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MONETARY SHOCKS IN MODELS WITH INATTENTIVE PRODUCERS

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### **ABSTRACT**

We study models where prices respond slowly to shocks because firms are rationally inattentive. Producers must pay a cost to observe the determinants of the current profit maximizing price, and hence observe them infrequently. To generate large real effects of monetary shocks in such a model the time between observations must be long and/or highly volatile. Previous work on rational inattentiveness has allowed for observation intervals which are either constant-but-long (e.g. Caballero (1989) or Reis (2006)) or volatile-but-short (e.g. Reis's (2006) example where observation costs are negligible), but not both. In these models, the real effects of monetary policy are small for realistic values of the average time between observations. We show that non-negligible observation costs produce both these effects: intervals between observations are both infrequent and volatile. This generates large real effects of monetary policy for realistic values of the average time between observations.

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# 1 Introduction

The sluggish propagation of new information from individual into aggregate prices is one of the key mechanisms behind the transmission of monetary shocks and the analysis of their real effects. We study micro founded models where prices respond slowly to shocks because firms must pay a fixed cost to observe the determinants of the profit maximizing price, as in the “rational inattentiveness” literature. The firm’s frequency of observation determines the velocity with which new information is embedded into prices, so that large real effects of monetary shocks require infrequent observations. For a given frequency of observation, more volatile times between observations increase the real effects, a result established by [Carvalho and Schwartzman \(2012\)](#) for the case of i.i.d. observation times. The models in [Caballero \(1989\)](#), and the baseline model of [Reis \(2006\)](#), yield optimally chosen constant observation times that produce infrequent adjustments but have no volatility. For a given frequency of observation these models yield real effects similar to Taylor’s exogenously staggered adjustment model, about half the size of what is produced by the Calvo model, or the exponentially distributed observations as used in [Mankiw and Reis \(2002\)](#) with the same frequency. For a special case with negligibly small observation costs [Reis \(2006\)](#) provides a micro foundation for random observation times.<sup>1</sup> Although the times between observations are volatile in this case, the frequency of observations is very high (due to the negligible costs), so that the real effects of monetary shocks are tiny.

Motivated to obtain larger effects of monetary shocks we study models where the intervals between optimally chosen observation times are both *long* and *volatile*. The first set of results derives the firms’ observation times as the solution to a fully specified profit maximization problem, with random and persistent variation in the observation costs. Obtaining long and volatile observation times, and hence larger effects of monetary policy, is to have observation costs that are both non-negligible and sufficiently persistent. Indeed, an important innovation of this paper is to produce from first principles optimal firm decisions that are persistent rather than i.i.d. as standard in the previous literature. We show that the process of the observation costs maps into the process for the optimally chosen observation times in a subtle non-linear way. For instance, a setting with i.i.d. observation costs will produce optimally chosen *constant* observation times. As the costs become sufficiently persistent, the optimal times become responsive to the realization of the observation cost. And hence the frequency of observation becomes volatile and persistent. The second set of results pertains to the

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<sup>1</sup>The case of negligibly small observation cost studied by [Reis \(2006\)](#) is one where for any given distribution of the observation cost  $\theta$ , the model is solved for a scaled process of the observation cost  $\theta' = c\theta$  where  $c \rightarrow 0$  is a scaling factor. In the scaled model, both the mean and volatility of the observation cost go to zero, but the coefficient of variation stays constant.

aggregation properties of an economy with a continuum of firms, for a given Markov process for the individual firm’s observation times. Methodologically, this aggregation result is new and generalizes the elementary renewal theorem in statistics where consecutive holding times (i.e. times between observations in our economic application) are i.i.d. to an environment where holding times follow a first order Markov process. The paper contains several new results, summarized below, that extend and complement the ones in the literature.

**Microfoundations of optimal observation times.** We use a simple general equilibrium model of price setting to derive the distribution of the optimal times between observations from the firm’s profit maximization problem. We assume that when the firm pays the observation cost it learns the current value of the production cost, a key piece of information for price setting, and also obtains a signal about future observation costs. The optimal choice for the time until the next observation is a function of the signal, a sufficient statistics for the future observation cost at every future date. We consider several cases, which are interesting both for their relationship to the literature, and for their substantive monetary policy implications.

The first case is one where the signals are such that the optimally chosen observation times are i.i.d. This case, while standard in the “rational inattentiveness” literature, has not yet been derived as the solution of an explicit profit maximization problem with random variation in the benefit/cost of acquiring information.<sup>2</sup> This case yields two novel insights. First, i.i.d. observation times arise only if the evolution of the observation costs is not independent of the firm’s decision to observe. Thus the firm gets a new cost every time it makes an observation. This requirement is specific to the i.i.d. case, and illustrates how special this arrangement is. The more natural setting in which the evolution of the costs is independent of the observation times will naturally give rise to the Markov observation times, discussed next. The second insight yielded by the model with i.i.d. observation times is that the optimal decision by the firm contains an option value argument: a firm that is facing a very low cost (for the next observation), has an incentive not to observe immediately, but rather to save the opportunity for a cheap observation for when it is most valuable in future.

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<sup>2</sup> An exception is the paper by [Woodford \(2008\)](#) who, in a different framework namely the one of “rational inattention”, characterizes optimal i.i.d. observation times. He assumes that keeping track of the state and *time elapsed* is costly. When the cost of keeping track of time is sufficiently large the optimal observation times become i.i.d. because the firm cannot base decisions on (any measure of) time and, therefore, base decisions exclusively on the current value of the noisy signals it receives, even though the process of interest is persistent. Each of the two frameworks has weaknesses and strengths in this particular application. On the one hand we view the assumption of no memory using the rational inattention model of Woodford as an extreme one. On the other hand in the rational inattention the design of the signals about the state space is chosen by the decision maker. Instead, in the rational inattentiveness model that we use, the signals are extreme: either complete or no information is revealed upon observation.

This implies that the support of the distribution of observation cost must include negative values in order to generate the arbitrarily small observation times that are common in the literature, such as the exponentially and i.i.d. distributed observations in [Mankiw and Reis \(2002\)](#). Both insights suggest that the i.i.d. case is rather special, for this reason we explore the more general, and more realistic case, which gives rise to persistent observation times.

The other major case explored in this paper is one where, unlike the i.i.d. case, the evolution of the costs is independent of the firm’s decision to observe. In this case the optimal decision rules imply that the times between successive observations form a Markov process. We characterize the properties of the observation cost process that leads to variability in the times between observations. This is important since, as mentioned, the variability of the times between observations maps into the size of the effect of a monetary shock on real output. Notice that variable observation times require both variability in costs and that the optimal decision rule is sensitive to the realization of the cost shock. An early contribution for volatile observation times was given by [Reis \(2006\)](#), who provides a micro foundation in the limiting case of negligible observation costs. His model makes the times between observations random, and hence volatile, but it produces very frequent observations (due to the small costs), so that the real effects of monetary shocks are tiny. We provide a global characterization of the decision rule, which allows for both negligible (reproducing Reis’ result) as well as non-negligible observation costs. The latter are essential to get infrequent observations. Interestingly, we show that for any given variability of the observation costs, the *persistence* of the costs affects the sensitivity of the optimal decision rule with respect to the cost. And hence the persistence of the observation costs determines the variability of times between observations. Higher persistence amplifies the variability of optimal observation times and, therefore, the size of the real effects of monetary shocks. On the one hand we show that for any given (non-negligible) expected value of the observation costs, the volatility of observation times converges to zero if persistence is sufficiently low. In the limiting case where the observation costs are i.i.d., the volatility of times between observations converges to zero and the real effects of monetary shocks are those of a model with constant times between observations. On the other hand, the real effects of monetary shocks can be substantially larger, close to those of a Calvo model with a realistic parametrization, with observation cost that are non-negligible and sufficiently persistent.

**Results on aggregation.** Another set of results pertains to the aggregation properties of an economy with “inattentive producers”. We denote by  $H(t'|t)$  the right CDF of the optimally chosen times larger than  $t'$ , conditional on a gap of time equal to  $t$  between the preceding two observations. This distribution is derived simply from the assumptions about

the observation costs, the signals and the optimal decision rules described above. While in our model the distribution  $H(t'|t)$  is the outcome of the solution to the firms' problem facing random and persistent variability in observation cost, our results on aggregation are independent of the specific model micro founding a given  $H$ . In particular, for a given  $H$ , we aggregate the times until the next observation across a continuum of firms to characterize the stationary cross-sectional distribution of the times until the next observation, which we denoted by  $Q$ . The distribution  $Q$  is key in determining the propagation of monetary shocks as it determines the time it takes before a given fraction of firms makes at least one observation and, therefore, incorporates new information into prices.<sup>3</sup> We show that the density  $q$  associated to the cross-sectional distribution  $Q$  is given by  $q(t) = R K(t)$ , where  $R$  is a constant (the unconditional frequency of observations per unit of time), and  $K$  is the invariant distribution of the Markov process formed by the time elapsed between consecutive observations.<sup>4</sup> And so by the law of large numbers,  $K(t)$  measures the cross-sectional fraction of firms that have waited at least  $t$  periods since the last observation.

We use this result to illustrate several applications. First, we show that the cumulative output response increases with both the *average* and the *coefficient of variation* of times between consecutive observations, as measured in the cross section according to the distribution  $K$ . This finding generalizes a result by [Carvalho and Schwartzman \(2012\)](#), obtained for the case of i.i.d. durations, to a framework where durations form a first order Markov process. This extension is important because quite special assumptions are needed to generate i.i.d. durations, as discussed above. This result is quantitatively relevant because persistent shocks, and the associated persistent times between observations, are essential to generate a sizable cross sectional variation of observation times from a reasonable variation in the observation cost.

Second, our analysis of the mapping from  $H$  to  $Q$  clarifies a related theoretical result in [Reis \(2006\)](#) concerning the aggregation of individual firms decision. Within the class of i.i.d. observation times, Reis derives conditions on the distribution of the individual firm's distribution of observation times that deliver an exponential cross sectional distribution  $Q$  as assumed e.g. in [Mankiw and Reis \(2002\)](#). We clarify that a necessary and sufficient condition to obtain an exponential cross sectional distribution  $Q$  is that the firm level distribution of observation times  $H$  must be exponential itself. Hence the exponential distribution of observation times, a case often used in the literature, turns out to be obtained as an optimal decision rule only in a very special case.<sup>5</sup>

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<sup>3</sup>[Reis \(2006\)](#) labels  $Q$  the "distribution of inattentiveness".

<sup>4</sup> Formally  $K(t)$  is the invariant cumulative distribution associated to the transition function  $H(t'|t)$ , as standard in the definition of the invariant distributions for a discrete-time continuous-state Markov process.

<sup>5</sup>Indeed, we solve analytically for the distribution of costs that implies an exponential distribution of

Third, under the assumption that price changes only occur upon observation dates, we derive a mapping between the cross sectional distribution of observation durations,  $K$ , and the kurtosis of the size-distribution of price changes.<sup>6</sup> This mapping can be used to calibrate the model.

The rest of the paper is organized as follows. For expositional convenience, we first derive in [Section 2](#) the mapping from the individual firm’s distribution of the times between observations  $H$  to the cross sectional distribution  $Q$ , and discuss three main applications of this new aggregation result. [Section 3](#) describes a general profit maximization problem, in the presence of a fixed cost of observing the state. The solution to this problem yields the optimal decision rules that micro founds the stochastic observation times of an individual firm. [Section 4](#) shows the general solution to the firm’s problem, in which case the optimal observation times form a first order Markov process. [Section 5](#) consider a particular setup for the firm’s problem, in which case the optimal observation times are i.i.d. [Section 6](#) concludes.

## 2 Aggregating the behavior of inattentive producers

This section characterizes the linkages between an individual firm’s behavior and the cross-sectional features that are important for modeling the propagation of an aggregate shock in an economy with “rationally inattentive” firms.

We are interested in an economy populated by firms that observe the state underlying their prices at randomly spaced times. The interpretation of “making an observation” is that the firm pays a cost to gather (aggregate, and process) information to be used to set prices. Upon each observation the firm chooses a new path for prices until the next observation.

The primitive for the analysis in this section is a distribution of the firm’s times between consecutive observations, as summarized by its right cumulative distribution function (CDF), which we denote by  $H$ .<sup>7</sup> Starting from a given  $H$ , this section aggregates the times between observations across firms to characterize the stationary *cross-sectional* distribution of the “times until the next observation”. That is, the fraction of firms that, at any point in time, will wait at least  $t$  units of time until the next observation. We denote the right-CDF of such distribution by  $Q(t)$ .

The distribution  $Q(t)$  is a key object in models with rational inattentive producers because

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observation times, which has two features that we find implausible: a large proportion of costs that are *negative* and a very large variability of observation costs.

<sup>6</sup>A micro foundation of price changes occurring only upon an observation dates can be the presence of menu costs. As shown in Proposition 1 of [Alvarez, Lippi, and Paciello \(2011\)](#), relatively small menu costs make price-plans not optimal in this environment at moderate inflation rates.

<sup>7</sup>The use of right CDF, i.e. the probability mass that the time between observations is *above* a given threshold, is more convenient algebraically than the traditional (left) CDF.

it determines the time left before a given fraction of firms will conduct an observation and adjust their behavior to the new information obtained. Therefore  $Q$  directly determines the time it takes for an aggregate shock to be incorporated into the information set of a given fraction of firms. And  $Q$  determines the speed at which a monetary shock affects the aggregate price level. In a stationary environment there is a constant rate of observations per unit of time, which we denote by  $R$ . In this section we will derive both  $Q$  and  $R$  as implied by a given  $H$ . In [Section 3](#) we will study a model where the distribution  $H$  is itself derived from the firm's optimal price setting choices subject to an information gathering friction.

Before providing an exhaustive formal analysis, we sketch two examples to fix ideas. The first one is a classic Taylor's model where each firm observes the state at deterministically spaced time intervals of length  $T$ . This assumption amounts to an i.i.d.  $H(t)$  distribution which is piecewise continuous: constant and equal to 1 for  $t \in [0, T)$ , and zero otherwise. It is easy to see how this  $H(t)$  implies a unique invariant cross-sectional distribution  $Q(t)$  of times until the next observation, with a density  $q(t)$  that is uniform for  $t \in (0, T)$ , and an observation rate given by  $R = 1/T$ .

The second example is the widely used Calvo pricing. It assumes that the distribution of times between observations for a firm follows a Poisson process, so that the probability that a firm waits at least  $t$  periods after adjusting is given by the exponential distribution  $H(t) = e^{-\lambda t}$  for  $t \in (0, \infty)$ . Aggregating a continuum of firms that follow this  $H(t)$  rule gives a cross section distribution of times until the next observation,  $Q(t)$ , that is also exponential, and a rate of observations per unit of time  $R = \lambda$ . While the mapping from  $H$  to  $Q$  for these simple examples can be solved using intuition, a general treatment is useful. It will allow us to extend the mapping along two important dimensions: first, to cases where the firm hazard rate of observations is not constant (so that  $H(t)$  is not exponential); second, and perhaps more interestingly, to allow for non i.i.d. observation times at the firm level. In the latter the distribution of the times  $t$  between observations is allowed to depend on the length  $s$  of the time since the last observation and denoted by  $H(t|s)$ . Markovian observation times appear to be a robust feature of a micro founded model, we thus see this case as an important extension of the i.i.d. observation times.<sup>8</sup>

We consider set-ups where the optimal decision of the firm will be such that, in a stationary environment, the time elapsed between consecutive observations forms a stationary Markov process. As already mentioned, the primitive of the analysis in this section is the *conditional* right-CDF  $H(t|s)$  for the times between consecutive observations of a firm. The function  $H : \mathbb{R}_+^2 \rightarrow [0, 1]$  satisfies the following: i)  $H(0|s) = 1$  for all  $s \geq 0$ , ii)  $H(t|s) \geq H(t'|s)$  for all

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<sup>8</sup>See [Alvarez, Lippi, and Paciello \(2011\)](#) for an example of a micro founded model where the optimal observations times are not i.i.d. We note that there is no systematic treatment of this problem in the literature.

$t' \geq t$ , iii)  $\lim_{t \rightarrow \infty} H(t|s) = 0$  for all  $s \geq 0$ , and iv)  $H(t|\cdot)$  is Borel-measurable, for all  $t \geq 0$ . The interpretation of  $H(t|s)$  is that if the gap between the preceding two observations is equal to  $s$ , then the probability that the time between the current and the next observation is at least  $t$  is  $H(t|s)$ . We assume that the stochastic process for observation times is independent across firms, and that the law of large number holds.

Let  $K(s)$  denote the invariant cumulative distribution function of the times between consecutive observations of length up to  $s$ . Formally,  $K(s)$  is the invariant distribution associated with the transition function  $H(t'|t)$ , as will be shown below. In particular, in an interval of time  $(t, t + dt)$  there are  $R dt$  observations that take place. Of those, a fraction  $K(s)$  is characterized by a gap between the preceding two observations greater or equal than  $s$ . Using the definition of  $K(s)$  we can compute the cross sectional distribution of times until the next observation, whose right CDF is<sup>9</sup>

$$Q(t) = -R \int_0^\infty \left( \int_0^\infty H(u+t|s) dK(s) \right) du = -R \int_t^\infty \int_0^\infty H(u|s) dK(s) du . \quad (1)$$

The expression is easy to interpret: the firms that will wait at least  $t$  periods before the next observation are those that drew an observation time larger than  $u + t$  some  $u$  periods ago. The integral inside the parenthesis measures those firms. The probability that a firm must wait this long depends on the gap  $s$  between the preceding two observations and is given by  $H(u + t, s)$ . In the integrand, this is then weighted by (the negative of)  $dK(s)$ , which gives the fraction of firms whose preceding two observations were separated by  $s$ . Integrating over both the time until the next observation,  $u$ , and the time between the preceding two observations,  $s$ , and multiplying by the frequency of observation,  $R$ , gives  $Q(t)$ . Note that the minus sign comes from using a *right* CDF for  $K(s)$ . The value of  $R$  is easily computed from [equation \(1\)](#), using that  $Q(0) = 1$ .

Differentiating this expression we obtain the density of the cross-sectional distribution of times until the next observation, as we summarize in the following lemma:

**LEMMA 1.** Given the conditional right-CDF,  $H$ , and its corresponding invariant distribution of times between observations,  $K$ , the density of the times until the next observation  $q(t)$  is:

$$q(t) \equiv -\frac{\partial Q(t)}{\partial t} = -R \int_0^\infty H(t|s) dK(s) \geq 0 , \quad (2)$$

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<sup>9</sup>Integration follows Lebesgue-Stieltjes.  $Q(t)$  is also referred to as distribution of the forward times.

where the number of observations per unit of time is

$$R = -\frac{1}{\int_0^\infty \int_0^\infty H(u|s) dK(s) du} \geq 0. \quad (3)$$

To complete the characterization of  $Q$  we need the (invariant) distribution of times between observations,  $K(s)$ . We distinguish between two cases: one where the distribution  $H$  is absolutely continuous, another where  $H$  is discrete. If  $H(\cdot|s)$  is absolutely continuous the invariant distribution  $K$  will be differentiable with density  $k(t) = -\partial K(t)/\partial t$  solving:

$$k(t) = \int_0^\infty h(t|s) k(s) ds \quad \text{for all } t \geq 0 \quad \text{where} \quad h(t|s) \equiv -\frac{\partial H(t|s)}{\partial t} \geq 0 \quad \text{for all } t, s \geq 0.$$

Likewise, since  $H$  is differentiable almost everywhere, the right-CDF of the invariant measure solves

$$K(t) = -\int_0^\infty H(t|s) dK(s) \quad \text{for all } t \geq 0. \quad (4)$$

The second case occurs when  $H(\cdot|s)$  is a step-function, so that there are countably many observation times  $\mathcal{T}$  with non-zero probabilities. Let  $\mathcal{T} = \{t_1, \dots, t_i, \dots, t_I\}$  with  $t_i < t_{i+1}$  for  $i \geq 1$  and  $h = [h_{t_i, t_j}]$  be a matrix defined as

$$h_{t_i, t_j} \equiv \lim_{t \uparrow t_i} H(t|t_j) - H(t_i|t_j) \geq 0.$$

Using this notation,  $\{k(t)\}_{t \in \mathcal{T}}$  is the corresponding invariant measure, i.e. a positive eigenvector associated with  $h$  solving:

$$k(t_i) = \sum_{t_j \in \mathcal{T}} h_{t_i, t_j} k(t_j) \quad \text{for all } t_i \in \mathcal{T}. \quad (5)$$

In this case the invariant measure is given by:

$$K(t) = \begin{cases} 1 & \text{if } t < t_1 \\ \sum_{t_i > t} k(t_i) & \text{if } t_1 \leq t_i < t_I \\ 0 & \text{if } t \geq t_I \end{cases}, \quad (6)$$

where the  $k(t_i)$  elements solve [equation \(5\)](#). We collect these results in the following lemma:

**LEMMA 2.** Given a conditional right-CDF  $H$  for the times between consecutive observations, the invariant distribution  $K$  is given by the solution of [equation \(4\)](#) if  $H(\cdot|s)$  is absolutely

continuous for each  $t$ , or by the solution of [equation \(6\)](#) if  $H(\cdot|s)$  is a step function.

