GLOBAL SUNSPOTS AND ASSET PRICES IN A MONETARY ECONOMY

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ABSTRACT

This paper constructs a simple model in which asset price fluctuations are caused by sunspots. Most existing sunspot models use local linear approximations: instead, I construct global sunspot equilibria. My agents are expected utility maximizers with logarithmic utility functions, there are no fundamental shocks and markets are sequentially complete. Despite the simplicity of these assumptions, I am able to go a considerable way towards explaining features of asset pricing data that have presented an obstacle to previous models that adopted similar assumptions. My model generates volatile persistent swings in asset prices, a substantial term premium for long bonds and bursts of conditional volatility in rates of return.

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1   Introduction

The representative agent (RA) model has been used by macroeconomists to understand business cycles for more than thirty years. This model, when supplemented by price rigidities and financial frictions, does a reasonable job of replicating the co-movements of consumption, investment, GDP and employment in past data (Smets and Wouters, 2003, 2007). But it fails badly when confronted with financial market facts (Cochrane, 2011).

This paper constructs a heterogenous agent general equilibrium model to explain asset prices. In this model, asset price fluctuations are caused by random shocks to the price level that reallocate consumption across two kinds of people. Asset prices are volatile and price dividend ratios are persistent even though there is no fundamental uncertainty and financial markets are sequentially complete. Following David Cass and Karl Shell (1983), I refer to the random variables that drive equilibria as 'sunspots'.

My work differs in three ways from standard asset pricing models. First, I allow for birth and death by exploiting Blanchard's (1985) concept of perpetual youth. Second, there are two types of people that differ in the rate at which they discount the future. Third, my model contains an asset, government debt, denominated in dollars. All three of these assumptions have appeared before in previous work.[1] My contribution is to combine them in a way that generates novel results.

Because I am interested in the effects of distributional shocks, my baseline model has no fundamental uncertainty of any kind. The model has a set of perfect foresight equilibria that are solutions to a difference equation which converges to a unique steady state. Because debt is denominated in dollars,

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[1] Farmer (2002a) develops a version of Blanchard's (1985) perpetual youth model with capital and aggregate uncertainty and Farmer (2002b) adds nominal government debt to this framework to explain asset price volatility. Farmer, Nourry, and Venditti (2012), Gârleanu, Kogan, and Panageas (2012) and Gârleanu and Panageas (2014) develop versions of the Blanchard framework with two agents. The results in the current paper rely on all three of these pieces; perpetual youth, multiple types and nominal debt.

the initial price level is indeterminate. And because the initial price level is indeterminate, there is more than one solution to the difference equation, each of which is an equilibrium, and each of which begins at a different initial point.

I exploit the indeterminacy of the set of the perfect foresight equilibria to construct a rational expectations equilibrium in which uncertainty is non-fundamental. The people in my model come to believe that the future price level is a random variable, driven by a sunspot, and they write financial contracts contingent on its realization. But the unborn cannot participate in the financial markets that open before they are born. As a consequence, sunspot shocks reallocate resources between people of different generations.

Most sunspot models add a shock to the perfect foresight equilibria of a model that has been linearized around an indeterminate steady state.[2] This method may be used to generate local sunspot equilibria but there is no guarantee that the sunspot solutions of a linear approximation are close to the equilibria of the original model once the variance of the shocks becomes large. In this paper, I exploit the nonlinear nature of my solution to compute global sunspot equilibria. This feature enables the model to generate a substantial term premium for assets that exhibit duration risk.

Although I model an endowment economy, the framework I provide can easily be extended to allow for production by adding capital and a labor market. If my explanation for asset price volatility is accepted, models that build on this framework have the potential to unify macroeconomics with finance theory in a simple and parsimonious way.

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[2] Farmer and Woodford (1984, 1997) is the first example of this type in a one dimensional model and Woodford (1986) extends the technique to higher dimensions. See Benhabib and Farmer (1999) and Farmer (1999), for further applications of this method to business cycle models.

2 Antecedents

An active body of scholars seek to explain asset price data using the representative agent model. Some of the modifications to this model that have been tried include richer utility specifications (Abel, 1990; Constantinides, 1990; Campbell and Cochrane, 1999) adding technology shocks with exogenous time-varying volatility (Bansal and Yaron, 2004), and assuming that technology is occasionally hit by rare disasters (Reitz, 1988; Barro, 2005, 2006; Wachter, 2013; Gabaix, 2012). Here, I take an alternative approach.

I draw on ideas developed at the University of Pennsylvania in the 1980s.[3] Using the term sunspots to refer to nonfundamental uncertainty, David Cass and Karl Shell (1983) showed that sunspots can have real effects on consumption, even in the presence of a complete set of financial markets. Using the term 'self-fulfilling prophecies' to refer to nonfundamental uncertainty, Costas Azariadis (1981) showed that nonfundamental shocks could be added to a DSGE model to drive business cycles. Drawing on both of these ideas, Roger Farmer and Michael Woodford (1984; 1997) combined self-fulfilling prophecies with indeterminacy to generate a model where sunspot shocks explain persistent fluctuations in GDP.

In this paper I move the sunspot research agenda forward by developing a model of asset pricing that represents an alternative to the widely used representative agent approach (Abel, 1990; Campbell and Cochrane, 1999; Bansal and Yaron, 2004).

My work is most closely related to four unpublished working papers, Farmer (2002a,b, 2014a) and Farmer, Nourry, and Venditti (2012). In Farmer (2002a) and Farmer (2002b) I constructed a perpetual youth model of the kind developed by Blanchard (1985). I added aggregate shocks, and I used the resulting framework to understand features of asset pricing data. The models developed in those papers exploited the existence of an indeterminate

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[3]Farmer (2014b) surveys the history of sunspots and self-fulfilling prophecies.

steady state, but they relied on the unrealistic feature that the equilibrium is dynamically inefficient.

In joint work with Carine Nourry and Alain Venditti (Farmer, Nourry, and Venditti, 2012) we thought we had solved the problem of dynamic inefficiency by constructing sunspot equilibria in a model with a unique perfect foresight equilibrium. Unfortunately, that turns out not to be the case as the putative equilibria we construct in that working paper fail to equate the marginal rates of substitution of each type of agent in every state. Consequently, the paper does not fulfil its claim to generate sunspot equilibria.

In this paper I combine ideas from all four of these working papers in a novel way. First, I build on the perpetual youth model of Blanchard (1985) as in Farmer (2002a). Second, I reintroduce nominally denominated government debt as in my (2002b) paper. Third, I exploit the idea that there are two types of agents, as in Farmer, Nourry, and Venditti (2012). And fourth, I introduce a technique to construct global sunspots that is a development of an idea first introduced in Farmer (2014a).

This is not the only paper to explore heterogenous agents models to understand asset pricing data. Gârleanu, Kogan, and Panageas (2012) build a two agent lifecycle model where the agents have recursive preferences but a common discount factor and they show that this model generates intergenerational shifts in consumption patterns that they call 'displacement risk'. In a related paper Gârleanu and Panageas (2014) study asset pricing in a continuous time stochastic overlapping generations model. In contrast to my work, these papers focus on fundamental equilibria and they adopt the common assumption of Epstein Zin preferences (Epstein and Zin, 1991, 1989).

Challe (2004) generates return predictability in an overlapping generations model and Guvenen (2009) constructs a production economy that he solves computationally. Constantinides and Duffie (1996) exploit cross-section heterogeneity of the income process to show that uninsurable income risk across consumers can potentially explain *any* observed process for asset

prices. Kubler and Schmedders (2011) construct a heterogenous agent overlapping generations model with sequentially complete markets. By dropping the rational expectations assumption, they are able to generate substantial asset price volatility. In a related paper, Feng and Hoelle (2014) generate large welfare distortions from sunspot fluctuations.

My work differs from these papers by providing a simple and analytically tractable model that provides a bridge between asset pricing models and business cycle models. In contrast to the now familiar assumption of Epstein Zin preferences, my agents are Von-Neumann Morgenstern expected utility maximizers with logarithmic utility functions. I abstract from fundamental shocks and I assume that markets are sequentially complete. Despite the simplicity of these assumptions, I am able to go a considerable way towards explaining features of asset pricing models.

3   The Main Ideas

I construct a model in which people have infinite horizons but finite lives. These people survive from one period to the next with an age-invariant probability. There are two types of people, one of which is more patient than the other. There is no production, and each type is endowed with a single commodity in every period. In the absence of money, the unique equilibrium of this model is characterized by a difference equation in a single state variable that converges to a unique steady state. The initial condition of this difference equation is the net indebtedness in the first period, of patient to impatient types.

In a steady state equilibrium, patient people consume less than their endowment when young and more than their endowment when old: impatient people consume more than their endowment when young and less than their endowment when old. In the steady state, there is an exponential age distribution of each type.

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I add a government to this model that consists of a treasury and a central bank. The treasury issues dollar denominated debt and, although money is used as a unit of account, no agent in the model holds money.[4] Each period, the treasury raises lump sum taxes that it uses to pay the interest on its debt and to roll over the principal. The central bank fixes the nominal interest rate at a constant.

The model possesses a set of perfect foresight equilibria that are solutions to a first order difference equation. There is more than one perfect foresight equilibrium because the initial price level is a free variable. I use this fact to construct a rational expectations equilibrium in which the price level is random. In this equilibrium, price-level fluctuations reallocate the tax burden of government debt between current and future generations. These fluctuations exist as equilibria, even in the presence of a complete set of state-contingent securities, because the unborn cannot insure against the state of the world into which they are born.

We have many theoretical models of sunspot equilibria, most of which cannot easily be mapped into real world data. This paper presents a model that combines standard assumptions about preferences with a plausible demographic structure to generate equilibria, driven by self-fulfilling prophecies, that exhibit many of the features that we see in real world asset markets.

