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ABSTRACT

The paper revisits the problem of wage bargaining between a firm and multiple workers. We show that the Subgame Perfect Equilibrium of the extensive-form game proposed by Stole and Zwiebel (1996a) does not imply a profile of wages and profits that coincides with the Shapley values as claimed in their classic paper. We propose an alternative extensive-form bargaining game, the Rolodex Game, that follows a simple and realistic protocol and that, under some mild restrictions, admits a unique Subgame Perfect Equilibrium generating a profile of wages and profits that are equal to the Shapley values. The vast applied literature that refers to the Stole and Zwiebel game to give a game-theoretic foundation to the use of the Shapley values as the outcome of the bargain between a firm and multiple workers should instead refer to the Rolodex game.

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1 Introduction

We revisit the problem of wage bargaining between a firm and multiple employees. The standard axiomatic solution for this type of multilateral bargaining problem is provided by Shapley (1953), who derives a simple formula for the expected payoffs to each agent starting from some desirable properties of any solution. The classic game-theoretic analysis of the bargaining problem between a firm and multiple employees is provided by Stole and Zwiebel (1996a and 1996b), who propose an extensive-form game with a simple and realistic protocol that admits, as its unique Subgame Perfect Equilibrium, a profile of wages and profits that coincides with the Shapley values. The Stole and Zwiebel bargaining game (henceforth, the SZ game) has been applied widely in the labor-search literature[1], where wages are not pinned down by competition because, due to search frictions, a firm and its employees have to spend time or other resources in order to find alternative trading partners (see, e.g., Cahuc, Marque and Wasmer 2008, Ebell and Haefke 2009, Helpman, Itskhoki and Redding 2010, Elsby and Michaels 2013, Acemoglu and Hawkins 2014, Helpman and Itskhoki 2015).

The paper contains two findings. First, we show that the SZ bargaining game does not support the Shapley values as a Subgame Perfect Equilibrium. Second, we propose an extensive-form bargaining game between a firm and its employee that follows a simple and realistic protocol and that, under some mild restrictions, admits as its unique Subgame Perfect Equilibrium a profile of wages and profits that coincides with the axiomatic solution by Shapley. We refer to this game as the "Rolodex Game," after the rotating file device used to store business contact information.

In the first part of the paper, we characterize the solution to the SZ bargaining game. The game includes a firm and n workers, who are placed in some arbitrary order from 1 to n. The game proceeds as a finite sequence of pairwise bargaining sessions between the firm and one of the workers. A bargaining session follows the same protocol as in Binmore, Rubinstein and Wolinsky (1986), i.e. the firm and the worker alternate in making proposals about the employee's wage and, after every rejection, there is some probability of a breakdown. The bargaining session may either end with an agreement over some wage, or with a breakdown. In case of agreement, the firm enters a bargaining session with the next worker in line. In case of breakdown, the employee exits the game and the whole bargaining process starts over with one less worker. When the firm reaches an agreement with all the workers who are

[1] Before the publication of Stole and Zwiebel (1996a,b), the labor search literature had dealt with bargaining between a firm and multiple workers in a reduced form way by simply positing some wage equations without foundations in either axiomatic or strategic bargaining theory (see, e.g., Bertola and Caballero 1994, Andolfatto 1996 or Smith 1999).

still in the game, the game ends, the agreed-upon wages are paid out and production takes place. As usual in the game-theoretic literature on bargaining, the focus is on the Subgame Perfect Equilibrium of the game in the limit as the probability of breakdown goes to zero.

Theorem 2 in Stole and Zwiebel (1996a), in conjunction with Theorem 4, claims that the unique Subgame Perfect Equilibrium of the game is such that every worker earns the same wage and that this common wage is the worker's Shapley value. In particular, Theorem 2 claims that the unique Subgame Perfect Equilibrium wages are given by what they call the stable wage profile, and Theorem 4 establishes the equivalence between the stable wage profile and the worker's Shapley value. Stole and Zwiebel derive the stable wage profile in a heuristic way, by informally describing a bargaining environment and then conjecturing some properties of the solution. This heuristic approach allows them to derive the stable wage profile in an intuitive and simple way. The SZ game is meant to formalize the heuristic arguments and provide a rigorous game-theoretic foundation for the stable wage profile and, thus, for the Shapley values.

We prove that Theorem 2 is wrong, as workers who are in different places in the initial ordering earn different wages and, even on average, these wages are different from the worker's Shapley value. There is a simple intuition for this result. When the firm enters a bargaining session with the last worker, it takes as given the wage agreements with all the previous workers. These wages do not affect the payoff to the firm if the negotiation with the last worker breaks down, as in this case the bargaining process starts over. However, these wages do affect the payoff to the firm if the firm reaches an agreement with the last worker, as in this case the wages are actually paid out. For this reason, the wage agreements with the previous workers affect the gains from trade between the firm and the last worker and, in turn, the last worker's wage. For example, if the firm agreed to pay the second-to-last worker one more dollar, its gains from trade with the last worker are one dollar lower and, hence, the firm and the last worker settle for a wage that is 50 cents lower. When the firm and the second-to-last worker bargain, they understand the effect of their agreement on the wage of the last worker. In particular, they understand that the marginal cost to the firm from paying the second-to-last worker an extra dollar is only 50 cents. For this reason, the second-to-last worker is able to extract a higher wage than the last worker. Similarly, when the firm and the third-to-last worker bargain, they understand that if they agree to a higher wage, the wage of the last two workers will be lower. For this reason, the third-to-last worker is able to extract an even higher wage than the second-to-last worker. Since workers who bargain first can basically hold the firm up and capture some of its gains from trade with the workers who bargain later, they obtain more than their expected marginal contribution to production (i.e., their Shapley value) and the firm obtains less than its Shapley value.

Formally, we prove that (under a reasonable tie-breaking assumption), there exists a unique Subgame Perfect Equilibrium of the SZ game. In this equilibrium, the wage earned by a worker is strictly decreasing in the worker's position in the initial ordering. In particular, the gains from trade accruing to the first worker are one half of the total surplus, where total surplus is defined as the firm's output net of the sum of the workers' outside options and the firm's profit with one less worker. The gains from trade accruing to the second worker are one fourth of the total surplus. The gains from trade accruing to the third worker are one eight of the total surplus, etc... The gains from trade accruing to the firm are a fraction $1 - \sum_{i=1}^{n} 1/2^i = 1/2^n$ of the total surplus. In contrast, the Shapley values are such that the gains from trade accruing to each worker and to the firm are equal to a fraction $1/(n+1)$ of the total surplus. Therefore, the equilibrium wages differ from the Shapley values for a particular realization of the ordering of workers, as well as in expectation across any distribution of orderings.

The literature has proposed several extensive-form bargaining games that implement (in expectation) the Shapley values as a Subgame Perfect Equilibrium. Gul (1986) considers a game with n agents, each holding an asset that can be fruitfully used in production in conjunction with the others. Trade occurs through a sequence of bilateral random meetings. When two agents meet, one of them is randomly selected to make a take-it-or-leave-it offer to the other. If the offer is accepted, the buyer remains in the market and the seller exits. If the offer is rejected, both agents remain in the market. The game ends when one agent acquires all the assets. While this game does implement the Shapley values, it does not conform to the structure of a typical labor market. Indeed, in the context of a labor market, the game implies that sometimes a worker sells his labor to another worker, and that at other times a firm sells its capital to a worker.

Hart and Mas-Colell (1996) consider a game with n agents. At each round, one of the agents is randomly chosen to propose an allocation. If all the other agents agree to the proposed allocation, the game ends and the allocation is implemented. If one or more of the other agents rejects the proposed allocation, the game continues. With some probability, the agent who proposed the allocation that was rejected leaves the game. With complementary probability, the agent remains in the game. In either case, another agent is randomly chosen to propose an allocation. While the Hart Mas-Colell game implements the Shapley values, it does not represent a realistic description of the bargaining process between a firm and its employees. Indeed, according to this game, there would be instances in which a worker proposes a wage not only for himself, but also for all of his coworkers.

More recently, De Fontenay and Gans (2014) propose a non-cooperative pairwise bargaining game between agents in a network. The setting is very general, allowing for externalities and incomplete networks. Each agent bargains bilaterally with every other agent to whom he is connected via the network. He does not carry out these negotiations by himself, but rather delegates a different negotiator to each pairwise negotiation. These negotiations take place simultaneously and according to the protocol of Binmore, Rubinstein, and Wolinsky (1986). Delegates are given instructions by their delegator prior to any negotiations. During negotiations a delegate does not receive any information about the actions taken in other pairwise negotiations, even those conducted on behalf of their own delegator. However, if any pairwise negotiation ends in a breakdown, this becomes public knowledge. For some specification of the off-equilibrium beliefs, De Fontenay and Gans establish the existence of a Perfect Bayesian Equilibrium of this imperfect information game such that the payoff of each agent is related to the Myerson-Shapley value. In the absence of externalities and if the network is complete, it reduces to the Shapley value. The delegated-negotiator model appears appropriate for negotiations between firms, although even in this setting the absence of any communication between delegates of the same firm has been pointed to as a weakness (see, e.g., Crawford and Yurukoglu 2012). In the labor market context, however, the assumption that the firm uses a different negotiator for each employee and that these negotiators do not communicate unless there is a breakdown is less appealing.

The above observations motivate the second part of the paper. There we introduce the Rolodex game, a novel extensive-form bargaining game between a firm and its workers that follows a reasonable protocol with perfect information and that, under some mild restrictions, has a unique Subgame Perfect Equilibrium where the workers' wages and the firm's profit coincide with the Shapley values.

The Rolodex game includes a firm and n workers, who are initially placed in some order from 1 to n. The game proceeds as a finite sequence of pairwise bargaining sessions between the firm and one of the workers. Each bargaining session involves the same protocol. The worker makes a wage offer. If the firm accepts the offer, it moves onto bargaining with the worker who, among those who have yet to reach an agreement, is next in the order. If the firm rejects the offer, the negotiation breaks down with some probability. Otherwise, the negotiation continues with the firm making a counteroffer. If the worker accepts the counteroffer, the firm moves onto bargaining with the next worker. If the worker rejects the counteroffer, the negotiation breaks down with some probability. Otherwise, the worker moves to the end of the line of workers who have yet to reach an agreement. The firm enters a bargaining session with the worker who is now first among those without an agreement. Whenever there is a breakdown, the worker exits the game and the whole bargaining process starts over with one less worker. When the firm reaches an agreement with all the workers who are still in the game, the bargaining process comes to an end. We refer to this as the Rolodex game because the firm cycles through the workers without agreement, rather than bargaining with each one of them until it either reaches an agreement or there is a breakdown.

We show that, under some mild restrictions, there is a unique Subgame Perfect Equilibrium to the Rolodex game. In this equilibrium, each worker earns the same wage and the common wage is equal to the worker's Shapley value. There is a simple intuition behind these results. Whenever a worker rejects the counteroffer of the firm, he becomes the last worker in line. Hence, a worker at any position in the line follows the same acceptance strategy as the last worker in the line, even though the firm's marginal cost from paying him a higher wage is lower than the firm's marginal cost from paying the last worker a higher wage. For this reason, all workers earn the same wage as the last one. Moreover, the wage of the last worker is such that his gains from trade are one half of the total surplus net of the wage of the other workers. When these two properties are put together, it is immediate to show that each worker and the firm capture a fraction $1/(n+1)$ of the total surplus. These payoffs are indeed the Shapley values.

2 The Stole and Zwiebel Game

2.1 Environment and Preliminaries

We begin by describing the extensive form of the bargaining game proposed by Stole and Zwiebel (1996a). We shall refer to it as the SZ game. The players in the SZ game are a firm and $n \geq 1$ workers. If the firm employs $k \in \{0, 1, ...n\}$ of the n workers and pays them wages w_1, w_2, $\ldots w_k$, its payoff is $y_k - w_1 - w_2 - ...w_k$, where y_k denotes the value of the output produced by the firm with k employees. We assume that y_k is strictly increasing and concave in k, i.e. $y_k < y_{k+1}$ and $y_{k+1} - y_k > y_{k+2} - y_{k+1}$ for $k = 0, 1, 2, ...$ Workers are ex-ante identical. If a worker is hired by the firm at the wage w, his payoff is w. If the worker is not hired by the firm, his payoff is $b \geq 0$, where b might represent the value of employment at some other firm or the value of unemployment.

The workers are placed in some arbitrary, but fixed order from 1 to n. The game consists of a finite sequence of bilateral bargaining sessions between the firm and one of the workers. The game starts with a bargaining session between the firm and the first worker in the order. The bargaining session may end either with an agreement over the worker's wage or with a breakdown. If the bargaining session ends with an agreement, the firm enters a bargaining session with the next worker in the order. If the bargaining session ends with a breakdown, the worker permanently exits the game. In this case, the bargaining game starts over, in the sense that all the previous agreements between the firm and the workers are erased and the firm enters a bargaining session with the worker who, among those still in the game, is first in the order. The game ends when the firm reaches an agreement with all the workers who are still in the game. When this happens, the firm pays the agreed upon wage to each of these workers and production takes place.

Each bargaining session follows the same alternating-offer protocol as in Binmore, Rubinstein and Wolinsky (1986, henceforth BRW). The session begins with the worker making a wage offer to the firm. If the firm accepts the offer, the session ends and the firm goes onto bargaining with the next worker in the order. If the firm rejects the offer, the negotiation breaks down with probability p and continues with probability $1 - p$, with $p \in (0, 1)$. If the negotiation continues, the firm makes a counteroffer to the worker. If the worker accepts the counteroffer, the bargaining session ends. Otherwise, the negotiation breaks down with probability p and continues with probability $1 - p$. If the negotiation does not break down, the bargaining session continues with the worker and the firm alternating in making wage offers until there is either an acceptance or a breakdown.

As in Stole and Zwiebel (1996a), we shall focus on Subgame Perfect Equilibria (henceforth, SPE). In order to characterize the set of SPE of the SZ game, it is useful to recall the solution to the BRW game when an increase in the wage transfers utility from the firm to the worker at a constant rate. When an increase in the wage transfers utility 1 for 1, we are in the canonical case of *perfectly transferable utility.* When an increase in the wage transfers utility at a rate different than 1 for 1, we say that *utility is non-perfectly transferable.* We are interested in this distinction because, as we shall see in the next subsection, the mistake in the proof of Theorem 2 of Stole and Zwiebel (1996a) is to apply the solution of the BRW game with perfectly transferable utility to an environment where utility is not transferred 1 for 1.