The results for existence and uniqueness of invariant distributions for a discrete-time Markov process give the exact conditions for existence and uniqueness of the invariant distribution  $K$ . We combine the results of [Lemma 1](#) and [Lemma 2](#) and obtain the following proposition.

**PROPOSITION 1.** Let  $K(t)$  be the invariant (right) cumulative distribution of times between consecutive observations as in [equation \(4\)](#) or in [equation \(6\)](#). The density  $q(t)$  of the stationary cross-sectional distribution of times until the next observation is proportional to  $K(t)$ :

$$q(t) = R K(t) \quad \text{for all } t \geq 0 \quad \text{and where} \quad R = \frac{1}{\int_0^\infty K(s) ds} . \quad (7)$$

The right CDF  $Q(t)$  is readily found by integration using that  $Q(0) = 1$ .

For a proof see [Appendix A](#). The results of [Proposition 1](#) substantially generalize the results in the literature about the way the economy aggregates starting from a given individual firm distribution of observation times  $H(t|s)$ . Existing results in the literature (e.g. [Reis \(2006\)](#), [Carvalho and Schwartzman \(2012\)](#)) are based on the case of i.i.d. observations, i.e.  $H(t|s_1) = H(t|s_2)$  for all  $s$  and all pairs  $(s_1, s_2)$ . The interpretation of [Proposition 1](#) is simple: in a cross section there is a mass  $K(t)$  of firms who will wait at least  $t$  periods before the next observation. The parameter  $R$ , measuring the average frequency of observation, scales the density proportionately. Notice that the function  $q(t)$  is a density function even in the cases in which the invariant distribution of times between consecutive observations  $K(t)$  does not have a density, as in the case where  $K(t)$  is a step function (e.g. the example of Taylor's model discussed above) or, more generally, the case discussed in [equation \(6\)](#).

## 2.1 Three applications of [Proposition 1](#)

Next we discuss three applications of [Proposition 1](#). In the first one we show that the cross sectional distribution of times between observations,  $K(t)$ , is a key determinant of the real effects of monetary shocks. The second application is literature driven: we derive necessary and sufficient conditions under which the firm distribution of observation times  $H(t|s)$  yields a distribution  $K(t)$  that is both i.i.d. and exponential, a case studied by [Reis \(2006\)](#) and extensively used in applications following [Mankiw and Reis \(2002\)](#). The third application studies the mapping from the distribution  $K(t)$  to the size distribution of price changes. This mapping can be used to identify the relevant statistics about the distribution  $K(t)$  using data on the size distribution of price changes.

### 2.1.1 The distribution of observations and the effect of monetary shocks

The cross sectional distribution of the times until the next observation,  $Q(t)$ , is a key ingredient in the propagation of monetary shocks into prices and output. We summarize this effect in a single statistic which measures the real cumulative effect of a monetary shock,  $\mathcal{M}$ :

$$\mathcal{M} \equiv \int_0^{\infty} Q(t) dt .$$

Because  $Q(t)$  measures the mass of firms that do not make an observation for at least  $t$  periods after the shock, then  $\mathcal{M}$  is the mass of firms that will not update their prices cumulated over all future horizons. This value ranges between zero, when all firms adjust immediately so that  $Q(t) = 0$  for all  $t \geq 0$ , to infinity when firms never adjust so that  $\lim_{t \rightarrow \infty} Q(t) = 1$ .

It can be easily shown that in a framework like the one of [Golosov and Lucas \(2007\)](#),  $\mathcal{M}$  is proportional to the size of the real effects of a money supply shock (see [Appendix D](#) for details). In particular, if an unexpected permanent shock increases, once and for all, the level of the money supply by  $\delta$  log points, then  $\int_{t_0}^{\infty} \log(M_t/P_t) dt \approx \delta \mathcal{M}$  (where  $M_t$  is the stock of money and  $P_t$  is aggregate price level). In such a framework, the statistic  $\mathcal{M}$  measures the (cumulative) time it takes for any shock to be incorporated into the agents' information set and, therefore, into prices. Smaller  $Q(t)$  (for any  $t$ ) causes information about the monetary shock to be incorporated faster into the aggregate price level. This, in turn delivers a smaller cumulated response of real balances  $M_t/P_t$  and, therefore, output. Accordingly, we treat  $\mathcal{M}$  as a proxy measure of the real effects of a monetary shock.

This measure  $\mathcal{M}$  is completely characterized by the mean and variance of the cross-sectional distribution of times between observations:

**PROPOSITION 2.** The statistic  $\mathcal{M} \equiv \int_0^{\infty} Q(t) dt$  is given by

$$\mathcal{M} = \frac{E_K(t)}{2} (1 + (CV_K(t))^2) , \tag{8}$$

where  $E_K(t)$  and  $CV_K(t)$  are, respectively, the average duration and coefficient of variation of times elapsed between consecutive observations.

For a proof see [Appendix A](#). Both the mean and the variance of the time between consecutive observations have a first order effect on  $\mathcal{M}$  and therefore on the real effects of monetary shocks (as shown in [Appendix D](#)). Notice that the results of [Proposition 2](#) are independent of the assumptions about the primitive distribution of times between observations of the firm,  $H$ .

A result similar to [Proposition 2](#) has been obtained by [Carvalho and Schwartzman \(2012\)](#)

for i.i.d. observation times. In our framework this corresponds to the case where  $H(t|t_1) = H(t|t_2)$  for all  $t$  and all pairs  $(t_1, t_2)$ . [Proposition 2](#) extends [Carvalho and Schwartzman's \(2012\)](#) results to a more general environment that allows for observation times to follow a Markov process. Such a situation may arise when the variability of observation times is not attributable to a single source, such as the size of innovations as well as the persistence of value of observations.

To illustrate the economics of [Proposition 2](#) by means of a simple example, consider an economy where each firm draws an observation of length  $\underline{T}$  with probability  $1 - p$ , and otherwise an observation of length  $\bar{T} > \underline{T}$ , independently of the length of the last observation. The distribution of the firm's observation times is given by  $\hat{H}(t) \equiv H(t|t')$  for all  $t > 0$ , where:

$$\hat{H}(t) = \begin{cases} 1 & \text{if } t \in [0, \underline{T}] \\ p & \text{if } t \in (\underline{T}, \bar{T}] \\ 0 & \text{otherwise} \end{cases} .$$

Given that observations are i.i.d. over time, the cross-sectional distribution of times between consecutive observations equals the firm's distribution of times between observations. That is,  $K(t) = \hat{H}(t)$  for all  $t$  (as implied by [Lemma 2](#)). Let  $\hat{T} \equiv (1 - p)\underline{T} + p\bar{T}$  denote the average length of time between consecutive observations. Using [Proposition 1](#) we obtain that  $R = 1/\hat{T}$  and the invariant (right) CDF of the cross-sectional times until the next observation is

$$Q(t) = \begin{cases} 1 - t/\hat{T} & \text{if } t \in [0, \underline{T}] \\ 1 - \underline{T}/\hat{T} - p(t - \underline{T})/\hat{T} & \text{if } t \in (\underline{T}, \bar{T}] \\ 0 & \text{otherwise} \end{cases} . \quad (9)$$

To understand [equation \(9\)](#) it is useful to compare two example economies. In the first economy, indexed by a subscript 1,  $p_1 = 1$  and  $\underline{T}_1 = \bar{T}_1 = \hat{T}$ . Then  $H_1(t)$  is degenerate; the time between consecutive observations is always  $\hat{T}$ . This is a case studied, for instance, by [Caballero \(1989\)](#) and [Reis \(2006\)](#). In the second economy, indexed by a subscript 2,  $p_2 \in (0, 1)$ , so that  $\underline{T}_2 < \hat{T} < \bar{T}_2$ , and observation times are truly random. As the mean time between consecutive observations is the same in the two economies, the distribution of observation times in economy 2 is obviously a mean-preserving spread of those in economy 1. And so [Proposition 2](#) implies that the more variable times between observations in economy 2 cause larger real effects of a monetary shock.

This example is also simple enough to make clear the economics of [Proposition 2](#). Intuitively,  $Q_2(t)$  decays more slowly than  $Q_1(t)$  (at least for  $t > \underline{T}_2$ ), so that the cumulative

effect summarized in  $\mathcal{M}$  is larger. Both functions start from the same point  $Q(0) = 1$  but in economy 2 there is a longer tail of possible observation times. This tail boosts the real effect of a monetary shock by delaying the response of some firms, slowing the incorporation of the shock into prices and so prolonging the real effect.

It is easy to see this formally either by using [equation \(8\)](#) or integrating  $Q(t)$  directly to give  $\mathcal{M} = (\hat{T}/2) \times [(1/p) (1 - \underline{T}/\hat{T})^2 + (\underline{T}/\hat{T}) (2 - \underline{T}/\hat{T})]$ . This shows how, for a given average time between observations  $\hat{T}$ , the size of the real effects of a monetary shock decreases with  $p$  and  $\underline{T}$ . Notice that  $\mathcal{M}$  is minimized at  $p = 1$ , where it takes the value of  $\hat{T}/2$ . This corresponds to economy 1 discussed above. A mean preserving spread of times between consecutive observations increases the real effects of monetary shocks.

### 2.1.2 The case of exponential i.i.d. observations

The case where observation times are i.i.d. has been first studied by [Reis \(2006\)](#). This case occurs when  $H(t|s)$  does *not* depend on  $s$ , so we denote it simply by  $\hat{H}(t)$ . Using [Proposition 1](#) we obtain that the invariant distribution  $K$  coincides with  $H$  and thus:

$$q(t) = R \hat{H}(t) \text{ for all } t \geq 0, \quad (10)$$

where the number of observations per unit of time is given by  $R = 1/\int_0^\infty \hat{H}(t) dt$ .<sup>10</sup> An immediate implication of [equation \(10\)](#) is that:

**COROLLARY 1.** Let the observation times be i.i.d. so that  $H(t|s) = \hat{H}(t)$  for each  $s \geq 0$ . The cross sectional distribution of the times until the next observation,  $Q$ , is exponential if and only if the firm's distribution of the times between observations,  $\hat{H}$  is exponential.

The proof follows since an exponential density is proportional to its right CDF, a property shared by  $q(t) \propto \hat{H}(t)$  in [equation \(10\)](#). [Corollary 1](#) clarifies an important result in [Reis's \(2006\)](#), namely his Proposition 6, in which the result is obtained under the same assumptions of this section about the distribution  $H$  (i.e. firms observation times being i.i.d. so that  $H(t'|t) = \hat{H}(t)$  for all  $t'$  and  $t$ ), but where the requirement that the distribution  $H$  has to be itself exponential in order for the cross-sectional distribution  $Q$  to be exponential is not as explicit.<sup>11</sup> This clarification is important because the case of an exponential cross-sectional distribution  $Q$  has been extensively used in the “rational inattentiveness” literature that followed the seminal work by [Mankiw and Reis \(2002\)](#). We conclude that the conditions for an

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<sup>10</sup>Note that this is the case considered for a renewal process, where  $Q$  is the forward recurrent time as, for instance, in [Karlin and Taylor \(1998\)](#).

<sup>11</sup>In private correspondence Reis has clarified that this result follows from his definition of stationarity in [Reis's \(2006\)](#) Definition 1.

exponential cross-sectional distribution  $Q$  are rather restrictive:  $\hat{H}(t)$  can only be exponential. [Section 5.1](#) below will further explore this case by establishing what primitive assumptions about the firm information and the distribution of observation costs are necessary for the firm distribution of observation times to be exponential.

### 2.1.3 From the distribution of observations to the distribution of price changes

We conclude this section by deriving the mapping from the cross sectional distribution of times between consecutive observations,  $K$ , to the size distribution of price changes. This mapping is useful to identify statistics on the distribution  $K$  from observable statistics on the distribution of price changes. Given the result of [Proposition 2](#) we are particularly interested in identifying the variability of observation times. We assume that a firm's optimal frictionless price,  $p_t^*$ , evolves according to a Brownian motion,  $d \log(p_t^*) = \mu_p dt + \sigma dB_t$ , where  $B_t$  is the realization of an idiosyncratic Wiener process. In [Section 3](#) we will show that this process for  $p_t^*$  arises naturally in a standard model where monopolistic firms face a constant elasticity downward sloping demand, produce output with a production technology which is linear in labor, and labor productivity follows a Brownian motion.

In absence of additional costs that prevent the firm from adjusting prices between observation dates, the presence of a non-zero drift in  $p_t^*$  implies that prices would change continuously in our model as the firm indexes its price to the expected growth in  $p_t^*$  between observation dates. This is a common element in models of rational inattentiveness that lack a physical cost of price adjustment. A robust pattern in the data is, however, that prices change infrequently. A simple way to obtain infrequent price changes in our model is to assume that  $\mu_p = -\sigma^2/2$ , implying that  $p_t^*$  is a martingale.<sup>12</sup> As a result, price changes only occur when new information arrives, so that the frequency of price changes coincide with the frequency of observations. In this specification of the model, the cross-sectional invariant distribution of log-price changes,  $\Delta \log(p)$ , is given by a mixture of normals indexed by  $t$ , where the mixture has density corresponding to the cross sectional distribution of times between consecutive observations, i.e.  $k(t) = -K'(t)$ , and each of the normals has mean and variance  $(0, \sigma^2 t)$ . The next lemma characterizes two useful moments of the size distribution of price changes:

**LEMMA 3.** Let the target price  $p_t^*$  be a martingale, then the variance of the log-price changes is

$$\text{Var}(\Delta \log(p)) = \sigma^2 E_K(t) \tag{11}$$

where  $E_K(t) = \int_0^\infty K(s) ds = 1/R$  is the average duration of time between observations, and

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<sup>12</sup>See [Alvarez, Lippi, and Paciello \(2011\)](#) for a model where information acquisition costs are combined with physical costs of adjusting prices to obtain infrequent price adjustments as well as infrequent observations.

the kurtosis of the log-price changes is equal to:

$$Kurt(\Delta \log(p)) = 3 [(CV_K(t))^2 + 1], \quad (12)$$

where  $CV_K(t)$  is the coefficient of variation of times between consecutive observations (taken with respect to the distribution  $K$ ).

See [Appendix A](#) for the proof. [Lemma 3](#) is useful for two reasons. First, combined with the results of [Lemma 2](#), it allows to obtain statistics about the cross-section distribution of price changes from the primitive assumptions about the individual firm distribution of times between observations  $H$ . Second, combined with observable statistics on the distribution of price changes, it allows to identify the coefficient of variation in observation times from the kurtosis of the the distribution of price changes, as well as to identify the volatility of the target price from the variance and frequency of price changes. For instance, if we use a kurtosis of price changes equal to 4, consistent with the estimates by [Alvarez, Le Bihan, and Lippi \(2014\)](#), we obtain a coefficient of variation equal to  $CV_K(t) = \sqrt{1/3}$ . Combining such estimate for  $CV_K(t)$  with the results of [Proposition 2](#), we obtain that the cumulative real effects of a permanent innovation to money supply is about 35% larger than the corresponding figure predicted by models with deterministic observation times (where  $CV_K(t) = 0$ , such as [Caballero \(1989\)](#) and [Reis \(2006\)](#)), and about 1/3 smaller than predicted by models with exponential observation times (where  $CV_K(t) = 1$ , such as [Mankiw and Reis \(2002\)](#)).

### 3 A firm problem leading to optimal observation times

This section studies a model of the firm's optimal decision for observation times. This problem gives rise to a Markov process for observations that can be represented by the conditional distribution  $H(t'|t)$ , the primitive of our analysis in [Section 2](#). We set up the firm's decision problem in a relatively general formulation that encompasses several cases of interest in which the duration of the optimally chosen observation times are constant, or follow a first-order Markov process ([Section 4](#)), or are i.i.d. ([Section 5](#)). We characterize the firm's optimal decision in these different cases, and use it to discuss the key determinants of the variation in observation times and, therefore, the real effects of monetary shocks. The distinction between models of constant observations (e.g. [Caballero](#)) and models of random observations is important since the variability of observations increases the real effects of monetary shocks ([Proposition 2](#)). Within the class of models that deliver variation in observation times, the case of Markov persistent durations of the time between observations is appealing both because we think it is a realistic representation of firms' behavior and

because, for a given variability of the observation costs, more persistent durations of the time between observations increase the real effects of monetary shocks. The case of i.i.d. observations is interesting mostly because a large literature has focused on its implications for macroeconomic dynamics as discussed above. Despite its widespread use, the “rational inattentiveness” literature has not provided a characterization of optimal i.i.d. observations originating from random variation in the benefit/cost of acquiring information. This paper fills that gap and shows what features of the primal problem, such as variance and persistence of the observation costs, are relevant to determine the real effects of a monetary shock. This section sets up a general environment where the firm’s optimal stopping times can be studied. [Section 4](#) discusses the general case of Markovian observation times. [Section 5](#) specializes the model to obtain i.i.d. observation times, a standard benchmark in the literature.

The underlying economy is a simple variant of [Golosov and Lucas \(2007\)](#). Firms take two types of decisions: i) they decide when to acquire information, and ii) set the price at which they are willing to sell their good. In particular, there is a continuum of firms, each producing a variety denoted by  $i$ , and facing a downward sloping demand given by  $C(P_{i,t}/W_t) = A (P_{i,t}/W_t)^{-\eta}$ , where  $A > 0$  is a constant,  $\eta > 1$  is the price-elasticity,  $P_{i,t}$  is the nominal price of variety  $i$ , and  $W_t$  is the nominal wage in period  $t$ . The production technology of each variety is linear in labor with idiosyncratic labor productivity  $z_{i,t}$ . It is well known that in this environment the frictionless optimal pricing strategy of each firm is given by a constant markup over nominal marginal cost, given by  $W_t/z_{i,t}$ . The (log) of the labor productivity follows a brownian motion with drift  $\phi$  and volatility  $\sigma$ :  $d\log(z_{i,t}) = \phi dt + \sigma dB_{i,t}$ , where  $B_{i,t}$  is the realization of an idiosyncratic Wiener process. We also assume that the product of the firm is replaced (“dies”) with a constant probability per unit of time equal to  $\lambda$ , and a new product is born with productivity  $z = 1$ . From a modeling point of view the exogenous substitutions allow for an invariant distribution of firm’s productivities and for bounded output. Finally, we assume that the nominal wage grows at a constant rate  $\mu$ , so that its dynamics between  $t$  and  $t + T$  are given by  $W_{t+T} = W_t e^{\mu T}$ .

**Information gathering and price changes.** In the spirit of the rational inattentiveness literature we assume that to gather information about the nominal marginal cost the firm must pay a fixed information-gathering cost (see e.g. [Reis \(2006\)](#)), an activity we refer to as an *observation*. Hence, at observation times, there is a discrete change in the information about the nominal marginal cost. While acquiring the information is costly, firms can change prices at any time at no cost.<sup>13</sup> Price changes between observation dates are referred to in

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<sup>13</sup>This differs from the models in which firms face a joint cost of observing the state and adjusting the price as in [Bonomo and Carvalho \(2004\)](#), where the two costs are associated, or [Alvarez, Lippi, and Paciello \(2011\)](#) where the firm can decide to selectively incur each of the costs.

the literature as *price plans*: such price changes are based upon the information gathered in the last observation and the law of motion of the relevant states.