4   The Structure of the Model

This section lays out the structure of the model.   Sections 4.1 – 4.4 explain the environment and Section 4.5 discusses an important implication of the absence of intergenerational transfers. This assumption implies that the model is non-Ricardian in the sense of Barro (1974).

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[4]This assumption is widely used in monetary models and can be thought of the limiting case of an economy where the medium of exchange function of money is small (Woodford, 1998).

4.1 Assumptions about people, apples and trees

There are two types of people. Each type is endowed with one unit of a unique perishable commodity in every period in which he is alive; I call this an apple. The wealth of a person in the year of his birth is equal to the discounted present value of his apples. I call this a tree.

People have logarithmic preferences and discount factors β_1 and β_2. Type 1 people are more patient than type 2 people. This assumption is represented by the inequalities,

$$0 < \beta_2 < \beta_1 < 1. \tag{1}$$

People of each type die with probability $1 - \pi$ and when a person dies he is replaced by a new person of the same type. The model contains μ_i type i people, where $\sum_i \mu_i = 1$, hence, there is a constant population of measure 1.

4.2 Assumptions about uncertainty

Uncertainty in period t is indexed by a random variable S_t with compact support $\boldsymbol{S}$

$$S_t \in \boldsymbol{S}.$$

I refer to a $\tau-$period sequence S_t^τ as a $\tau-$period *history* with root S_t,

$$S_t^\tau = \{S_t, S_{t+1}, ...S_\tau\}.$$

The root is the initial date-state pair and a history, S_t^τ is a $\tau - t$ dimensional random variable with support $\boldsymbol{S}_t^{\tau-t}$.

In the remainder of the paper, I will drop t subscripts to cut down on notation. Instead, I will use the notation x to refer to $x_t(S)$ and $x(S')$ to refer to $x_{t+1}(S')$. All real date t variables are functions of the current realization of S.

4.3 Assumptions about the asset markets

Asset markets are sequentially complete. Three assets are actively traded; Arrow securities, government debt, and trees.

An Arrow security costs $Q(S')$ apples at date t and pays one apple at date $t+1$ if and only if state S' occurs. In aggregate, people of type i hold $a_i(S)$ type S securities at date t in state S. The quantities of each security demanded by each type may be positive or negative.[5]

Government debt costs $Q^N D'$ dollars at date t and is a claim to D' dollars at date $t+1$. Because the dollar price of apples is a random variable, the real return to government debt is also random.

A tree costs p_k apples at date t and delivers one apple every period in which the issuer of the asset remains alive. The price of a tree is computed recursively from the pricing equation,

$$p_k = 1 + \pi E\left[p_k\left(S'\right)Q\left(S'\right)\right].$$

The term π appears in this expression to reflect the fact that the tree will be worthless next period with probability $(1-\pi)$. This reflects the probability that the person issuing the claim has died.

Let $Q\left(S_t^T\right)$ be the price today of a claim to one apple in history S_t^T. I assume that

$$\liminf_{T\to\infty} Q\left(S_t^T\right) = 0, \qquad \text{for all } S_t^\tau.$$

This is the stochastic generalization, for this economy, of the assumption that the interest rate is greater than the growth rate and it rules out equilibria that are dynamically inefficient.

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[5]Because I make assumptions that allow me to aggregate the consumption decision of each type, I do not refer to the asset holdings of individual agents. But in fact these asset holdings display a rich pattern of heterogeneity. Asset holdings depend not just on type, but also on the state into which a person was born.

4.4 Assumptions about government

Government consists of a central bank and a treasury. The treasury issues dollar denominated one-period debt and faces the budget constraint

$$Q^N D' = D - \tau p, \tag{2}$$

at every date t where τ is the proportional tax rate and p is the dollar price of an apple.

The central bank sets the gross interest rate equal to a constant, that I denote by R^N, where $R^N \equiv 1/Q^N$, in every period. A monetary policy rule of this kind is called *passive*. The treasury issues sufficient nominal debt to roll over its existing debt, net of tax revenues. A fiscal policy of this kind is called *active*.[6]

Michael Woodford (1995), has shown that, in representative agent economies, the combination of an active fiscal policy and a passive monetary policy leads to a unique equilibrium price level. This result is known as the fiscal theory of the price level and it *does not hold* in the model I develop in this paper.[7]

Dividing Equation (2) by p and multiplying and dividing the left-hand side by p', we can write the following expression for the evolution of government debt

$$Q'b' = b - \tau, \tag{3}$$

where

$$b \equiv \frac{D}{p}, \quad \text{and} \quad Q' = \frac{p'}{pR^N}.$$

In Section 6.2 I will combine Equation (3) with a difference equation in p_k and b that arises from the assumption that the marginal rates of substitution

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[6]This terminology is due to Leeper (1991).

[7]The fiscal theory of the price level treats the government budget constraint as a valuation equation. For a given net present value of tax revenues, there is a unique price level for which the budget is exactly balanced. That is not true in my model. Instead, variations in the price level redistribute the tax burden of the debt between the current generation and future generations.

of each type are equal state by state. This leads to two difference equations in two variables, p_k and b. For any given initial condition, pinned down by the initial price of apples, these difference equations fully characterize the set of perfect foresight equilibria.

4.5 Aggregate wealth and Ricardian equivalence

The aggregate wealth of the private sector consists of the after tax value of existing trees, plus the value of government debt.

I will use the symbol W to represent aggregate private wealth,

$$W = p_k - T + b, \tag{4}$$

and the symbols τ and T to represent the tax rate and the tax obligations of current generations. T and τ are related by the identity,

$$T \equiv p_k \tau.$$

Because the economy is closed, government debt is the liability of private agents. But some of the people who will repay that debt have not yet been born.[8] Using $\bar{T}$ to represent the tax liability of future generations, the net present value of the government's assets must equal the net present value of its liabilities,

$$b \equiv T + \bar{T}. \tag{5}$$

Note however that

$$b \neq T.$$

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[8]Gârleanu, Kogan, and Panageas (2012) refer to the risk introduced by incomplete participation as 'displacement risk'. In their work, all uncertainty is fundamental. Farmer, Nourry, and Venditti (2012) also cite incomplete participation as a reason for the existence of sunspot equilibria. However, their paper does not allow for a nominal asset. As a consequence, the equilibrium in Farmer, Nourry, and Venditti (2012) is unique. I am indebted to Pawel Zabczyk, Markus Brunnermeir and Valentin Haddad for discussions which helped me to clarify this issue.

This model is non-Ricardian in the sense of Barro (1974) because future, as yet unborn generations, are partially liable for the debts incurred by the treasury on behalf of the current generation.

The fact that the model is non-Ricardian depends, not just on demographics, but also on the assumption that there are no active intergenerational transfers. This is an important assumption because it allows me to construct sunspot equilibria in which people born into different sunspot states have different utilities. One might think that, if people cared for their children, they would make asset market trades on their behalf that would eliminate the effects of nonfundamental uncertainty. That argument is incorrect.

In order for asset market trades to eliminate sunspot uncertainty it must be possible for a person to leave his children with positive bequests in some states of nature and with negative bequests in others. Although these trades would never be observed on the equilibrium path, their conceptual existence is required in order to enforce uniqueness of the fundamental equilibrium. The fact that western legal codes prohibit debt bondage is sufficient to rule out trades of this kind.

5   Household choice

In this section I solve individual maximization problems and, in Sections 6 and 8, I put the solutions to these problems together with the market clearing conditions to characterize equilibria.

5.1   Utility maximization as a recursive problem

Agents have logarithmic preferences and an agent of type i solves the problem

$$J_i\left[W_i\right] = \max_{\{a_i(S')\}} \left\{\log C_i + \pi\beta_i EJ\left[W_i'\left(S'\right)\right]\right\},$$

such that

$$\sum_{S'} \pi Q\left(S'\right) W_i'\left(S'\right) + C_i \leq a_i\left(S\right), \tag{6}$$

and

$$W_i \equiv a_i\left(S\right).$$

Here, $J_i\left[W_i\right]$ is the maximum attainable utility by household i with wealth W_i at date t in state S, and a_i is the holding of a type i agent of security S. In the period of his birth, the wealth of a person of type i is equal to

$$W_{i,0} = p_k\left(S\right)\left(1 - \tau\right). \tag{7}$$

5.2   Annuities and the lifecycle

Because this is an economy in which the set of agents is changing over time, I must keep track of peoples' assets when they die. I follow Blanchard (1985) by assuming that there exist complete annuities markets. The term π multiplies each security price in Equation (6) because a person who holds a positive amount of security S' simultaneously purchases an annuities contract. He earns a return greater than the market return in state S' in return for leaving his assets to the annuities company in the event of his death. Similarly, a person who borrows security S' is required to purchase a life insurance policy that discharges his debt in the event of his death.

5.3   Consumption demand functions

I have made three strong assumptions. First, every person has the same probability of death, independent of his current age. Second, preferences are logarithmic, and third, markets are sequentially complete. The first two of these assumptions are common to all models that use Blanchard's 1985 perpetual youth model. The third assumption, of sequentially complete markets, allows me to easily solve my model when there are two types of people.

I show, in Appendix A, that these assumptions imply that the aggregate consumption of the two types are linear functions of their wealth.

$$AC_1 = W_1 \equiv \alpha_1 (S), \qquad BC_2 = W_2 \equiv \alpha_2 (S), \qquad (8)$$

where the parameters A and B are functions of the discount factors, β_i and of the survival probability, π,

$$A = \frac{1}{1 - \beta_1 \pi}, \qquad B = \frac{1}{1 - \beta_2 \pi}.$$

The assumption that type 1 agents are more patient than type 2 agents implies that

$$A > B.$$

6   An expression for the pricing kernel

A pricing kernel, m', is a random variable with the property that the price of any asset can be computed as the expected value of its product with m'. In this section I derive an expression for the pricing kernel as a function of three variables; the current price of tree, the current value of government debt, and the future value of government debt. An important part of this derivation, is the assumption that people born in the future cannot participate in the asset markets.