Lemma 1: *Consider the BRW alternating-offer game between a firm and a worker. The payoff to the worker in case of agreement at the wage w is w, and the payoff to the worker in case of breakdown is b. The payoff to the firm in case of agreement at the wage w is $y - w - t(w)$, where $t(w) = \alpha - \beta w$ and $\beta \in [0, 1]$, and the payoff to the firm in case of* *breakdown is* z*. (i) If* $y - z - b - t(b) < 0$*, any SPE is such that the firm and the worker do not reach an agreement; (ii) If* $y - z - b - t(b) \geq 0$*, the unique SPE*[2] *is such that the firm and the worker immediately reach an agreement over the wage*

$$w = b + \frac{1}{(2-p)(1-\beta)} \left[y - z - b - t(b) \right]. \tag{1}$$

Proof: The result follows immediately from Proposition 4.2 in Muthoo (1999). ■

A few comments about Lemma 1 are in order. When $\beta = 0$, we are in the perfectly transferable utility case where a one dollar increase in the wage lowers the payoff of the firm by 1 and raises the payoff of the worker by 1. When $\beta > 0$, we are in the imperfectly transferable utility case, where a one dollar increase in the wage lowers the payoff of the firm by $1 - \beta$, and raises the payoff of the worker by 1. For any $\beta \in [0, 1]$, any SPE of the game is such that the firm and the worker do not reach an agreement if there are no gains from trade, i.e. if $y - z - b - t(b) < 0$. If the gains from trade are positive, i.e. $y - z - b - t(b) \geq 0$, the unique SPE is such that the firm and the worker immediately agree to the wage in (1). Given this wage, the gains from trade accruing to the worker and the firm are, respectively, given by

$$\begin{aligned} w - b &= \frac{1}{(2-p)(1-\beta)} [y - z - b - t(b)], \\ y - w - t(w) - z &= \frac{1-p}{2-p} [y - z - b - t(b)]. \end{aligned} \tag{2}$$

As the probability of breakdown p goes to zero, the solution to the BRW game coincides with the axiomatic Nash bargaining solution. In fact, the axiomatic Nash bargaining solution is given by the wage that maximizes the product of the worker's gains from trade and the firm's gains from trade, i.e.

$$\max_{w} [w - b][y - z - w - t(w)]. \tag{3}$$

The solution to the maximization problem in (3) is

$$w = b + \frac{1}{2(1-\beta)} \left[y - z - b - t(b) \right]. \tag{4}$$

[2] To be precise, there are multiple SPE of the BRW game when the gains from trade are zero. All of the SPEs are payoff equivalent, but some of them involve agreement and some do not. For the remainder of the Section, we restrict attention to the SPE in which agreement takes place instantaneously when the gains from trade are zero.

Given the above wage, the gains from trade accruing to the worker and the firm are, respectively, given by

$$
\begin{aligned}
w - b &= \frac{1}{2(1-\beta)}\,[y - z - b - t(b)], \\
y - w - t(w) - z &= \frac{1}{2}\,[y - z - b - t(b)].
\end{aligned}
\tag{5}
$$

It is immediate to see that, in the limit for p going to zero, the outcome (1) and the payoffs (2) of the BRW bargaining game coincides with the outcome (4) and the payoffs (5) of axiomatic Nash bargaining.

The above results are all very well known. However, we wanted to repeat them here to point out the following fact. When utility is perfectly transferable, the limit of the solution to the BRW game for $p \to 0$ (as well as the axiomatic Nash bargaining solution) is such that the gains from trade accruing to the firm are equal to the gains from trade accruing to the worker. When utility is not perfectly transferable, the solution to the BRW game (as well as the axiomatic Nash bargaining solution) is such that the ratio of the gains accruing to the worker to those accruing to the firm is equal to $1/(1-\beta)$, which is different than 1 and strictly increasing in β. Intuitively, the marginal benefit to the worker from being paid an extra dollar is 1, and the marginal cost to the firm from paying the worker an extra dollar is $1-\beta$. The higher is β, the lower is the firm's marginal cost relative to the worker's marginal benefit, the stronger is the worker's bargaining position and, hence, his relative gains from trade. Unless the marginal cost and the marginal benefit are equal, i.e. $\beta = 0$, the gains from trade accruing to the worker are different than those accruing to the firm.

2.2 SZ Game with Two Workers

Let us begin the analysis by introducing some notation. We shall refer to $\Gamma_n^n(0)$ as the subgame in which the firm is left with n workers, it has yet to reach an agreement with all of the workers, and it is about to enter a bargaining session with the first one in line. We denote with π_n the payoff to the firm in this game, and with $w_{n,i}$ the payoff to the i-th of n workers. Clearly, the SZ game between the firm and n workers is the subgame $\Gamma_n^n(0)$. We shall refer to $\Gamma_k^n(s)$ as the subgame in which there are n workers left in the game, $n-k$ of them have reached an agreement with the firm for wages summing up to s, k workers have yet to reach an agreement with the firm, and the firm is about to start a bargaining session with the first of those k workers. We denote with $w_{k,i}^n(s)$ the equilibrium wage of the i-th of the k workers without agreement, and we denote with $t_k^n(s)$ the sum of wages of the k workers without agreement.

In order to gain some intuition, we study the SZ game with two workers. We solve the game backwards. First, we characterize the outcome of the subgame $\Gamma_1^1(0)$ in which, after a breakdown with one of the workers, the firm enters a bargaining session with the other one. If the bargaining session ends with the firm and the worker agreeing to the wage w, the payoff to the firm is $y_1 - w$, and the payoff to the worker is w. If the bargaining session ends with breakdown, the payoff to the firm is y_0, and the payoff to the worker is b. In either case, when the bargaining session ends so does the subgame. The subgame is the same as the BRW game and we can characterize its outcome using Lemma 1. In particular, assuming that the gains from trade $y_1 - y_0 - b$ are positive, the unique SPE of the subgame is such that the firm and the worker immediately reach an agreement over the wage

$$w_{1,1} = b + \frac{1}{2-p}\left[y_1 - y_0 - b\right]. \tag{6}$$

In turn, this implies that the firm's equilibrium payoff of the subgame is

$$\pi_1 = y_0 + \frac{1-p}{2-p}\left[y_1 - y_0 - b\right]. \tag{7}$$

When bargaining with the only worker left in the game, the marginal cost to the firm from paying the worker an extra dollar and the marginal benefit to the worker from being paid an extra dollar are both equal to 1. Hence, utility is perfectly transferrable and the ratio between the gains from trade accruing to the worker, $w_{1,1} - b$, relative to those accruing to the firm, $\pi_1 - y_0$, is equal to $1/(1-p)$, which as discussed in the comments to Lemma 1, converges to 1 in the limit for $p \to 0$.

Second, we characterize the outcome of the subgame $\Gamma_1^2(w_1)$ in which, after reaching an agreement with the first worker over some wage w_1, the firm enters a bargaining session with the second worker. If the bargaining session terminates with the firm and the second worker agreeing to the wage w_2, the game comes to an end. In this case, the payoff to the firm is $y_2 - w_1 - w_2$ and the payoff to the second worker is w_2. If the bargaining session ends with breakdown, the second worker exits, all previous agreements are erased and the firm enters a bargaining session with the one remaining worker. In this case, the payoff to the firm is given by π_1 in (7) and the payoff to the second worker is b. Overall, the bargaining session between the firm and the second worker has the same protocol and payoff structure as the BWR game and, hence, we can characterize its equilibrium outcome using Lemma 1. In particular, if $w_1 > y_2 - \pi_1 - b$, any SPE involves a breakdown between the firm and the second worker. If $w_1 \leq y_2 - \pi_1 - b$, the unique SPE is such that the firm and the second worker immediately reach an agreement over the wage

$$w_{1,1}^2(w_1) = b + \frac{1}{2-p}\left[y_2 - w_1 - \pi_1 - b\right]. \tag{8}$$

When the firm has yet to reach an agreement with only one worker, the marginal cost to the firm from paying the worker an extra dollar and the marginal benefit to the worker from being paid an extra dollar are both equal to 1. Hence, utility is perfectly transferrable and the ratio between the gains from trade accruing to the worker, $w_{1,1}^2(w_1) - b$, and those accruing to the firm, $y_2 - w_1 - w_{1,1}^2(w_1) - \pi_1$, is equal to $1/(1-p)$, which again converges to 1 in the limit for $p \to 0$.

Notice that the outcome of the bargaining session between the firm and the second worker in (8) depends on w_1, i.e. the wage agreed upon by the firm and the first worker. In fact, while w_1 does not affect the firm's payoff in case of disagreement with the second worker (as in this case, the firm will renegotiate the wage of the first worker), it does negatively affect the firm's payoff in case of agreement with the second worker (as, in this case, the wage w_1 will be paid out). For this reason, the equilibrium wage of the second worker depends on w_1. In particular, the wage $w_{1,1}^2(w_1)$ paid to the second workers—which is also equal to the sum of wages $t_1^2(w_1)$ paid by the firm to the workers following the first one—is a function of w_1 of the form

$$t_1^2(w_1) = \alpha_1 - \beta_1 w_1, \tag{9}$$

where the coefficient β_1 is given by

$$\beta_1 = \frac{1}{2-p}.$$

Third, we characterize the outcome of the subgame $\Gamma_2^2(0)$ in which the firm has yet to reach an agreement with both workers. To this aim, consider the bargaining session between the firm and the first worker. If the firm and the first worker agree to a wage $w_1 \leq y_2 - \pi_1 - b$, the firm and the second worker immediately agree to the wage $w_{1,1}^2(w_1)$ and the game comes to an end. In this case, the payoff to the firm is $y_2 - w_1 - t_1^2(w_1)$ and the payoff to the first worker is w_1. If the firm and the first worker agree to a wage $w_1 > y_2 - \pi_1 - b$, the firm and the second worker do not reach an agreement. In this case, the firm and the first worker renegotiate and achieve payoffs of π_1 and w_1 respectively. Finally, if the firm and the first worker do not reach an agreement, the firm is left with the second worker only. In this case, the firm achieves a payoff of π_1 and the first worker achieves a payoff of b.

The bargaining session between the firm and the first worker does not have the same payoff structure as the BRW game, because the wage that the firm and the first worker agree upon does not affect their payoffs if it leads to a breakdown between the firm and the second worker. However, assume that, whenever indifferent, the firm chooses to reject any wage demand from the first worker that would lead to a breakdown with the second worker and, similarly, the firm chooses not to make any counteroffer that would lead to a breakdown with the second worker. Under this tie-breaking assumption, we show that the outcome of the bargaining session is the same outcome as in BRW (see Appendix A). In particular, as long as $y_2 - b - t_1^2(b) - \pi_1 \geq 0$ or equivalently $y_2 - \pi_1 - 2b \geq 0$, the unique SPE is such that the firm and the first worker immediately agree to the wage

$$w_{2,1} = b + \frac{1}{1-p}\left[y_2 - \pi_1 - b - t_1^2(b)\right]. \tag{10}$$

In the bargaining session between the firm and the first of two workers, utility is not perfectly transferable as $\beta_1 > 0$. Indeed, the first worker's marginal benefit from receiving a higher wage is 1, while the firm's marginal cost is $(1-p)/(2-p)$, as paying the first worker an extra dollar reduces the gains from trade between the firm and the second worker by a dollar and, in turn, the second worker's wage by $1/(2-p)$ dollars. Therefore, the outcome of the bargaining session is such that the gains from trade accruing to the worker are not equal to those accruing to the firm. Instead, the ratio between the gains from trade accruing to the worker, $w_{2,1} - b$, and those accruing to the firm, $y_2 - w_{2,1} - t_1^2(w_{2,1}) - \pi_1$, is given by $(2-p)/(1-p)^2$, which converges to 2 in the limit for $p \to 0$.

Now we can summarize the outcome of the SZ game between the firm and two workers. If $y_2 - \pi_1 - 2b \geq 0$, the unique SPE is such that the firm reaches an immediate agreement with the first worker for the wage $w_{2,1}$ in (10). Since $w_{2,1} \leq y_2 - \pi_1 - b$, the firm then reaches an immediate agreement with the second worker for a wage $w_{2,2} = w_{1,1}^2(w_{2,1})$. After substituting out $t_1^2(b)$ in (10), we find that the wage (and payoff) of the first worker is

$$w_{2,1} = b + \frac{1}{2-p}\left[y_2 - \pi_1 - 2b\right]. \tag{11}$$

In turn, we can solve for the wage (and payoff) of the second worker as

$$w_{2,2} = b + \frac{1-p}{(2-p)^2}\left[y_2 - \pi_1 - 2b\right]. \tag{12}$$

Finally, we can solve for the payoff of the firm as

$$\begin{aligned} \pi_2 &= y_2 - w_{2,1} - w_{2,2} \\ &= \pi_1 + \left(\frac{1-p}{2-p}\right)^2 \left[y_2 - \pi_1 - 2b\right] \end{aligned} \tag{13}$$

In the limit for $p \to 0$, the payoffs to the workers and the firm are given by

$$\begin{aligned} w_{2,1} &= b + \tfrac{1}{2}\left[y_2 - \pi_1 - 2b\right] \\ &= b + \tfrac{1}{2}\left[y_2 - y_1 - b\right] + \tfrac{1}{4}\left[y_1 - y_0 - b\right], \\ w_{2,2} &= b + \tfrac{1}{4}\left[y_2 - \pi_1 - 2b\right] \\ &= b + \tfrac{1}{4}\left[y_2 - y_1 - b\right] + \tfrac{1}{8}\left[y_1 - y_0 - b\right], \\ \pi_2 &= \pi_1 + \tfrac{1}{4}\left[y_2 - \pi_1 - 2b\right] \\ &= y_0 + \tfrac{1}{4}\left[y_2 - y_1 - b\right] + \tfrac{5}{8}\left[y_1 - y_0 - b\right]. \end{aligned} \tag{14}$$

Several remarks about the payoffs in (14) are in order. First, the equilibrium wage of the first worker is higher than the equilibrium wage of the second worker. This is intuitive. If the firm pays the first worker an extra dollar, the gains from trade between the firm and the second worker will be 1 dollar lower and, hence, the wage of the second worker will be 50 cents lower. Overall, the marginal cost to the firm from paying the worker an extra dollar is only 50 cents. In contrast, the marginal cost to the firm from paying the second worker an extra dollar is 1 full dollar. Therefore, the utility between the firm and the first worker is transferred at the rate of 1 to 2, while the utility between the firm and the second worker is transferred at the rate of 1 to 1. For this reason, the first worker captures twice as much total surplus than the second worker, where total surplus is defined as $y_2 - \pi_1 - 2b$.

Second, the payoffs in (14) are different from the Shapley values, which are given by[3]

$$\begin{aligned} w_2^* &= b + \tfrac{1}{3}\left[y_2 - \pi_1 - 2b\right] \\ &= b + \tfrac{1}{3}\left[y_2 - y_1 - b\right] + \tfrac{1}{6}\left[y_1 - y_0 - b\right], \\ \pi_2^* &= \pi_1 + \tfrac{1}{3}\left[y_2 - \pi_1 - 2b\right] \\ &= y_0 + \tfrac{1}{3}\left[y_2 - y_1 - b\right] + \tfrac{2}{3}\left[y_1 - y_0 - b\right]. \end{aligned} \tag{15}$$

The equilibrium payoffs are not only different from the Shapley values in realization, as in equilibrium the two workers receive different payoffs even though their Shapley values are identical. The equilibrium payoffs are also different from the Shapley values after taking expectations over random orderings of workers, as for every realized ordering the equilibrium payoff to the firm is π_2 and different from its Shapley value π_2^*. Indeed, the first worker captures one half of the total surplus and the second worker captures one fourth of the total surplus and, hence, the firm's payoff is equal to one fourth of the total surplus. In contrast, the Shapley value of the firm is such that the firm captures one third of the total surplus.