**The value of information.** We first develop notation to quantify the value of information. For expositional simplicity, we drop the time and firm sub-indexes. The firm's period profits, scaled by the nominal wage, are given by

$$\Pi(p, z) \equiv C(p) \left( p - \frac{1}{z} \right), \quad \text{where } p \equiv \frac{P}{W}, \quad (13)$$

so that real profits are a function of the ratio of the nominal price to the nominal wage,  $p$ , and of productivity,  $z$ . When prices are set conditional on perfect information of the current productivity  $z$  the expected profits are

$$\Pi^*(z) \equiv \max_p \Pi(p, z) = A \left( \frac{1}{z(\eta - 1)} \right)^{1-\eta} \eta^{-\eta}, \quad \text{with maximizer } p^*(z) = \frac{\eta}{\eta - 1} \frac{1}{z}$$

where  $p^*(z)$  denotes the instantaneous profit maximizing price as a function of productivity. We notice that  $p^*(z)$  is the standard markup pricing of monopolistic competition.

To highlight the role of imperfect information, we compare the  $T$ -periods ahead expected profits under perfect information with the expected profits under imperfect information. Let  $L(\cdot; T|z)$  denote the CDF of  $T$ -periods ahead productivity conditional on the current productivity,  $z$ . Given the assumptions on the process for  $z$ ,  $L(\cdot; T|z)$  is a log-normal distribution. In the case of perfect information, the  $T$ -periods ahead expected profit evaluated conditional on current productivity  $z$  and the optimal pricing function  $p^*(z')$ , is given by

$$\int_0^\infty \Pi^*(z') dL(z'; T | z) = \Pi^*(z) e^{bT}, \quad \text{where } b \equiv (\eta - 1)(\phi - \sigma^2/2) + \eta(\eta - 1)\sigma^2/2. \quad (14)$$

Alternatively, in the case of imperfect information the expected profits  $T$  periods ahead, conditional on current productivity  $z$ , is a function of the best estimate of productivity  $z'$ :

$$\hat{\Pi}(T, z) \equiv \max_p \int_0^\infty \Pi(p, z') dL(z'; T | z), \quad (15)$$

with maximizer  $\hat{p}(T, z)$  given by

$$\hat{p}(T, z) = \frac{\eta}{\eta - 1} \int_0^\infty \frac{1}{z'} dL(z'; T | z) = \frac{\eta}{\eta - 1} \frac{1}{z} e^{(\sigma^2/2 - \phi)T}. \quad (16)$$

The optimal price  $\hat{p}(T, z)$  is thus given by the markup  $\eta/(\eta - 1)$  times the  $T$  periods ahead

expected value of the marginal cost  $1/z'$  (normalized by wages), conditional on the last observed productivity  $z$ . We notice that  $\hat{p}(0, z) = p^*(z)$ . Equation (16) illustrates the workings of price plans: at each time  $T$  following the last observation, the firm charges the price that maximizes expected profits. The nominal price of the firm grows at the rate  $\mu + \sigma^2/2 - \phi$ , where  $\mu$  is the component reflecting nominal wage growth (recall that we are using  $p = P/W$  as the control variable), and the  $\sigma^2/2 - \phi$  reflects the expected growth of productivity. Thus, if  $\mu - \phi + \sigma^2/2 \neq 0$  then the optimal price plan implies that prices change continuously. On the other hand, if the level of the nominal marginal cost is a martingale, i.e.  $\mu - \phi + \sigma^2/2 = 0$ , then the optimal price plan features a constant nominal price between consecutive observations.

Using equation (15) into equation (13) we can write  $\hat{\Pi}(T, z)$  as

$$\hat{\Pi}(T, z) = \Pi^*(z) e^{aT}, \quad \text{where } a \equiv (\eta - 1)(\phi - \sigma^2/2). \quad (17)$$

The two expected profits in equation (14) and equation (17) are taken with respect to the same  $T$ -periods horizon. The difference between the two expressions lies in the information used to set prices. When prices are set based upon  $T$ -periods-old information the expected profits grow at the rate  $a$ . Instead when prices are set with complete information the expected profits grow at the rate  $b > a$ . Hence  $B \equiv b - a = \eta(\eta - 1)\sigma^2/2$  captures the rate at which the benefit of acquiring information increases as a function of the time elapsed since the last observation.<sup>14</sup> Notice that the value of information equals (one half) the curvature in the profit function,  $\eta(\eta - 1)$ , times the incremental uncertainty  $\sigma^2$ . Intuitively, a higher the volatility of productivity innovations ( $\sigma$ ) lowers the information content of past observations, increasing the value of information. Likewise, a higher demand elasticity ( $\eta$ ) boosts the impact of a given error in pricing on the firm's profits, thus increasing the value of information.

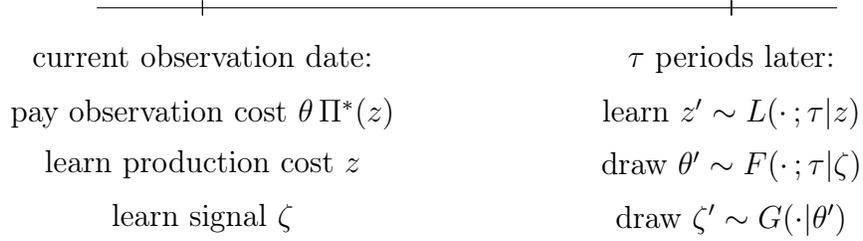
**The cost of gathering information.** We assume that acquiring information about the level of the production cost  $1/z$  requires the payment of an observation cost. The economic interpretation of the observation cost captures the time cost of decision making in firms when gathering and aggregating information as well as the physical cost of acquiring the information needed to make the price decision (e.g. Zbaracki et al. (2004), Reis (2006)). The firm incurs an observation cost any time it acquires information about the state. Upon payment of the observation cost, the firm learns the current value of  $z$ .<sup>15</sup> No information on the state arrives

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<sup>14</sup>Abel, Eberly, and Panageas (2007) also solve for optimal observation times in a framework where the benefit of doing an observation increases at a constant rate as a function of time elapsed.

<sup>15</sup>Notice the difference with the rational inattention literature that developed after Sims (2003) where there is a constraint on the amount of information that can be processed per period, and agents typically process a limited amount of information every period.

Figure 1: Time line



between observation dates. We assume that the observation cost is equal to  $\theta \Pi^*(z)$ , so that it is proportional to the value of the static monopolist profit under complete information. This assumption serves two purposes. First, this assumption reduces the technical difficulty of the problem. The two expected profits in [equation \(14\)](#) and [equation \(17\)](#) show that the benefits of information are proportional to  $\Pi^*(z)$ . Making the costs scale similarly leaves the entire firm's problem homogeneous in  $\Pi^*(z)$ , and so simplifying the analysis. Second, when  $z$  is persistent, this assumption is also necessary to produce i.i.d. times between observations as we will do in [Section 5](#).

We will call  $\theta$  the observation cost. We allow for  $\theta$  to be random, with a finite lower bound  $\underline{\theta}$ . The processes of the observation cost  $\theta$  and of the production cost  $z$  are assumed to be independent. Immediately after paying the observation cost, the firm not only learns the current value of  $z$ , but also receives a signal  $\zeta \in [\underline{\zeta}, \infty)$  which is informative about the future realizations of  $\theta$ . In particular, the signal  $\zeta$  summarizes all the information about the value of the observation cost to be paid  $\tau$  periods from now, for any  $\tau \geq 0$ . Mathematically we write  $F(\theta'; \tau|\zeta)$  to be the CDF of the observation cost  $\theta' \in [\underline{\theta}, \infty)$  to be paid  $\tau$  periods after the current observation, conditional on the signal  $\zeta$ . The dependence of the distribution  $F$  on  $\tau$  allows the distribution of the observation cost  $\theta'$  to vary with the time elapsed between observations. In practice, we will work with a specification where the signal  $\zeta$  is proportional to the expected cost of the next observation for any given  $\tau$ , i.e.  $E[\theta'; \tau|\zeta] \propto \zeta$ . Upon the next observation, when a particular cost  $\theta'$  is realized, a new signal  $\zeta' \in [\underline{\zeta}, \infty)$  is drawn from the CDF  $G(\cdot|\theta')$ . The timeline in [Figure 1](#) describes the structure of the observation cost  $\theta$ , the associated signal  $\zeta$  and production cost  $z$ , for an observation date that occurs  $\tau$  periods after the current observation.

The functions  $F$  and  $G$  fully characterize the process for the observation cost, and provide enough flexibility to cover cases discussed in the literature as well as generalizations that we find useful. For instance, the model of deterministic observation times studied by [Caballero](#)

(1989) and Reis (2006) is encompassed by our framework if the signal is uninformative about the future observation cost, which is the case if  $F(\theta', \tau_0 | \zeta_0) = F(\theta', \tau_1 | \zeta_1)$  for all  $\theta'$  and all pairs  $(\tau_0, \zeta_0)$ . In this case, the distribution  $G$  is irrelevant because, given that the signal is uninformative, the mechanism to obtain the new signal is irrelevant.<sup>16</sup>

Another case discussed in the literature is one where the firm's observation times are i.i.d., as proposed by Reis (2006). Our model provides a foundation to i.i.d. observation times: the firm has to draw a signal about the future observation cost that is both informative about the next observation cost and independent of all other shocks (including the current value of the observation cost). In this case, the particular form of the distribution  $G$  is relevant. Formally, observation times are i.i.d. in our model if and only if  $G(\zeta' | \theta_0) = G(\zeta' | \theta_1)$  for all  $\zeta'$  and all pairs  $\theta_1, \theta_0$ . The distribution  $F$  shapes the precision of the signal. Finally, the more general case where  $G(\zeta' | \theta_0) \neq G(\zeta' | \theta_1)$  for at least some  $\zeta'$  and some pairs  $\theta_1 \neq \theta_0$  allows us to extend our analysis to the case of observation times correlated over time, a case which we find more reasonable than the i.i.d. assumption.

**The firm's Bellman equation.** Without loss of generality we consider the problem of the firm right after it has paid the observation cost, when it is deciding the length of the time until the next observation which we denote as  $\tau$ . The state of the firm at this date consists of both the signal  $\zeta$  and the productivity  $z$ . The value of the firm  $V(\zeta, z)$ , i.e. the expected discounted value of the profits net of the observation cost, solves the following recursive problem

$$V(\zeta, z) = \max_{\tau \in \mathbb{R}_+} \int_0^\tau e^{-(\rho+\lambda)s} \hat{\Pi}(s, z) ds + \tag{18}$$

$$+ e^{-(\rho+\lambda)\tau} \int_{-\infty}^\infty \int_{\underline{\theta}}^\infty \left( -\theta' \Pi^*(z') + \int_{\underline{\zeta}}^\infty V(\zeta', z') dG(\zeta' | \theta') \right) dF(\theta' ; \tau | \zeta) dL(z' ; \tau | z),$$

where real profits are discounted at rate  $\rho + \lambda$ , with  $\rho$  capturing the agent's preferences discount rate in the set-up of interest (described in the appendix of Alvarez and Lippi (2014)), and  $\lambda$  is due to the substitution of products. The firm chooses the time  $\tau \geq 0$  between the current and the next observation, where  $\tau = \infty$  represents the case of never observing. The first term on the right-hand side of equation (18) reflects the expectations of cumulated profits until the next observation. Equations (14)-(17) imply that the s-periods ahead expected profits,  $\hat{\Pi}(s, z)$ , scale with  $\Pi^*(z)$ , i.e.  $E[\theta_\tau \Pi^*(z_\tau) | \zeta_0, z_0] = E[\theta_\tau | \zeta_0] e^{b\tau} \Pi^*(z_0)$ . The second term on the right-hand side of equation (18) refers to the discounted expectation of the

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<sup>16</sup>In Caballero (1989) and Reis (2006) the cost is fixed, which amounts to a degenerate  $F$ . Our setup thus delivers a small generalization: absent an informative signal, observation times will be constant even if  $F$  is not degenerate.

continuation value upon the next observation,  $V(\zeta', z')$ , conditional on the next observation date taking place  $\tau$  periods from now, the current productivity being  $z$ , and the current signal being  $\zeta$ . The integrals on the right-hand side of [equation \(18\)](#) reflect the expectation of  $V(\zeta', z')$  with respect to the possible realizations of  $z'$  and  $\zeta'$  (which also depend on the realization of  $\theta'$ ).

We use that both expected profits and observation cost scale with  $\Pi^*(z)$ , together with the independence of  $z'$  and  $\theta'$ , to express the value function as  $V(\zeta, z) = v(\zeta) \Pi^*(z)$ , where the function  $v(\zeta)$  solves the following simpler recursive problem:

$$v(\zeta) = \max_{\tau \in \mathbb{R}_+} \int_0^\tau e^{-(\rho+\lambda-a)t} dt + e^{-(\rho+\lambda-b)\tau} \int_{\underline{\theta}}^\infty \left( \int_{\underline{\zeta}}^\infty v(\zeta') dG(\zeta' | \theta') - \theta' \right) dF(\theta'; \tau | \zeta) . \quad (19)$$

We denote by the function  $\tau(\zeta)$  the optimal policy that solves the firm's problem. Notice that this problem has relatively few parameters. The functions  $v(\cdot)$  and  $\tau(\cdot)$  depend only on three scalars ( $a$ ,  $b$  and  $\rho + \lambda$ ), and two functions ( $G$  and  $F$ ). The first term on the right-hand side of [equation \(19\)](#) contains the value of expected profits using only current information. The second term contains the value of expected profits after a new observation, as well as the expected future observation cost. The optimal choice of  $\tau$  balances the value of information (summarized by  $b > a$ ) against the expected cost of a new observation  $\theta'$ . An immediate result of [equation \(19\)](#) is that the optimal time between observations only depends on the realization of the signal  $\zeta$ , and not on productivity  $z$ . Finally, in order to have a finite value of a firm in the frictionless case, we assume that  $\lambda + \rho > b$ . Hence the value function is bounded below and above as follows:

$$0 < \underline{v} \equiv \frac{1}{\rho + \lambda - a} \leq v(\zeta) \leq \bar{v} \equiv \frac{1}{\rho + \lambda - b} < \infty \text{ for all } \theta \in (0, +\infty),$$

where the lower bound is obtained by setting  $\tau = \infty$ , while the upper bound is obtained in the perfect information case when observation is always costless, and the firm observes continuously.

**Obtaining  $H(t|t)$ .** We can use the solution to the firm's problem  $\tau(\zeta)$ , and the properties of the process governing the signal  $\zeta$ , to derive the conditional right-CDF for the times between consecutive observations, which in [Section 2](#) was denoted by  $H(\cdot|t)$ . In particular, assume that  $\tau(\cdot)$  is strictly increasing in  $\zeta$ , a property which will be satisfied in our applications, and let  $\zeta = \hat{\zeta}(t)$  denote the mapping from an observation of length  $t$  to the associated signal  $\zeta$ , where  $\hat{\zeta}(t) = \tau^{-1}(t)$  is the inverse function associated to the optimal policy  $\tau(\cdot)$ . For

simplicity, consider the case with no substitutions, i.e.  $\lambda = 0$ .<sup>17</sup> In this case, the distribution  $H$  is given by  $H(t' | t) = \int_{\underline{\theta}}^{\infty} (1 - G(\hat{\zeta}(t') | \theta')) dF(\theta'; t | \hat{\zeta}(t))$ , which equals the probability of drawing a signal  $\zeta' \geq \hat{\zeta}(t')$  at the next observation, conditional on having drawn a signal  $\zeta = \hat{\zeta}(t)$  at the last observation, which occurred  $t$  periods ago. This requires integrating over all possible realizations of the future observation cost  $\theta'$ , since the realization of  $\theta'$  determines the distribution from which the signal  $\zeta'$  will be drawn upon the next observation.

### 3.1 The case of constant optimal observation times

This subsection presents a specification of the observation cost process which implies that the optimal (consecutive) observation times have equal duration. This case serves as a useful benchmark, and is a limiting case, of the more general case of random observations that will be considered in the next sections. In particular, we assume that signals are uninformative about future observation costs, so that firms face the same problem upon every observation and choose the same time to the next observation. We assume that  $F(\theta, \tau_0 | \zeta_0) = F(\theta, \tau_1 | \zeta_1) \equiv \hat{F}(\theta)$  for all  $\theta$  and all pairs  $(\tau_0, \zeta_0)$  and  $(\tau_1, \zeta_1)$  and that  $G$  is arbitrary. The assumption on  $F$  means that the signal is uninformative about future observation costs. Note that the signal could be uninformative and yet  $\hat{F}(\theta)$  does not need to be degenerate. The function  $G$  can be left unspecified because, given that the signal is uninformative, the mechanism to obtain the new signal is irrelevant. Using these assumptions on  $F$  in [equation \(19\)](#), one obtains that  $\tau(\zeta) = \hat{\tau}$  and  $v(\zeta) = \hat{v}$ , so neither function depends on  $\zeta$ . In particular, letting the expected observation cost  $E[\theta] = \int_0^{\infty} \theta d\hat{F}(\theta) > 0$ , the Bellman equation becomes:

$$\hat{v} = \max_{\tau \in \mathbb{R}_+} \int_0^{\tau} e^{-(\rho+\lambda-a)t} dt + e^{-(\rho+\lambda-b)\tau} (\hat{v} - E[\theta]) . \quad (20)$$

The analysis that follows characterizes the comparative statics of the optimal decision rule, summarized by the constant  $\hat{\tau}$ , as a function of the model parameter  $E[\theta]$ , i.e. the expected value of the observation cost. To do so, as well as for future use, we define the function  $\tilde{\tau}(\cdot)$  so that the optimal policy is  $\hat{\tau} = \tilde{\tau}(E[\theta])$ . The function  $\tilde{\tau}(\cdot)$  is

$$\tilde{\tau}(x) = \begin{cases} \frac{1}{B} [\ln(\bar{v}) - \ln(\tilde{v}(x) - x)] & \text{if } 0 < x < \underline{v} \\ \infty & \text{if } x \geq \underline{v} \end{cases} , \quad (21)$$

$$\text{where } \tilde{v}(x) = \begin{cases} \text{smallest solution of } \tilde{v} = \underline{v} + B \underline{v} \bar{v} \left(\frac{\tilde{v}-x}{\bar{v}}\right)^{\frac{1}{B\underline{v}}} & \text{if } 0 < x < \underline{v} \\ \underline{v} & \text{if } x \geq \underline{v} \end{cases} , \quad (22)$$

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<sup>17</sup>The case with substitutions follows a similar logic, but is algebraically more involved (see [Appendix B](#)).

and  $B \equiv b - a = \eta(\eta - 1)\sigma^2/2$  summarizes the value of information. The following proposition collects these results (see [Appendix A](#) for the proof):

**PROPOSITION 3.** Assume that signals are not informative, i.e.  $F(\theta, \tau_0 | \zeta_0) = F(\theta, \tau_1 | \zeta_1) \equiv \hat{F}(\theta)$  for all  $\theta$  and all pairs  $(\tau_0, \zeta_0)$  and  $(\tau_1, \zeta_1)$ , and  $E[\theta] > 0$ . The optimal observation time is given by  $\hat{\tau} = \hat{\tau}(E[\theta])$ , which is strictly increasing in  $E[\theta]$  for  $0 < E[\theta] < \underline{v}$ , and infinite for  $E[\theta] \geq \underline{v}$ .

The optimal time between observations is increasing in the expected observation cost: the larger the expected observation cost, the more time must elapse before the benefit of an observation is large enough to induce the firm to pay the observation cost. As the expected value of the observation costs becomes small, formally as  $E[\theta] \downarrow 0$ , the optimal observation time is approximated by the square root function  $\hat{\tau} \approx \sqrt{2E[\theta]/B}$ . This limiting case in which  $E[\theta] \downarrow 0$  reproduces [Reis's \(2006\)](#) in Proposition 5, where a perturbation method was used to derive an approximation for  $\hat{\tau}$ . [Proposition 3](#) generalizes Reis's results by characterizing the solution away from the case where  $E[\theta]$  is small. These results will be useful in the next section to characterize the determinants of the shape of the optimal observation times in general as a function of the expected value of the observation cost shocks and of their persistence.

