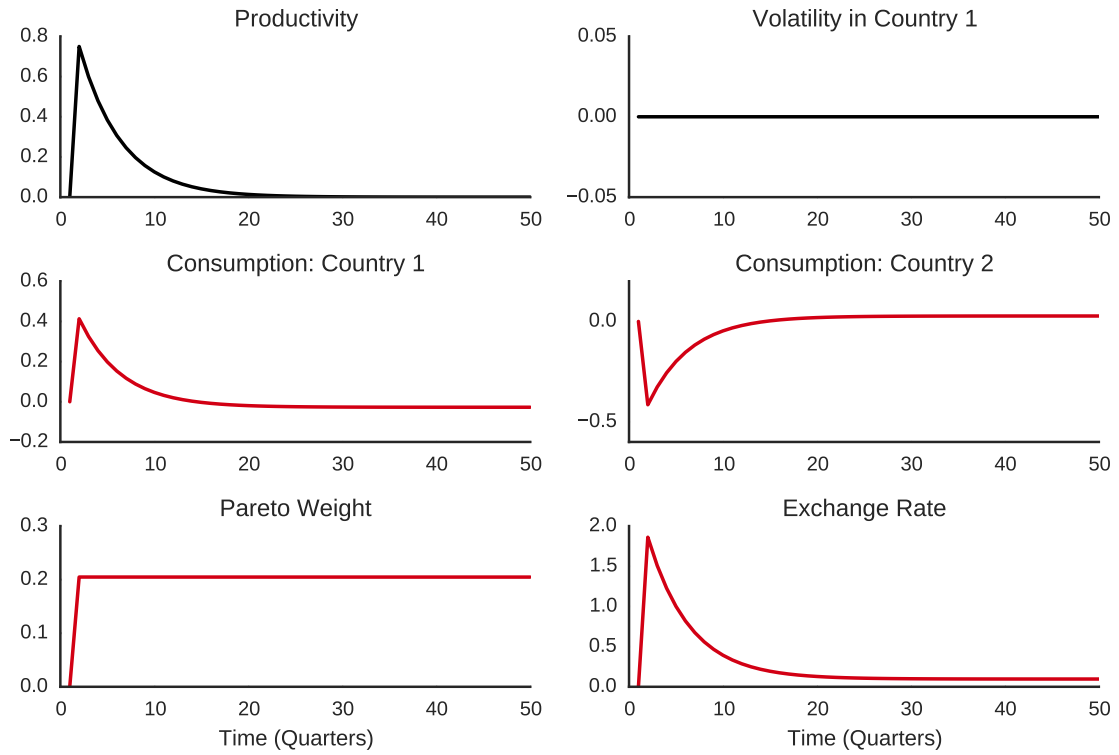


**Figure 8.** Dynamics of the real exchange rate. The lines represent autocorrelation functions for the real exchange rate ( $\log e_t$ ) in models with additive ( $\alpha = \rho = -1$ ) and recursive ( $\alpha = -9$ ) preferences.





**Figure 9.** Responses of variables to an impulse in relative productivity  $\log \hat{z}_t = (1/2)(\log z_{1t} - \log z_{2t})$  in country 2. The impulse takes place at date  $t = 1$ . Responses are reported as percent deviations from mean values.



**Figure 10.** Responses of variables to an impulse in volatility  $v_t$ . The impulse takes place at date  $t = 1$ . Responses are reported as percent deviations from mean values.