Finally, Theorem 2 in Stole and Zwiebel (1996a) states that the unique SPE of the bargaining game is such that the payoffs to the firm and to the workers are given by (15). However, the theorem is incorrect because we have established that the unique SPE of the SZ

[3] Theorem 4 in Stole and Zwiebel (1996a) proves that the Shapley values can be written as in (15).

game features the payoffs in (14), which are different from the Shapley values in (15).[4] The mistake in the proof of Theorem 2 in Stole and Zwiebel (1996a) is the failure to recognize that the wage negotiated by the firm and a worker affects the value to the firm from reaching an agreement with the following workers and, in turn, it affects the following workers' wages. For this reason, the marginal cost to the firm from paying a worker a higher wage is less than the marginal benefit to the worker from being paid a higher wage. Hence, utility is not perfectly transferable and the solution to the BRW bargaining session (as well as the axiomatic Nash bargaining solution) does not equate the gains from trade accruing to the worker to those accruing to the firm, which is what is incorrectly assumed in the proof of Theorem 2 in Stole and Zwiebel (1996a).

2.3 SZ Game with n Workers

The qualitative properties of the outcome of the SZ game with two workers generalize to the case of an arbitrary number of workers. The following proposition contains the characterization of the unique SPE of the subgame $\Gamma_n^n(0)$ in which the firm has yet to reach an agreement with all of the n workers remaining in the game.

Proposition 1: *Consider the subgame $\Gamma_n^n(0)$. (i) If $y_n - \pi_{n-1} - nb < 0$, any SPE is such that the firm does not reach an agreement with all of the n workers. The payoff to the firm is given by $\pi_n = \pi_{n-1}$, with $\pi_0 = y_0$. (ii) If $y_n - \pi_{n-1} - nb \geq 0$, the unique SPE is such that the firm immediately reaches an agreement with all of the n workers. The payoff to the firm is given by*

$$\pi_n = \pi_{n-1} + \left(\frac{1-p}{2-p}\right)^n [y_n - \pi_{n-1} - nb] \text{, with } \pi_0 = y_0. \tag{16}$$

The payoff to the i-th worker is given by

$$w_{n,i} = b + \frac{1}{2-p}\left[y_n - \left(\sum\nolimits_{j=1}^{i-1} w_{n,j}\right) - \pi_{n-1} - (n+1-i)b\right]. \tag{17}$$

For $n = 1$, Proposition 1 holds as the payoffs in (16) and (17) boil down to the equilibrium payoffs of the BRW game. For $n = 2$, Proposition 1 holds as the payoffs in (16) and (17) are those derived in the previous subsection. In what follows we are going to prove that Proposition 1 holds for a generic n by induction. That is, we are going to prove that if the

[4] To be more precise, we proved that there is a unique SPE in which, when indifferent, the firm rejects any offer from the first worker (and does not make any counteroffer to the first worker) that induces to a breakdown in negotiations with the second worker. However, it is easy to show that the payoffs in Stole and Zwiebel (1996a) do not constitute an SPE (see Appendix B).

proposition holds for the subgame $\Gamma_n^n(0)$, it also holds for the subgame $\Gamma_{n+1}^{n+1}(0)$ where the firm has yet to reach an agreement with all of the $n+1$ workers left in the game.

Central to the characterization of the equilibrium of $\Gamma_{n+1}^{n+1}(0)$ is the following lemma.

Lemma 2: *Consider the subgame $\Gamma_k^{n+1}(s)$ in which the firm has $n+1$ workers, it has yet to reach an agreement with $k \leq n+1$ workers, and it has agreed to wages summing up to s with the first $n+1-k$ workers. (i) If $y_{n+1} - s - \pi_n - kb < 0$, any SPE is such that the firm does not reach an agreement with all the k remaining workers; (ii) If $y_{n+1} - s - \pi_n - kb \geq 0$, the unique SPE is such that the firm reaches an immediate agreement with each of the k remaining workers. The sum of the wages paid to the k remaining workers is*

$$t_k^{n+1}(s) = kb + \left[1 - \left(\frac{1-p}{2-p}\right)^k\right] [y_{n+1} - s - \pi_n - kb] \,. \tag{18}$$

For $k = 1$, Lemma 2 holds as the payoffs in (18) are the same as those in the BRW game. We prove that Lemma 2 holds for any $k \leq n+1$ by induction. That is, we prove that, if Lemma 2 holds for some arbitrary $k \leq n$, then it also holds for $k+1$. To this aim, we consider the subgame $\Gamma_{k+1}^{n+1}(s)$, in which the firm has $n+1$ employees, it has yet to reach an agreement with $k+1$ of them and it has agreed to wages summing up to s with the first $n-k$ workers. As usual, we characterize the solution to this subgame by backward induction.

First, consider the subgame $\Gamma_n^n(0)$ in which, after a breakdown in negotiations between the firm and the first of the $k+1$ workers without agreement, bargaining starts over between the firm and the n workers left in the game. Since we have conjectured that Proposition 1 holds when the firm has n workers, the SPE payoff of the firm in this subgame is uniquely determined and given by π_n.

Second, consider the subgame $\Gamma_k^{n+1}(s + w_1)$ in which, after the firm has reached an agreement at some wage w_1 with the first worker without an agreement, the firm starts bargaining with the other k workers without an agreement. Since we conjectured that Lemma 2 holds when the firm has $n+1$ workers and has yet to reach an agreement with k of them, there is a unique SPE to this subgame. In particular, if $w_1 > \overline{w}_{k+1}^{n+1}(s) \equiv y_{n+1} - s - \pi_n - kb$, the SPE is such that the firm does not reach an agreement with all of the k remaining workers. In this case, the firm's payoff is π_n. If $w_1 \leq \overline{w}_{k+1}^{n+1}(s)$, the SPE is such that the firm immediately reaches an agreement with all of the k remaining workers. In this case, the firm's payoff is $y_{n+1} - s - w_1 - t_k^{n+1}(s + w_1)$.

Third, we characterize the solution to the subgame $\Gamma_{k+1}^{n+1}(s)$. Consider the bargaining session between the firm and the first of the $k+1$ workers without an agreement. If the firm and the worker do not reach an agreement, the worker exits the game and the firm enters the subgame $\Gamma_n^n(0)$. In this case, the payoff to the firm is π_n and the payoff to the worker is b. If the firm and the worker agree to a wage $w_1 > \overline{w}_{k+1}^{n+1}(s)$, the firm enters the subgame $\Gamma_k^{n+1}(s+w_1)$ with negative gains from trade. In this case, the payoff to the firm is π_n and the payoff to the worker is the wage earned by the $(n-k)$-th worker in the game with n workers. Finally, if the firm and the worker agree to a wage $w_1 \leq \overline{w}_{k+1}^{n+1}(s)$, the firm enters the subgame $\Gamma_k^{n+1}(s+w_1)$ with positive gains from trade. In this case, the firm reaches an agreement with all the other workers, the payoff to the firm is $y_{n+1} - s - w_1 - t_k^{n+1}(s+w_1)$ and the payoff to the worker is w_1. Notice that $t_k^{n+1}(s+w_1)$ is of the form

$$t_k^{n+1}(s+w_1) = \alpha_k - \beta_k w_1, \tag{19}$$

where the coefficient β_k is given by

$$\beta_k = 1 - \left(\frac{1-p}{2-p}\right)^k .$$

The bargaining session between the firm and the first of the $k+1$ workers without agreement does not have the same payoff structure as the BRW game because, if the firm and the worker agree to a wage $w_1 > \overline{w}_{k+1}^{n+1}(s)$, their payoffs do not depend on w_1. However, assume that, whenever indifferent, the firm chooses to reject any wage demand from the first worker that would lead to a breakdown with the second worker and, similarly, the firm chooses not to make any counteroffer that would lead to a breakdown with the second worker. Under this tie-breaking assumption, we show that the outcome of the bargaining session is the same outcome as in BRW (see Appendix A). It then follows from Lemma 1 that, if $y_{n+1} - s - \pi_n - b - t_k^{n+1}(s+b) < 0$ or equivalently $y_{n+1} - s - \pi_n - (k+1)b < 0$, any SPE is such that the firm and the worker do not reach an agreement. In contrast, if $y_{n+1} - s - \pi_n - b - t_k^{n+1}(s+b) \geq 0$ or equivalently $y_{n+1} - s - \pi_n - (k+1)b \geq 0$, the unique SPE is such that firm and the first worker immediately reach an agreement over the wage

$$w_{k+1,1}^{n+1}(s) = b + \frac{1}{(2-p)(1-\beta_k)}\left[y_{n+1} - s - \pi_n - b - t_k^{n+1}(s+b)\right]. \tag{20}$$

In the bargaining session between the firm and the first of $k+1$ workers without agreement, utility is not perfectly transferable. Indeed, the first worker's marginal benefit from receiving a higher wage is 1, while the firm's marginal cost from paying him a higher wage is $(1-p)^k/(2-p)^k$, as paying the first worker an extra dollar reduces the sum of wages paid to the k following workers by $\beta_k = 1-(1-p)^k/(2-p)^k$ dollars. Therefore, the outcome of the bargaining session does not equate the gains from trade accruing to the worker to those accruing to the firm. Instead, as discusses in the comments to Lemma 1, the ratio between the worker's gains from trade and the firm's is $(2-p)^k/(1-p)^{k+1}$, which converges to 2^k in the limit for $p \to 0$.

We can now summarize the characterization of the subgame $\Gamma_{k+1}^{n+1}(s)$. If $y_{n+1} - s - \pi_n - (k+1)b < 0$, any SPE is such that the firm and the first worker do not reach an agreement. If $y_{n+1} - s - \pi_n - (k+1)b \geq 0$, any SPE is such that the firm and the first worker immediately reach an agreement over the wage $w_{k+1,1}^{n+1}(s)$ in (20). Substituting $t_k^{n+1}(s+b)$ into (20), we can write the wage $w_{k+1,1}^{n+1}(s)$ as

$$w_{k+1,1}^{n+1}(s) = b + \frac{1}{2-p}\left[y_{n+1} - s - \pi_n - (k+1)b\right]. \tag{21}$$

Since $w_{k+1,1}^{n+1}(s) \leq \overline{w}_{k+1}^{n+1}(s)$, the firm then reaches an immediate agreement with the remaining k workers for wages totaling up to

$$t_k^{n+1}(s + w_{k+1,1}^{n+1}(s)) = kb + \frac{1-p}{2-p}\left[1 - \left(\frac{1-p}{2-p}\right)^k\right]\left[y_{n+1} - s - \pi_n - (k+1)b\right]. \tag{22}$$

The sum $t_{k+1}^{n+1}(s)$ between the wage paid by the firm to the first worker, $w_{k+1,1}^{n+1}(s)$, and the wages paid to the remaining k workers, $t_k^{n+1}(s + w_{k+1,1}^{n+1}(s))$, is equal to

$$t_{k+1}^{n+1}(s) = (k+1)b + \left[1 - \left(\frac{1-p}{2-p}\right)^{k+1}\right]\left[y_{n+1} - s - \pi_n - (k+1)b\right]. \tag{23}$$

These results establish that, if Lemma 2 holds for some $k \leq n$, it also holds for $k+1$. Since the lemma trivially holds for $k = 1$, this means that it holds for any generic $k \leq n+1$. We have thus completed the proof of Lemma 2.

Letting $k = n+1$ and $s = 0$ in Lemma 2, we can characterize the payoffs of the subgame $\Gamma_{n+1}^{n+1}(0)$. In particular, if $y_{n+1} - \pi_n - (n+1)b < 0$, any SPE is such that the firm does not reach an agreement with all of its $n+1$ workers. In this case, the payoff to the firm is given by $\pi_{n+1} = \pi_n$. If $y_{n+1} - \pi_n - (n+1)b \geq 0$, the unique SPE is such that the firm immediately reaches an agreement with all of its $n+1$ workers. In this case, the payoff to the firm is given by

$$\pi_{n+1} = \pi_n + \left(\frac{1-p}{2-p}\right)^{n+1}\left[y_{n+1} - \pi_n - (n+1)b\right]. \tag{24}$$

The payoff to the i-th worker is given by

$$w_{n+1,i} = b + \frac{1}{2-p}\left[y_{n+1} - \left(\sum_{j=1}^{i-1} w_{n+1,j}\right) - \pi_n - (n+2-i)b\right]. \tag{25}$$

The above results show that, if Proposition 1 holds for some n, it also holds for $n+1$. Since the proposition holds for $n = 1$, this means that it holds for any generic $n = 2, 3...$ We have thus completed the proof of Proposition 1.

We are now in the position to characterize the solution of the SZ game in the limit as the probability of breakdown goes to zero.

Theorem 1: (Stole and Zwiebel game). *Consider the SZ game between the firm and n workers. Assume that the total surplus is positive, i.e. $y_n - \pi_{n-1} - nb > 0$. In the limit for $p \to 0$, the unique SPE of the game is such that the payoff π_n to the firm is given by the difference equation*

$$\pi_j = \pi_{j-1} + \frac{1}{2^j}\left[y_j - \pi_{j-1} - jb\right], \text{ for } j = 1, 2, ...n, \tag{26}$$

with initial condition $\pi_0 = y_0$. The payoff $w_{n,i}$ to the i-th of n workers is given by

$$w_{n,i} = b + \frac{1}{2^i}\left[y_n - \pi_{n-1} - nb\right]. \tag{27}$$

Proof: It is straightforward to show that if $y_n - \pi_{n-1} - nb > 0$ then $y_j - \pi_{j-1} - jb > 0$ for $j = 1, 2, ...n-1$. From this observation and Proposition 1, it follows that π_j is given by (16) for $j = 1, 2, ...n$ and $w_{n,i}$ is given by (17) for $i = 1, 2, ...n$. Taking the limit of (16) and (17) for $p \to 0$, we obtain (26) and (27). ■

Theorem 1 shows that the properties of the solution to the SZ game with 2 workers generalize nicely to the game with n workers. First, the worker's payoffs are decreasing with respect to the order in which they bargain with the firm. In general, the i-th worker to bargain with the firm captures a share of the total surplus that is twice as large as the one captured by the $(i+1)$-th worker. This is intuitive. If the firm pays the i-th worker an extra dollar, the gains from trade between the firm and the remaining $n-i$ workers decline and the wage of all of these workers falls by a total of $1 - 1/2^{n-i}$ dollars. Hence, the firm's cost from paying the i-th worker an extra dollar is $1/2^{n-i}$. If the firm pays the $(i+1)$-th worker an extra dollar, the gains from trade between the firm and the remaining $n-i-1$ workers decline and the wage of all of these workers falls by a total of $1 - 1/2^{n-i-1}$ dollars. Hence, the firm's cost from paying the $(i+1)$-th worker an extra dollar is $1/2^{n-i-1}$. Since the cost to the firm from paying the i-th worker an extra dollar is half of the cost from paying the $(i+1)$-th worker an extra dollar, the i-th worker captures twice as much of the total surplus as the $(i+1)$-th worker.