6.1   Marginal rates of substitution

Let m_i be the marginal rate of substitution of a type i person who is alive in two consecutive periods. When preferences are logarithmic and markets are complete, the marginal rates of substitution of each type are equal to the ratios of their consumptions in consecutive date-state pairs, weighted by the

discount rate and the probability that they will survive,

$$m_1 = \frac{\pi \beta_1 c_1(S)}{c_1^O(S')}, \quad \text{and} \quad m_2 = \frac{\pi \beta_2 c_2(S)}{c_2^O(S')}, \quad \text{for all} \quad S. \tag{9}$$

Note that this equation holds for all possible values of S and S'. In the remainder of the paper, I will drop the dependence of current period variables on S to keep the notation more readable.

I am using lower-case c_i to represent the consumption of an individual person of type i, and upper case C_i to mean the aggregate consumption of all people of type i. The superscript O on the term c_i^O indexes a person who was alive in the previous period.

Following this convention, $c_i^O(S')$ is the consumption, next period in state S', of a type i person who is still alive and $C_i^O(S')$ is the aggregate consumption of all of these people. I show in Appendix B, that Equation (9) can be aggregated across people and that the ratios of consumptions of each type in two consecutive periods obeys the same equation as individual marginal utilities,

$$m_1 = \frac{\pi \beta_1 C_1}{C_1^O(S')}, \quad \text{and} \quad m_2 = \frac{\pi \beta_2 C_2}{C_2^O(S')}. \tag{10}$$

The price of an Arrow security, $Q'(S')$, is related to the marginal rate of substitution of each type, m_i, by the expression

$$Q'(S') = \chi(S') m_i(S'), \quad \text{for} \quad i \in \{1, 2\}, \tag{11}$$

where $\chi(S')$ is the probability that state S' will occur and the random variable

$$m_i(S') \equiv \frac{Q'(S')}{\chi(S')}, \tag{12}$$

is the pricing kernel.

I seek an expression for $m_i(S')$ as a function of the components of wealth at consecutive dates. I show in Appendix B, that the numerators and de-

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nominators of Equations (10) can be expressed as affine functions of p_k, b, p_k' and b',

$$C_i = \theta_{0,i} + \theta_{1,i} \left[p_k \left(1 - \tau \right) + b \right], \tag{13}$$

$$C_i^O \left(S' \right) = \eta_{0,i} + \eta_{1,i} p_k' \left(S' \right) \left(1 - \tau \right) + \eta_{2,i} b \left(S' \right), \tag{14}$$

where the coefficients of these equations are functions of the deep parameters, β_1, β_2 and π.

Although the coefficients of C_i depend only on the sum, $p_k + b$, the terms $p_k \left(S' \right)$ and $b \left(S' \right)$ appear in the expression for $C_i^O \left(S' \right)$ with different coefficients. This important property follows from the fact that the newborns next period do not hold government debt. It is important because the fact that $\eta_{1,i}$ and $\eta_{2,i}$ are different implies that variations in the composition of wealth between trees and government debt will influence the pricing kernel.

6.2 The pricing kernel

Combining equations (10), (13) and (14), one can derive an expression for the pricing kernel as a function of b, p_k and $b' \left(S' \right)$. First, note that trade in asset markets links $p_k' \left(S' \right)$ to $b' \left(S' \right)$, b and p_k. I will describe this dependence by a function $\psi \left(\cdot \right)$.

To derive this function I exploit the fact that in a competitive equilibrium with complete markets, the marginal rates of substitution of each type must be equal in every state. Combining equations (10), (13) and (14), these marginal rates of substitution can be written as functions of p_k, b, p_k' and b',

$$m_1 \left[p_k, b, p_k' \left(S' \right), b' \left(S' \right) \right] = m_2 \left[p_k, b, p_k' \left(S' \right), b' \left(S' \right) \right]. \tag{15}$$

Solving (15) for p_k' leads to the definition of the function $\psi \left(\cdot \right)$,

$$p_k' \left(S' \right) = \psi \left[p_k, b, b' \left(S' \right) \right], \tag{16}$$

Next, I seek an expression for the pricing kernel, which I describe by a

function $\phi\left(\cdot\right)$. Replacing Equation (16) in the either of the functions $m_i\left(\cdot\right)$, for $i \in \{1, 2\}$ we obtain the following definition,

$$\phi\left[p_k, b, b'\left(S'\right)\right] \equiv m_i\left[p_k, b, \psi\left[p_k, b, b'\left(S'\right)\right], b'\left(S'\right)\right]. \tag{17}$$

The pricing kernel is related to the price of an Arrow security by the identity,

$$\phi\left(p_k, b, b'\right) \equiv \frac{Q\left(S'\right)}{\chi\left(S'\right)}. \tag{18}$$

7   Characterizing equilibria

An equilibrium is a possibly stochastic sequence $\{b, p_k\}$, that satisfies the following pair of stochastic difference equations,

$$p'_k = \psi\left[p_k, b, b'\left(S'\right)\right], \tag{19}$$

$$b = \tau + E_s\left\{\phi\left[p_k, b, b'\left(S'\right)\right]b'\left(S'\right)\right\}. \tag{20}$$

I derived Equation (19) in Section 6.2, Equation (16). Equation (20) is a valuation equation for the current value of government debt.

Suppose first, that we consider only perfect foresight equilibria. These equilibria are characterized by non-stochastic sequences that satisfy equations (19) and (20). I show in this section, that for a calibrated version of this model, there is a single steady state solution to (19) and (20) and that this steady state is a saddle. The *saddle path,* also called the *stable manifold,* is a one-dimensional manifold of points with the property that trajectories that begin on this manifold converge to the steady state (Guckenheimer and Holmes, 1983).

In economics, we are used to associating saddle-paths with uniqueness of equilibrium. For example, the canonical real business cycle model can be described as a first order difference equation in consumption and capital

that has a unique saddlepath stable steady state. This model is associated with an initial condition for the capital stock. For every value of capital, there is a unique value of consumption that places the system on the stable manifold. The stable branch of the saddle, plus the initial value of capital, fully characterizes the unique perfect foresight equilibrium.

The model I have described here is different. Although there is a unique stable saddle path, the initial value of outstanding government debt depends on the initial price level: And this can take on a continuum of values in an open set. Because of the dependence of the initial condition on the dollar price of apples, this model is associated with a continuum of perfect foresight equilibria. An equilibrium is characterized by a sequence that begins at an arbitrary point on the stable manifold and converges to the unique steady state over time.

7.1 Perfect foresight equilibria

To study the properties of a perfect foresight equilibrium, I define a variable m'

$$m' = \phi\left(p_k, b, b'\right), \tag{21}$$

and, in Appendix C, I derive a transformation of variables that rewrites perfect foresight solutions to equations (19) and (20) as an equivalent system in the variables $\{b, m\}$.[9]

The transformed system has the form,

$$\left[\begin{array}{c} m' - F\left(m, b\right) \\ b' - G\left(m, b\right) \end{array}\right] = 0. \tag{22}$$

In a separate Appendix, available online, I publish the code used to solve the model and I show that, for the parameter values used in a calibrated

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[9]This transformation is convenient because, given an expression for the evolution of the sequence $\{m\}$, I can price any asset by computing the conditional expectation of its return with m'.

version of the model, there exists a unique feasible steady state, $\{\bar{m}, \bar{b}\}$ that satisfies the equation

$$\begin{bmatrix} \bar{m} - F\left(\bar{m}, \bar{b}\right) \\ \bar{b} - G\left(\bar{m}, \bar{b}\right) \end{bmatrix} = 0. \tag{23}$$
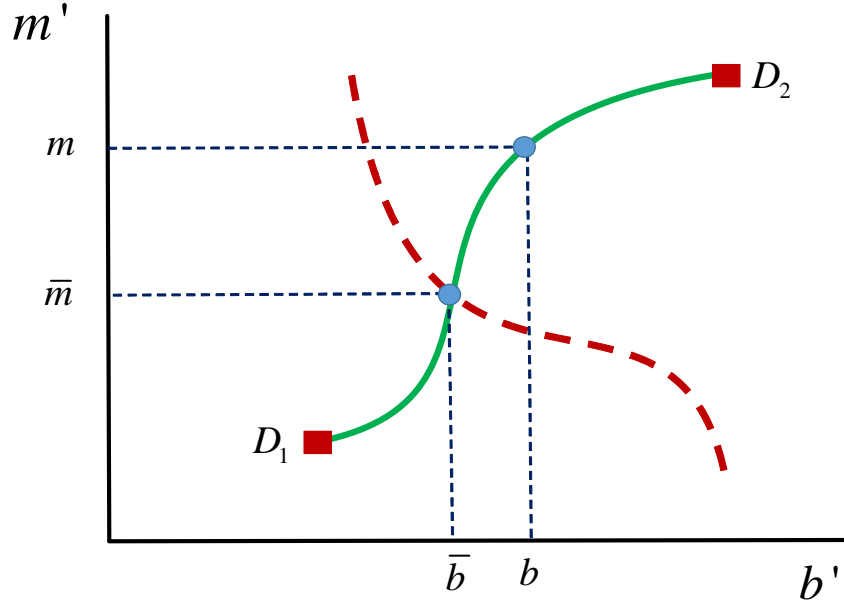
Further, this steady state is a saddle point.