## 4 Optimal Markov observation times

The results of [Proposition 2](#) imply that the case of constant observation times studied in the literature and discussed in the previous section predicts the smallest real effects of monetary shocks for given average time between observations  $E_K(\tau) = \hat{\tau}$ , as the associated coefficient of variation of observations is  $CV_K(\tau) = 0$ . Studying the underpinnings of variability in observation times is important because such variability increases the size of the real effects of monetary shocks. This section presents a specification of the process for the observation cost implying that optimal observation times form a first order Markov process. The solution to the firm's problem will provide a mapping from the realization of the observation cost to the optimal time between consecutive observations,  $\tau$ .

We will use this mapping to obtain a relationship from the coefficient of variation of observation costs,  $CV(\theta)$ , to the coefficient of variation of observation times,  $CV_K(\tau)$ . This mapping is useful to appreciate the ability of the model to generate a given variation in observation times, for a given variation of the observation costs.

## 4.1 Solving for the observation time with a specific cost distribution

We assume that the observation cost  $\theta \geq 0$  follows a continuous time process, and the firm learns about the current value of  $\theta$  only upon the time it observes. The observation cost is modeled in a way that parallels the production cost. Both of them follow a continuous time stochastic process whose distributions are independent of whether and when the firm chooses to observe them.<sup>18</sup> A special case of the process for the observation cost is the one where the observation cost is constant until a new value is drawn from the distribution  $\hat{F}(\cdot)$  with support on  $(0, \infty)$ . The cumulative distribution in this case is given by

$$F(\theta', \tau | \theta) = e^{-\frac{\tau}{\kappa}} \mathbf{1}_{\theta' \geq \theta} + (1 - e^{-\frac{\tau}{\kappa}}) \hat{F}(\theta') \quad , \quad \kappa < \infty . \quad (23)$$

The arrival rate of the new value is given by the constant  $1/\kappa$  (i.e. it is a Poisson process), and it is independent of the current value of  $\theta$ . We will refer to  $\kappa$  as the parameter governing the persistence of observation cost, with a higher  $\kappa$  associated with a higher persistence. We assume that the current realization of the observation cost is the best and only predictor of future realizations of the observation cost. Formally we assume that  $\zeta = \theta$ , implying  $G(\zeta | \theta) = \mathbf{1}_{\zeta \geq \theta}$ . Without loss of generality we can replace  $\zeta$  with  $\theta$  as the argument of the firm policy and value function, i.e.  $\tau(\theta)$  and  $v(\theta)$ .

Given the process for  $\theta$  implied by [equation \(23\)](#), we can rewrite the Bellman equation in [equation \(19\)](#) as:

$$v(\theta) = \max_{\tau \in \mathbb{R}_+} \int_0^\tau e^{-(\rho+\lambda-a)t} dt + e^{-(\rho+\lambda-b)\tau} \left( e^{-\frac{\tau}{\kappa}} (v(\theta) - \zeta) + (1 - e^{-\frac{\tau}{\kappa}}) (E[v] - E[\theta]) \right) , \quad (24)$$

where  $E[v] \equiv \int_0^\infty v(\theta) d\hat{F}(\theta)$  and  $E[\theta] \equiv \int_0^\infty \theta d\hat{F}(\theta)$ . The next two propositions characterize the shape of the decision rule  $\tau(\theta)$  that solves the problem in [equation \(24\)](#): the general message is that the shape crucially depends on two parameters, the expected value,  $E(\theta)$ , and the persistence,  $\kappa$ , of the observation cost. As shown by [Reis's \(2006\)](#) proposition 4 (on the optimal decision rule around a situation when planning is costless), it will be true that as the expected value of observation costs becomes tiny, i.e. as  $E(\theta) \rightarrow 0$ , the decision rule,  $\tau(\theta)$ , is a square root formula for any given level of persistence  $\kappa$ . The next proposition extends that characterization around small values of the cost realization  $\theta$ . For instance, it shows that, for any given expected value of the costs,  $E(\theta) > 0$ , there will be a threshold for the persistence  $\kappa$  such that, if the persistence is low enough, unlike a square root formula,

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<sup>18</sup>The assumption that the evolution of the observation costs is independent of the decision to observe will need to be abandoned to generate i.i.d. observation times, as discussed in [Section 5](#).

the firm chooses a strictly positive time until the next observation even if the observation cost is arbitrarily small:

PROPOSITION 4. Assume that  $E[\theta] < \underline{v}$ , then,

- (i) for any  $0 < \kappa < \infty$  we have  $\tau(\theta) > 0$  and  $\tau'(\theta) > 0$  if  $\theta > 0$ , with  $\lim_{\theta \rightarrow \infty} \tau(\theta) = \infty$  and  $\lim_{\theta \rightarrow \infty} \tau'(\theta) = 0$ ;
- (ii) let  $\underline{\kappa} \equiv E[\theta]/(B\underline{v}) < \bar{\kappa}$ , then for all  $\kappa < \underline{\kappa}$  we have  $\lim_{\theta \rightarrow 0^+} \tau(\theta) > 0$ ;
- (iii) let  $\bar{\kappa} \equiv \underline{v} + E[\theta]/(\bar{v}B)$ , then for all  $\kappa \geq \bar{\kappa}$  we have  $\lim_{\theta \rightarrow 0^+} \tau(\theta) = 0$  and  $\lim_{\theta \rightarrow 0^+} \tau'(\theta) = +\infty$ .

The proposition holds for a sufficiently small level of the expected future costs, i.e. for  $E[\theta] < \underline{v}$  (see [Appendix A](#) for the proof). If this condition is not satisfied, the optimal time between observations diverges: the expected cost of the next observation is so high that the firm is better off not observing. [Figure 2](#) illustrates the results of our proposition through numerical examples that reproduce the average frequency and size of price changes estimated by [Nakamura and Steinsson \(2008\)](#). In these examples, and in the ones that will follow later in the paper, we will concentrate on the case in which the nominal marginal cost is a martingale (i.e.  $\mu - \phi + \sigma^2/2 = 0$ ) because, as it is well documented, prices change infrequently.<sup>19</sup>

The first point of the proposition states that the time between consecutive observations is strictly positive provided that  $\theta > 0$ , since the firm would face unbounded losses in a finite period of time if it were to observe the state continuously. Moreover, point (i) also states that the optimal observation time is strictly increasing in the current value of the observation cost  $\theta$ , since a higher  $\theta$  signals a higher expected value of the future observation cost.<sup>20</sup>

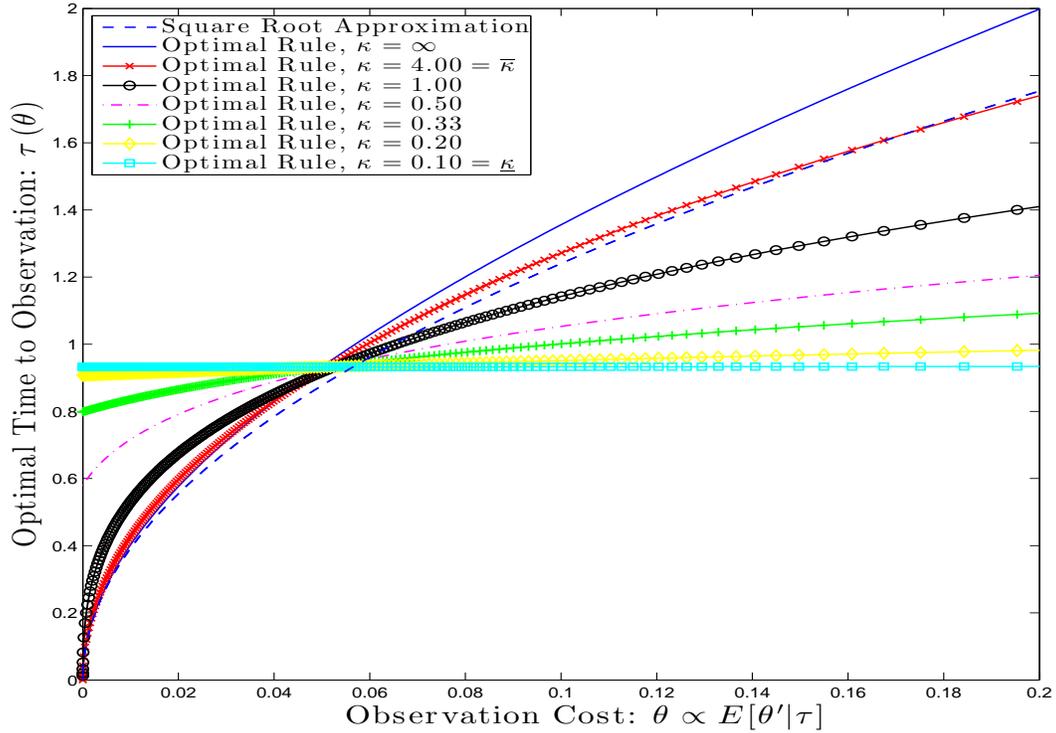
The next two points of the proposition are crucial to see how the shape of the optimal rule at small values of  $\theta$  depends on the persistence of the cost shocks,  $\kappa$ . Point (ii) states a sufficient condition on the persistence parameter  $\kappa$  such that if the persistence is low enough (i.e.  $\kappa < \underline{\kappa}$ ) then the optimal time between observations converges to a strictly positive value as the observation cost converges to zero. Intuitively, as the persistence of observation costs decreases, the current observation cost becomes a bad predictor of future observation costs. When the persistence is sufficiently low, such expectation will be dominated by the

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<sup>19</sup>This is a meaningful benchmark also because if one were to add a separate cost of adjusting prices, the optimal policy would indeed feature constant prices between consecutive observations as long as the drift in the marginal cost is not too large (see [Alvarez, Lippi, and Paciello \(2011\)](#) for a rigorous characterization).

<sup>20</sup>This can be seen noting that  $E[\theta'; \tau|\theta] = e^{-\frac{\tau}{\kappa}}\theta + (1 - e^{-\frac{\tau}{\kappa}})E[\theta]$  increases with  $\theta$  for any time  $\tau$ .

Figure 2: The optimal observation times rule  $\tau(\theta)$ , as persistence  $\kappa$  varies



Note: The figure reports the optimal time to the next observation  $\tau(\theta)$ , in years, as a function of the signal value  $\zeta = \theta$  on the horizontal axis, and for different values of persistence, as measured by  $\kappa$ . The distribution  $\hat{F}$  is assumed to be exponential with average observation cost  $E[\theta] = 0.05$ . Depending on persistence  $\kappa$ , this cost implies between 1.1 and 1.4 observations per year on average. We set the drift  $\mu - \phi + \sigma^2/2 = 0$  to obtain infrequent price changes, and  $\sigma = 0.114$  to match an average size of price changes equal to 10%. The other parameters are  $\eta = 5$ ,  $\lambda = 0.25$  and  $\rho = 0.02$ . For more details see [Appendix C](#). The dashed line plots the function  $\sqrt{2\theta/B}$ .

unconditional mean, i.e.  $E[\theta] > 0$ , to the point that even when the observation cost is tiny the firm will still choose to wait for a strictly positive time before making another observation.

A different rule obtains if the persistence parameter  $\kappa$  is sufficiently high (i.e.  $\kappa \geq \bar{\kappa}$ ), a condition described under point (iii). In this case the optimal time between observations converges to zero as the observation cost converges to zero (with an infinite slope at zero). The reason is that when persistence is high the current observation cost is a good predictor of the future cost, thus the decision rule becomes responsive to the current cost.

## 4.2 Two extreme cases

To further illustrate the key role of the persistence of observation costs for optimal observation times, the next proposition considers two extreme cases for  $\kappa$ : permanent observation cost, i.e.  $\kappa \rightarrow \infty$ ; and i.i.d. observation cost, i.e.  $\kappa \rightarrow 0$  (see [Appendix A](#) for the proof). This proposition is informative about the shape of the decision rule, not just at small values of  $\theta$  as done before, showing that it converges pointwise to the decision rule in the problem with constant time between observations when  $\kappa \rightarrow \infty$ .

**PROPOSITION 5.** Assume that  $E[\theta] < \underline{v}$ , and let  $\tau(\theta; \kappa)$  and  $v(\theta; \kappa)$  denote, respectively, the solution to the firm problem and associated value function in [equation \(24\)](#) conditional on a given persistence parameter  $\kappa$ . Then we have that:

- (i) for any  $\theta \geq 0$ ,  $\lim_{\kappa \rightarrow \infty} \tau(\theta; \kappa) = \tilde{\tau}(\theta)$  and  $\lim_{\kappa \rightarrow \infty} v(\theta; \kappa) = \tilde{v}(\theta)$  where the functions  $\tilde{\tau}(\cdot)$  and  $\tilde{v}(\cdot)$  are given by [equations \(22\)-\(21\)](#);
- (ii) for any  $\theta > 0$ ,  $\lim_{\kappa \rightarrow 0} \frac{\partial \tau(\theta; \kappa)}{\partial \theta} = \frac{\partial v(\theta; \kappa)}{\partial \theta} = 0$ .

Point (i) shows that when the observation cost is permanent, i.e.  $\kappa \rightarrow \infty$ , the optimal observation time is given by the function  $\tilde{\tau}(\theta)$ , in [equation \(21\)](#). One can use this expression to show that the optimal policy is approximated by  $\tilde{\tau}(\theta) = \sqrt{2\theta/B} + o(\theta)$ , implying an elasticity of observation times to observation costs of 1/2 in a neighborhood of  $\theta = 0$ . Although the functional form  $\tilde{\tau}(\theta)$  is the same as in [Section 3.1](#), it is important to notice that the resulting dynamics are very different. Because  $\theta$  is Markov here, the conditional mean varies across firms, and hence the time between observations is not equal across firms. In the previous example, the expected value of  $\theta$  was constant both over time and across firms.

Point (ii) shows that as the cost becomes i.i.d. (i.e.  $\kappa \rightarrow 0$ ) the optimal observation time is insensitive to the realization of the current observation cost and converges to a constant, just as in [Section 3.1](#). Intuitively, when  $\kappa \rightarrow 0$  the current observation cost is uninformative about the future observation cost, while its unconditional expectation  $E[\theta]$  is the main determinant

of the optimal time to the next observation and the rule becomes flat. In [Figure 2](#) we illustrate the effects of the persistence parameter  $\kappa$  on the shape of the decision rule  $\tau(\theta)$  at parameters of the model that imply an average frequency of price changes of about 1.3 adjustments/observations per year, and an average size of price changes equal to 10% (see [Appendix C](#) for more details on the parameters choice). It is apparent that the decision rule has a square-root like shape only when the persistence of the cost shocks is sufficiently high. Otherwise the decision rule is flat and unresponsive to changes in the costs.

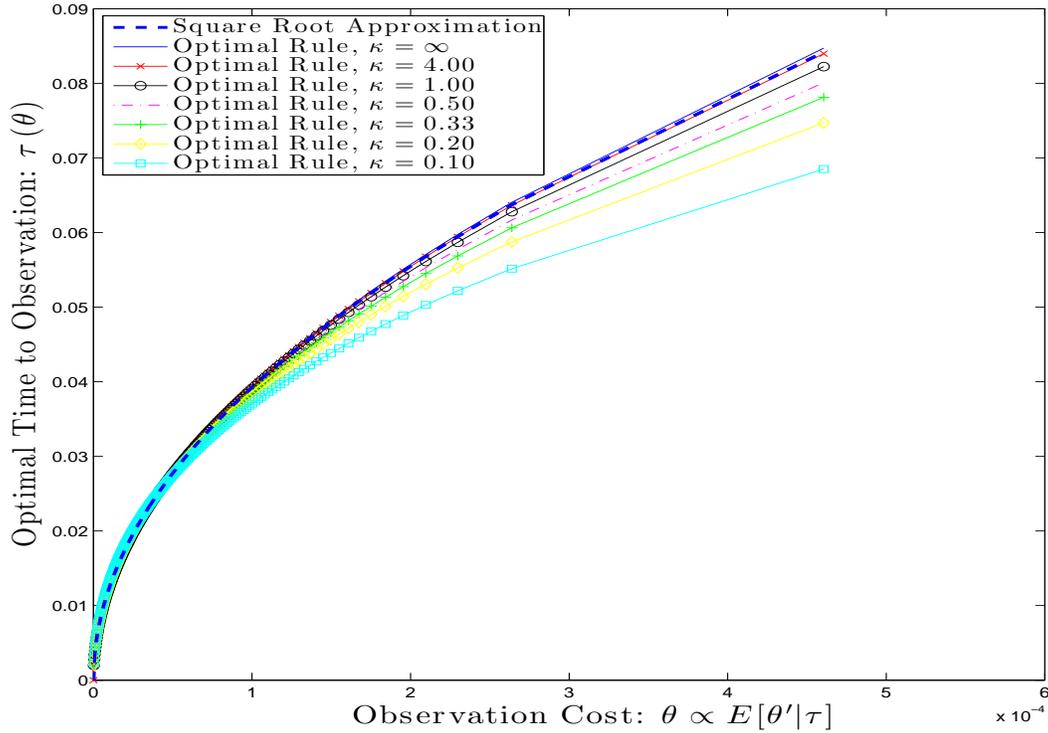
### 4.3 The scale of observation cost and the optimal observation times

The results of [Proposition 4](#) and [Proposition 5](#) relate to the findings of [Reis's \(2006\)](#) Proposition 4, which shows that the optimal time between observations is (in our notation) approximately equal to  $\tau(\theta) \approx \sqrt{2\theta/B}$  for  $E[\theta] \rightarrow 0$ . We have extended that characterization to a more general case when  $E[\theta]$  is not necessarily close to zero, and show how the shape of the optimal observation times depends both on the average observation cost  $E[\theta]$ , and on the persistence of the observation cost,  $\kappa$ . The extension to non negligible  $E[\theta]$  is important: in the case of negligible costs (i.e. when  $E[\theta] \rightarrow 0$ , a situation Reis refers to as “costless planning”) the mean frequency of observation will be very high (diverging in the limit), so that the economy becomes one with flexible prices. A new and important result of this extension is to show that, when the costs are non-negligible, the optimal firm behavior is well approximated by a square root formula only if the observation costs are sufficiently persistent. Notice in particular that our propositions imply that, as the realizations of the observation cost converge to zero ( $\theta \downarrow 0$ ), the optimal observation time will not converge to zero if the persistence is low enough ( $\kappa < \underline{\kappa}$ ), and that  $\tau(\theta)$  will converge to a constant as the observation cost process converges to i.i.d. ( $\kappa \rightarrow 0$ ).<sup>21</sup> Our results highlight the crucial role of  $E[\theta]$  for the shape of  $\tau(\theta)$ : the thresholds  $\underline{\kappa}$  and  $\bar{\kappa}$  are increasing in  $E[\theta]$ . To highlight the role of the scale of observation costs, in [Figure 3](#) we plot the optimal  $\tau(\theta)$  when, everything else being equal, we scale the distribution of observation costs  $\hat{F}$  so that  $E[\theta]$  is 1/1000 smaller than in the case of [Figure 2](#). It is shown that the optimal time to the next observation is indeed well approximated by the square root expression, even at very low levels of persistence. However, the negligible observation costs used in the case of [Figure 3](#) imply a sizeable increase in the frequency of information gathering, from about 1.3 observation per year (in the baseline case) to about 40 observations per year. In such a case, the quantitative behavior of the model is close to flexible prices.

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<sup>21</sup>It is immediate to see the difference with the square root approximation where the observation time converges to zero with an elasticity of 1/2 as the observation cost  $\theta$  converges to zero.

Figure 3: The optimal observation times as all costs gets negligible:  $E[\theta] \rightarrow 0$



Note: The figure reports the optimal time to the next observation  $\tau(\zeta)$ , in years, as a function of the signal value  $\zeta$  on the horizontal axis, and for different values of persistence, as measured by  $\kappa$ . The distribution  $\hat{F}$  is assumed to be exponential with average observation cost  $E[\theta] = 0.05/1000$ : such negligible costs implies about 40 observations per year on average. We set the drift  $\mu - \phi + \sigma^2/2 = 0$  to obtain infrequent price changes, and  $\sigma = 0.114$  to match an average size of price changes equal to 10%. The other parameters are  $\eta = 5$ ,  $\lambda = 0.25$  and  $\rho = 0.02$ . For more details see [Appendix C](#).