Second, the payoffs in (26)-(27) are different from the Shapley values. In fact, the Shapley value π_n^* of the firm is given by the difference equation

$$\pi_j^* = \pi_{j-1}^* + \frac{1}{j+1}\left[y_j - \pi_{j-1}^* - jb\right], \text{ for } j = 1, 2, ...n, \tag{28}$$

with initial condition $\pi_0^* = y_0$. The Shapley value w_n^* of each worker is given by

$$w_n^* = b + \frac{1}{n+1}\left[y_n - \pi_{n-1}^* - nb\right]. \tag{29}$$

For a particular ordering of workers, the payoffs in the SZ game are different from the Shapley value, as every worker is paid a different wage even though they all have the same Shapley value. Moreover, in the expectation over any distribution of orderings, the payoffs in the SZ game are different from the Shapley values, as π_n is different from π_n^*. More precisely, in the SZ game, the first worker captures half of the total surplus, the second worker captures one fourth of the total surplus, and so on and so forth. The firm is left with $1/2^n$ of the total surplus. In contrast, the Shapley values are such that each worker captures a fraction $1/(n+1)$ of the total surplus and so does the firm.

Finally, the payoffs in (26)-(27) are different from those reported by Stole and Zwiebel (1996a). In fact, Theorem 2 in Stole and Zwiebel (1996a) states that the unique SPE of the bargaining game is such that the equilibrium payoffs are equal to the Shapley values. However, Theorem 1 above shows that the unique SPE is such that the equilibrium payoffs are those in (26)-(27), which differ from the Shapley values. Therefore, Theorem 2 in Stole and Zwiebel is incorrect for every number of workers strictly greater than one.[5]

Even though the SZ game does not deliver the Shapley values as equilibrium payoffs, the solution of the game is of some interest as the protocol is quite natural. Hence, it is useful to discuss some of its features. It is easy to verify by induction that, for any number of workers n, the payoff to the firm in the SZ game is strictly smaller than the Shapley value. This is intuitive because, in the SZ game, the first worker can hold up the firm and capture some of its gains from trade with the other workers and, hence, obtain a wage that is higher than his expected marginal contribution (i.e., his Shapley value). Next, notice that, if the firm chooses how many workers to hire and wages are set according to the SZ game, it will choose N such that[6] $\pi_N - \pi_{N-1} \doteq 0$ or, equivalently, such that $y_N - \pi_N - Nb \doteq 0$. In

[5] More precisely, we proved that there is a unique SPE in which, when indifferent, the firm rejects any offer from the first worker (and does not make any counteroffer to the first worker) that induces to a breakdown in negotiations with the second worker. It is easy to show that the payoffs in Stole and Zwiebel (1996a) do not constitute an SPE.

[6] Following Stole and Zwiebel (1996a), we use the notation $x_N - x_{N-1} \doteq 0$ as shorthand for $x_N - x_{N-1} \geq 0$ and $x_{N+1} - x_N \leq 0$.

contrast, if wages are given by the worker's Shapley values, the firm will choose N^* such that $\pi^*_{N^*} - \pi^*_{N^*-1} \doteq 0$ or, equivalently, such that $y_{N^*} - \pi^*_{N^*} - N^* b \doteq 0$. Since $\pi_n < \pi^*_n$, it follows immediately that the firm will hire more workers if the wages are set according to the SZ game. In turn, we know from Stole and Zwiebel (1996a), that if the wages are given by the Shapley values, the firm will hire more workers than it is efficient as hiring an extra worker not only increases output, but it also lowers the wage of the inframarginal workers. Thus, there will be even more overhiring when wages are set according to the SZ game than when they are set given by the workers' Shapley values.

3 The Rolodex Game

3.1 Environment and Preliminaries

In this section, we propose a novel bargaining game between a firm and multiple workers, which we refer to as the Rolodex game. The players in the game are a firm and n identical workers. If the firm employs k workers and pays them wages w_1, w_2, ...w_k, its payoff is $y_k - w_1 - w_2 - ...w_k$, where y_k denotes the value of the output produced by the firm with k employees. We assume that y_k is strictly increasing and concave in k, i.e. $y_k < y_{k+1}$ and $y_{k+1} - y_k > y_{k+2} - y_{k+1}$ for $k = 0, 1, 2, ...$ If a worker is hired by the firm at the wage w, his payoff is w. If the worker is not hired by the firm, his payoff is $b \geq 0$, where b might represent the value of employment at some other firm or the value of unemployment.

The workers are initially placed in some arbitrary order from 1 to n. The game consists of a finite sequence of bargaining sessions between the firm and one of the workers. The game starts with a bargaining session between the firm and the first worker in the order. The game ends when the firm has reached an agreement with all the workers left in the game. When this happens, the firm pays the agreed-upon wage to each of the workers and production takes place.

Each bargaining session involves one round of offer and counteroffer. The session starts with the worker making a wage offer. If the firm accepts the offer, the firm enters a bargaining session with the worker who, among those that have yet to reach an agreement, is next in the order. If the firm rejects the offer, negotiations break down with probability p and continue with probability $1 - p$. If negotiations break down, the worker exits the game, all past agreements are erased and the whole bargaining process starts over with the workers who are still in the game being placed in some random order.[7] If negotiations continue, the firm

[7] We assume that the remaining workers are placed in some random order only for the sake of concreteness.

makes a counteroffer. If the worker accepts the counteroffer, the firm enters a bargaining session with the worker who, among those that have yet to reach an agreement, is next in the order. If the worker rejects the offer, negotiations break down with probability p and continue with probability $1-p$. If the negotiations continue, the worker takes the last place in the order of workers who have yet to reach an agreement, and the firm enters a bargaining session with the worker who is now first in the order among those who are still without agreement.

It is useful to compare the Rolodex game with the SZ game. In the SZ game, a bargaining session between a firm and a worker continues until the firm and the worker reach an agreement or until the worker exits the game. Under this bargaining protocol, the worker can reject the firm's counteroffers without consequence on his position in the line of workers. For this reason, the worker can take advantage of the fact that his wage lowers the wage paid by the firm to the workers who follow him and, in equilibrium, workers at the front of the line earn higher wages than workers at the end of the line. In the Rolodex game, a worker moves to the end of the line if he rejects the firm's counteroffer. Under this bargaining protocol, any worker is in the same strategic position as the last worker in the line. As we shall see in the next pages, this implies that every worker ends up earning the same wage and this common wage is the Shapley value.

It is also useful to compare the Rolodex game with the bargaining games in Gul (1986), Hart and Mas-Colell (1996) and De Fontenay and Gans (2014). The games in Gul and Hart and Mas-Colell have equilibrium payoffs equal to the Shapley values. However, in the protocol of both of these games, the firm and each of the workers have a symmetric role. In the Gul game, each bargaining session involves two randomly selected players, which may be a worker and the firm—in which case the firm buys the labor of the worker—but also may be two workers—in which case one worker buys the labor of the other worker. In the Hart and Mas-Colell game, each player has an equal probability of proposing an entire allocation. This player may be the firm—in which case, the firm offers a wage to each of the workers—but it also may be a worker—in which case, the worker demands a wage for himself and for his coworkers. In contrast, in the protocol of the Rolodex game, the firm and the worker have different roles. In particular, every bilateral bargaining session involves the firm and one of the workers, and the object of the bargain is the wage of the worker at hand. We believe that the Rolodex game conforms better to a real-world labor negotiation. The game in De

All the results in this Section continue to hold if the remaining workers are placed in their original order, in the order at the time of breakdown, or in any other order. Indeed, we do not make use of the assumption about the workers' ordering in any of the proofs.

Fontenay and Gans assumes that the firm bargains with each worker through a different delegate, and that the outcome of each bargain is observed only by the delegate and the worker directly involved unless it is a breakdown.[8] In contrast, in the Rolodex game, the firm bargains directly with every worker and all the outcomes are publicly observed.

3.2 Rolodex Game with Two Workers

Let us begin the analysis of the Rolodex game by introducing some notation. We shall refer to $\Gamma_n^n(0)$ as the subgame in which the firm is left with n workers, it has yet to reach an agreement with any of the workers, and it is about to enter a bargaining session with the first one in line. We denote with π_n the payoff to the firm in this game, and with $w_{n,i}$ the payoff to the i-th of n workers. Clearly, the Rolodex game between the firm and n workers is the subgame $\Gamma_n^n(0)$. We shall refer to $\Gamma_k^n(s)$ as the subgame in which there are n workers left in the game, $n-k$ of them have reached an agreement with the firm for wages summing up to s, k workers have yet to reach an agreement with the firm, and the firm is about to start a bargaining session with the first of those k workers. We denote with $w_{k,i}^n(s)$ the equilibrium wage of the i-th of the k workers without agreement, and with $t_k^n(s)$ the sum of wages of the k workers without agreement.

As we did for the SZ game, we shall focus on Subgame Perfect Equilibria. Here, however, we restrict attention to SPE with two additional properties. First, we restrict attention to Markov SPE, i.e. SPE such that the players follow the same strategies whenever they are in subgames with the same payoff relevant states (n, k, s). Second, we restrict attention to SPE without delay, i.e. SPE such that, in any subgame where the gains from trade are positive, the firm reaches an immediate agreement with all the remaining workers. For the sake of brevity, we shall refer to the SPE with these two properties as Markov SPE.[9]

In order to build some intuition, it is useful to consider the Rolodex game with 2 workers. We solve the game backwards. First, we characterize the outcome of the subgame $\Gamma_1^1(0)$ in which, after a breakdown in negotiations with one worker, the firm starts bargaining with the other one. The subgame begins with the worker making an offer. If the offer is rejected the firm makes a counteroffer. If the counteroffer is rejected, the worker moves to the end of

[8] Notice also that the firm would have an incentive to hide the fact that the bargain between a delegate and a worker ended with a breakdown.

[9] The restriction to Markov SPE is common in the literature (see, e.g., Gul 1989 and Hart and Mas-Colell 1996). The restriction to SPE without delay is also common (see, e.g., Muthoo 1999). The second restriction greatly simplifies the expositions. However, we suspect that there are no Markov SPE with delay when the gains from trade are strictly positive.

the line of workers without agreement. However, since there are no other workers left, the worker gets to make another offer right away. If the worker and the firm eventually reach an agreement at the wage w, the firm's payoff is $y_1 - w$ and the worker's payoff is w. If the firm and the worker do not reach an agreement, the firm's payoff is y_0 and the worker's payoff is b. The protocol described above boils down to the protocol of the BRW game. Hence, assuming that the gains from trade $y_1 - y_0 - b$ are positive, the unique (Markov) SPE of the subgame is such that the firm and the worker immediately reach an agreement over the wage

$$w_{1,1} = b + \frac{1}{2-p}\left[y_1 - y_0 - b\right]. \tag{30}$$

In turn, this implies that the firm's equilibrium payoff in the subgame is

$$\pi_1 = y_0 + \frac{1-p}{2-p}\left[y_1 - y_0 - b\right]. \tag{31}$$

Second, we characterize the outcome of the subgame $\Gamma_1^2(w_1)$ in which, after the firm and the first worker have reached an agreement at some arbitrary wage w_1, the firm starts bargaining with the second worker. The subgame begins with the second worker making an offer. If the offer is rejected, the firm makes a counteroffer. If the counteroffer is rejected, the worker moves to the end of the line of workers who have yet to reach an agreement with the firm. Since there are no other workers with whom the firm has yet to agree, the worker gets to make another offer right away. If the firm and the second worker eventually agree on a wage w_2, the firm's payoff is $y_2 - w_1 - w_2$ and the worker's payoff is w_2. If the firm and the second worker do not reach an agreement, the worker exits the game and the firm enters a bargaining session with the first worker. In this case, the firm's payoff is π_1 and the worker's payoff is b. Overall, the protocol and the payoff structure of the bargaining session between the firm and the second worker are the same as in the BRW game. Hence, if the gains from trade $y_2 - \pi_1 - w_1 - b$ are negative, any (Markov) SPE is such that the firm and the second worker do not reach an agreement. If the gains from trade are positive, the unique (Markov) SPE is such that the firm and the second worker immediately agree to the wage

$$w_{1,1}^2(w_1) = b + \frac{1}{2-p}\left[y_2 - w_1 - \pi_1 - b\right]. \tag{32}$$

Since the second worker is the only worker without an agreement in the $\Gamma_1^2(w_1)$ subgame, $t_1^2(w_1)$ is equal to $w_{1,1}^2(w_1)$.

Third, we characterize the outcome of the subgame $\Gamma_2^2(0)$ in which the firm has yet to reach an agreement with both workers. To this aim, consider the bargaining session between the firm and the first worker. We analyze this bargaining session backwards. That is, we first characterize the optimal response of the worker to an arbitrary counteroffer of the firm. Then, we solve for the optimal counteroffer of the firm. Next, we characterize the optimal response of the firm to an arbitrary offer of the worker. Finally, we solve for the optimal offer of the worker. We carry out the analysis for the case in which there are strictly positive gains from trade, i.e. $y_2 - \pi_1 - b - t_1^2(b) > 0$ or, equivalently, $y_2 - \pi_1 - 2b > 0$.

As a preliminary step, we establish some properties of the wage $w_{2,1}$ that the firm and the first worker agree upon. First, notice that $w_{2,1}$ must be greater than b, as the worker never finds it optimal to accept or offer a wage that is lower than his outside option. Second, notice that $w_{2,1}$ must be smaller than $y_2 - \pi_1 - b$. In fact, if $w_{2,1} > y_2 - \pi_1 - b$, the firm would not reach an agreement with the second worker and $w_{2,1}$ could not be part of an SPE in which the firm trades without delay with both employees. Third, notice that $w_{2,1}$ must be greater than $pb + (1-p)w_{1,1}^2(w_{2,1})$. To see why this is the case, note that when the firm makes a counteroffer, the worker can always attains a payoff of $u_w = pb + (1-p)w_{1,1}^2(w_{2,1})$ by rejecting the counteroffer, moving to the end of the line of workers who have yet to reach an agreement, and then agree to the wage $w_{1,1}^2(w_{2,1})$. This implies that, in an SPE without delay, the firm's payoff when making a counteroffer cannot be greater than $u_f = y_2 - u_w - t_1^2(u_w)$ and that the firm will always be willing to accept any offer w such that $y_2 - w - t_1^2(w) \geq u_f$. Since $y_2 - w - t_1^2(w)$ is decreasing in w and the firm accepts any offer w such that $y_2 - w - t_1^2(w) \geq u_f$, the worker can always attain a payoff of u_w. In a Markov SPE without delay, this implies that $w_{2,1} \geq pb + (1-p)w_{1,1}^2(w_{2,1})$.