Figure 1: The set of pefect foresight equilibria

Figure 1 depicts the dynamical system $\{m, b\} \rightarrow \{m', b'\}$ defined by equation (22). The axes of this figure represent values of $\{b, m\}$. The map defined in Equation (22) sends every point in this space to some other point. The upward sloping solid curve is the stable manifold and the downward sloping dashed curve is the unstable manifold.

The stable manifold is a set $D = [D_1, D_2]$ and a function $g : D \rightarrow \mathbb{R}$,

$$b = g\left(m\right), \tag{24}$$

with the property that every point that begins on this manifold follows a

first order difference equation $f : D \to D$,

$$m' = f(m), \tag{25}$$

that converges to the steady state $\{\bar{b}, \bar{m}\}$.

7.2 Why there are multiple perfect foresight equilibria

The stable manifold, $g(m)$ is one of two solutions to the functional equation,

$$m' = F[m, g(m)] \equiv f(m),$$
$$g(m') = G[m, g(m)] \equiv g[f(m)]. \tag{26}$$

The other is the unstable manifold (the dashed curve on Figure 1). All feasible bounded trajectories must start on, and remain on, the stable manifold.

In the first period of the model, type 1 people enter the period with a net claim on type 2 people that I represent by $a_{1,0}$. This initial condition imposes a linear restriction on the three variable $a_{1,0}$, $p_{k,0}$ and b_0,

$$\delta_0 + \delta_1 b_0 + \delta_2 p_{k,0} = a_{1,0}, \tag{27}$$

where δ_0, δ_1 and δ_2 are functions of the deep parameters. After transforming the system to the new coordinates $\{m, b\}$, Equation (27) implicitly imposes a linear restriction on $a_{1,0}$, m_0 and b_0. The trajectory that originates at $\{m_0, b_0\}$, calculated by iterating the equation,

$$m' = f(m), \tag{28}$$

characterizes a perfect foresight equilibrium.

If government debt were denominated in units of apples, Equation (27) would define a unique value of $p_{k,0}$ for every value of initial indebtedness, $a_{1,0}$. Instead, when debt is denominated in dollars, there are many feasible

initial pairs $\{p_{k,0}, b_0\}$, all of which lie on the stable manifold and all of which satisfy the initial condition, Equation (27).

8   Rational expectations equilibria

In this section, I show how to construct a set of rational expectations equilibria by randomizing over the perfect foresight equilibria of the underlying model. In these equilibria, people form self-fulfilling beliefs about the distribution of future prices.

8.1   Randomizations over perfect foresight equilibria

In the finite Arrow-Debreu model there is, generically, a finite odd number of equilibria. But one cannot construct new stochastic equilibria by randomizing across the existing perfect foresight equilibria. This is a direct implication of the first welfare theorem which asserts that every competitive equilibrium is Pareto optimal. Because people are assumed to be risk averse, they would always prefer the mean of a gamble to the gamble itself. And, in the case of sunspot fluctuations, that mean is available.

That result breaks down when there is incomplete participation in asset markets as a consequence of birth and death (Cass and Shell, 1983). When there is incomplete participation in the asset markets, one *can* construct randomizations across the perfect foresight equilibria of the model that are themselves equilibria.

To construct equilibria of this kind, I generate sequences of random variables $\{m, b, p_k\}$ that satisfy the equations,

$$p'_k (S') = \psi \left[p_k (S'), b, b' (S') \right], \tag{29}$$

$$b = \tau + E \left[b' (S') \, m' (S') \right], \tag{30}$$

$$m' (S') \equiv \phi \left[p_k (S'), b, b' (S') \right]. \tag{31}$$

Because there are multiple perfect foresight equilibria, there are multiple possible values of p_k', b' and m' from the perspective of date t. In a stationary environment, people come to understand that the future price is a random variable and they form beliefs that are indexed to an observable shock, S'. This shock is a sunspot that is unrelated to fundamentals.

8.2   Beliefs and sunspots

What coordinates beliefs on a sunspot equilibrium? Suppose that Mr. A and Mr. B believe the writing of an influential financial journalist, Mr. W. Mr. W writes a weekly column for the fictitious *Lombard Street Journal* and his writing is known to be an uncannily accurate prediction of asset prices. Mr. W only ever writes two types of article; one of them, his optimistic piece, has historically been associated with a 10% increase in the price of trees. His second, pessimistic piece, is always associated with a 10% fall in the price of trees.

Mr. A and Mr. B are both aware that Mr. W makes accurate predictions and, wishing to insure against wealth fluctuations, they use the articles of Mr. W to write a contract. In the event that Mr. W writes an optimistic piece, Mr. A agrees, in advance, that he will transfer wealth to Mr. B. In the event that Mr. W writes a pessimistic piece, the transfer is in the other direction. These contracts have the effect of ensuring that Mr. W's predictions are self-fulfilling.[10] How can that be an equilibrium?

There are three groups of people involved in any potential trade. Patient agents alive today, impatient agents alive today, and agents of both types who will be born tomorrow. Fluctuations in the price of trees cause a wealth redistribution from the newly born to the existing generations. This

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[10]I have shown in Farmer (2002c), that self-fulfilling beliefs can be enforced by what I call a 'belief function'; a new fundamental that has the same methodological status as preferences, technology and endowments. In Farmer (2012), I estimated a model in which the belief function is a primitive and I showed that it fits post-war US data better than a standard New-Keynesian model.

wealth redistribution operates by a transfer of tax obligations to or from the unborn. Because the existing agents have different propensities to consume out of wealth, they choose to change their net obligations to each other in different ways depending on whether the transfer from the unborn is positive or negative. In a rational expectations equilibrium, the different behaviors of Mr. A and Mr. B are self-fulfilling.

8.3 Constructing a sunspot equilibrium with two future states

I will construct a rational expectations equilibrium in which agents correctly forecast that only two values, $b'(s_a)$ and $b'(s_b)$ will occur in equilibrium. I will call the event of an optimistic forecast by Mr. W, event $\{s_a\}$ and a pessimistic forecast, event $\{s_b\}$. Suppose further that the economy begins in a perfect foresight equilibrium in which

$$\bar{m} = f(\bar{m}), \qquad \text{and} \qquad \bar{b} = g(\bar{b}). \tag{32}$$

At some special date T, everybody correctly forecasts that, at date $T+1$, the price level, and therefore the value of b', will take one of two values, $b'(s_a)$ or $b'(s_b)$. Recognizing that these events will occur, those people alive at date T trade a pair of Arrow securities that sell for prices Q'_a and Q'_b where

$$\frac{Q'_a}{\chi_a} = \phi\left[p_k, \bar{b}, b'(s_a)\right], \qquad \text{and} \qquad \frac{Q'_b}{\chi_b} = \phi\left[p_k, \bar{b}, b'(s_b)\right]. \tag{33}$$

The events $S' = s_a$ and $S' = s_b$ occur with known probabilities, χ_a and χ_b, and the pricing kernel, $\phi(\cdot)$, equates the marginal rates of substitution of types 1 and 2, state by state.[11]

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[11] Not all values of $b'(s_a)$ and $b'(s_b)$ are feasible; feasibility requires that the consumption of both groups is non-negative ate date $t+1$. That condition places an upper bound on the value of b' and, therefore, a lower bound on p'. There is, however, an open set of feasible values for $b'(S')$, each of which is associated with a different value, next period, for the

By assumption, the equilibrium up to and including date T has the property that $m = \bar{m}$ and $b = \bar{b}$ and the equilibria from date $T + 1$ on satisfy the equations $m' = f(m)$ and $b' = g(m')$. Although there are two possible values of m at date $T + 1$, at all other dates, the economy is in a perfect foresight equilibrium.

The real value of debt at date T, equal to $\bar{b}$, must satisfy the valuation equation,

$$\bar{b} - \tau = \chi_a \phi\left[p_k, \bar{b}, b'(s_a)\right] b'(s_a) + \chi_b \phi\left[p_k, \bar{b}, b'(s_b)\right] b'(s_b). \tag{34}$$

For two arbitrary values of $b'(S')$, Equation (34) defines the value of p_k.[12]

There is a further condition to be verified. For arbitrary choice of $b'(S')$, there is no guarantee that the system remains on the stable manifold in both states. That condition adds the additional restriction, that

$$b'(S') = g\left[m'(S')\right], \tag{37}$$

and it implies that $b'(s_a)$ cannot be chosen independently of $b'(s_b)$. To guarantee that equation (37) holds, I will require that

$$E_s\left[m'\right] = f(\bar{m}), \tag{38}$$

where

$$m'(S') \equiv \phi\left[p_k, \bar{b}, b'(S')\right]. \tag{39}$$

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price of commodities, $p(S')$.

[12] Because Walras law holds, Equation (34) can be stated equivalently as a valuation equation for trees,

$$p_k = 1 + \pi\left\{\chi_a \phi\left[p_k, \bar{b}, b'(s_a)\right] p_k'(s_a) + \chi_b \phi\left[p_k, \bar{b}, b'(s_b)\right] p_k'(s_b)\right\}, \tag{35}$$

where

$$p_k'(s) = \psi\left[p_k, \bar{b}, b'(s)\right], \quad s \in \{a, b\}. \tag{36}$$

Equations (37) and (38) imply that, if both continuation values are on the stable manifold, we cannot choose $b'(s_a)$ independently of $b'(s_b)$. We are, however, free to pick an arbitrary value for $b'(s_a)$ and an arbitrary probability $\chi(s_a)$. There is a lot of flexibility in defining a rational expectations equilibrium, even in the case of only two states.

8.4 Constructing a sunspot equilibrium: the general case

This section generalizes the idea of randomizing over perfect foresight equilibria to the case in which people come to believe that there are many possible continuation values for the price level at every possible date.