## 4.4 The variation of costs and observation times

We conclude from this analysis that combinations of low average observation costs and / or a high persistence of the observation cost are associated with a higher elasticity of observation times with respect to the observation costs. This result is useful to determine the mapping from the variation of observation costs to the variation of observation times. The next proposition uses these results to characterize the coefficient of variation of observation times (excluding substitutions) in terms of the coefficient of variation of the observation cost (see [Appendix A](#) for the proof).

**PROPOSITION 6.** Let  $E(\theta) \rightarrow 0$  and the coefficient of variation of observation cost be such that  $CV(\theta) > 0$  as  $E(\theta) \rightarrow 0$ . Let  $CV_K(\tau)$  denote the coefficient of variation of the times between consecutive observations conditional on no substitutions. We have that

$$CV_K(\tau) = \frac{1}{2} CV(\theta) + o\left(\sqrt{E(\theta)}\right) . \quad (25)$$

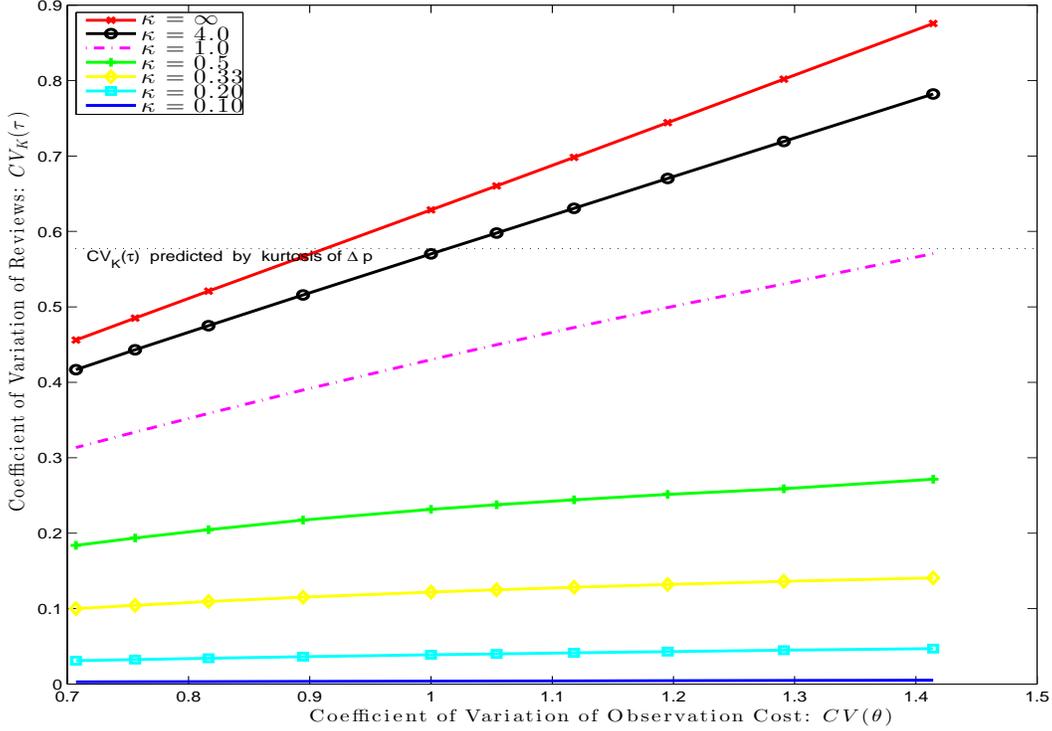
An increase in the coefficient of variation of observation costs affects the cross-sectional variation of observation times with a coefficient of 1/2. The result follows from using that, as shown by [Reis's \(2006\)](#) Proposition 4, the optimal policy is approximated by  $\tilde{\tau}(\theta) \approx \sqrt{2\theta/B}$  as  $E[\theta] \rightarrow 0$ , so that the elasticity of the time between observations to the observation cost is 1/2. However, as we showed above, the elasticity of observation times to observation cost may be much lower when  $E[\theta]$  is away from zero. In order to study the mapping from the variation in observation cost to the variation in observation times in the more general case when  $E[\theta]$  is away from zero. We assume that the distribution  $\hat{F}$  is a Gamma distribution with shape parameter  $\alpha_1$  and scale parameter  $\alpha_2$ . The parameter  $\alpha_1$  determines the coefficient of variation of the observation cost  $\theta$ . The parameter  $\alpha_2$  is chosen so that the average frequency of observations is 1.3 on a yearly basis. Together with the assumption of no drift in nominal marginal cost, this matches the estimated frequency of price changes in the U.S. data. We notice that targeting  $1/E[\tau] = 1.3$  implies an implicit target for  $E[\theta]$ , an important parameter in the analysis above to determine the elasticity of optimal observation times to observation cost.

In [Figure 4](#) we plot the model-implied coefficient of variation of observation times ( $CV_K(\tau)$  on the vertical axis) as a function of the the coefficient of variation of observation costs, for different values of  $\kappa$ . The figure also plots the level of variation in observation times that is consistent with the kurtosis of the empirical distribution of price changes.<sup>22</sup> This shows

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<sup>22</sup> This uses an estimated value for the kurtosis of price changes around 4, consistent with [Alvarez, Le](#)

Figure 4: From the variation of observation costs to the variation of observation times



Note: The figure reports the coefficient of variation of times between observations (including substitutions) as a function of the coefficient of variation of observation cost, and for different values of persistence, as measured by  $\kappa$ . The distribution  $\hat{F}$  is assumed to be  $Gamma(\alpha_1, \alpha_2)$ , where  $\alpha_1$  determines the coefficient of variation of observation costs upon a new draw, and  $\alpha_2$  is chosen to match an yearly average frequency of observations/adjustments equal to 1.3. We set the drift  $\mu - \phi + \sigma^2/2 = 0$  to obtain infrequent price changes, and  $\sigma = 0.114$  to match an average size of price changes equal to 10%. The other parameters are  $\eta = 5$ ,  $\lambda = 0.25$  and  $\rho = 0.02$ . For more details see [Appendix C](#).

that the model can predict variation in observation times consistent with the distribution of price changes only when persistence in the observation cost is sufficiently large. With high persistence the model predicts a sizable variation in observation times even when the variation in observation cost is moderate.<sup>23</sup>

We conclude this section by stressing that the process of optimal observation times does not inherit the properties of the process of the observation cost. In fact as the process of observation cost converges to i.i.d., i.e.  $\kappa \rightarrow 0$ , the optimal time between observations converges to a constant. The economic motivation is that when  $\kappa \rightarrow 0$  the current observation cost is uninformative about the cost of the next observation, so that the firm plans according to its best estimate which is the expected value  $E[\theta]$ . An immediate implication of these

Bihan, and Lippi (2014), and the results in [Lemma 3](#), which yields the coefficient of variation  $CV_K(\tau) = 0.58$ .

<sup>23</sup>In the case of  $\kappa = \infty$ , the required standard deviation of observation cost is of the same order of magnitude of its mean, i.e.  $CV(\theta) \approx 1$ .

result is that the model with a Markovian stochastic process of observation cost cannot generate i.i.d. observation times, which has been a main focus of the “rational inattentiveness” literature (see Reis (2006), Mankiw and Reis (2002), Carvalho and Schwartzman (2012)). We want to remark that this is not a feature of our particular specification of the distribution function  $F$ , but it is a more general result. In order for observations to be i.i.d. over time, we need an environment where the signal about the future observation costs is *both* informative *and* independent of all other shocks (including the current observation cost). This structure cannot be achieved with an environment where the observation cost follows a stochastic process that evolves independently of the observation decisions of the firm. We will therefore consider a specification of the observation cost process that delivers i.i.d. observations times in the next section.

## 5 Optimal i.i.d. observation times

This section presents a specification of the firm problem where the optimal consecutive observation times are i.i.d. random variables. We are interested in this case mostly because a large literature has investigated the implications of i.i.d. observation times for the real effects of monetary shocks, such as Mankiw and Reis (2002) or more recently Carvalho and Schwartzman (2012). The solution to the firm problem will provide a mapping from the realization of the observation costs to the optimal time between consecutive observations,  $\tau$ . As in the previous section, we will use this mapping to obtain a relationship from the coefficient of variation of observation costs  $CV(\theta)$  to the coefficient of variation of observation times,  $CV_K(\tau)$ , and hence to the size of the real effects of monetary shocks.

As shown in the previous section optimal i.i.d. observation times cannot be produced in an environment where the observation cost follows a stochastic process that evolves independently of the observation decisions of the firm. Therefore in this section we have to assume that the firm receives a signal about the future observation costs that is both informative *and* independent of all other shocks (including the current observation cost). Formally, the observation times are i.i.d. over time if and only if  $G(\zeta' | \theta_0) = G(\zeta' | \theta_1)$  for all  $\zeta'$  and all pairs  $\theta_1, \theta_0$ , and hence  $G(\zeta' | \theta) \equiv \hat{G}(\zeta')$  for all  $\zeta'$  and  $\theta$ . The signal  $\zeta'$  has to be independent of the current realization of the observation cost,  $\theta'$ , otherwise  $\zeta'$  would also be correlated with the previous realization of the signal,  $\zeta$ , violating the requirement that observation times are i.i.d.; this is because  $\theta'$  and  $\zeta$  are correlated as  $\zeta$  is informative about  $\theta'$  by assumption.

The specification of the distribution  $F(\theta', \tau | \zeta)$  determines the predictability of the next observation cost  $\theta'$  conditional on the realization of the signal  $\zeta$ , and on the time elapsed

since the signal was received,  $\tau$ . A simple tractable case is one where such distribution is

$$F(\theta', \tau | \zeta) = e^{-\gamma\tau} \mathbf{1}_{\theta' \geq \zeta} + (1 - e^{-\gamma\tau}) \hat{F}(\theta'). \quad (26)$$

The parameter  $\gamma$  determines the information content of the signal, which is perfectly precise if  $\gamma = 0$ , and has no information as  $\gamma \rightarrow \infty$ . Notice that a higher value of the signal  $\zeta$  is associated with a higher expected observation cost:  $E[\theta' | \zeta, \tau] = e^{-\gamma\tau} \zeta + (1 - e^{-\gamma\tau}) \int \theta' d\hat{F}(\theta')$ . In this case the realization of the signal coincides with the cost of the next observation with a probability that decreases as time elapses, otherwise the future observation cost will be drawn from the distribution  $\hat{F}$  (independent of  $\zeta$ ). This specification of  $F$  captures the idea that, as time passes, the precision of the information about the future observation cost depreciates, while preserving tractability.

Given [equation \(26\)](#) and the assumption  $G(\zeta' | \theta) \equiv \hat{G}(\zeta')$  for all  $\zeta'$  and  $\theta$ , we can rewrite the firm problem in [equation \(19\)](#) as:

$$v(\zeta) = \max_{\tau \in \mathbb{R}_+} \int_0^\tau e^{-(\rho+\lambda-a)t} dt + e^{-(\rho+\lambda-b)\tau} \left( E[v] - \left( e^{-\gamma\tau} \zeta + (1 - e^{-\gamma\tau}) E[\theta] \right) \right), \quad (27)$$

where  $E[v] \equiv \int v(\zeta') d\hat{G}(\zeta')$  and  $E[\theta] \equiv \int \theta' d\hat{F}(\theta')$ .<sup>24</sup> The proposition below characterizes the decision rule in this case (see [Appendix A](#) for the proof). We will use such characterization to compare to the decision rule obtained in the case of persistent observation cost in the previous section.

**PROPOSITION 7.** Assume that  $F(\cdot)$  is given by [equation \(26\)](#) with  $0 < \gamma < \infty$  and that  $\underline{v} > E[\theta]$ . Then: (i) there exists a  $\zeta^*$  so that  $0 < \tau(\zeta) < \infty$  and  $\tau'(\zeta) > 0$  for all  $\zeta^* \leq \zeta < \infty$ ; moreover if  $\zeta^* > \underline{\zeta}$ , then  $\tau(\zeta) = 0$  for all  $\underline{\zeta} \leq \zeta < \zeta^*$ ; (ii) if  $\underline{\zeta} \geq 0$ , then  $\tau(\underline{\zeta}) > 0$  and  $\zeta^* = \underline{\zeta}$ .

Point (i) of the proposition states that the time between observations,  $\tau$ , increases with the signal about the future observation cost,  $\zeta$ , and does so strictly for values above a threshold  $\zeta^*$ . Intuitively, a higher realization of  $\zeta$  is associated to a higher expected observation cost, so the firm delays the time of the next observation. When the realization of such signal is low enough, i.e.  $\zeta < \zeta^*$ , the expectation of the next observation cost is so low that the firm finds it optimal to observe immediately. Point (ii) establishes that  $\zeta^*$  must be negative for the firm to exercise the observation immediately. In fact, if the lower bound of the support of  $\zeta$  is positive, i.e.  $\underline{\zeta} \geq 0$ , then  $\tau(\underline{\zeta}) > 0$ . For instance, even if  $\zeta = 0$  so that the cost of observing immediately would be zero, then  $\tau(0) > 0$ . We notice that this is true for all values of  $\gamma$ .

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<sup>24</sup>Note that the particular specification of the distribution  $\hat{F}$  is immaterial for this problem, as only the associated mean  $E[\theta]$  matters.

The economics of  $\tau(0) > 0$  is very different from the case of persistent observation cost in the previous section. Here, the firm waits a strictly positive period of time before making another observation even if the cost of observing (again) immediately is zero because of an option value argument: if the observation occurs immediately the opportunity to potentially exercise an observation at a low cost would be wasted because there is no correlation between consecutive observation costs. Due to the nature of i.i.d. observation times, the setup of the problem is such that each observation is associated with an i.i.d. draw for the distribution of the observation cost to be paid upon the next observation. A firm that draws a signal of low (future) observation cost will wait for the best moment to use this option. The option value of waiting will turn out to be relevant to determine the smallest time between observations. For instance, one important implication is that, in order to generate a positive probability of arbitrarily small durations of the time between observations –e.g. in an exponential distribution– the distribution of observation costs needs to have positive mass on negative costs, i.e.  $\underline{\theta} < 0$  and  $\int_{\underline{\theta}}^0 d\hat{G}(x) > 0$ .<sup>25</sup> This result is interesting as a large part of the literature that developed after [Mankiw and Reis \(2002\)](#) has focused on quantifying the predictions of models of rational inattentiveness in which the distribution of the times between observations indeed features positive mass on arbitrarily small durations. We will come back to this issue in [Section 5.1](#). More interestingly, the option value of waiting also reduces the variability of observation times that follows from the variability of observation cost. This in turn reduces the ability of such model to produce large variation in observation times and, therefore, large real effects of monetary shocks. We discuss this in more details later.

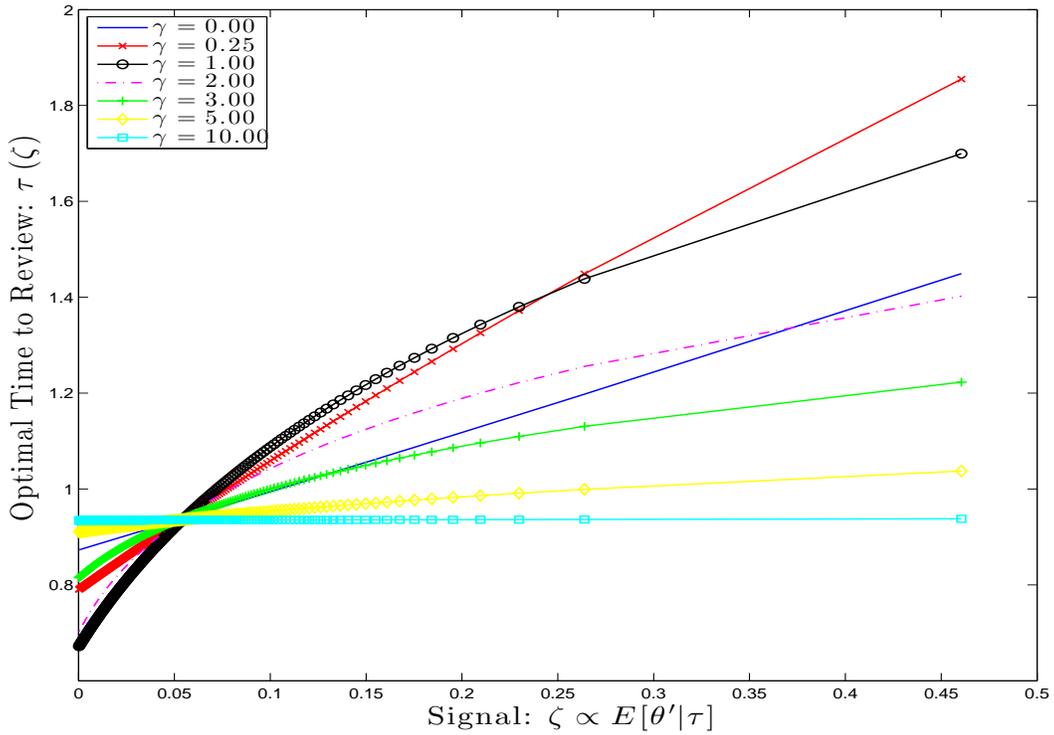
Next we consider two special cases for the informativeness of the signal, for which we have an (almost) closed form solution for the optimal policy. In the case where  $\gamma = \infty$ , the signal carries no information about the observation cost to be paid, and hence the model coincides with the case characterized in [Proposition 3](#), where the policy  $\tau(\cdot)$  does not depend on the realization of  $\zeta$ . At the other extreme,  $\gamma = 0$ , the signal is perfectly informative about the cost to be paid at the next observation. The next two propositions describe the optimal policy and the mapping from the variability of observation cost to the variability of observation times in the  $\gamma = 0$  case.

**PROPOSITION 8.** Let  $F(\theta', \tau|\zeta) = \mathbf{1}_{\theta' \geq \zeta}$  for all  $\zeta$ , i.e.  $\gamma = 0$  in [equation \(26\)](#). The optimal

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<sup>25</sup>Notice that a negative observation cost is not allowed in the model where the costs are Markovian because that would give the firm the possibility of unbounded profits by simply collecting the fees infinitely many times as long as the cost remains negative.

Figure 5: Optimal i.i.d. observation times as a function of the signal on the observation cost



Note: The figure plots the optimal time to the next observation  $\tau(\zeta)$ , in years, as a function of the signal value  $\zeta$  on the horizontal axis, and for different values of precision, as measured by  $1/\gamma$ . The distribution  $\hat{G}$  is assumed to be exponential with average  $E[\zeta] = 0.05$ , and we further assume  $E[\theta] = E[\zeta]$ . We set the drift  $\mu - \phi + \sigma^2/2 = 0$  to obtain infrequent price changes, and  $\sigma = 0.114$  to match an average size of price changes equal to 10%. The other parameters are  $\eta = 5$ ,  $\lambda = 0.25$  and  $\rho = 0.02$ . For more details see [Appendix C](#).

policy  $\tau(\zeta)$  is given by

$$\tau(\zeta) = \begin{cases} 0 & \text{if } \underline{\zeta} \leq \zeta \leq E[v] - \bar{v} \\ \frac{1}{B} [ \ln(\bar{v}) - \ln(E[v] - \zeta) ] & \text{if } E[v] - \bar{v} < \zeta < E[v] \text{ ,} \\ \infty & \text{if } \zeta \geq E[v] \end{cases} \quad (28)$$

where the expected value  $E[v] = \int_{\underline{\zeta}}^{\infty} v(\zeta') d\hat{G}(\zeta')$  is the solution of an explicit equation that is displayed in the proof. If  $\underline{\zeta} \geq 0$ , then: (i)  $E[v] < \bar{v}$  and  $\tau(\zeta) > 0$  for all  $\zeta \geq \underline{\zeta}$ ; (ii) if  $\hat{G}$  is first order stochastically higher, then  $E[v]$  is smaller; and (iii) if  $\hat{G}$  is second order stochastically higher, then  $E[v]$  is higher.