Given the preliminary results above, we can solve for the optimal response of the worker to some arbitrary counteroffer by the firm. Suppose that the firm makes a counteroffer $w \leq y_2 - \pi_1 - b$ to the worker. The worker finds it optimal to accept w if and only if

$$w \geq pb + (1-p)w_{1,1}^2(w_{2,1}). \tag{33}$$

The condition above is easy to understand. The left-hand side of (33) is the worker's payoff if he accepts the counteroffer of the firm. The right-hand side of (33) is the worker's expected payoff is he rejects the counteroffer of the firm. With probability p, there is a breakdown and the worker exits the game. In this case, the worker's payoff is b. With probability $1-p$, the worker remains in the game but moves to the end of the line of workers without agreement. In this case, the worker's payoff is $w_{1,1}^2(w_{2,1})$. In fact, the firm will reach an agreement with the next worker for a wage of $w_{2,1}$ and, when the firm returns to bargaining with the first worker, there will be an immediate agreement over the wage $w_{1,1}^2(w_{2,1})$. As we shall see below, the firm never finds it optimal to make a counteroffer $w > y_2 - \pi_1 - b$, whether the worker accepts it or not.

The firm chooses its counteroffer taking as given the worker's acceptance strategy. If the firm makes a counteroffer $w \leq y_2 - \pi_1 - b$ that satisfies (33), it attains a payoff of $y_2 - w - t_1^2(w)$. In fact, the worker accepts the counteroffer w and the firm immediately reaches an agreement with the next worker for the wage $t_1^2(w)$. If the firm makes a counteroffer $w \leq y_2 - \pi_1 - b$ that violates (33), it attains a payoff of $p\pi_1 + (1-p)\left[y_2 - w_{2,1} - t_1^2(w_{2,1})\right]$. In fact, the worker rejects the counteroffer w. Then, with probability p, the worker exits the game and the firm is left with only one worker. In this case, the firm's payoff is π_1. With probability $1-p$, the worker remains in the game, but moves to the end of the line of workers without agreement. In this case, the firm's payoff is $y_2 - w_{2,1} - t_1^2(w_{2,1})$. Finally, if the firm makes a counteroffer $w > y_2 - \pi_1 - b$, it attains a payoff of π_1 if the worker accepts the counteroffer, and a payoff of $p\pi_1 + (1-p)\left[y_2 - w_{2,1} - t_1^2(w_{2,1})\right]$ if the worker rejects the counteroffer.

The counteroffer that maximizes the payoff of the firm is

$$w_c = pb + (1-p)w_{1,1}^2(w_{2,1}). \tag{34}$$

It is easy to show why this is the case. Recall that $w_c \leq w_{2,1}$ and $w_{2,1} \leq y_2 - \pi_1 - b$, and notice that at least one inequality is strict as $w_c = w_{2,1} = y_2 - \pi_1 - b$ contradicts the assumption of strictly positive gains from trade. Since $w_c < y_2 - \pi_1 - b$ and satisfies condition (33), the firm's payoff from making the counteroffer w_c is

$$y_2 - w_c - t_1^2(w_c). \tag{35}$$

Now, consider the firm's payoff from making some alternative counteroffer w', with $w' \leq y_2 - \pi_1 - b$. If the firm makes a counteroffer $w' > w_c$, condition (33) is satisfied and the firm attains a payoff of

$$y_2 - w' - t_1^2(w') < y_2 - w_c - t_1^2(w_c). \tag{36}$$

If the firm makes a counteroffer $w' < w_c$, condition (33) is violated and the firm attains a payoff of

$$\begin{aligned} & p\pi_1 + (1-p)\left[y_2 - w_{2,1} - t_1^2(w_{2,1})\right] \\ \leq\ & y_2 - w_{2,1} - t_1^2(w_{2,1}) \\ \leq\ & y_2 - w_c - t_1^2(w_c), \end{aligned} \tag{37}$$

where at least one of the two inequalities in (37) is strict. The first inequality follows from the fact that $y_2 - w - t_1^2(w) \geq \pi_1$ for all $w \leq y_2 - \pi_1 - b$ and $w_{2,1} \leq y_2 - \pi_1 - b$. The second inequality follows from the fact that $y_2 - w - t_1^2(w)$ is strictly decreasing in w and $w_c \leq w_{2,1}$. At least one of the two inequalities is strict, as they both hold as equalities only when $w_c = w_{2,1} = y_2 - \pi_1 - b$, which is a possibility we have already ruled out. Now, consider the firm's payoff from making some alternative counteroffer w', with $w' > y_2 - \pi_1 - b$. If the firm makes a counteroffer $w' > y_2 - \pi_1 - b$ which is accepted, it attains a payoff of

$$\begin{aligned} \pi_1 &\leq p\pi_1 + (1-p)\left[y_2 - w_{2,1} - t_1^2(w_{2,1})\right] \\ &< y_2 - w_c - t_1^2(w_c), \end{aligned} \tag{38}$$

where the first inequality follows from the fact that $y_2 - w_{2,1} - t_1^2(w_{2,1})$ is greater than π_1, and the second inequality follows from (37). If the firm makes a counteroffer $w' > y_2 - \pi_1 - b$ which is rejected, it attains a payoff of $p\pi_1 + (1-p)\left[y_2 - w_{2,1} - t_1^2(w_{2,1})\right]$, which is strictly smaller than $y_2 - w_c - t_1^2(w_c)$. We have thus established that w_c is the counteroffer that maximizes the payoff to the firm.

Given the characterization of the optimal counteroffer w_c, we can find the optimal acceptance strategy of the firm to an arbitrary wage offer by the worker. Suppose that the worker makes an offer $w \leq y_2 - \pi_1 - b$. The firm finds it optimal to accept w if and only if

$$y_2 - w - t_1^2(w) \geq p\pi_1 + (1-p)\left[y_2 - w_c - t_1^2(w_c)\right]. \tag{39}$$

The condition above is easy to understand. The left-hand side of (39) is the payoff to the firm from accepting the offer w. If the firm accepts the offer of the first worker, it starts bargaining with the second worker and immediately reaches an agreement at the wage $w_{1,1}^2(w)$. Hence, the firm's payoff from accepting the offer w is $y_2 - w - t_1^2(w)$. The right-hand side of (39) is the payoff to the firm from rejecting the offer w. If the firm rejects the offer, the worker exits with probability p. In this case, the firm is left with one worker and its payoff is π_1. With probability $1-p$, the worker remains in the game and the firm makes him a counteroffer w_c. The worker accepts it and the firm and the second worker reach an immediate agreement at the wage $w_{1,1}^2(w_c)$. Hence, the firm's payoff from rejecting the offer w is $p\pi_1 + (1-p)\left[y_2 - w_c - t_1^2(w_c)\right]$. Now, suppose that the worker makes an offer $w > y_2 - \pi_1 - b$. In this case the firm rejects the offer w, as (38) guarantees that the firm's payoff from rejecting, $p\pi_1 + (1-p)\left[y_2 - w_c - t_1^2(w_c)\right]$, is strictly greater than the firm's payoff from accepting, π_1.

The worker chooses the offer w taking as given the firm's acceptance strategy. If the worker makes an offer $w \leq y_2 - \pi_1 - b$ that satisfies (39), the firm accepts the offer and reaches an immediate agreement with the other worker. In this case, the worker's payoff is w. If the worker makes an offer $w \leq y_2 - \pi_1 - b$ that violates (39), the firm rejects the offer w. Then, with probability p, the worker exits the game. With probability $1-p$, the firm makes the counteroffer w_c, which the worker accepts, and then it reaches an immediate agreement with the other worker. In this case, the worker's payoff is $pb + (1-p)w_c$. Finally, if the worker makes an offer $w > y_2 - \pi_1 - b$, the firm rejects the offer and the worker's payoff is $pb + (1-p)w_c$.

The offer that maximizes the worker's payoff is w_o such that

$$y_2 - w_o - t_1^2(w_o) = p\pi_1 + (1-p)\left[y_2 - w_c - t_1^2(w_c)\right]. \tag{40}$$

It is easy to verify that this is the case. Notice that w_o is strictly greater than w_c and strictly smaller than $y_2 - \pi_1 - b$. In fact, since $y_2 - w_c - t_1^2(w_c) > \pi_1$, the right-hand side of (40) is strictly greater than π_1 and strictly smaller than $y_2 - w_c - t_1^2(w_c)$. The left-hand side of (40) is strictly decreasing in w_o and takes the value $y_2 - w_c - t_1^2(w_c)$ for $w_o = w_c$, and the value π_1 for $w_o = y_2 - \pi_1 - b$. Hence, the offer w_o that equates the left and the right-hand sides of (40) is strictly greater than w_c and strictly smaller than $y_2 - \pi_1 - b$. Since $w_o < y_2 - \pi_1 - b$ and it satisfies (39), the worker's payoff from making the offer w_o is w_o. If the worker makes an offer $w' < w_o$, condition (39) is satisfied. Hence, the worker's payoff is $w' < w_o$. If the worker makes an offer $w' > w_o$, either condition (39) is violated or $w' > y_2 - \pi_1 - b$. In either case, the firm rejects the offer and the worker's payoff is $pb + (1-p)w_c < w_o$. Thus we have established that w_o is the offer that maximizes the payoff to the worker.

We are now in the position to explicitly solve for the optimal counteroffer of the firm, w_c, and the optimal offer of the worker, w_o. Using (32) to substitute out $w_{1,1}^2(w_{2,1})$ in (34), we find that w_c is given by

$$w_c = b + \frac{1-p}{2-p}\left[y_2 - \pi_1 - 2b\right] - \frac{1-p}{2-p}\left[w_{2,1} - b\right]. \tag{41}$$

Using (32) to substitute out $w_{1,1}^2(w_o)$ in (40), we find that w_o is given by

$$w_o = b + p\left[y_2 - \pi_1 - 2b\right] + (1-p)\left[w_c - b\right]. \tag{42}$$

Using the fact that in a Markov SPE $w_o = w_{2,1}$, we can use the above equations to find that $w_{2,1}$ is given by

$$w_{2,1} = b + \frac{1}{1 + (1-p) + (1-p)^2}\left[y_2 - \pi_1 - 2b\right]. \tag{43}$$

This completes the characterization of the Rolodex game $\Gamma_2^2(0)$ between the firm and 2 workers. To summarize, if the gains from trade are strictly positive, i.e. $y_2 - \pi_1 - 2b > 0$, the unique Markov SPE has the following features. The firm and the first worker reach an immediate agreement at the wage $w_{2,1}$ in (43). The firm and the second worker reach an immediate agreement at the wage $w_{2,2} = t_1^2(w_{2,1})$ given by

$$w_{2,2} = b + \frac{1-p}{1+(1-p)+(1-p)^2}\left[y_2 - \pi_1 - 2b\right]. \tag{44}$$

The profit of the firm $\pi_2 = y_2 - w_{2,1} - w_{2,2}$ is given by

$$\pi_2 = \pi_1 + \frac{(1-p)^2}{1+(1-p)+(1-p)^2}\left[y_2 - \pi_1 - 2b\right]. \tag{45}$$

In the limit for p going to zero, the equilibrium payoff to the first worker is

$$\begin{aligned} w_{2,1} &= b + \tfrac{1}{3}\left[y_2 - \pi_1 - 2b\right] \\ &= b + \tfrac{1}{3}\left[y_2 - y_1 - b\right] + \tfrac{1}{6}\left[y_1 - y_0 - b\right]. \end{aligned} \tag{46}$$

The equilibrium payoff to the second worker is

$$\begin{aligned} w_{2,2} &= b + \tfrac{1}{3}\left[y_2 - \pi_1 - 2b\right] \\ &= b + \tfrac{1}{3}\left[y_2 - y_1 - b\right] + \tfrac{1}{6}\left[y_1 - y_0 - b\right]. \end{aligned} \tag{47}$$

The equilibrium payoff to the firm is

$$\begin{aligned} \pi_2 &= \pi_1 + \tfrac{1}{3}\left[y_2 - \pi_1 - 2b\right] \\ &= y_0 + \tfrac{1}{3}\left[y_2 - y_1 - b\right] + \tfrac{2}{3}\left[y_1 - y_0 - b\right]. \end{aligned} \tag{48}$$

The payoffs in (46)-(48) are equal to the Shapley values to the workers and the firm. Hence, the Rolodex game offers a game-theoretic foundation to the standard cooperative solution to the bargaining problem between a firm and two workers. The Rolodex game follows a natural protocol, in which the firm participates in every bilateral negotiation and, in any bilateral negotiation, only the wage of the participating worker is discussed. Moreover, the Rolodex game is one of perfect information, where there is no need to make assumptions about off-equilibium beliefs.

There is a simple intuition behind the equivalence of the equilibrium payoffs of the Rolodex game and the Shapley values. First, notice that every worker earns the same wage as the last worker in line. Indeed, a worker at any position in the line knows that if he rejects the firm's counteroffer he will become the last worker in line and, for this reason, he has the same outside option and earns the same wage as the last worker in line. That is, $w_{2,1} = w_{2,2} = w_2$. Second, notice that the wage of the last worker is such that the gains from trade accruing to the worker are equal to the gains from trade accruing to the firm, as the last worker's wage does not affect the wage agreement with any other worker. Under the assumption that a disagreement causes the bargaining game to start over, the wage of the last worker is such that $w_{2,2} - b = y_2 - w_{2,1} - w_{2,2} - \pi_1$. Combining the two observations above, we find that $w_2 - b = y_2 - \pi_1 - 2w_2$. As explained in Stole and Zwiebel (1996a), the solution to this equation is equal to the worker's Shapley value.

3.3 Rolodex Game with n Workers

The properties of the solution of the Rolodex game with 2 workers generalize to the case of an arbitrary number of workers. The following proposition contains the characterization of the unique Markov SPE of the subgame $\Gamma_n^n(0)$ in which the firm has yet to reach an agreement with all of the n workers remaining in the game.