To generate sunspot equilibria, I randomize over the set of perfect foresight equilibria. Every perfect foresight equilibrium is uniquely characterized by an initial value of the pricing kernel, m_0 and a pair of functions, $f : D \to D$ and $g : D \to \mathbb{R}$ such that

$$m' = f(m), \qquad b = g(m), \qquad \text{and,} \qquad b_0 = g(m_0), \tag{40}$$

where the functions $f(\cdot)$ and $g(\cdot)$ are defined as solutions to the operator equation

$$
\begin{aligned}
m' &= F[m, g(m)] \equiv f(m), \\
g(m') &= G[m, g(m)] \equiv g[f(m)].
\end{aligned}
\tag{41}
$$

To construct a rational expectations equilibrium, I select continuation values $m'(S')$ with the property that

$$E[m'] = f(m), \tag{42}$$

and I determine p_k from the valuation equation,

$$p_k = 1 + \pi E_s[m'p'_k], \tag{43}$$

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where

$$p'_k = \psi\left[p_k, g\left(m\right), g\left(m'\right)\right].\qquad(44)$$

The expectation in (42) is taken with respect to a probability measure that defines the properties of the equilibrium. In my simulations, I used a Beta distribution, but that is only one of many possibilities. In a sunspot model, the beliefs of people about future outcomes are self-fulfilling and the probability measure over outcomes is a primitive of the model.

9 Global numerical approximations to equilibria

In this section I introduce a new method for computing sunspot-driven rational expectations equilibria. The usual method of computing sunspot equilibria proceeds by linearizing a dynamic stochastic general equilibrium model around an indeterminate steady state and adding random shocks to the resulting linear system (Farmer, 1999; Woodford, 1986). This method produces a valid approximation to the equilibria of a non-linear model but the accuracy of the approximation decreases as the variance of the shocks becomes larger. In this section, I show how to construct a higher order global approximation that remains valid for shocks that move the pricing kernel over the entire range of its support.

9.1 The method described

To construct global sunspot equilibria, I map the pricing kernel into the interval $[0, 1]$ and I assume that, for any value of m, the variable m' has a Beta distribution with mean $f\left(m\right)$, where $m' = f\left(m\right)$ is the stable manifold of the map (22). That assumption implies that in any given period, people believe that m' is a random variable with support D for every value of $m \in D$. In words, however well the economy is doing today, there is always positive

probability that the next period will be associated with an extreme value in which the discount factor is at its upper or lower bound.

The Beta distribution, (Johnson, Kotz, and Balakrishnan, 1995, Chapter 21), is characterized by two parameters, α and β and if m' has a beta distribution, its conditional expectation is given by the expression,

$$E\left[m' \mid m\right] = \frac{\alpha}{\alpha + \beta}.$$

Alternatively, one may parameterize the Beta distribution by the mean μ and the 'sample size', V, where

$$\alpha = V\mu, \quad \text{and} \quad \beta = V\left(1 - \mu\right).$$

By modeling m' as a Beta distributed random variable, I am able to capture in a parsimonious way, the idea that people believe that equilibria will be selected by the psychology of market participants.

One possible approach to modeling sunspots would be to fix the sample size, V. This leads to the following dependence of the parameters α and β on m,

$$\alpha\left(m\right) = Vf\left(m\right), \quad \beta\left(m\right) = V\left(1 - f\left(m\right)\right).$$

This approach is problematic since for m close to D_1 or D_2, probability mass piles up at the boundaries. It seems desirable to retain the property that the distribution has a single interior peak, a condition that requires that α and β are both greater than 1. For that reason, I chose to let V be state dependent.

In my simulations, I chose a parameter $k > 1$ and I picked V such that

$$V\left(m\right) = k \max\left[\frac{1}{f\left(m\right)}, \frac{1}{1 - f\left(m\right)}\right]. \tag{45}$$

This choice of V guarantees that α and β are both greater than 1, and hence

the distribution has a single interior mode.

Figure 2 depicts the distribution of m' for three different values of m. The figure is drawn for the choice of $k = 4$, which corresponds to my baseline calibration. Higher values of k generate pictures with the same qualitative features but with a lower variance for each distribution.
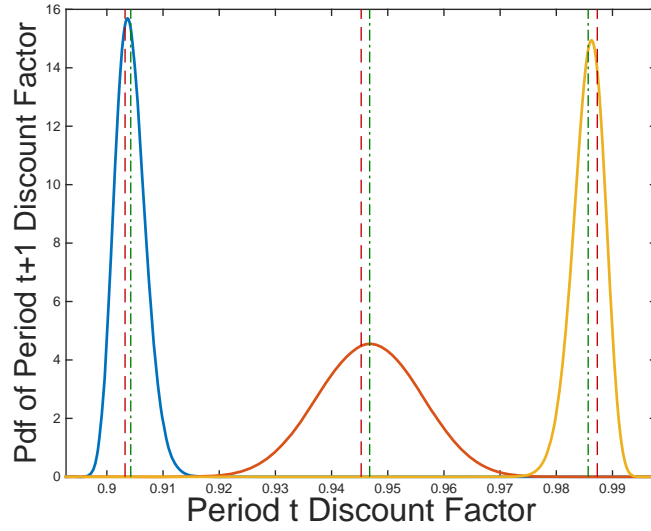

Figure 2: The Distribution of m' when $k = 4$

The three dashed vertical red lines on Figure 2 depict values of m. I chose values,

$$m = \left[\begin{array}{ccc} 0.903 & 0.945 & 0.988 \end{array} \right],$$

which correspond to the midpoint of the support of m' and a distance of 0.01 from each end.

The dot-dashed vertical green lines depict the function $f(m)$. These correspond to the values,

$$f(m) = \left[\begin{array}{ccc} 0.904 & 0.947 & 0.987 \end{array} \right].$$

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For each value of m the associated single-peaked curve is the Beta distribution associated with that realization of m', with mean $f(m)$. Notice that the variance of m' is greater when m is in the center of the set D than at either end. This property is dictated by three assumptions. First, m' has a Beta distribution, second, m' has full support for every m, and third, the distribution of m' has a single interior mode.

9.2 Calibrating the model to understand the behavior of asset prices

We have many examples of sunspot models. The interesting question is whether a calibrated version of a sunspot model can help us understand the behavior of asset prices. To address this question, I calibrated the model to the parameter values reported in Table 1.

Table 1

Parameter Description	Parameter Name	Parameter Value
Survival probability	π	0.98
Fraction of type 1 in the population	μ_1	0.5
Gross nominal interest rate	R^N	1.05
Discount factor of type 1	β_1	0.98
Discount factor of type 2	β_2	0.90
Variance parameter	k	4
Primary surplus	τ	0.02

The parameter π is the probability that a person will survive into the subsequent period and, when $\pi = 0.98$, the typical person has an expected life of 50 years. I arrived at this number by splitting the age distribution of the US population into quintiles and weighting each quintile by life expectancy using mortality tables.

The choice of μ_1 to be 0.5 was arbitrary. I did, however, conduct robustness checks and the results I report below are not sensitive to alternative choices.

To describe monetary policy, I chose $R^N = 1.05$. That choice is frequently cited by central bankers as the 'normal value' for interest rates and it is consistent with a safe real rate of 3% and an inflation target of 2%. This choice has no effect on the equilibrium behavior of b and m since these are real variables. It will, however, influence the behavior of the inflation rate.[13]

The parameters β_1 and β_2, affect the steady state discount factor and one can show that

$$\beta_2 < \bar{m} < \beta_1.$$

I chose values of 0.98 and 0.9 by experimenting with the model to find values that led to a mean safe rate of 3%. The gap between these two discount factors determines the possible range of sunspot fluctuations and it needs to be relatively large if the model is to have a hope of capturing observed asset price movements.

Table 2

Variable Name	Parameter Name	Parameter Value
Equilibrium discount factor	$\bar{m}$	0.97
Equilibrium government debt	$\bar{b}$	0.69
Equilibrium asset price	$\bar{p}_k$	20.6
Return to a tree	R^R	1.03
Return to debt	R^S	1.03

For any value of the support of m', the parameter k determines the variance of the sunspot distribution for any given m. I experimented with differ-

---

[13]In my simulations, I have assumed that all government debt is one-period. In that case, a policy of fixing the interest rate leads to price level volatility of a similar magnitude to that of asset price volatility. More generally, if debt is of longer maturity, movements in the pricing kernel will cause revaluations in the price of long bonds and a more muted response of prices.

ent values of k and chose $k = 4$ to match asset returns with an approximate range of plus or minus 20%. Higher values of k lead to lower asset return volatility and lower values lead to higher volatility. Finally, I chose a value of τ of 2% to match the mean post-war primary government budget surplus.

The calibration of Table 1 implies the steady state values for $\bar{m}, \bar{b}, \bar{p}_k, R^R$ and R^S, reported in Table 2. Here, R^R and R^S are the real gross returns to holding a tree, or to holding government debt, in the non-stochastic steady state. These are the same and both are equal to 1.03, corresponding to a real interest rate of 3%.

9.3 Approximate global solutions

A perfect foresight solution to the model is characterized by a set D and a pair of functions $f(m) : D \to D$ and $g(m) : D \to \mathbb{R}$ such that

$$
\begin{aligned}
m' &= F[m, g(m)] \equiv f(m), \\
g(m') &= G[m, g(m)] \equiv g[f(m)],
\end{aligned}
\tag{46}
$$

for all $m \in D$. To solve these equations I used Chebyshev collocation as described in Judd (1998). That method converts the operator equation, (46), into a non-linear algebraic equation in the coefficients of two unknown polynomials $\hat{f}(m)$ and $\hat{g}(m)$. These polynomials approximate the functions $f(m)$ and $g(m)$ and by increasing the number of terms in the polynomial, one can achieve an arbitrary close approximation to f and g. In practice, I used polynomials of order 4.