See [Appendix A](#) for the proof. When  $\gamma = 0$  the optimal policy  $\tau(\zeta)$  is known in closed form up to the value of the expected value function  $E[v]$ . It features all the properties of the case of imperfectly informative signals ( $\gamma > 0$ ), with the exception of being characterized by an asymptote, so that if  $\zeta \geq E[v]$ , then  $\tau = +\infty$ . This is because with  $\gamma = 0$  the expected cost of the next observation does not revert to its unconditional mean as time elapses, so that if it is large enough ( $\zeta \geq E[v]$ ) making an observation will never be optimal for the firm. Points (ii) and (iii) of the proposition highlight how the properties of the distribution of signals, which coincide with observation costs in this case, affect the value of the firm.<sup>26</sup>

The expression in [equation \(28\)](#) is particularly useful to study the elasticity of  $\tau(\zeta)$  with respect to  $\zeta$ . In particular, we note that  $\tau'(\zeta) = 1/[B(E[v] - \zeta)]$  for  $\zeta \in (E[v] - \bar{v}, E[v])$ . This implies that when the benefit of information  $B$  and/or the continuation value  $E[v]$  are larger, then the slope of the optimal policy with respect to the observation cost is smaller. This suggests that economies with a higher frequency of i.i.d. observations are characterized by a lower sensitivity of observation times to variation in observation costs.<sup>27</sup> The next proposition uses the analytical results of [Proposition 8](#) to characterize the coefficient of variation of observation times (excluding substitutions) in terms of the coefficient of variation of the observation cost in the case of i.i.d. observations with perfect predicability (see [Appendix A](#) for the proof).

**PROPOSITION 9.** Let  $F(\theta', \tau|\zeta) = \mathbf{1}_{\theta' \geq \zeta}$  for all  $\zeta$ , i.e.  $\gamma = 0$  in [equation \(26\)](#), and  $G(\zeta'|\theta) = \hat{G}(\zeta')$  for all  $\zeta'$  and  $\theta$ . Moreover, let  $E(\zeta) = \int_{\underline{\zeta}}^{\infty} \zeta d\hat{G}(\zeta)$  and the coefficient of variation of observation cost be such that  $CV(\zeta) > 0$  as  $E(\zeta) \rightarrow 0$ . Let  $CV_K(\tau)$  denote the

<sup>26</sup>We notice that in this case the value of  $E[\theta]$  is immaterial as observation costs are never drawn from the distribution  $\hat{F}$ .

<sup>27</sup>Note that both the values of  $B$  and  $E[v]$  are positively associated to the average frequency of observations: as  $B$  increases, observing more frequently is optimal; if the average frequency of observations is optimally higher, then  $E[v]$  increases and gets closer to its upper bound  $\bar{v}$ .

coefficient of variation of times between observations conditional on no substitutions. We have that

$$CV_K(\tau) = \frac{1}{2}(\rho + \lambda - b)\bar{\tau} CV(\zeta) + o\left(\sqrt{E(\zeta)}\right), \quad (29)$$

where  $\bar{\tau} = \sqrt{2E(\zeta)/(b-a)}$  is a measure of the mean time between observations.

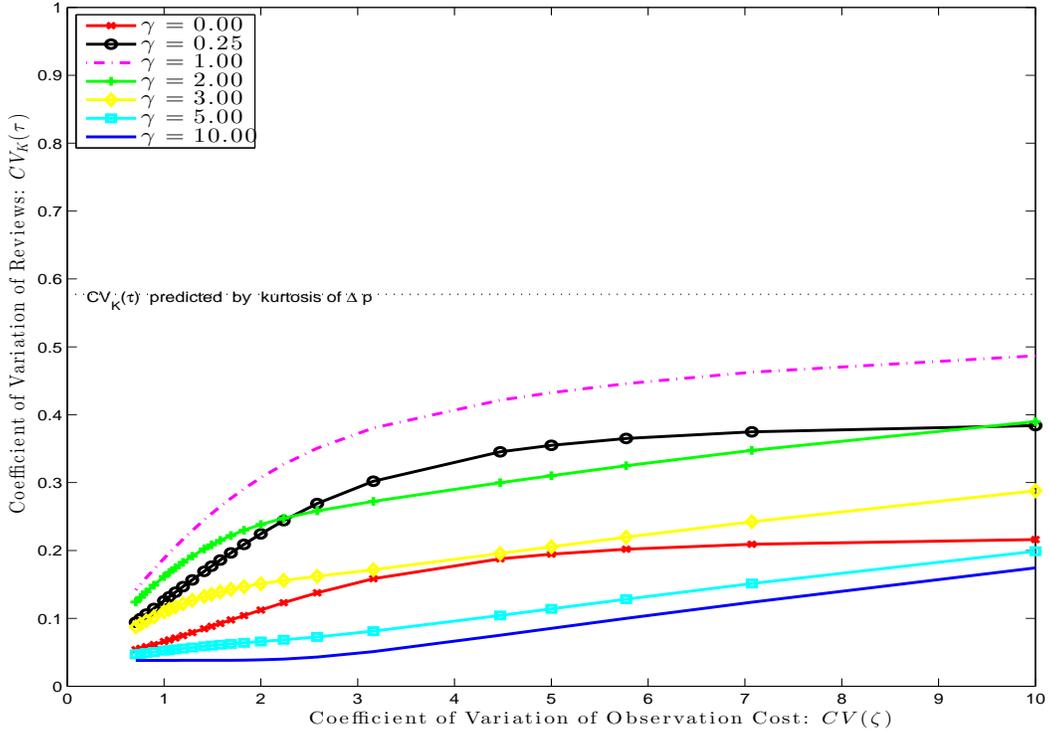
The proposition shows that the mapping from  $CV(\zeta)$  to  $CV_K(\tau)$  depends on two terms. The first term, i.e.  $(\rho + \lambda - b)$ , is related to the discount rate in [equation \(27\)](#) and typically has the order of magnitude of the interest rate. The second term, i.e.  $\bar{\tau} = \sqrt{2E(\theta)/(b-a)}$ , is of the same order of magnitude of the average duration of observation times. For instance, an economy with a yearly discount rate of 5% and an average time between observations of one year, would be characterized by  $CV_K(\tau)/CV(\zeta) \approx 0.025$ . Such a small slope of the variability in observation times with respect to observation costs is explained by the option value argument discussed above.

In order to study the mapping from the variation in observation cost to the variation in observation times in the more general case when signals are not perfectly informative  $\gamma > 0$ , we do the following exercise. We assume that the distribution  $\hat{G}$  is a Gamma distribution with shape parameter  $\alpha_1$  and scale parameter  $\alpha_2$ . The parameter  $\alpha_1$  determines the coefficient of variation of the observation cost  $\theta$ , while the parameter  $\alpha_2$  is chosen so that the average frequency of observations is 1.3 on a yearly basis which, together with the assumption of no drift in nominal marginal cost, matches the estimated frequency of price changes in the U.S. data. In [Figure 6](#) we plot the model-implied coefficient of variation of observation times ( $CV_K(\tau)$  on the vertical axis) as a function of the coefficient of variation of observation costs, for different values of  $\gamma$ . The figure also plots the level of variation in observation times that is consistent with the kurtosis of the distribution of price change, as explained in [Section 4](#). The results clearly show that a much higher variability in observation costs is needed to produce a given variability in observation times compared to the case with persistent shocks (see [Figure 4](#) for comparison), for all values of  $\gamma$ . We conclude that the model of i.i.d. observations has a hard time generating large real effects of monetary shocks: for the parametrization presented in [Figure 6](#) there is no level of  $CV(\zeta)$ , no matter how large, that is able to produce the variation of  $CV_K(\tau)$  consistent with the kurtosis of the distribution of price changes.

## 5.1 An application: micro-founding exponential i.id. observations

In this section we use our results to solve a “reverse engineering problem”: for given distribution of times between consecutive observations  $K$  as defined in [equation \(4\)](#), we find parameters for the firm problem, including the distribution of the observation cost, so that

Figure 6: Coefficient of variation of observation times with i.i.d. observation cost



Note: The figure reports the coefficient of variation of times between observations (including substitutions) as a function of the coefficient of variation of signals about future observation cost,  $\zeta$ , and for different values of precision, as measured by  $1/\gamma$ . The distribution  $\hat{F}$  is assumed to be  $Gamma(\alpha_1, \alpha_2)$ , where  $\alpha_1$  determines the coefficient of variation of observation costs upon a new draw, and  $\alpha_2$  is chosen to match an yearly average frequency of observations/adjustments equal to 1.3. We set  $E[\theta] = E[\zeta]$ . We set the drift  $\mu - \phi + \sigma^2/2 = 0$  to obtain infrequent price changes, and  $\sigma = 0.114$  to match an average size of price changes equal to 10%. The other parameters are  $\eta = 5$ ,  $\lambda = 0.25$  and  $\rho = 0.02$ . For more details see [Appendix C](#).

aggregating the resulting optimal decision rules  $H(t|t)$  we obtain the target distribution  $K$ . While we could do this for any  $K$ , a particular interesting application, given the results by [Mankiw and Reis \(2002\)](#), is to use an i.i.d. exponential  $K$ . The aim of this is to identify the type of micro-foundation and the parameters of the firm problem in [Section 5](#) that would be consistent with this popular framework.

Let  $\hat{K}$  denote the invariant distribution of times between consecutive observations conditional on no substitution. We note that the invariant distribution of observation times  $K$  including substitutions (which are exponentially distributed with parameter  $\lambda$  by assumption) is exponential if and only if  $\hat{K}$  is exponential, and satisfies  $K(t) = e^{-\lambda t} \hat{K}(t)$  for each  $t \geq 0$ . The next proposition describes the distribution of observation cost  $\hat{G}$  that microfounds an exponential distribution of observation times, where  $\hat{K}(t) = \exp(-\xi t)$  for some  $\xi > 0$ , in the case where observations are i.i.d. and signals are perfectly informative, i.e.  $\gamma = 0$ .

**PROPOSITION 10.** Consider the case when observation times are i.i.d. with  $\gamma = 0$ . Let the invariant exponential distribution of times between consecutive observations (excluding substitutions) be  $\hat{K}(t) = \exp(-\xi t)$  with parameter  $\xi > 0$ . Then the density of the observation costs (and signals) is a displaced beta distribution  $Beta(\alpha_l, \alpha_r)$  with left parameter  $\alpha_l = 1$ , right parameter  $\alpha_r = \xi/B$ , and support  $(\underline{\zeta}, \underline{\zeta} + \bar{v})$ , so that the density  $\hat{g}(\cdot) = \hat{G}'(\cdot)$  is given by:

$$\hat{g}(\zeta) = \frac{\xi}{\bar{v} B} \left( \frac{E[v] - \zeta}{\bar{v}} \right)^{\frac{\xi}{B} - 1} \quad \text{for } \zeta \in (\underline{\zeta}, \underline{\zeta} + \bar{v}), \quad (30)$$

where the expected value function and the lower bound of the distribution of cost are

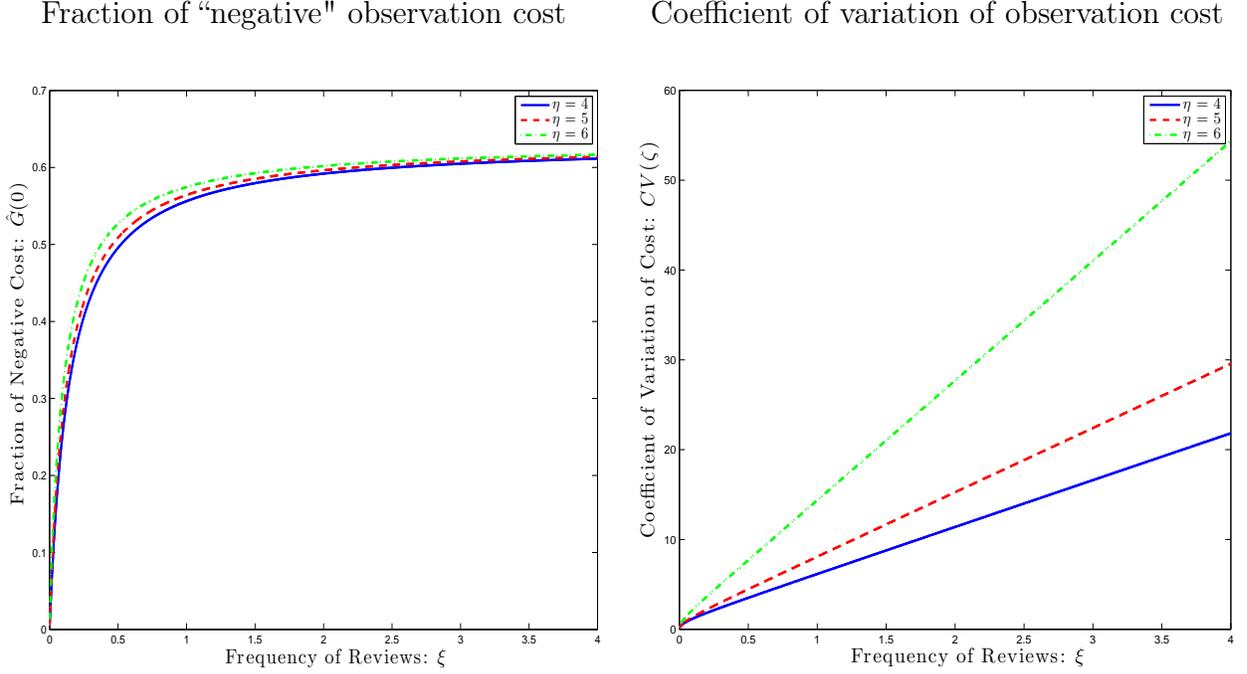
$$E[v] = \underline{v} + (\bar{v} - \underline{v}) \frac{\xi \underline{v}}{1 + \xi \underline{v}}, \quad \underline{\zeta} = E[v] - \bar{v}. \quad (31)$$

The implied fraction of negative cost  $\zeta \leq 0$  and the coefficient of variation of cost are

$$\hat{G}(0) = 1 - \left( \frac{\underline{v}}{\bar{v}} + \left( 1 - \frac{\underline{v}}{\bar{v}} \right) \frac{\xi \underline{v}}{1 + \xi \underline{v}} \right)^{\frac{\xi}{B}} > 0, \quad \text{and } CV(\zeta) = \sqrt{\frac{\xi/B}{2 + \xi/B} \left( \frac{1 + \xi \underline{v}}{\underline{v}/\bar{v}} \right)}. \quad (32)$$

See [Appendix A](#) for the proof. This proposition shows that if observation times are exponential, the observation costs have to be distributed as a displaced beta distribution. The shape of this distribution is determined by  $\xi/B - 1$ , which depends on the ratio of the exponential parameter  $\xi$  to the benefits of information:  $B = \eta(\eta - 1) \sigma^2/2$ . The support of the cost is the interval  $(E[v] - \bar{v}, E[v])$ . Since the exponential distribution has positive probability for arbitrarily small values of observation times, the option value argument of

Figure 7: Exponential i.i.d. observation times: properties of  $\hat{G}$  as a function of  $\xi$  and  $\eta$



Note: The figure reports the fraction of negative cost (left panel) and the coefficient of variation of cost (right panel) as a function of the frequency of observations on the horizontal axis, for three different values of  $\eta = \{4, 5, 6\}$ , associated to the distribution of [Proposition 10](#). We set  $E[\theta] = E[\zeta]$ . We set the drift  $\mu - \phi + \sigma^2/2 = 0$  to obtain infrequent price changes, and  $\sigma = 0.114$  to match an average size of price changes equal to 10% when the frequency of observations/price adjustments is 1.3 on a yearly base. The discount rate is  $\rho = 0.02$ . For more details see [Appendix C](#).

[Proposition 7](#) implies that the lower bound is negative. The expression for the fraction of negative cost  $\hat{G}(0)$  allows us to quantify the importance of this feature.

We next comment on the comparative static of the change in the distribution of the observation cost as we vary the parameter of the exponential distribution of observation times. Note that if  $\xi \rightarrow 0$  observations are very infrequent, then  $\underline{\zeta} \rightarrow 0$ , so that  $\hat{G}(0) \rightarrow 0$  and  $CV(\zeta) \rightarrow 0$ . In words, in order to have almost no observations, the distribution of the observation cost must be almost degenerate, concentrated around its upper bound  $\bar{\zeta} = \underline{v}$ . At the other extreme, if  $\xi \rightarrow \infty$  observations are very frequent, then  $\underline{\zeta} \rightarrow \bar{v}$ , the fraction  $\hat{G}(0) \rightarrow 1 - \exp(-1) \approx 0.63$  and  $CV(\zeta) \rightarrow +\infty$ . The general case moves monotonically between these two extremes, with both the fraction of negative costs,  $\hat{G}(0)$ , and the coefficient of variation,  $CV(\zeta)$ , being increasing functions of the frequency of observations  $\xi$ .

In the left panel of [Figure 7](#) we plot the fraction of negative costs as a function of the average frequency of observations  $\xi$ , for selected parameter values. This figure plots three

lines, corresponding to three values of  $\eta$ , which change the value of acquiring information captured by  $B$ . Note that for  $\xi \approx 1.3$ , which roughly corresponds to the average frequency of price changes per year in the US, then  $\hat{G}(0) \approx 0.55$ , so that more than half of the observation costs are negative. We interpret this as saying that the option value effect is very large for this model to be quantitatively realistic, since negative observation cost have no direct economic interpretation. We also note that the value of  $\eta$  has marginal impact on the value of  $\hat{G}(0)$ . Additionally in the right panel of [Figure 7](#) we plot the coefficient of variation of the observation cost as a function of the frequency of observations,  $\xi$ , and three values of  $\eta$ . For  $\xi \approx 1.3$ , given our parameter values, the coefficient of variation is about 10 at  $\eta = 5$ . We recall that we are measuring the observation cost ( $\zeta = \theta$ ) as a multiple of the frictionless yearly profit, implying that a large coefficient of variation can be associated to, in our view, unreasonably high values of the cost of an observation in units of profits. Finally, the coefficient of variation of cost increases with the value of  $\eta$ . Intuitively, as the value of information increases firms have incentives to observe sooner for a given realization of the observation cost. And so more variation in costs is needed to generate large enough durations of observations such that the observed outcomes are consistent with an exponential distribution.

## 6 Concluding remarks

We explored the micro foundations, as well as the aggregation, of models where the optimal price setting decision of the firm is subject to information gathering costs, as in the “rational inattentiveness” literature. Our analytical results unveil a few shortcomings of the current literature, as well as potentially fruitful avenues for future research. We showed that a natural modeling of the firm’s decisions, one that assumes independence of the cost process from the firm’s decisions to gather information, produces optimal observation times that are persistent and, for reasonable parameterizations, infrequent. These results, which deviate from existing models mainly in the fact that observation times are not i.i.d., are also useful for generating larger real effects of monetary policy.

Our theoretical characterization relies on a number of simplifying assumptions, whose importance seems of interest for future research. A limitation of our framework is the absence of physical costs of price adjustments, implying that prices change continuously in presence of a drift in nominal marginal cost. In [Alvarez, Lippi, and Paciello \(2011, 2012\)](#) we explore the optimal pricing decision of firms, and its aggregate implications, when firms face non-random observation and adjustment costs. An important result of these papers is that *price plans* are not optimal for reasonable parameterizations of the adjustment cost, so that prices change infrequently even in presence of a drift in marginal cost. A large literature (e.g. [Mankiw and](#)

Reis (2002)) has emphasized the role of price plans for the propagation of monetary shocks to inflation and output. As in this literature, we allow firms to use price plans at no cost. However, the monetary shock we study causes no predictable dynamics in nominal marginal cost so that price plans are not used by firms in response to it.

Our results on the aggregation of firms actions and propagation of monetary shocks rely, for tractability, on a framework where there are no strategic complementarities in price setting, i.e. the profit-maximizing price of a firm is independent of the other firms' actions. While allowing for these strategic complementarities may increase the size of the real effects of monetary shocks, it is not easy to predict how they would affect the mapping from the cross sectional distribution of observation times to the size of the real effects of monetary shocks.

Finally, we have studied a model where, for simplicity, the cost/benefit of making an observation varies because of random variation in the observation cost. There are potentially other interesting sources of variation. For instance, relaxing the assumption that the cost of making an observation scales with firms' profits can deliver variability and persistence in observation times arising from variability and persistence of the productivity process. In alternative, adding a separate cost of adjusting prices (i.e. a menu cost) would cause firms to differ in the gap between the price they post and the optimal one. In this specification, variation in observation times may arise from variation across firms in the benefit, as opposed to the cost, of the next observation. Further investigating and measuring alternative sources of variation in the cost/benefit of observations is also left for future research.