Proposition 2: *Consider the subgame $\Gamma_n^n(0)$. (i) If $y_n - \pi_{n-1} - nb < 0$, any Markov SPE is such that the firm does not reach an agreement with all of the n workers. The payoff to the firm is given by $\pi_n = \pi_{n-1}$, with $\pi_0 = y_0$. (ii) If $y_n - \pi_{n-1} - nb \geq 0$, the unique Markov SPE is such that the firm immediately reaches an agreement with all of the n workers. The payoff to the firm is given by*

$$\pi_n = \pi_{n-1} + \frac{(1-p)^n}{\sum_{j=0}^{n}(1-p)^j}\left[y_n - \pi_{n-1} - nb\right], \text{ with } \pi_0 = y_0. \tag{49}$$

The payoff to the i-th worker is given by

$$w_{n,i} = b + \frac{1}{\sum_{j=0}^{n+1-i}(1-p)^j}\left[y_n - \sum_{j=1}^{i-1} w_{n,j} - \pi_{n-1} - (n+1-i)b\right]. \tag{50}$$

For $n = 1$, Proposition 2 holds as the payoffs in (49) and (50) boil down to the equilibrium payoffs of the BRW game. For $n = 2$, Proposition 2 holds as the payoffs in (49) and (50) coincide with those derived in the previous subsection. In the next pages, we are going to prove that Proposition 2 holds for a generic n by induction. That is, we are going to prove that if the proposition holds for the subgame $\Gamma_n^n(0)$, it also holds for the subgame $\Gamma_{n+1}^{n+1}(0)$ in which the firm has yet to reach an agreement with all of the $n+1$ workers left in the game.

Central to the characterization of the subgame $\Gamma_{n+1}^{n+1}(0)$ is the following lemma.

Lemma 3: *Consider the subgame $\Gamma_k^{n+1}(s)$ in which the firm has $n+1$ workers, it has yet to reach an agreement with $k \leq n+1$ workers, and it has agreed to wages summing up to s with the other $n+1-k$ workers. (i) If $y_{n+1} - s - \pi_n - kb < 0$, any Markov SPE is such that the firm does not reach an agreement with all of the k remaining workers; (ii) If $y_{n+1} - s - \pi_n - kb \geq 0$, the unique Markov SPE is such that the firm reaches an* *immediate agreement with each of the k remaining workers. The sum of the wages paid to the k remaining workers is*

$$t_k^{n+1}(s) = kb + \frac{\sum_{j=0}^{k-1}(1-p)^j}{\sum_{j=0}^{k}(1-p)^j}\left[y_{n+1} - s - \pi_n - kb\right]. \tag{51}$$

The wage paid to the first of the k remaining workers is

$$w_{k,1}^{n+1}(s) = b + \frac{1}{\sum_{j=0}^{k}(1-p)^j}\left[y_{n+1} - s - \pi_n - kb\right]. \tag{52}$$

The wage paid to the last of the k remaining workers is

$$w_{k,k}^{n+1}(s) = b + \frac{(1-p)^{k-1}}{\sum_{j=0}^{k}(1-p)^j}\left[y_{n+1} - s - \pi_n - kb\right]. \tag{53}$$

For $k = 1$, Lemma 3 holds as the payoffs in (51)-(53) are the same as in the BRW game. We prove that Lemma 3 holds for any $k \leq n+1$ by induction. That is, we prove that, if Lemma 3 holds for some arbitrary $k \leq n$, then it also holds for $k+1$. To this aim, we consider the subgame $\Gamma_{k+1}^{n+1}(s)$, in which the firm has $n+1$ employees, it has yet to reach an agreement with $k+1$ of them and it has agreed to wages summing up to s with the other $n-k$. As usual, we characterize the solution to this subgame by backward induction.

First, consider the subgame $\Gamma_n^n(0)$ in which, after a breakdown in negotiations between the firm and the first of the $k+1$ workers without agreement, bargaining starts over between the firm and the n workers left in the game. Since we have conjectured that Proposition 2 holds when the firm has n workers, the SPE payoff of the firm in this subgame is uniquely determined and given by π_n.

Second, consider the subgame $\Gamma_k^{n+1}(s + w_1)$ in which, after the firm has reached an agreement at some wage w_1 with the first worker without an agreement, the firm starts bargaining with the other k workers without an agreement. Since we conjectured that Lemma 3 holds when the firm has $n+1$ workers and has yet to reach an agreement with k of them, there is a unique Markov SPE to this subgame. In particular, if $w_1 > \overline{w}_{k+1}^{n+1}(s) \equiv y_{n+1} - s - \pi_n - kb$, any SPE is such that the firm does not reach an agreement with all of the k remaining workers. In this case, the firm's payoff is π_n. If $w_1 \leq \overline{w}_{k+1}^{n+1}(s)$, the unique SPE is such that the firm immediately reaches an agreement with all of the k remaining workers. In this case, the firm's payoff is $y_{n+1} - s - w_1 - t_k^{n+1}(s + w_1)$.

Third, we characterize the outcome of the subgame $\Gamma_{k+1}^{n+1}(s)$. To this aim, consider the bargaining session between the firm and the first of the $k+1$ workers without an agreement.

We analyze this bargaining session backwards. We first characterize the optimal response of the worker to an arbitrary counteroffer of the firm. We then solve for the optimal counteroffer of the firm. Next, we characterize the optimal response of the firm to an arbitrary offer of the worker. And, finally, we solve for the optimal offer of the worker. We carry out the analysis for the case in which there are strictly positive gains from trade, i.e. $y_{n+1}-s-\pi_n-b-t_k^{n+1}(s+b) > 0$ or equivalently $y_{n+1} - s - \pi_n - (k+1)b > 0$.

As in the case of two workers, let us establish some basic properties of the wage $w_{k+1,1}^{n+1}(s)$ that the firm negotiates with the first of the $k+1$ workers without agreement. First, $w_{k+1,1}^{n+1}(s)$ is greater than b, as the worker never finds it optimal to accept or offer a wage lower than his outside option. Second, $w_{k+1,1}^{n+1}(s)$ is smaller than $\overline{w}_{k+1}^{n+1}(s)$. In fact, if $w_{k+1,1}^{n+1}(s) > \overline{w}_{k+1}^{n+1}(s)$, the firm would not reach an agreement with all of the remaining workers and, hence, $w_{k+1,1}^{n+1}(s)$ could not be part of an SPE in which the firm trades without delay with all of its $k+1$ employees. Third, $w_{k+1,1}^{n+1}(s)$ is greater than $pb + (1-p)w_{k,k}^{n+1}(s + w_{k+1,1}^{n+1}(s))$. To see why, notice that when the firm makes a counteroffer, the worker can attains a payoff of $u_w = pb + (1-p)w_{k,k}^{n+1}(s + w_{k+1,1}^{n+1}(s))$ by rejecting such offer, moving to the end of the line, and agreeing to the wage $w_{k,k}^{n+1}(s + w_{k+1,1}^{n+1}(s))$ after the firm reaches an agreement with the other workers. This implies that, in an SPE without delay, the firm's payoff when making a counteroffer cannot be greater than $u_f = y_{n+1} - u_w - t_k^{n+1}(s + u_w)$ and that the firm will always be willing to accept any offer w such that $y_{n+1} - s - w - t_k^{n+1}(s+w) \geq u_f$. Since $y_{n+1} - s - w - t_k^{n+1}(s+w)$ is decreasing in w and the firm accepts any offer w such that $y_{n+1} - s - w - t_k^{n+1}(s+w) \geq u_f$, the worker can always attain a payoff of u_w. In a Markov SPE without delay, this implies that $w_{k+1,1}^{n+1}(s)$ is greater than $u_w = pb + (1-p)w_{k,k}^{n+1}(s + w_{k+1,1}^{n+1}(s))$.

Given the preliminary results above, we can solve for the optimal response of the worker to an arbitrary counteroffer by the firm. Suppose that the firm makes a counteroffer $w \leq \overline{w}_{k+1}^{n+1}(s)$ to the worker. The worker finds it optimal to accept w if and only if

$$w \geq pb + (1-p)w_{k,k}^{n+1}(s + w_{k+1,1}^{n+1}(s)). \tag{54}$$

The above condition is easy to understand. The left-hand side of (54) is the worker's payoff if he accepts the counteroffer of the firm. In fact, if the worker accepts $w \leq \overline{w}_{k+1}^{n+1}(s)$, the firm immediately reaches an agreement with all the other k workers and the worker is paid the agreed upon wage w. The right-hand side of (54) is the worker's payoff if he rejects the counteroffer of the firm. In fact, if the worker rejects w, he exits the game and attains the payoff b with probability p. With probability $1-p$, the worker moves to the end of the line of workers without an agreement. The firm immediately agrees with the next worker to a wage $w_{k+1,1}^{n+1}(s)$ and, then, it starts bargaining with the k remaining workers. From Lemma

3, it follows that the last of these k workers (who is the worker who rejected w) agrees to a wage $w_{k,k}^{n+1}(s + w_{k+1,1}^{n+1}(s))$. Hence, the right-hand side of (54) is the worker's payoff if he rejects the counteroffer of the firm. As we shall see below, the firm never finds it optimal to make a counteroffer $w > \overline{w}_{k+1}^{n+1}(s)$, whether the worker accepts it or not.

The firm chooses its counteroffer taking as given the acceptance strategy of the worker. If the firm makes a counteroffer $w \leq \overline{w}_{k+1}^{n+1}(s)$ such that condition (54) is satisfied, its payoff is $y_{n+1} - s - w - t_k^{n+1}(s+w)$. In fact, the worker accepts w and the firm immediately reaches an agreement with the k remaining workers for a total wage bill of $t_k^{n+1}(s+w)$. If the firm makes a counteroffer $w \leq \overline{w}_{k+1}^{n+1}(s)$ such that condition (54) is violated, its expected payoff is $p\pi_n + (1-p)\left[y_{n+1} - s - w_{k+1,1}^{n+1}(s) - t_k^{n+1}(s + w_{k+1,1}^{n+1}(s))\right]$. In fact, the worker rejects w. With probability p, the worker exits the game and the bargaining game starts over between the firm and the surviving n workers. In this case, the firm's payoff is π_n. With probability $1-p$, the worker moves to the end of the line of workers without agreement. Then, the firm reaches an agreement with the next worker for a wage of $w_{k+1,1}^{n+1}(s)$ and with the remaining k workers for wages summing up to $t_k^{n+1}(s + w_{k+1,1}^{n+1}(s))$. In this case, the firm's payoff is $y_{n+1} - s - w_{k+1,1}^{n+1}(s) - t_k^{n+1}(s + w_{k+1,1}^{n+1}(s))$.

The counteroffer that maximizes the payoff of the firm is

$$w_c = pb + (1-p)w_{k,k}^{n+1}(s + w_{k+1,1}^{n+1}(s)). \tag{55}$$

It is easy to show why this is the case. Recall that $w_c \leq w_{k+1,1}^{n+1}(s)$ and $w_{k+1,1}^{n+1}(s) \leq \overline{w}_{k+1}^{n+1}(s)$, and notice that at least one inequality is strict as $w_c = w_{k+1,1}^{n+1}(s) = \overline{w}_{k+1}^{n+1}(s)$ contradicts the assumption of strictly positive gains from trade. Since $w_c < \overline{w}_{k+1}^{n+1}(s)$ and satisfies condition (54), the firm's payoff from making the counteroffer w_c is

$$y_{n+1} - s - w_c - t_k^{n+1}(s + w_c). \tag{56}$$

Now, consider the firm's payoff from making some alternative counteroffer $w' \leq \overline{w}_{k+1}^{n+1}(s)$. If the firm makes a counteroffer $w' > w_c$, condition (54) is satisfied and the firm attains a payoff of

$$y_{n+1} - s - w' - t_k^{n+1}(s + w') < y_{n+1} - s - w_c - t_k^{n+1}(s + w_c). \tag{57}$$

If the firm makes a counteroffer $w' \leq \overline{w}_{k+1}^{n+1}(s)$ such that $w' < w_c$, condition (54) is violated and the firm attains a payoff of

$$\begin{aligned} & p\pi_n + (1-p)\left[y_{n+1} - s - w_{k+1,1}^{n+1}(s) - t_k^{n+1}(s + w_{k+1,1}^{n+1}(s))\right] \\ \leq\ & y_{n+1} - s - w_{k+1,1}^{n+1}(s) - t_k^{n+1}(s + w_{k+1,1}^{n+1}(s)) \\ \leq\ & y_{n+1} - s - w_c - t_k^{n+1}(s + w_c), \end{aligned} \tag{58}$$

where at least one of the two inequalities in (58) is strict. The first inequality follows from the fact that $y_{n+1} - s - w - t_k^{n+1}(s+w) \geq \pi_n$ for all $w \leq \overline{w}_{k+1}^{n+1}(s)$ and $w_{k+1,1}^{n+1}(s) \leq \overline{w}_{k+1}^{n+1}(s)$. The second inequality follows from the fact that $y_{n+1} - s - w - t_k^{n+1}(s+w)$ is strictly decreasing in w and $w_c \leq w_{k+1,1}^{n+1}(s)$. At least one of the two inequalities is strict, as they are both equalities only when $w_c = w_{k+1,1}^{n+1}(s) = \overline{w}_{k+1}^{n+1}(s)$, a possibility we have already ruled out.

Now, consider the firm's payoff from making some alternative counteroffer $w' > \overline{w}_{k+1}^{n+1}(s)$. If the firm makes a counteroffer $w' > \overline{w}_{k+1}^{n+1}(s)$ which is accepted, it attains a payoff of

$$\begin{aligned} \pi_n \quad &\leq p\pi_n + (1-p)\left[y_{n+1} - s - w_{k+1,1}^{n+1}(s) - t_k^{n+1}(s + w_{k+1,1}^{n+1}(s))\right] \\ &< y_{n+1} - s - w_c - t_k^{n+1}(s+w_c), \end{aligned} \tag{59}$$

where the first inequality follows from the fact that $y_{n+1} - s - w - t_k^{n+1}(s+w)$ is greater than π_n for $w = w_{k+1,1}^{n+1}(s)$, and the second inequality follows from (58). If the firm makes a counteroffer $w' > \overline{w}_{k+1}^{n+1}(s)$ which is rejected, it attains the same payoff as in the first line of (58), which is strictly smaller than $y_{n+1} - s - w_c - t_k^{n+1}(s+w_c)$. Thus, we have established that w_c is the counteroffer that maximizes the payoff to the firm.

Given the characterization of w_c, we can find the optimal acceptance strategy of the firm to an arbitrary wage offer by the worker. Suppose that the worker makes an offer $w \leq \overline{w}_{k+1}^{n+1}(s)$. The firm finds it optimal to accepts w if and only if

$$y_{n+1} - s - w - t_k^{n+1}(s+w) \geq p\pi_n + (1-p)\left[y_{n+1} - s - w_c - t_k^{n+1}(s+w_c)\right]. \tag{60}$$

The above condition is easy to understand. The left-hand side of (60) is the firm's payoff from accepting the offer. If fact, if the firm accepts an offer $w \leq \overline{w}_{k+1}^{n+1}(s)$, it immediately reaches an agreement with the k remaining workers for wages summing up to $t_k^{n+1}(s+w)$. Hence, if the firm accepts the offer $w \leq \overline{w}_{k+1}^{n+1}(s)$, its payoff is given by the left-hand side of (60). The right-hand side of (60) is the firm's expected payoff from rejecting the offer. In fact, if the firm rejects w, with probability p, the worker exits and the bargaining game starts over with the n surviving workers. In this case, the firm's payoff is π_n. With probability $1-p$, the worker remains in the game and the firm makes him the counteroffer w_c. In this case, the firm's payoff is $y_{n+1} - s - w_c - t_k^{n+1}(s+w_c)$. Hence, if the firm rejects the offer $w \leq \overline{w}_{k+1}^{n+1}(s)$, its expected payoff is given by the right-hand side of (60). Now, suppose that the worker makes an offer $w > \overline{w}_{k+1}^{n+1}(s)$. In this case the firm rejects the offer, as (59) guarantees that the firm's payoff from rejecting, $p\pi_n + (1-p)\left[y_{n+1} - s - w_c - t_k^{n+1}(s+w_c)\right]$, is strictly greater than the firm's payoff from accepting, π_n.