To compute the boundaries of the set D, I derived an expression for the aggregate consumption of each type, as functions of m and b and I solved Equation (47)

$$
C_1[D_1, g(D_1)] = 0, \quad C_2[D_2, g(D_2)] = 0.
\tag{47}
$$

This calculation gave me the two points where one or the other type consumes

the entire endowment of the economy. For the calibration from Table 1, the lower boundary, D_1 is equal to 0.893 and the upper boundary, D_2, is equal to 0.998. When $m = D_1$, type 2 agents consume all of GDP. When $m = D_2$, type 1 agents consume everything.
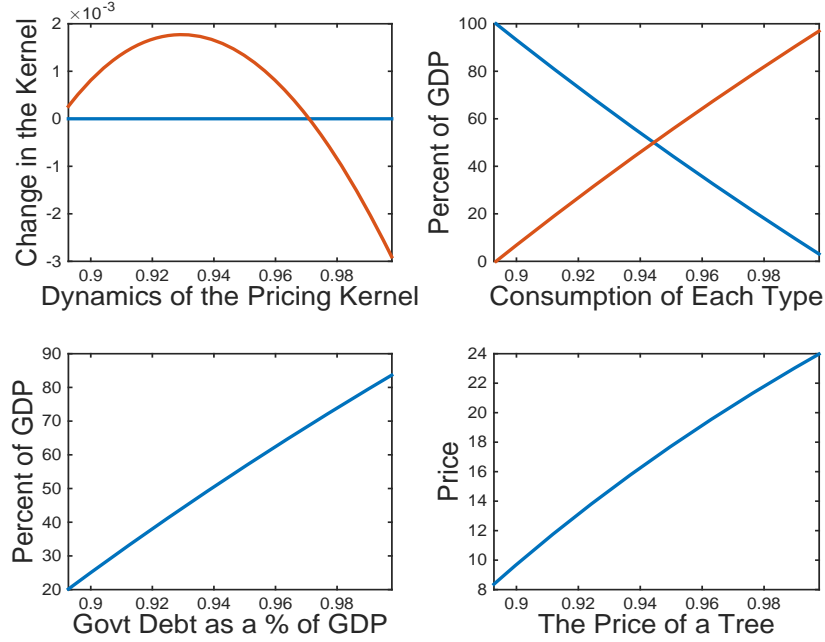

Figure 3: Some properties of the global solution

The top left panel of Figure 3 graphs the function $\hat{f}(m) - m$, on the vertical axis as a function of m on the horizontal axis. The point where the curve crosses the zero axis corresponds to the steady state $\bar{m} = 0.97$ and the range of m is defined by the set D. I have graphed the change in m as a function of m, rather than m' as a function of m, because in a plot of m' against m, it is difficult to discern the difference between m' and the 45 degree line.

The top right panel of Figure 2 graphs the consumption of type 1 people, this is the upward sloping curve, and the consumption of type 2 people, this is the downward sloping curve. The lower left panel of Figure 3 is the function

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$b = \hat{g}(m)$. This panel shows that, when the primary surplus is 2% of GDP, government debt can attain values between 20% and 85% of GDP.

The lower right panel of Figure 3 is the price of a tree as a function of m. This panel demonstrates that, for the calibration in Table 1, the price of a tree varies between 8 and 24. This fact is significant since p_k determines the lifetime wealth of a newborn. A person born into the world when $p_k = 24$ will be three times better off during his life than a person born into the world when $p_k = 8$.

10   Explaining data with a global sunspot model

Representative agent models are difficult to square with three features of asset price data. 1) Asset prices are persistent and volatile and price dividend ratios are mean reverting. 2) Aggregate consumption is smooth but the return to a riskless asset is five hundred basis points less than the return to the stock market. 3) Asset price volatility is non-constant and non-Gaussian, and models that assume that asset prices are log normally distributed with time-invariant volatility are rejected decisively by the data (Bollerslev, Engle, and Nelson, 1994).

This section presents a series of graphs that depict the characteristics of data simulated from the global sunspot model and it demonstrates that these simulations go a long way towards explaining all three of these features.

10.1   Excess volatility

To simulate data, I initialized $m_0 = \bar{m}$ and I generated 60 years of data by drawing a sequence of Beta distributed random variables that obey the recursion,

$$m' = B\left[V(m), \hat{f}(m)\right],$$

where $B(V, \mu)$ is the beta distribution parameterized by sample size V and mean μ. I chose V to be a function of m, using the method described in Section 9.1, Equation (45).
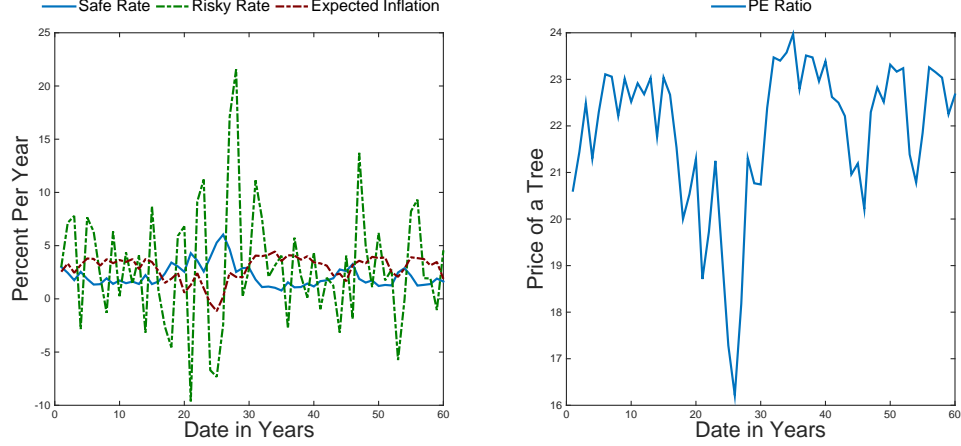

Figure 4: Sixty years of simulated data

Figure 4 plots the data generated from a single 60 year simulation. The right panel is the price of a tree. This represents a claim to one tree and one apple next period and with probability π and nothing with probability $1 - \pi$.

To construct the series for p_k, I used the definitions of the functions $\phi(\cdot)$, $\psi(\cdot)$ and the bond valuation equation, (20), to derive a function $\xi(\cdot):$ $D \times \mathbb{R} \to \mathbb{R}_+$

$$p_k' = \xi\left[m', g\left(m'\right)\right], \tag{48}$$

and I solved the asset price valuation equation

$$p_k = 1 + \pi \int_{D_1}^{D_2} \xi\left[m', \hat{g}\left(m'\right)\right] \Pr\left[m'; V\left(m\right), \hat{f}\left(m\right)\right] dm', \tag{49}$$

where $\Pr\left[m'; V\left(m\right), \hat{f}\left(m\right)\right]$ is the density function of a Beta distributed random variable, defined over the set D, with mean $\hat{f}\left(m\right)$ and sample size $V\left(m\right)$.

If we think of the death of the tree as a random event, p_k represents an equity claim to one apple every period with a default probability in any given period of $1 - \pi$. Under that interpretation, the price of the tree is the model counterpart of the average price-earning ratio in the stock market. In the simulated series of sixty years of data reported in Figure 4, the model PE ratio varies between 24 and 16 and is both persistent and mean reverting.

The left panel of Figure 4 plots the safe rate, the risky rate and the expected inflation rate. The safe rate is the return to a claim to one apple for sure and the risky rate is the return to buying a tree and selling it one period later. The risky rate is defined by the equation,

$$r^R \equiv 100 \left(\frac{\pi p_k'}{p_k - 1} - 1 \right). \tag{50}$$

and the real safe return r^S is equal to

$$r^S \equiv 100 \left(\frac{1}{E[m']} - 1 \right),$$

where

$$E[m'] \equiv f(m).$$

Expected inflation is defined as

$$\Pi \equiv R^N E[m'],$$

where $R^N = 1.05$.

Table 3 reports the means and standard deviations of r^S and r^R for this draw of sixty years of data, along with the Sharpe ratio, defined as

$$\text{Sharpe} = \frac{r^R - r^S}{\sigma^r},$$

where σ^r is the standard deviation of r^R.

Table 3:	Safe Rate	Risky Rate	Sharpe Ratio
Mean	2.15	3.04	
Std. Dev.	1.07	5.7	0.16

Two features stand out from this simulation. First; the risky return is highly volatile fluctuating in this sample between a high of 22% and a low of -10%. Second; the return from buying a long claim and holding it for a year has a return which is almost 1% higher than the riskless rate. The fact that asset prices are volatile and mean reverting, even when aggregate consumption is constant, is the first feature of the data that I set out to understand. The following section probes more deeply into the second feature, the ability of this model to understand the equity premium puzzle.

10.2   The Sharpe ratio, the equity premium and the term premium

In the US data, the mean return to equity has been, on average, 5% higher than the return to government bonds. Because it is possible to leverage returns through borrowing, finance economists focus instead on a different statistic; the Sharpe ratio. The Sharpe ratio, defined as the excess return on a risky asset divided by its standard deviation, has varied in US data between 0.25 and 0.5 depending on the time period and the frequency over which it is measured (Cochrane, 2001).

Figure 4 suggests my model can explain part, but not all, of the equity premium. In one simulated data series of 60 years, the excess return was approximately 1% and the Sharpe ratio was 0.16. This fact raises several questions. First; is the result a fluke?