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## A Appendix: Proofs

**Proof.** (of [Proposition 1](#)) Consider the case where  $H(\cdot|s)$  is absolutely continuous for all  $s$ . We use a guess and verify strategy, so we substitute [equation \(7\)](#) into [equation \(2\)](#) and write

$$R K(T) = -R \int_0^\infty H(T|s) k(s) ds \text{ for all } T \geq 0. \quad (33)$$

Note that this expression holds for  $T = 0$  since  $K(0) = H(0|s) = 1$  for all  $s$  since both  $K$  and  $H(\cdot|s)$  are right-CDFs. Next, differentiate both sides of [equation \(33\)](#) to obtain:

$$k(T) = \int_0^\infty h(T|s) k(s) ds \quad \text{for all } T \geq 0, \quad (34)$$

which is exactly the definition of an invariant distribution given in [equation \(1\)](#). Consider now the case where  $H$  has positive mass on countably many values, note that for each  $T$  we can write:

$$\begin{aligned} q(T) &= R \sum_{j=1}^{\infty} H(T|t_j) k(t_j) = R \sum_{j=1}^{\infty} \sum_{\{i:t_i \geq T\}} h_{ij} k(t_j), \\ &= R \sum_{\{i:t_i \geq T\}} \sum_{j=1}^{\infty} h_{ij} k(t_j) = R \sum_{\{i:t_i \geq T\}} k(t_i) = R K(T), \end{aligned}$$

where the first equality uses that times between consecutive observations take countably-many values, the second uses the definition of  $h$ , the third permutes the summations, and the last one uses the definition of an invariant distribution. Note that the density of the time until the next observation is piece-wise linear, with downward jumps at the observation times  $T = t_i$  for all  $i \geq 1$ .  $\square$

**Proof.** (of [Proposition 2](#)) First we show that

$$\mathcal{M} \equiv \int_0^\infty Q(t) dt = R \int_0^\infty \int_t^\infty K(s) ds dt = R \int_0^\infty s K(s) ds$$

We write  $A(J) \equiv \int_0^J \int_t^J K(s) ds dt$  and  $B(J) \equiv \int_0^J s K(s) ds$  for all  $J \geq 0$ . Note  $A(0) = B(0) = 0$ . Second,  $B'(J) = JK(J)$  and  $A'(J) = \int_0^J K(J) dt = JK(J)$ , hence  $A(J) = B(J)$  and taking  $J \rightarrow \infty$  we obtained the desired result.

Second let the unconditional variance of  $\tau$  be

$$Var(\tau) \equiv - \int_0^\infty s^2 K'(s) ds - \left( - \int_0^\infty s K'(s) ds \right)^2 = 2 \int_0^\infty s K(s) ds - \left( \int_0^\infty K(s) ds \right)^2,$$

where the second equality comes from integrating by parts. The latter, together with the definition of  $R$ , i.e.  $\int_0^\infty K(s) ds = 1/R = E[\tau]$  implies that  $\int_0^\infty s K(s) ds = 1/2 Var(\tau) + (E[\tau])^2$ , which immediately implies the result.  $\square$

**Proof.** (of [Lemma 3](#)) Consider first the distribution of log price changes of a firm observing/adjusting  $\tau$  periods after the last observation/adjustment,  $\Delta \log(p(\tau))$ . Conditional

on the next observation taking place in  $\tau$  periods, the distribution of log-price changes upon the next observation/adjustment is normal with mean and variance equal to 0 and  $\sigma^2 \tau$  respectively. Let index the distribution of  $\Delta \log(p(\tau))$  by  $\tau$ . Recall that the kurtosis of a random variable  $x$  with zero mean is defined as  $Kurt(x) = m_4(x)/(m_2(x))^2$ , where  $m_s(x) = \mathbb{E}[x - \mathbb{E}(x)]^s$  is the  $s^{th}$  centered moment of  $x$ . Therefore, the second and fourth centered moments of  $\Delta \log(p(\tau))$  are equal to  $m_2(\tau) = \sigma^2 \tau$  and  $m_4(\tau) = 3 \sigma^4 \tau^2$  respectively. Consider now the cross-sectional distribution of price changes,  $\Delta \log(p)$ . Such distribution is given by the mixture of normals arising from the different firms drawing different times to the next observation,  $\tau$ , each with density  $k(\tau)$ .

The second moment of such mixture of normals is:

$$\int m_2(\tau)k(\tau)d\tau = \sigma^2 \int \tau k(\tau)d\tau.$$

The fourth moment of such mixture of normals is given by the weighted average of the fourth moments of each normal,  $m_4(\tau)$ :

$$\int m_4(\tau)k(\tau)d\tau = 3 \sigma^4 \int \tau^2 k(\tau)d\tau.$$

Thus the kurtosis of the distribution of log-price changes is given by:

$$Kurt(\Delta \log(p)) = \frac{3 \left( \int \tau^2 k(\tau) d\tau \right)}{\left( \int \tau k(\tau) d\tau \right)^2} = \frac{3 (Var(\tau) + (E[\tau])^2)}{(E[\tau])^2} = 3 ((CV(\tau))^2 + 1) . \quad \square$$

**Proof.** (of [Proposition 3](#)) Taking the first order condition with respect to  $\tau$  in the right hand side of the Bellman equation  $\hat{\tau} = \frac{1}{B}[\ln(\bar{v}) - \ln(\hat{v} - E[\theta])]$ . Using this to eliminate  $\tau$  from the Bellman equation we obtain

$$\hat{v} = \begin{cases} \bar{v} & \text{if } E[\theta] = 0 \\ \text{smallest solution of } \hat{v} = \underline{v} + B \underline{v} \bar{v} \left( \frac{\hat{v} - E[\theta]}{\bar{v}} \right)^{\frac{1}{B\underline{v}}} & \text{if } 0 < E[\theta] < \underline{v} , \\ \underline{v} & \text{if } E[\theta] \geq \underline{v} \end{cases} , \quad (35)$$

which can be written as:  $x = s(x; \hat{\theta}, \alpha) \equiv \underline{v}/\bar{v} + \alpha \left( x - \hat{\theta} \right)^{\frac{1}{\alpha}}$ , where  $x \equiv \hat{v}/\bar{v}$  and  $\alpha = B\underline{v} < 1$  under our maintained assumptions. We are looking for a solution with  $\underline{v}/\bar{v} \leq x \leq 1$ . Note that for  $\hat{\theta} = 0$  then there is a unique solution  $\hat{v} = \bar{v}$  –to see this note that  $s(x; \hat{\theta}, \alpha)$  is strictly convex in  $x$  and that  $s'(1, 0, \alpha) = 1$ . For  $\hat{\theta} > 0$  there is also a solution of this equation with

$x > 1$ , which does not correspond to a solution for the value function. For values of  $\hat{\theta}$  where there are two solutions, the smallest one is in the desired range. Since increases in  $\hat{\theta}$  shift horizontally the function  $s$  then the smallest solution decreases with  $\hat{\theta}$ . Since increases in  $\alpha$  shift the function  $s$  up, then the smallest solution increases with  $\alpha$ . Finally, if  $\hat{\theta} = \underline{v}/\bar{v}$  the smallest solution is  $\hat{v} = \underline{v}$ . If  $\hat{\theta} > \underline{v}/\bar{v}$  then there is only one solution of  $x = s(x)$  which is not the solution of the value function, and hence in this case  $\hat{\tau} = +\infty$  and  $\hat{v} = \underline{v}$ .  $\square$

**Proof.** (of [Proposition 4](#)) We first prove part (i). We start by arguing that with  $\kappa > 0$ , then  $\tau(\theta) > 0$  for all  $\theta > 0$ . This follows because if  $\tau(\theta) = 0$  with  $\theta > 0$  then the expected time until there is a change is strictly positive, and at each time there is a strictly positive cost, so the expected discounted cost diverges to  $+\infty$ .

Next we derive the first order condition to the firm problem and show it is sufficient for an optimum. For this, note that the derivative of the objective function in the Bellman [equation \(24\)](#) with respect to  $\tau$  is equal to  $e^{-(\rho+\lambda-a)\tau} [M(\tau) - N(\tau, \theta)]$  where these functions are given by:

$$\begin{aligned} M(\tau) &\equiv 1 - (\rho + \lambda - b)e^{B\tau} (E[v] - E[\theta]) \quad \text{and} \\ N(\tau, \theta) &\equiv (\rho + \lambda - b + 1/\kappa)e^{(B-1/\kappa)\tau} [v(\theta) - \theta - (E[v] - E[\theta])] . \end{aligned} \quad (36)$$

First, we note several properties of  $M$  and  $N$  at  $\theta > 0$ : (a) for (i) to hold, i.e.  $\tau(\theta) > 0$  for any  $\theta > 0$ , it must be the case that  $M(0) - N(0, \theta) > 0$ ; (b)  $M(\tau) - N(\tau, \theta)$  is continuous in  $\tau$  and  $\lim_{\tau \rightarrow \infty} M(\tau) - N(\tau, \theta) = -\infty$  for all  $\theta < \infty$ . From (a) and (b) we obtain that there is at least one finite solution to the first order condition,  $\tau(\theta) < \infty$ , for all  $\theta < \infty$ . Next, we show that there is a unique local maximum. Differentiating the first order condition with respect to  $\tau$  we have  $S(\tau) \equiv M_\tau(\tau) - N_\tau(\tau, \theta) = -B + \kappa^{-1} (1 - (\rho + \lambda - b)e^{B\tau} (E[v] - E[\theta]))$ . We prove now that the objective function has at least one local maximum where the first order condition holds. Let's denote by  $\tau_1$  the smallest local maximum. Notice that  $S(\tau_1) < 0$  by definition of a local maximum. If there would be another local maximum  $\tau_2 > \tau_1$ , then there must be a value of  $\tau_m \in (\tau_1, \tau_2)$  that is a local minimum requiring  $S(\tau_m) \geq 0$ . But notice that the function  $S(\tau)$  is strictly decreasing in  $\tau$  which is a contradiction of  $S(\tau_m) \geq 0 > S(\tau_1)$  as  $\tau_m > \tau_1$ . Finally, notice that as  $\theta \rightarrow \infty$ , then  $\tau(\theta) \rightarrow \infty$ . This follows from [equation \(36\)](#) given that  $\theta \rightarrow \infty$  implies that  $N(\tau, \theta) \rightarrow -\infty$  and the first order condition can be satisfied only if  $\tau \rightarrow +\infty$  so that also  $M(\tau)$  diverges to  $-\infty$ .

Next we use the first order condition to prove  $\tau'(\theta) > 0$  if  $\theta > 0$  and  $\lim_{\theta \rightarrow \infty} \tau'(\theta) = 0$ .

The implicit function theorem gives

$$\frac{\partial \tau}{\partial \theta} = \frac{N_\theta}{M_\tau - N_\tau} > 0,$$

where  $N_\theta = (\rho + \lambda - b + 1/\kappa)e^{(B-1/\kappa)\tau} (v'(\theta) - 1) < 0$  because  $v'(\cdot) \leq 0$ , and  $M_\tau - N_\tau < 0$  because  $\tau(\theta)$  is a maximum. Finally,  $\lim_{\theta \rightarrow \infty} \tau'(\theta) = 0$  follows because  $M_\tau - N_\tau$  diverges to  $-\infty$  while  $N_\theta$  either converges to zero or diverges to  $-\infty$  but at a lower rate than  $M_\tau - N_\tau$  as  $\theta$  converges to zero.

Next, we prove part (iii). We argue that if  $1/\kappa$  is low enough then  $\lim_{\theta \rightarrow 0^+} \tau(\theta) = 0$ . We notice that if  $\tau = 0$  solves the first order condition to the firm problem at  $\theta = 0$  then the firm value is given by  $v(0) = \frac{1+1/\kappa(E[v]-E[\theta])}{\rho+\lambda-b+1/\kappa}$ ; substituting this expression for  $v(0)$  into [equation \(36\)](#) we obtain that indeed  $\tau(0) = 0$  is an extreme point. We are left to show that  $\tau(0) = 0$  is a maximum which requires the second derivative at  $\tau = 0$  to be negative,  $S(0) = -B + \kappa^{-1}(1 - (\rho + \lambda - b)(E[v] - E[\theta])) < 0$ . By using  $E[v] > 1/(\rho + \lambda - a)$ , a sufficient condition for the latter is  $-B + \kappa^{-1}[1 - (\rho + \lambda - b)(1/(\rho + \lambda - a) - E[\theta])] < 0$ , implying  $\kappa > \bar{\kappa} \equiv \frac{B+(\rho+\lambda-a)(\rho+\lambda-b)E[\theta]}{B(\rho+\lambda-a)}$ . Finally, if  $\lim_{\theta \rightarrow 0^+} \tau(\theta) = 0$  then  $\lim_{\theta \rightarrow 0^+} \tau'(\theta) = +\infty$  follows because  $N_\theta(\tau; \theta) = (\rho + \lambda - b + 1/\kappa)e^{(b-a-1/\kappa)\tau} (v'(\theta) - 1)$  diverges as  $\theta \rightarrow 0^+$  because  $v'(\theta) = \frac{-e^{-(\rho+\lambda-b+1/\kappa)\tau(\theta)}}{1-e^{-(\rho+\lambda-b+1/\kappa)\tau(\theta)}}$  diverges while  $M_\tau - N_\tau$  converge to a finite negative value.

Finally, we prove part (ii). As above, if  $\lim_{\theta \rightarrow 0^+} \tau(\theta) = 0$  then the firm value is given by  $\lim_{\theta \rightarrow 0^+} v(\theta) = \frac{1+\kappa^{-1}(E[v]-E[\theta])}{\rho+\lambda-b+1/\kappa}$ . By monotonicity of the value function we must have  $\lim_{\theta \rightarrow 0^+} v(\theta) > E[v]$ , which implies  $\frac{1+\kappa^{-1}(E[v]-E[\theta])}{\rho+\lambda-b+1/\kappa} > E[v]$  and  $E[\theta]/\kappa < 1 - (\rho + \lambda - b)E[v]$ . By using  $E[v] > \underline{v}$ , we obtain that a necessary condition for  $\tau(0) = 0$  is that  $E[\theta]/\kappa < \frac{b-a}{\rho+\lambda-a}$ . Therefore, if  $\kappa \leq \underline{\kappa} \equiv \frac{(\rho+\lambda-a)E[\theta]}{B}$  then  $\lim_{\theta \rightarrow 0^+} \tau(\theta) > 0$ . Notice that  $E[\theta] < \underline{v}$  guarantees that  $\bar{\kappa} > \underline{\kappa}$ .  $\square$

**Proof.** (of [Proposition 5](#)) We first prove part (i). Using the first order condition in [equation \(36\)](#) we have that  $\lim_{\kappa \rightarrow \infty} M(\tau; \kappa) - N(\tau, \theta; \kappa) = 1 - (\rho + \lambda - b)e^{B\tau}(\lim_{\kappa \rightarrow \infty} v(\theta; \kappa) - \theta)$ . Then solving the first order condition immediately implies that  $\lim_{\kappa \rightarrow \infty} \tau(\theta; \kappa) = \tilde{\tau}(\theta)$  and  $\lim_{\kappa \rightarrow \infty} v(\theta; \kappa) = \tilde{v}(\theta)$  where the functions  $\tilde{\tau}(\cdot)$  and  $\tilde{v}(\cdot)$  are given by [equations \(22\)-\(21\)](#).

We now prove part (ii). We first establish the following:

**LEMMA 4.**  $\forall \theta > 0$  if  $0 < \lim_{\kappa \rightarrow 0} \tau(\theta; \kappa)$  then  $\lim_{\kappa \rightarrow 0} \frac{\partial v(\theta; \kappa)}{\partial \theta} = \lim_{\kappa \rightarrow 0} \frac{\partial \tau(\theta; \kappa)}{\partial \theta} = 0$ .

We prove the first limit in this lemma using the envelope theorem and the assumed lower bounds for the limit of  $\tau$ :  $\lim_{\kappa \rightarrow 0} \frac{\partial v(\theta; \kappa)}{\partial \theta} = \lim_{\kappa \rightarrow 0} 1/(1 - e^{(\rho+\lambda-b+1/\kappa)\tau(\theta; \kappa)}) = 0$ . To prove the second limit in the lemma we take the limit of the first order condition for  $\tau$ , namely

equation (36),

$$\frac{\partial \tau(\theta; \kappa)}{\partial \theta} = \frac{(\rho + \lambda - b + 1/\kappa)e^{(B-1/\kappa)\tau(\theta; \kappa)} \left( \frac{\partial v(\theta; \kappa)}{\partial \theta} - 1 \right)}{-B + \kappa^{-1} (1 - (\rho + \lambda - b)e^{B\tau(\theta; \kappa)} (E[v(\theta; \kappa) - \theta]))},$$

The limit of the numerator is zero given the result we just proved on the derivative of the value function. The limit of the denominator is infinite given that for each  $\theta, \kappa$  we have  $\underline{v} < v(\theta; \kappa) < \bar{v}$ . This concludes the proof of the lemma.

We complete the proof of the proposition by adding a second lemma, which establishes that the hypothesis of the lemma holds.