The worker chooses the offer w taking as given the firm's acceptance strategy. If the worker makes an offer $w \leq \overline{w}_{k+1}^{n+1}(s)$ that satisfies (60), the worker's payoff is w as the firm accepts the offer and reaches an immediate agreement with all the k remaining workers. If the worker makes an offer $w \leq \overline{w}_{k+1}^{n+1}(s)$ that violates (60), the worker's expected payoff is $pb + (1-p)w_c$. In fact, the firm rejects the offer w. Then, with probability p, the worker exits the game and achieves the payoff b. With probability $1-p$, the worker remains in the game and the firm makes him the acceptable counteroffer w_c. Similarly, if the worker makes an offer $w > \overline{w}_{k+1}^{n+1}(s)$, the firm rejects the offer and the worker's expected payoff is $pb + (1-p)w_c$.

The worker finds it optimal to make to the firm the offer w_o such that

$$y_{n+1} - s - w_o - t_k^{n+1}(s + w_o) = p\pi_n + (1-p)\left[y_{n+1} - s - w_c - t_k^{n+1}(s + w_c)\right]. \qquad (61)$$

It is easy to verify that this is the case. To this aim, notice that w_o is greater than w_c and smaller than $\overline{w}_{k+1}^{n+1}(s)$. In fact, since $y_{n+1} - s - w_c - t_k^{n+1}(s + w_c) > \pi_n$, the right-hand side of (61) is strictly greater than π_n and strictly smaller than $y_{n+1} - s - w_c - t_k^{n+1}(s + w_c)$. The left-hand side of (61) is strictly decreasing in w_o, it takes the value $y_{n+1} - s - w_c - t_k^{n+1}(s + w_c)$ for $w_o = w_c$, and it takes the value π_n for $w_o = \overline{w}_{k+1}^{n+1}(s)$. Hence, the w_o that equates the left and the right-hand side of (61) is strictly greater than w_c and strictly smaller than $\overline{w}_{k+1}^{n+1}(s)$. Now, notice that, since the offer w_o satisfies (60) and it is smaller than $\overline{w}_{k+1}^{n+1}(s)$, the firm accepts it and the worker's payoff is w_o. In contrast, if the worker makes an offer $w' < w_o$, the firm accepts it and the worker's payoff is $w' < w_o$. Similarly, if the worker makes an offer $w' > w_o$, the firm rejects it, as either w' violates (60) or $w' > \overline{w}_{k+1}^{n+1}(s)$. In this case, the worker's payoff is $pb + (1-p)w_c$, which we argued is smaller than w_o. Thus, we have established that w_o is the offer that maximizes the payoff to the worker.

Now, we are in the position to explicitly solve for the optimal counteroffer of the firm, w_c, and the optimal offer of the worker, w_o. Using (53) to substitute out $w_{k,k}^{n+1}(s + w)$ in (55), we find that w_c is given by

$$w_c = b + \frac{(1-p)^k}{\sum_{j=0}^{k}(1-p)^j}\left\{[y_{n+1} - s - \pi_n - (k+1)b] - \left[w_{k+1,1}^{n+1}(s) - b\right]\right\}. \qquad (62)$$

Using (51) to substitute out $t_k^{n+1}(s + w)$ in (61), we find that w_o is given by

$$w_o = b + p\left[y_{n+1} - s - \pi_n - (k+1)b\right] + (1-p)\left[w_c - b\right]. \qquad (63)$$

Using the fact that in a Markov SPE $w_o = w_{k+1,1}^{n+1}(s)$, we can use the above equations to find that $w_{k+1,1}^{n+1}(s)$ is given by

$$w_{k+1,1}^{n+1}(s) = b + \frac{1}{\sum_{j=0}^{k+1}(1-p)^j}\left[y_{n+1} - s - \pi_n - (k+1)b\right]. \qquad (64)$$

We have completed the characterization of the subgame $\Gamma^{n+1}_{k+1}(s)$. If the gains from trade are positive, i.e. $y_{n+1} - s - \pi_n - (k+1)b > 0$, the unique Markov SPE is such that the firm and the first of the $k+1$ remaining workers reach an immediate agreement at the wage $w^{n+1}_{k+1,1}(s)$, and the firm and the i-th of the $k+1$ remaining workers reach an immediate agreement at the wage $w^{n+1}_{k+1,i}(s) = w^{n+1}_{k+2-i,1}(s + \sum_{j=1}^{i-1} w^{n+1}_{k+1,j}(s))$. The wage $w^{n+1}_{k+1,1}(s)$ paid to the first worker is given by (64). The wage $w^{n+1}_{k+1,k+1}(s) = w^{n+1}_{k,k}(s + w^{n+1}_{k+1,1}(s))$ paid to the last worker is given by

$$w^{n+1}_{k+1,k+1}(s) = b + \frac{(1-p)^k}{\sum_{j=0}^{k+1}(1-p)^j}\left[y_{n+1} - s - \pi_n - (k+1)b\right]. \tag{65}$$

The sum of wages $t^{n+1}_{k+1}(s) = w^{n+1}_{k+1,1}(s) + t^{n+1}_{k}(s + w^{n+1}_{k+1,1}(s))$ paid to the $k+1$ workers is given by

$$t^{n+1}_{k+1}(s) = (k+1)b + \frac{\sum_{j=0}^{k}(1-p)^j}{\sum_{j=0}^{k+1}(1-p)^j}\left[y_{n+1} - s - \pi_n - (k+1)b\right]. \tag{66}$$

If the gains from trade between the firm and the remaining $k+1$ workers are equal to zero, it is easy to verify that the unique Markov SPE also involves immediate agreement at the wages (65) and (66). Finally, if the gains from trade between the firm and the remaining $k+1$ workers are strictly negative, it is easy to verify that any (Markov) SPE is such that the firm does not reach an agreement with all of the $k+1$ workers. These observations show that, if Lemma 3 holds for some $k < n+1$, it also holds for $k+1$. Since the lemma trivially holds for $k = 1$, this means that it holds for any generic k. We have thus concluded the proof of Lemma 3.

Letting $k = n+1$ and $s = 0$ in Lemma 3, we can characterize the payoffs of the subgame $\Gamma^{n+1}_{n+1}(0)$. In particular, if $y_{n+1} - \pi_n - (n+1)b > 0$, the unique Markov SPE is such that the firm immediately reaches an agreement with all of its $n+1$ employees. In this case, the payoff to the firm is given by

$$\pi_{n+1} = \pi_n + \frac{(1-p)^{n+1}}{\sum_{j=0}^{n+1}(1-p)^j}\left[y_{n+1} - \pi_n - (n+1)b\right]. \tag{67}$$

The payoff to the i-th worker is given by

$$w_{n+1,i} = b + \frac{1}{\sum_{j=0}^{n+2-i}(1-p)^j}\left[y_{n+1} - \sum_{j=1}^{i-1} w_{n+1,j} - \pi_n - (n+2-i)b\right]. \tag{68}$$

The above results show that, if Proposition 2 holds for some n, it also holds for $n+1$. Since the proposition holds for $n = 1$, this means that it holds for any generic $n = 2, 3...$ We have thus completed the proof of Proposition 2.

We are now in the position to characterize the solution of the Rolodex game in the limit as the probability of breakdown goes to zero.

Theorem 2: (Rolodex game). *Consider the Rolodex game between the firm and n workers. Assume that the total surplus is positive, i.e. $y_n - \pi_{n-1} - nb > 0$. In the limit for $p \to 0$, the unique Markov SPE of the game is such that the payoff π_n to the firm is given by the difference equation*

$$\pi_j = \pi_{j-1} + \frac{1}{j+1}\left[y_j - \pi_{j-1} - jb\right], \text{ for } j = 1, 2, ...n, \tag{69}$$

with initial condition $\pi_0 = y_0$. The payoff $w_{n,i}$ to the i-th of n workers is given by

$$w_{n,i} = b + \frac{1}{n+1}\left[y_n - \pi_{n-1} - nb\right]. \tag{70}$$

Proof: It is straightforward to show that if $y_n - \pi_{n-1} - nb > 0$ then $y_j - \pi_{j-1} - jb > 0$ for $j = 1, 2, ...n-1$. From this observation and Proposition 2, it follows that π_j is given by (49) for $j = 1, 2, ...n$ and $w_{n,i}$ is given by (50) for $i = 1, 2, ...n$. Taking the limit of (49) and (50) for $p \to 0$, we obtain (69) and (70). ■

Theorem 2 shows that, also in the case of a generic number of workers, the equilibrium payoffs of the Rolodex game converge to the Shapley values when the probability of a breakdown following a rejection goes to zero.

4 Conclusions

In this paper, we revisited the bargaining problem between a firm and n workers. In the first part of the paper, we analyzed the extensive-form bargaining game proposed by Stole and Zwiebel (1996a). We proved that the equilibrium of the SZ game is not such that all workers are paid their Shapley value, as incorrectly claimed by Stole and Zwiebel (1996a). Indeed, we showed that the unique Subgame Perfect Equilibrium of the SZ game is such that the first worker to bargain with the firm captures twice as much surplus as the second worker, who in turn captures twice as much surplus as the third worker, etc... The firm captures a fraction $1/2^n$ of the surplus. These payoffs are different from the Shapley values both for a particular realization of the initial ordering of workers, as well as in expectation over any distribution of orderings. In the second part of the paper, we presented an alternative extensive-form game, which we dubbed the Rolodex game. The Rolodex game follows a protocol that is sensible in the context of a wage negotiation between a firm and its workers. Moreover, the

Rolodex game admits, under some mild restrictions, a unique Subgame Perfect Equilibrium in which the profile of workers' wages and the profit of the firm coincide with the Shapley values.

We believe that there are two important results in the paper. First, a large number of papers in the labor/search literature have adopted the Shapley values as the solution to the bargaining problem between a firm and multiple workers and have referred to the SZ game for the game-theoretic foundation of that solution. Our paper shows that the reference to the SZ game is unwarranted and, instead, the literature should refer to the Rolodex game. Subject to replacing the SZ game with the Rolodex game, the conclusions reached in this applied literature are still valid. Second, we believe that the correct characterization of the solution of the SZ game is of interest on its own. Indeed, one might envision situations in which a firm needs to reach an agreement with one supplier before it can start bargaining with the next one. In these situations, we show that the upstream suppliers are in a superior bargaining position relative to the downstream suppliers and, in equilibrium, they end up extracting a larger fraction of the total surplus. This property of equilibrium suggests that it is important to investigate the actions that suppliers can take in order to bargain with a producer earlier rather than later.

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Appendix

A SZ and BRW Stage Games: Equivalence

Consider the subgame $\Gamma_{k+1}^{n+1}(s)$ in which there are $n+1$ workers left in the game, $n-k$ of them have reached an agreement with the firm for wages summing up to s, $k+1$ workers have yet to reach an agreement with the firm, and the firm is about to start a bargaining session with the first of those $k+1$ workers.

We want to characterize the outcome of the bargaining session between the firm and the first of the $k+1$ workers without agreement. As discussed in the main text, if the bargaining session ends with the firm and the worker agreeing to a wage $w \leq \overline{w}_{k+1}^{n+1}(s)$, with $\overline{w}_{k+1}^{n+1}(s) \equiv y_{n+1} - s - \pi_n - kb$, the firm reaches an agreement with all the k remaining workers for wages summing up to $t_k^{n+1}(s+w)$. Hence, in this case, the payoff to the firm is $y_{n+1} - w - t_k^{n+1}(s+w)$ and the payoff to the worker is w. If the bargaining session ends with the firm and the worker agreeing to a wage $w > \overline{w}_{k+1}^{n+1}(s)$, the firm does not reach an agreement with the following worker. In this case, the bargaining process starts over with n workers; the payoff to the firm is π_n and the payoff to the worker is $w_{n,n+1-k}$. Finally, if the bargaining session ends with a breakdown, the payoff to the firm is π_n and the payoff to the worker is b.

We focus on Subgame Perfect Equilibria subject to a tie-breaking rule. In particular, whenever the firm is indifferent, we assume that it chooses to reject an offer $w_o > \overline{w}_{k+1}^{n+1}(s)$ that leads to a breakdown with one of the other workers without agreement. Similarly, whenever the firm is indifferent, we assume that it does not make a counteroffer $w_c > \overline{w}_{k+1}^{n+1}(s)$ that leads to a breakdown with one of the other workers without agreement. In order to carry out the analysis, it is useful to introduce some additional notation. Consider the subgame that starts with the worker making an offer to the firm, and denote as m_W and M_W the mimimum and the maximum payoff to the worker among all SPEs that satisfy the tie-breaking rule. Consider the subgame that starts with the firm making a counteroffer to the worker, and denote as m_F and M_F the minimum and the maximum payoff to the firm among all SPEs that satisfy the tie-breaking rule. Finally, it is useful to define the function $\phi(x)$ as the solution with respect to w of the equation

$$y_{n+1} - s - w - t_k^{n+1}(s+w) = x.$$

That is,

$$\phi(x) = b + [y_{n+1} - s - \pi_n - (k+1)b] - \left(\frac{2-p}{1-p}\right)^k [x - \pi_n]\,. \qquad \text{(A1)}$$

In what follows, we show that there is a unique SPE satisfying the tie-breaking rule and that such SPE has the same solution as the BRW game. The proof involves three intermediate claims, which closely follow the steps in Chapter 3 of Muthoo (1999). We carry out the analysis under the assumption that there are strictly positive gains from trade, i.e. $y_{n+1} - s - \pi_n - b - t_k^{n+1}(b) > 0$ or, equivalently, $y_{n+1} - s - \pi_n - (k+1)b > 0$.