The average Sharpe ratio in 60 years of simulated data is a random variable and because asset returns are so volatile, its standard deviation is high. To examine the ability of my model to produce a high Sharpe ratio, I simulated $6,000$ draws of 60 years of data and I plotted the empirical frequency

distributions of the riskless rate, the mean return to holding a tree and the Sharpe ratio. The results are graphed in Figure 5.
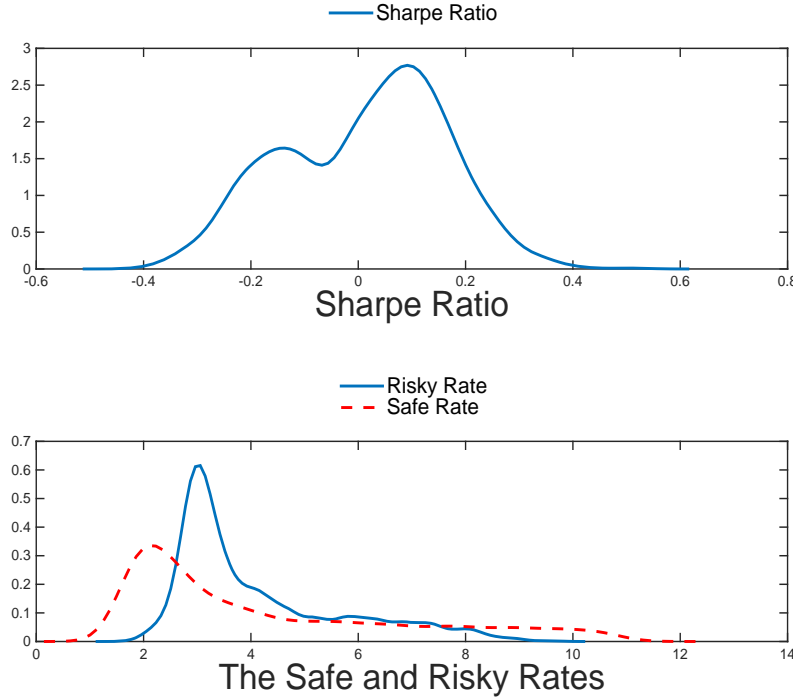

Figure 5: The Sharpe Ratio and the Returns to Safe and Risky Assets

The lower panel of Figure 5 plots the distributions over these 6,000 draws of the average safe and risky returns in 60 year time series. The dashed line, is the safe return and the solid line is the risky rate. This figure shows that a 1% equity premium is not a fluke; it is characteristic of the invariant distribution of returns.

The upper panel of Figure 5 plots the distribution of Sharpe ratios in these simulations. Although the modal risky rate is 1% higher than the safe rate for this model, there is a huge dispersion in average Sharpe ratios even when data are averaged over sixty years. Note, from the lower panel, that in the right tail of the distribution, the average safe rate exceeds the

risky rate. It is this feature that explains the bimodal distribution of Sharpe ratios in the upper panel of the figure. These data demonstrate that that the dominant mode of the distribution has a Sharpe ratio of 0.1 and that there is a non-trivial probability of observing a Sharpe ratio of 0.2 or higher. In this sample 9.3% of the draws had a Sharpe ratio of 0.2 or greater.

Andrew Abel (1999) has pointed to the important distinction between the equity premium and the term premium. The equity premium is the excess return to holding a long dated claim to an uncertain income stream such as equity. The term premium is the excess return to holding a long-dated claim to a safe income stream such as a thirty year treasury bond. Abel finds that about 1/4 of the equity premium puzzle can be attributed solely to the term premium, a finding that is consistent with the data generated by my model.

Is this a success? Partially. The model has logarithmic preferences, expected utility and no fundamental uncertainty; and yet, it is able to generate a substantial Sharpe ratio. It seems likely that a version of this model that allows for more risk aversion and aggregate fundamental uncertainty will be able to do much better in this dimension.

10.3   Endogenous conditional volatility

Traditional asset pricing models rely on time varying volatility to explain asset prices (Bansal and Yaron, 2004). The fact that asset prices display bursts of volatility was highlighted by the ARCH and GARCH models of Engle (1982) and Bollerslev (1986) and has since become a staple feature of asset pricing models.

In much of the finance literature, conditional volatility is introduced by assuming that shocks to dividend growth are driven by an exogenous stochastic process with a time-varying standard deviation. The model I develop in this paper generates *endogenous* conditional volatility.
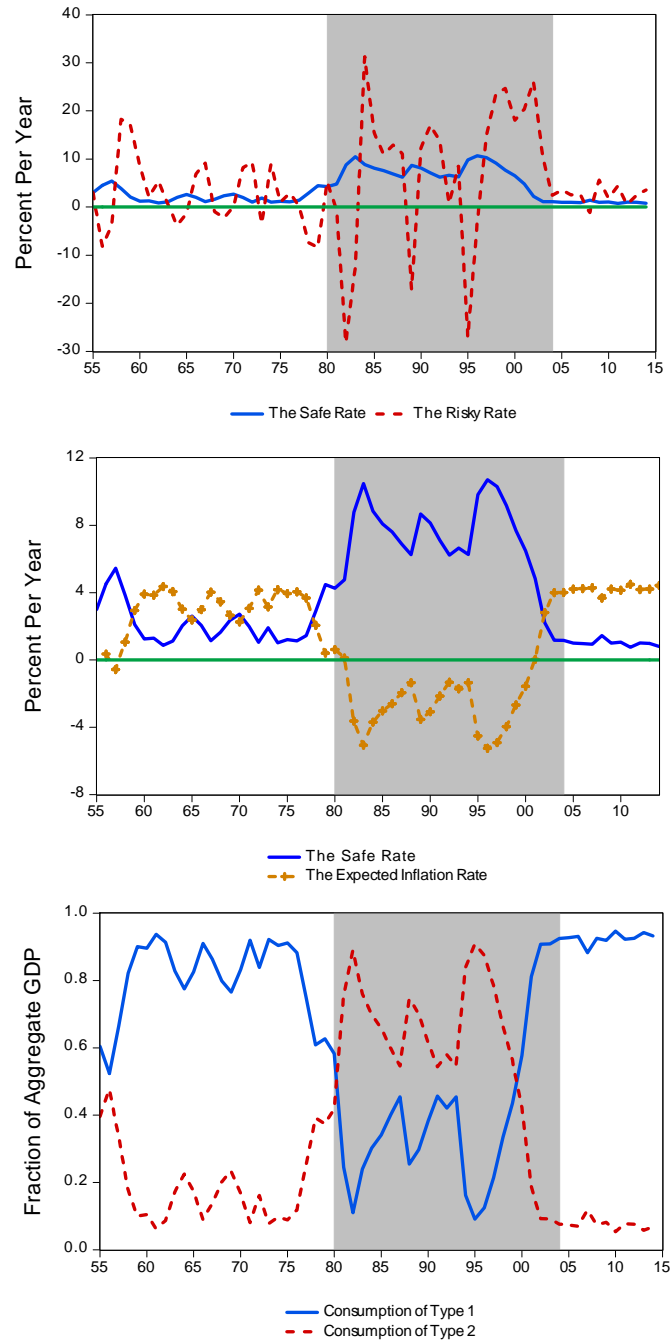
Figure 6: Time varying volatility

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The intuition for this result is contained in Figure 2 in which I plotted conditional Beta distributions for three different values of m. The assumption that expectations are rational requires that the pricing kernel should be mean reverting. The fact that the support of m' is bounded implies that the variance of m' is endogenously higher when m is in the middle of its support than when it is at either end.

If the discount factor strays towards the middle of its range following a large negative shock, there is an increased probability that it will be hit with an even larger negative shock that sends it towards the lower bound of its support. Once it reaches that region, the variance of future shocks falls and it takes a longer time to escape back towards the mean of the invariant distribution. This feature generates endogenous bursts of stochastic volatility.

One such burst is depicted in Figure 6. The top panel of this figure depicts the risky rate and the safe rate for one draw of sixty years of data beginning in 1955 and ending in 2015. The shaded region between observations 1980 and 2004 depicts an episode where the volatility of the return to a tree is higher than at other times. The middle panel blows up this picture and replaces the risky rate with the expected inflation rate. Notice that a period of high volatility is associated with a higher than average safe rate and a period of deflation.

The bottom panel of Figure 6 shows the consumption of types 1 and 2 over this period. Notice that a period of high volatility is associated with a reversal of the fraction of GDP consumed by each type. In normal times, the economy is close to the non-stochastic steady state. The risky rate is relatively smooth and the safe rate is low and stable. In these periods, patient people consume roughly 80% of GDP and impatient people consume the remaining 20%.

Occasionally, the economy is hit by a large shock which moves the stochastic discount factor away from its steady state and towards the middle

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of its support. Once it reaches this point, volatility increases and the safe return spikes. In the series generated in this example, the safe rate stays elevated at approximately 8% for a period of more than twenty years. Over this period, the consumption patterns of the patient and impatient people are reversed and it is the impatient people who consume the larger fraction of GDP. These results are suggestive of the Great Depression or the 2008 financial crisis.

11   Conclusion

In this paper, I have presented a theory that explains asset pricing data in a new way. In contrast to much of the existing literature in both macro-economics and finance, my work is based on the idea that most asset price fluctuations are caused by non-fundamental shocks to beliefs. My model produces data that display volatile asset prices, a sizeable term premium and bursts of time varying volatility. If one accepts the argument that a simpler explanation is a better one, the fact that I am able to reproduce these empirical facts in a model with logarithmic preferences and no fundamental shocks suggests that the model is on the right track.