Claim 1: *The payoff bounds m_W, M_W, m_F and M_F are such that*

$$\begin{aligned} &m_W \geq b, \quad M_W \leq \overline{w}_k^{n+1}(s), \\ &m_F \geq \pi_n, \quad M_F \leq y - s - b - t_k^{n+1}(s+b). \end{aligned} \tag{A2}$$

Proof: First, notice that the worker can always attain a payoff of b by making an offer $w_o = b$ and by rejecting any counteroffer $w_c < b$. Hence, $m_W \geq b$. Similarly, the firm can always attain a payoff of π_n by making a counteroffer $w_c = \overline{w}_k^{n+1}(s)$ and by rejecting any wage offer $w_o > \overline{w}_k^{n+1}(s)$. Hence, $m_F \geq \pi_n$. Next, suppose there is an SPE with $M_W > \overline{w}_{k+1}^{n+1}(s)$. Then, at some point, the worker must either make an offer $w_o > \overline{w}_{k+1}^{n+1}(s)$ that is accepted by the firm, or the firm must make a counteroffer $w_c > \overline{w}_{k+1}^{n+1}(s)$ that is accepted by the worker. In the first case, if the firm accepts the offer it attains a payoff of π_n, and if it rejects the offer it attains a payoff greater than $p\pi_n + (1-p)m_F \geq \pi_n$. Under the assumption that, in case of indifference, the firm rejects an offer that causes a breakdown with a subsequent worker, the firm always rejects an offer $w_o > \overline{w}_{k+1}^{n+1}(s)$. If the firm makes a counteroffer $w_c > \overline{w}_{k+1}^{n+1}(s)$ which is accepted by the worker, it attains a payoff π_n. If the firm makes instead a counteroffer $w_c' < \overline{w}_{k+1}^{n+1}(s)$, it attains a payoff non-smaller than π_n. Under the assumption that, in case of indifference, the firm does not make a counteroffer that causes a breakdown with a subsequent worker, the firm never finds it optimal to make a counteroffer $w_c > \overline{w}_{k+1}^{n+1}(s)$. Hence, $M_W \leq \overline{w}_{k+1}^{n+1}(s)$. Finally, suppose there is an SPE with $M_F > y - s - b - t_k^{n+1}(s+b)$. Then, after some history of play, the firm must either make a counteroffer offer $w_c < b$ that is accepted by the worker, or the worker must make an offer $w_o < b$ that is accepted by the firm. If the firm makes a counteroffer $w_c < b$, the worker's payoff from accepting is w_c and from rejecting is $pb + (1-p)m_W \geq b$. Hence, the worker will never accept a counteroffer $w_c < b$. If the worker makes an offer $w_o < b$ that is accepted, the worker's payoff is $w_o < b \leq m_W$. Hence, the worker will never make an offer $w_o < b$. Hence, $M_F \leq y - s - b - t_k^{n+1}(s+b)$. ■

Claim 2: *The payoff bounds m_W, M_W, m_F and M_F are such that*

$$\begin{aligned} &m_W \geq \phi(p\pi_n + (1-p)M_F), \\ &m_F \geq y_{n+1} - s - pb - (1-p)M_W - t_k^{n+1}(s + pb + (1-p)M_W). \end{aligned} \tag{A3}$$

Proof: Consider a subgame starting with the worker making an offer. Notice that the firm always accepts an offer $w \leq \overline{w}_{k+1}^{n+1}(s)$ such that

$$y_{n+1} - s - w - t_k^{n+1}(s + w) \geq p\pi_n + (1 - p)M_F. \tag{A4}$$

Since $p\pi_n + (1-p)M_F \geq \pi_n$, the offer w_o for which the left-hand side of the above inequality equals the right-hand side is smaller than $\overline{w}_{k+1}^{n+1}(s)$. Hence, if the worker makes the offer w_o, his payoff is

$$u_W = w_o \equiv \phi(p\pi_n + (1 - p)M_F). \tag{A5}$$

Since $m_W \geq u_W$, we have established the first part of the claim. Next, consider a subgame starting with the firm making a counteroffer. Notice that the worker always accepts a counteroffer $w \leq \overline{w}_{k+1}^{n+1}(s)$ such that

$$w \geq pb + (1 - p)M_W. \tag{A6}$$

Since $pb + (1 - p)M_W < \overline{w}_{k+1}^{n+1}(s)$, the counteroffer w_c for which the left-hand side of the above inequality equals the right-hand side is smaller than $\overline{w}_{k+1}^{n+1}(s)$. Hence, the payoff to the firm when making the counteroffer w_c is

$$u_F = y_{n+1} - s - pb - (1 - p)M_W - t_k^{n+1}(s + pb + (1 - p)M_W). \tag{A7}$$

Since $m_F \geq u_F$, we have established the second part of the claim. ■

Claim 3: *The payoff bounds m_W, M_W, m_F and M_F are such that*

$$\begin{aligned} &M_F \leq y_{n+1} - s - pb - (1 - p)m_W - t_k^{n+1}(s + pb + (1 - p)m_W), \\ &M_W \leq \phi(p\pi_n + (1 - p)m_F). \end{aligned} \tag{A8}$$

Proof: Consider a subgame starting with the firm making a counteroffer. In any SPE, the worker rejects every counteroffer $w_c < \underline{w}_c$, where

$$\underline{w}_c = pb + (1 - p)m_W \leq \overline{w}_{k+1}^{n+1}(s). \tag{A9}$$

If the SPE involves the worker accepting the counteroffer, the payoff to the firm cannot be greater than

$$u_F^a = y_{n+1} - s - \underline{w}_c - t_k^{n+1}(s + \underline{w}_c). \tag{A10}$$

If the SPE involves the worker rejecting the counteroffer and the continuation payoffs are u_W and v_F, the payoff to the firm is

$$\begin{aligned} u_F^r &= p\pi_n + (1 - p)v_F \\ &\leq p\pi_n + (1 - p)[y_{n+1} - s - u_W - t_k^{n+1}(s + u_W)] \\ &\leq y_{n+1} - s - u_W - t_k^{n+1}(s + u_W) \\ &\leq y_{n+1} - s - m_W - t_k^{n+1}(s + m_W) \\ &\leq y_{n+1} - s - \underline{w}_c - t_k^{n+1}(s + \underline{w}_c), \end{aligned} \tag{A11}$$

where the second line follows from the fact that $v_F \leq y_{n+1} - s - u_W - t_k^{n+1}(s + u_W)$, the third line follows from the fact that $u_W < M_W \leq \overline{w}_{k+1}^{n+1}(s)$ and hence $\pi_n \leq y_{n+1} - s - u_W - t_k^{n+1}(s + u_W)$, the fourth line follows from the fact that $m_W \leq u_W$, and the last line from the fact that $\underline{w}_c \leq m_W$. Overall, the payoff to the firm when making a counteroffer cannot be greater than

$$u_F \leq M_F \leq \max\{u_F^a, u_F^r\} = y_{n+1} - s - \underline{w}_c - t_k^{n+1}(s + \underline{w}_c). \tag{A12}$$

Next, consider a subgame starting with the worker making an offer. In any SPE, the firm rejects every offer $w_o > \overline{w}_o$, where

$$y_{n+1} - s - \overline{w}_o - t_k^{n+1}(s + \overline{w}_o) = p\pi_n + (1-p)m_F. \tag{A13}$$

If the SPE involves the firm accepting the counteroffer, the payoff to the worker cannot be greater than

$$u_W^a = \overline{w}_o = \phi(p\pi_n + (1-p)m_F). \tag{A14}$$

If the SPE involves the firm rejecting the offer and the continuation payoffs are u_F and v_W, the payoff to the worker is

$$\begin{aligned} u_W^r &= pb + (1-p)v_W \\ &\leq pb + (1-p)\phi(u_F) \\ &\leq \phi(u_F) \\ &\leq \phi(m_F) \\ &\leq \phi(p\pi_n + (1-p)m_F), \end{aligned} \tag{A15}$$

where the second line follows from the fact that $v_W \leq \phi(u_F)$, the third line follows from the fact that $b \leq \phi(u_F)$, the fourth line follows from the fact that $m_F \leq u_F$, and the last line from the fact that $\pi_n \leq m_F$. Overall, the payoff to the worker when making an offer cannot be greater than

$$u_W \leq M_W \leq \max\{u_W^a, u_W^r\} = \overline{w}_o. \tag{A16}$$

This completes the proof of the claim. ■

From Claims 2 and 3 and the definitions of t_k^{n+1} and ϕ, it follows that

$$\begin{aligned} m_W &\geq b + \frac{1}{2-p}\left[y_{n+1} - s - \pi_n - (k+1)b\right], \\ M_W &\leq b + \frac{1}{2-p}\left[y_{n+1} - s - \pi_n - (k+1)b\right], \\ m_F &\geq \pi_n + \left(\frac{1-p}{2-p}\right)^{k+1}\left[y_{n+1} - s - \pi_n - (k+1)b\right], \\ M_F &\leq \pi_n + \left(\frac{1-p}{2-p}\right)^{k+1}\left[y_{n+1} - s - \pi_n - (k+1)b\right]. \end{aligned} \tag{A17}$$

Since $m_W \leq M_W$ and $m_F \leq M_F$, the above inequalities imply

$$\begin{aligned} m_W &= M_W = b + \frac{1}{2-p}\left[y_{n+1} - s - \pi_n - (k+1)b\right], \\ m_F &= M_F = \pi_n + \left(\frac{1-p}{2-p}\right)^{k+1}\left[y_{n+1} - s - \pi_n - (k+1)b\right]. \end{aligned} \tag{A18}$$

That is, all SPE's starting with the worker making an offer give the same payoff for the worker, and all SPE's starting with the firm making a counteroffer give the same payoff to the firm.

We can now derive the equilibrium outcome of the bargaining session. The session starts with the worker making an offer w_o to the firm. The firm finds it optimal to accepts the offer if $w_o \leq \overline{w}_{k+1}^{n+1}(s)$ and

$$w_o \leq \phi(p\pi_n + (1-p)M_F). \tag{A19}$$

The worker chooses the offer taking as given the firm's acceptance strategy above. Denote w_o^* as $\phi(p\pi_n + (1-p)M_F)$. If the worker makes the offer w_o^*, the firm accepts it as w_o^* satisfies (A19) and it is strictly smaller than $\overline{w}_{k+1}^{n+1}(s)$. Hence, the worker's payoff is

$$\begin{aligned} w_o^* &= \phi(p\pi_n + (1-p)M_F) \\ &= b + \frac{1}{2-p}\left[y_{n+1} - s - \pi_n - (k+1)b\right]. \end{aligned} \tag{A20}$$

If the worker makes an offer $w_o < w_o^*$, the firm accepts it as w_o satisfies (A19) and it is smaller than $\overline{w}_{k+1}^{n+1}(s)$. Hence, the worker's payoff is

$$\begin{aligned} w_o &< \phi(p\pi_n + (1-p)M_F) \\ &= b + \frac{1}{2-p}\left[y_{n+1} - s - \pi_n - (k+1)b\right]. \end{aligned} \tag{A21}$$

If the worker makes an offer $w_o > w_o^*$, the firm rejects it as w_o either violates (A19) or it is greater than $\overline{w}_{k+1}^{n+1}(s)$. Hence, the worker's payoff is no greater than

$$\begin{aligned} & pb + (1-p)\phi(M_F) \\ = \quad & b + \frac{(1-p)^2}{2-p}\left[y_{n+1} - s - \pi_n - (k+1)b\right] \\ < \quad & b + \frac{1}{2-p}\left[y_{n+1} - s - \pi_n - (k+1)b\right]. \end{aligned} \tag{A22}$$

From the above expressions, it follows that the worker finds it optimal to make the offer w_o^*, which is immediately accepted by the firm.

We can now summarize our findings. When the gains from trade are strictly positive, i.e. $y_{n+1} - s - \pi_n - (k+1)b > 0$, the unique SPE (subject to the tie-breaking rule about offers and counteroffers leading to a breakdown between the firm and a subsequent worker) is such that the bargaining session ends immediately with an agreement at the wage w_o^*. When the gains from trade are strictly negative, it is straightforward to verify that the unique SPE is such that the bargaining session ends with a breakdown. When the gains from trade are zero, there are several payoff equivalent SPEs. As standard in the literature, we assume that the firm and the worker reach an immediate agreement at the wage w_o^*. Since w_o^* is the same wage outcome as in the BRW game, it follows that the unique SPE is such that the outcome of the bargaining session between the firm and the first of $k+1$ workers without agreement is the same as the outcome of the BRW game described in Lemma 1.

B SZ Payoffs not an SPE of SZ Game

We prove that the equilibrium strategies in Stole and Zwiebel (1996a) are not an SPE of the SZ game. For the sake of illustration, we present the result for the case of 2 workers.

Suppose that, as implied by Stole and Zwiebel (1996a), the outcome of the bargaining session between the firm and the first worker is given by the SPE of the BRW game with perfectly transferrable utility. That is, the worker always makes the acceptable wage offer

$$w_{2,1} = b + \frac{1}{2-p}\left[y_2 - \pi_1 - t_1^2(w_{2,1}) - b\right], \tag{B1}$$

where

$$t_1^2(w) = b + \frac{1}{2-p}\left[y_2 - w - \pi_1 - b\right]. \tag{B2}$$

Suppose that the firm makes some arbitrary counteroffer $w \leq y_2 - \pi_1 - b$ to the worker. Clearly, the worker finds it optimal to accept the counteroffer if and only if

$$\begin{aligned} w &\leq pb + (1-p)w_{2,1} \\ &= b + \frac{(1-p)^2}{(2-p)^2 - 1}\left[y_2 - \pi_1 - 2b\right]. \end{aligned} \tag{B3}$$

Given the worker's acceptance strategy in (B3), it is straightforward to verify that the firm finds it optimal to make the counteroffer

$$w_c = b + \frac{(1-p)^2}{(2-p)^2 - 1}\left[y_2 - \pi_1 - 2b\right]. \tag{B4}$$

In doing so, the firm attains a payoff of

$$\begin{aligned} u_F &= y_2 - w_c - t_1^2(w_c) \\ &= \pi_1 + \frac{1-p}{2-p}\left[\frac{(2-p)^2-(1-p)^2-1}{(2-p)^2-1}\right][y_2-\pi_1-2b]. \end{aligned} \tag{B5}$$

Now, suppose that the worker makes the firm some arbitrary offer $w \leq y_2 - \pi_1 - b$. The firm finds it optimal to accept the offer if and only if w is such that

$$y_2 - w - t_1^2(w) \geq p\pi_1 + (1-p)u_F. \tag{B6}$$

Therefore, if it is optimal for the worker to make an offer $w \leq y_2 - \pi_1 - b$, it must be the case that the worker makes the offer w_o such that

$$y_2 - w_o - t_1^2(w_o) = p\pi_1 + (1-p)u_F, \tag{B7}$$

which, after solving for w_o, gives

$$w_o = b + \left[1-(1-p)\frac{(2-p)^2-(1-p)^2-1}{(2-p)^2-1}\right][y_2-\pi_1-2b]\,. \tag{B8}$$

Notice that $w_{2,1} < y_2 - \pi_1 - b$, but $w_{2,1}$ is strictly smaller than w_o. Therefore, the worker could attain a strictly higher payoff by deviating from the strategy in (B1) and by offering the wage w_o.