My model is rich in its implications. It provides a simple theory of the pricing kernel that can be used to price other assets. The model is open to more rigorous econometric testing and its parameters can be estimated, rather than calibrated, using non-linear methods. It provides a theory of the term structure of interest rates that can be tested against observed bond yields and by adding a richer theory, in which output fluctuates as a consequence of labor supply or because of movements in the unemployment rate, the theory can be expanded to distinguish between the term premium and the equity premium. I view all of these extensions as grist for the mill of future research. Conducting these extensions is important because my model is not just a positive theory of asset prices; it is ripe with normative implications.

In my baseline calibration, I chose parameters to match key features of the data and I generated simulated data series that closely mimic observed interest rates and asset prices in the real world. In these simulations, asset price fluctuations cause Pareto inefficient reallocations of wealth between current and future generations and these reallocations lead to substantial fluctuations in welfare. If my model is correct, and these fluctuations are the main reason why asset prices move in the real world, stabilizing asset prices through monetary and fiscal interventions will be unambiguously welfare improving.

Appendix A: Optimal decision rules

Let $J_i(W)$ represent the value function of a person of type i. This function obeys the Bellman equation,

$$J_i[W_i] = \max_{\{W_i(S')\}} \left\{ \log\left[W_i - \sum_{S'} \pi Q(S') W_i'(S') \right] \right.$$
$$\left. + \pi \beta_i EJ[W_i'(S')] \right\}, \quad \text{(A1)}$$

where

$$W_i - \sum_{S'} \pi Q(S') W_i'(S') \equiv C_i. \quad \text{(A2)}$$

The unknown functions $J_i(W)$ must satisfy the following envelope condition,

$$J_{iW}[W_i] = \frac{1}{C_i}, \quad \text{(A3)}$$

and the Euler equations for each state

$$-\frac{\pi Q(S')}{C_1} + \beta_i \pi \chi(S') J_W[W'(S')] = 0. \quad \text{(A4)}$$

Since this is a logarithmic problem with complete markets I will guess that the value functions take the form

$$J_1(W) = A \log(W_1), \quad J_2(\alpha) = B \log(W_2), \tag{A5}$$

and verify this conjecture by finding values for the numbers A and B such that equations (A3) and (A4) hold. By replacing the unknown functions $J_i(\cdot)$ with their conjectured functional forms from Equation (A5) we arrive at Equations (A6) and (A7).

$$C_1 = \frac{W_1}{A}, \quad C_2 = \frac{W_2}{B}, \tag{A6}$$

$$C_1 = \frac{Q'(S') W_1'(S')}{\chi(S') \beta_i A}, \quad C_2 = \frac{Q'(S') W_2'(S')}{\chi(S') \beta_i B}. \tag{A7}$$

The two budget equations, for each type, (A2), together with the four first order conditions, (A6) – (A7), constitute six equations in the six unknowns, W_1', W_2', C_1, C_2, A and B. To solve these equations, substitute from (A7), state by state, into Equation (A2), use the fact that $\sum \chi(S') = 1$, and cancel C_i from each side to give the expressions

$$A = \frac{1}{1 - \beta_1 \pi}, \quad B = \frac{1}{1 - \beta_2 \pi}. \tag{A8}$$

Combining these solutions for A and B with (A6) gives the consumption rules that we seek.

Appendix B: Deriving an expression for the pricing kernel

In Appendix B we seek to establish that the first order condition

$$\chi(S')Q(S') = \frac{\pi\beta_i c_i}{c_i'^O(S')}, \tag{B1}$$

implies that

$$\chi(S')Q(S') = \frac{\theta_{0,i} + \theta_{1,i}\left[p_k(1-\tau) + b\right]}{\eta_{0,i} + \eta_{1,i}p_k'(S')(1-\tau) + \eta_{2,i}b'(S')}. \tag{B2}$$

The following argument follows closely from the argument developed in Farmer, Nourry, and Venditti (2011). We begin with some definitions. Let c_i^O be the consumption of a type i person who was alive in the previous period and let c_i^Y denote the consumption of a newborn of type i. Further, let C_i^Y be the aggregate consumption of all newborns of type i. To prove that (B2) follows from (B1), we must find expressions for c_i and $c_i'^O$ as functions of p_k and b.

The following steps imply that Equation (B1) must also hold not only for individuals, but also in aggregate. Multiplying both sides of (B1) by $c_i'^O(S')$ and adding up over all people of type i who are alive in two consecutive periods gives the expression,

$$\chi(S')Q(S')C_i'^O = \pi\beta_i C_i. \tag{B3}$$

Rearranging, leads to the expression.

$$\chi(S')Q(S') = \frac{\pi\beta_i C_i}{C_i'^O(S')}. \tag{B4}$$

This establishes the claim following Equation (9) in Section 6.1.

Goods and asset market clearing imply

$$C_1 + C_2 = 1, \tag{B5}$$

and

$$W_1 + W_2 = p_k (1 - \tau) + b. \tag{B6}$$

Combing these equations with the solutions for consumption from Appendix A, we have that,

$$C_1 = \frac{[p_k (1 - \tau) + b - B]}{A - B}, \quad \text{and} \quad C_2 = \frac{[A - p_k (1 - \tau) - b]}{A - B}. \tag{B7}$$

It follows that the coefficients of the numerators of (B2) are given by the following definitions,

$$\theta_{1,0} \equiv \frac{-B \beta_1 \pi}{A - B}, \qquad \theta_{1,1} \equiv \frac{\beta_1 \pi}{A - B}, \tag{B8}$$

$$\theta_{2,0} \equiv \frac{A \beta_2 \pi}{A - B}, \qquad \theta_{2,1} \equiv \frac{-\beta_2 \pi}{A - B}. \tag{B9}$$

Next we seek expressions for the denominator of Equation (B2).

The aggregate consumption of all type i people alive in period $t + 1$ can be decomposed into the consumption of those who were alive in period t and the consumption of the newborns. Let $\mathcal{A}_t$ be the index set of all type i people alive at date t and let $\mathcal{N}_{t+1}$ be the index set of all type i newborns at date $t + 1$. Using these definitions,

$$\sum_{\mathcal{A}_{t+1}} c_i' (S') = \pi \sum_{\mathcal{A}_t} c_i'^{O} (S') + \sum_{\mathcal{N}_{t+1}} c_i'^{Y} (S'), \tag{B10}$$

where π premultiplies the first term on the right-side of this expression to reflect the fact a fraction $1 - \pi$ of the previous generations have died. We can

rewrite Equation (B10), using the definitions of C_i', $C_i'^Y$ and $C_i'^O$, as follows,

$$C_i'^O (S') = \frac{C_i' (S') - C_i'^Y (S')}{\pi}. \tag{B11}$$

Now we seek an expression for $C_i'^Y (S)$ as a function of wealth. There are $1 - \pi$ newborns of each type, each of whom consumes a fraction of his wealth. These facts lead to the equations,

$$C_1'^Y (S') = A^{-1} p_k (S') (1 - \tau) (1 - \pi), \tag{B12}$$

and

$$C_2'^Y (S') = B^{-1} p_k (S') (1 - \tau) (1 - \pi), \tag{B13}$$

which determine the aggregate consumptions of newborns of each type. Combining (B12) and (B13) with (B11), making use of (B7), leads to the expressions we seek,

$$C_1'^O (S') = \frac{[p_k (S') (1 - \tau) + b (S') - B]}{\pi (A - B)} - \frac{p_k (S') (1 - \tau) (1 - \pi)}{\pi A}, \tag{B14}$$

and

$$C_2'^O (S') = \frac{[A - p_k (S') (1 - \tau) - b (S')]}{\pi (A - B)} - \frac{p_k (S') (1 - \tau) (1 - \pi)}{\pi B}. \tag{B15}$$

These equations express the denominators of Equations (B2) as functions of the components of wealth. It follows that the coefficients $\eta_{i,0}, \eta_{i,1}$ and $\eta_{i,2}$ from Equation (14) in Section 6.1 are defined as,

$$\eta_{1,0} \equiv \frac{-B}{\pi (A - B)}, \quad \eta_{1,1} \equiv (1 - \tau) \left[\frac{1}{\pi (A - B)} - \frac{(1 - \pi)}{\pi A} \right], \tag{B16}$$

$$\eta_{1,2} \equiv \frac{1}{\pi (A - B)},$$

and

$$\eta_{2,0} \equiv \frac{A}{\pi(A-B)}, \quad \eta_{2,1} \equiv -(1-\tau)\left[\frac{1}{\pi(A-B)} + \frac{(1-\pi)}{\pi B}\right], \quad (B17)$$

$$\eta_{2,2} \equiv \frac{-1}{\pi(A-B)}.$$

Appendix C: Transforming variables

We seek to derive a map $\{b, m\} \to \{b', m'\}$ given the functions ψ and ϕ,

$$p'_k = \psi(p_k, b, b'), \quad (C1)$$

$$m' = \phi(p_k, b, b'), \quad (C2)$$

and the government budget equation,

$$b = m'b' + \tau. \quad (C3)$$

Equations (C1)–(C3) constitute three equations in the three unknowns b, p_k and p'_k which may be solved to find three functions

$$b = \theta_1(b', m'), \quad p'_k = \theta_2(b', m') \quad \text{and} \quad p_k = \theta_3(b', m'). \quad (C4)$$

Substituting $\theta_2(\cdot)$ and $\theta_3(\cdot)$ from (C4) into (C1),

$$\theta_2(b', m') = \psi[\theta_3(b', m'), b, b']. \quad (C5)$$

Solving equations (C3) and (C5) for m' and b' as functions of m and b leads to the functions we seek,

$$m' = F(m, b), \quad (C6)$$

$$b' = G(m, b). \quad (C7)$$

The existence of the functions F and G is not guaranteed for all parameter

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values. The online Appendix provides code to compute F and G for my baseline calibration and to establish numerically that equations (C6) and (C7) have a unique steady state for which debt and the consumptions of each group are non-negative.

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