**LEMMA 5.** We show that for each  $\theta > 0$  we have  $\lim_{\kappa \rightarrow 0} \tau(\theta; \kappa) > 0$ .

We prove this lemma by contradiction, supposing that  $\lim_{\kappa \rightarrow 0} \tau(\theta; \kappa) = 0$ . Define  $p(\theta) = \lim_{\kappa \rightarrow 0} e^{-\tau(\theta; \kappa)/\kappa}$ . Taking the limit on both sides of equation (24) and rearranging terms gives

$$(1 - p(\theta)) \lim_{\kappa \rightarrow 0} v(\theta, \kappa) = -p(\theta)\theta + (1 - p(\theta))E \left[ \lim_{\kappa \rightarrow 0} v(\theta, \kappa) - \theta' \right]$$

where we used Lebesgue dominated convergence exchanging the expected value and the limit. The case in which  $p(\theta) = 1$  yields an immediate contradiction with  $\theta > 0$ . Consider now the complementary case in which  $p(\theta) < 1$  we get

$$\lim_{\kappa \rightarrow 0} v(\theta; \kappa) = E \left[ \lim_{\kappa \rightarrow 0} v(\theta'; \kappa) - \theta' \right] - \frac{p(\theta)}{1 - p(\theta)} \theta.$$

Since  $\forall \kappa$  we have that  $v(\theta; \kappa)$  is decreasing in  $\theta$  then it is easy to show that there exist a  $\bar{\theta} > 0$  such that for  $\theta \in [0, \bar{\theta})$  we have  $\lim_{\kappa \rightarrow 0} v(\theta; \kappa) > E \left[ \lim_{\kappa \rightarrow 0} v(\theta'; \kappa) - \theta' \right] - \frac{p(\theta)}{1 - p(\theta)} \theta$  arriving to a contradiction. Hence we have shown that  $\tau(\theta; \kappa) > 0$  for all  $\theta \in [0, \bar{\theta})$ . Finally using that for each  $\kappa$  the function  $\tau(\theta, \kappa)$  is strictly decreasing in  $\kappa$  for  $\theta \leq \bar{\theta} < E[\theta]$ , we have that  $\lim_{\kappa \rightarrow 0} \tau(\theta, \kappa)$  is weakly decreasing, and thus  $\lim_{\kappa \rightarrow 0} \tau(\theta, \kappa) > 0$  for all  $\theta > 0$ . This completes the proof of the second lemma.  $\square$

**Proof.** (of Proposition 6) Using the results of Reis's (2006) Proposition 4, we have that as  $E[\theta] \rightarrow 0$ ,  $\tau(\theta) \rightarrow \tilde{\tau}(\theta) = \sqrt{\frac{2\theta}{B}}$ . Let  $\bar{\theta} \equiv E[\theta]$ . Assuming a constant strictly positive coefficient of variation  $\sqrt{\nu} > 0$ , so that  $Var(\theta) = \nu \bar{\theta}^2$ , and using the square root approximation, we have  $E[\tau] = E[\tau(\bar{\theta}) + \tilde{\tau}'(\bar{\theta})(\theta - \bar{\theta})] + o(\bar{\theta}) = \tilde{\tau}(\bar{\theta}) + o(\bar{\theta})$  and

$Var(\tau) = (\tilde{\tau}'(\bar{\theta}))^2 Var(\theta) + o(Var(\theta))$  we obtain

$$\begin{aligned} \frac{Var(\tau)}{(E[\tau])^2} &= \frac{1}{2\bar{\theta}/(b-a) + o(\bar{\theta})} \left( \left( \frac{2\bar{\theta}}{b-a} + o(\bar{\theta}) \right)^{-1/2} \frac{1}{b-a} \right)^2 Var(\theta) + \frac{o(\nu \bar{\theta}^2)}{2\bar{\theta}/(b-a) + o(\bar{\theta})} \\ &= \frac{1}{4} \frac{Var(\theta)}{(E[\theta])^2} + o(\bar{\theta}). \end{aligned}$$

**Proof.** (of [Proposition 7](#)) The derivative of the objective function in the Bellman [equation \(27\)](#) with respect to  $\tau$  is equal to  $e^{-(\rho+\lambda-a)\tau} [M(\tau) - N(\tau, \zeta)]$  where these functions are given by:

$$\begin{aligned} M(\tau) &\equiv 1 - (\rho + \lambda - b)e^{B\tau} [E[v] - E[\theta]] \quad \text{and} \\ N(\tau, \zeta) &\equiv (\rho + \lambda + \gamma - b)e^{(B-\gamma)\tau} [E[\theta] - \zeta]. \end{aligned} \tag{37}$$

First, we note two properties of  $M$  and  $N$ : (a) if  $\gamma > 0$ ,  $\lim_{\tau \rightarrow \infty} M(\tau) - N(\tau, \zeta) = -\infty$  for any  $\zeta$ ; (b)  $N$  is strictly decreasing in  $\zeta$  for all  $\tau < \infty$ , while  $M$  does not vary with  $\zeta$ . We distinguish two cases. The first case is if  $M(0) - N(0, \underline{\zeta}) > 0$  then (b) implies that  $M(0) - N(0, \zeta) > 0$  for all  $\zeta > \underline{\zeta}$ . The first order condition then implies that  $\tau(\zeta) > 0$  for all  $\zeta$  and  $\zeta^* = \underline{\zeta}$ . If instead,  $M(0) - N(0, \underline{\zeta}) \leq 0$ , then  $\zeta^* \geq \underline{\zeta}$  with  $\tau(\zeta) = 0$  and  $v(\zeta) = E[v] - \zeta$  for all  $\zeta < \zeta^*$ . For  $\zeta \geq \zeta^*$ , we have that  $M(0) - N(0, \zeta) \geq 0$  which when combined with (a) it implies that the solution to the firm problem is given by the solution to the first order condition at a value  $0 < \tau(\zeta) < \infty$ . Moreover, by differentiating the first order condition at the interior maximum we obtain

$$\frac{\partial \tau(\zeta)}{\partial \zeta} = \frac{N_\zeta}{M_\tau - N_\tau} > 0,$$

where  $N_\zeta(\tau; \zeta) = -(\rho + \lambda + \gamma - b)e^{-(\gamma-b+a)\tau} < 0$ , and  $M_\tau - N_\tau \leq 0$  by the definition of a maximum. The second case is if  $M(0) - N(0, \underline{\zeta}) > 0$ . In this case, the optimal  $\tau(\zeta)$  is always strictly bigger than zero. Differentiating the first order condition with respect to  $\tau$  we have  $S(\tau) \equiv M_\tau(\tau) - N_\tau(\tau, \zeta) = -B + \gamma(1 - (\rho + \lambda - b)e^{B\tau}[E[v] - E[\theta]])$ . We prove now that the objective function has at most one interior local maximum. Let's denote by  $\tau_1$  the smallest local maximum. Notice that  $S(\tau_1) \leq 0$  by definition of a local maximum. If there would be another local maximum  $\tau_2 > \tau_1$ , then there must be a value of  $\tau_m \in (\tau_1, \tau_2)$  that is a local minimum requiring  $S(\tau_m) \geq 0$ . But notice that the function  $S(\tau)$  is strictly decreasing in  $\tau$  which is a contradiction of  $S(\tau_m) \geq 0 > S(\tau_1)$  as  $\tau_m > \tau_1$ .

To prove point that  $\tau(\zeta)$  is not bounded assume for a contradiction that  $\tau(\zeta)$  has an

upper bound. Then, provided that  $\gamma < \infty$ , there exists a  $\zeta$  large enough for which the value function is arbitrarily negative, and in particular smaller than  $\underline{v}$ , implying that the upper bound is not optimal. Once we have established that  $\lim_{\zeta \rightarrow \infty} \tau(\zeta) = \infty$ .

To prove point (ii) assume by contradiction that  $\underline{\zeta} \geq 0$  and  $\tau(\underline{\zeta}) = 0$  then: if  $\underline{\zeta} > 0$ ,  $v(\underline{\zeta}) = E[v] - \underline{\zeta} < E[v]$ , which is a contradiction because  $v(\underline{\zeta}) \geq E[v]$  as  $v(\zeta)$  is decreasing in  $\zeta$ ; if  $\underline{\zeta} = 0$ , then  $v(\underline{\zeta}) = E[v]$  which is only possible if  $v(\zeta)$  is constant. But this is possible only if  $\tau(\zeta) = \infty$  for all  $\zeta$ , which is contradiction given the arguments above. By definition of  $\zeta^*$ , given that  $\tau(\underline{\zeta}) > 0$  it follows that  $\zeta^* = \underline{\zeta}$ .

Finally notice that a stochastically higher distribution of cost  $\hat{G}$  implies a weakly lower value function  $v(\zeta)$  for each  $\zeta$  and hence a lower value of  $E[v]$ . For any interior optimum we have

$$\frac{\partial \tau(\zeta)}{\partial E[v]} = -\frac{M_{E[v]}}{M_\tau - N_\tau} < 0,$$

since  $M_\tau - N_\tau < 0$  and  $M_{E[v]} < 0$ .  $\square$

**Proof.** of **Proposition 8** We first prove the part (i) of the proposition in steps.

1. As  $\gamma \rightarrow 0$  in the first order condition in **equation (37)** we have that

$$\lim_{\gamma \rightarrow 0} M(\tau; \gamma) - N(\tau, \zeta; \gamma) = e^{-(\rho+\lambda-a)\tau} (1 - e^{B\tau} [\rho + \lambda - b] [E[v] - \zeta]). \quad (38)$$

**Equation (28)** is then obtained by setting the expression above equal to zero for a finite positive value of  $\tau$ . If this expression is negative at  $\tau = 0$ , then we set  $\tau = 0$ , while we set  $\tau = \infty$  if it is strictly positive for all finite  $\tau$ .

2. Using the optimal policy in **equation (28)** in the Bellman equation **(27)** the value function takes the following form

$$v(\zeta) = \begin{cases} E[v] - \zeta & \text{if } \underline{\zeta} \leq \zeta \leq E[v] - \bar{v} \\ \underline{v} + B \underline{v} \bar{v} \left( \frac{E[v] - \zeta}{\bar{v}} \right)^{\frac{\rho+\lambda-a}{B}} & \text{if } E[v] - \bar{v} < \zeta < E[v] \\ \underline{v} & \text{if } \zeta \geq E[v] \end{cases}, \quad (39)$$

Integrating both sides of **equation (39)** with respect to  $\hat{G}$  we obtain

$$\begin{aligned} E[v] &= \underline{v} \left( 1 - \hat{G}(E[v] - \bar{v}) \right) + B \underline{v} \bar{v} \int_{E[v] - \bar{v}}^{E[v]} \left( \frac{E[v] - x}{\bar{v}} \right)^{\frac{\rho+\lambda-a}{B}} d\hat{G}(x) + \\ &+ \bar{v} \int_{\underline{\zeta}}^{E[v] - \bar{v}} \left( \frac{E[v] - x}{\bar{v}} \right) d\hat{G}(x). \end{aligned} \quad (40)$$

3. If  $\underline{\zeta} \geq 0$  then  $\hat{\tau}(\underline{\zeta}) > 0$ . Note that  $E[v] < \bar{v}$  since the right hand side is the value of the case of not observation cost. Hence  $\hat{\tau}(\underline{\zeta}) > 0$ , as we argued for the general case.
4. If  $\underline{\zeta} \geq 0$  [equation \(40\)](#) becomes

$$E[v] = \underline{v} + B \underline{v} \bar{v} \int_{\underline{\zeta}}^{E[v]} \left( \frac{E[v] - \zeta}{\bar{v}} \right)^{\frac{\rho + \lambda - a}{B}} d\hat{G}(\zeta) . \quad (41)$$

The first and second derivatives of the rhs of the last equation w.r.t  $E[v]$  are

$$\begin{aligned} 1 > & [(\rho + \lambda) - b]^{\frac{\rho + \lambda - a}{B} - 1} \int_{\underline{\zeta}}^{E[v]} (E[v] - \zeta)^{\frac{\rho + \lambda - a}{B} - 1} d\hat{G}(\zeta) \geq 0, \\ & [(\rho + \lambda) - b]^{\frac{\rho + \lambda - a}{B} - 1} \left( \frac{\rho + \lambda - a}{B} - 1 \right) \int_{\underline{\zeta}}^{E[v]} (E[v] - \zeta)^{\frac{\rho + \lambda - a}{B} - 2} d\hat{G}(\zeta) \geq 0. \end{aligned}$$

Note that the first derivative is zero at  $E[v] = \underline{\zeta}$ . If  $E[v] < 1/(\rho + \lambda - b)$  the first derivative is strictly increasing and strictly smaller than one. Hence there is at most one solution for [equation \(41\)](#) with  $E[v] < 1/(\rho + \lambda - b)$ . If there is a second intersection where  $E[v]$  satisfies [equation \(41\)](#) it must be at a point where  $E[v] > 1/(\rho + \lambda - b)$  since the slope must be larger than one. Given that the rhs of [equation \(41\)](#) is convex in  $E[v]$  there are at most two intersection. Now we show that there is at least one intersection. Consider the case where the distribution  $\hat{G}$  is concentrated at  $\zeta = 0$ . Then there is a unique intersection at  $E[v] = \bar{v}$ . Next, for any other  $\hat{G}$ , fixing a given  $E[v]$ , the rhs of [equation \(41\)](#) is smaller than in the case of  $\hat{G}$  concentrated. Hence, there is always an intersection with slope smaller than one.

5. The solution of the foc is a maximum. Consider a case where  $E[v] - \underline{\zeta} < \zeta < E[v]$ , then the second derivative of the right hand side of the Bellman equation, evaluated at [equation \(28\)](#) gives:

$$e^{-(\rho + \lambda - a)\tau(\zeta)} \left( -(\rho + \lambda - a) + e^{B\tau(\zeta)} [-(\rho + \lambda) + b]^2 [E[v] - \zeta] \right) = e^{-(\rho + \lambda - a)\tau(\zeta)} B < 0.$$

6. Since  $\mathbb{E} \left[ \max(E[v] - \zeta, 0)^{\frac{\rho + \lambda - a}{B}} \right]$  is the expected value of a convex function of  $\zeta$  for a fixed value of  $E[v]$ , a mean preserving spread in  $\hat{G}(\cdot)$  increases its value. Hence a mean preserving spread in the distribution of  $\zeta$  increases the rhs of [equation \(41\)](#) for each  $E[v]$ , and thus increases the value of the intersection.

□

**Proof.** (of [Proposition 9](#)) Let  $\bar{\theta} \equiv \mathbb{E}(\theta)$  and  $\bar{\tau} = \tau(\bar{\theta})$ . Integrating the value function in [equation \(27\)](#) for the case of zero variance of observation cost gives

$$v(\bar{\theta}) - \bar{\theta} = \frac{(1 - e^{-(\rho+\lambda-a)\bar{\tau}}) / (\rho + \lambda - a) - \bar{\theta}}{1 - e^{-(\rho+\lambda-b)\bar{\tau}}}$$

Combining the right hand side of this equation with [equation \(28\)](#) allows us to write  $v(\bar{\theta}) - \bar{\theta} = \frac{e^{-(b-a)\bar{\tau}}}{\rho+\lambda-b}$  which then gives

$$\tau'(\bar{\theta}) = \frac{\rho + \lambda - b}{b - a} e^{(b-a)\bar{\tau}} = \frac{\rho + \lambda - b}{b - a} e^{\sqrt{2\bar{\theta}(b-a)}} = \frac{\rho + \lambda - b}{\eta(\eta - 1)\sigma^2/2} e^{\sqrt{2\bar{\theta}\eta(\eta-1)\sigma^2/2}} \quad (42)$$

where the last equality uses the square root formula derived above for the limit case of no variance. Next, assume a constant strictly positive coefficient of variation  $\sqrt{\nu} > 0$ , so that  $Var(\theta) = \nu\bar{\theta}^2$ . When  $\nu \approx 0$ ,  $\tau(\theta)$  is approximately given by  $\sqrt{2\bar{\theta}/B}$  so that  $(\mathbb{E}(\tau))^2 = 2\bar{\theta}/(b-a) + o(\bar{\theta})$ . Using [equation \(42\)](#), and  $Var(\tau) = (\tau'(\bar{\theta}))^2 Var(\theta) + o(Var(\theta))$  we obtain

$$\begin{aligned} \frac{Var(\tau)}{(\mathbb{E}(\tau))^2} &= \left( \frac{\rho + \lambda - b}{b - a} \right)^2 \left( 1 + \sqrt{2\bar{\theta}(b-a) + o(\bar{\theta})} + \frac{1}{2!} (2\bar{\theta}(b-a) + o(\bar{\theta})) + \dots \right)^2 \frac{Var(\theta)}{2\bar{\theta}/(b-a) + o(\bar{\theta})} \\ &\quad + \frac{o(\nu\bar{\theta}^2)}{2\bar{\theta}/(b-a) + o(\bar{\theta})} \\ &= \left( \frac{\rho + \lambda - b}{(b-a)} \right)^2 \left( 1 + o(\sqrt{\bar{\theta}}) \right) \frac{Var(\theta)}{2\bar{\theta}/(b-a) \left( 1 + \frac{o(\bar{\theta})}{2\bar{\theta}/(b-a)} \right)} + o(\bar{\theta}) \\ &= \left( \frac{\rho + \lambda - b}{(b-a)} \right)^2 \frac{Var(\theta)}{\bar{\theta}^2} \frac{(b-a)\bar{\theta}}{2} \left( 1 + o(\sqrt{\bar{\theta}}) \right) \left( 1 - \frac{o(\bar{\theta})}{\bar{\theta}} \right) + o(\bar{\theta}) \\ &= (\rho + \lambda - b)^2 \nu \frac{\bar{\theta}}{2(b-a)} \left( 1 + o(\sqrt{\bar{\theta}}) - \frac{o(\bar{\theta})}{\bar{\theta}} - \frac{o(\bar{\theta}^{\frac{3}{2}})}{\bar{\theta}} \right) + o(\bar{\theta}) \\ &= (\rho + \lambda - b)^2 \nu \frac{\bar{\theta}}{2(b-a)} + o(\bar{\theta}) \end{aligned}$$

which, using  $\nu = \frac{Var(\theta)}{(\bar{\theta})^2} > 0$ , gives [equation \(29\)](#).

**Proof.** (of [Proposition 10](#)) The cumulative distribution function of observation cost is given by  $\hat{G}(\zeta; E[v]) = 1 - K(\tau(\zeta; E[v]))$ . By differentiating the last equation with respect to  $\zeta$ , we obtain

$$\hat{g}(\zeta; E[v]) = -K'(\tau(\zeta; E[v])) \frac{\partial \tau(\zeta; E[v])}{\partial \zeta} \quad \text{for all } \zeta > \underline{\zeta}. \quad (43)$$

Then by replacing the optimal policy and its derivative we obtain

$$\begin{aligned}
\hat{g}(\zeta; E[v]) &= \frac{\xi \exp(-\xi \tau(\zeta))}{B (E[v] - \zeta)} \\
&= \frac{\xi \exp\left(\frac{\xi}{B} \log([\rho + \lambda - b] [E[v] - \zeta])\right)}{B [E[v] - \zeta]} \\
&= a_0 \frac{\exp(\log(a_1^{a_0} [E[v] - \zeta]^{a_0}))}{[E[v] - \zeta]} = a_0 \frac{a_1^{a_0} [E[v] - \zeta]^{a_0}}{[E[v] - \zeta]} = a_0 a_1^{a_0} [E[v] - \zeta]^{a_0-1}
\end{aligned}$$

where  $a_0 = \frac{\xi}{B}$  and  $a_1 = \rho + \lambda - b$ .

$$E[v] = \underline{v} + B \underline{v} \bar{v} \int_{E[v]-\bar{v}}^{E[v]} \left( \frac{E[v] - \zeta}{\bar{v}} \right)^{\frac{\rho+\lambda-a}{B}} \hat{g}(\zeta; E[v]) d\zeta . \quad (44)$$

Finally, by using [equation \(30\)](#) we obtain

$$E[v] = \underline{v} + B \underline{v} \bar{v} \frac{\xi}{\bar{v} B} \int_{E[v]-\bar{v}}^{E[v]} \left( \frac{E[v] - \zeta}{\bar{v}} \right)^{\frac{\rho+\lambda-a+\xi}{B}-1} d\zeta , \quad (45)$$

which gives

$$E[v] = \underline{v} + B \underline{v} \bar{v} \frac{\xi \underline{v}}{1 + \xi \underline{v}} = \underline{v} + (\bar{v} - \underline{v}) \frac{\xi \underline{v}}{1 + \xi \underline{v}} .$$

The distribution  $\zeta$  is then a displaced Beta distribution, i.e.  $\zeta = c + dz$  where  $z$  has a standard beta distribution with shape parameters  $(\tilde{a}, \tilde{b}) = 1, \xi/B$ . The displaced parameters are  $c = E[v] - \bar{v}$  and  $d = \bar{v}$ . Hence the coefficient of variation of  $\zeta$ , denoted by  $cv(\zeta)$ , is:

$$cv(\zeta) = \sqrt{\frac{\xi/B}{2 + \xi/B} \frac{1}{1 - (1 + \xi/B)(\bar{v} - E[v])/\bar{v}}},$$

and using the expression for  $E[v]$  we obtain  $\bar{v} - E[v] = (\bar{v} - \underline{v})/(1 + \xi \underline{v})$ , and then

$$cv(\zeta) = \sqrt{\frac{\xi/B}{2 + \xi/B} \frac{1}{1 - \left(\frac{1+\xi/B}{1+\xi \underline{v}}\right) \left(\frac{\bar{v}-\underline{v}}{\bar{v}}\right)}},$$

which using that  $\underline{v} = (1/B)(\bar{v} - \underline{v})/\bar{v}$  then becomes

$$cv(\zeta) = \sqrt{\frac{\xi/B}{2 + \xi/B} \left( \frac{1 + (\xi/B)(\bar{v} - \underline{v})/\bar{v}}{\underline{v}/\bar{v}} \right)} = \sqrt{\frac{\xi/B}{2 + \xi/B} \left( \frac{1 + \xi \underline{v}}{\underline{v}/\bar{v}} \right)}$$

□

## B Obtaining the distribution $H(t'|t)$ from the firm policy

We can use the solution to the firm problem  $\tau(\cdot)$ , and the properties of the process governing the signal  $\zeta$  to derive the conditional right-CDF for the times to the next observation, which in [Section 2](#) was denoted by  $H(\cdot|t)$ . In particular, let  $\zeta = \hat{\zeta}(t)$  denote the mapping from an observation of length  $t$  to the associated signal  $\zeta$ , where  $\hat{\zeta}(t) = \tau^{-1}(t)$ . Let also  $J(\zeta'; s | \zeta) \equiv \int_{\underline{\theta}}^{\infty} (1 - G(\zeta' | \theta')) dF(\theta'; s | \zeta)$  denote the right-CDF of drawing the signal  $\zeta'$  upon an observation taking place  $s$  periods after the last observation, where the firm drew a signal equal to  $\zeta$ . Consider the right-CDF for the planned time to the next observation,  $t'$ , conditional on the last planned observation time being equal to  $t$ , which we denote by  $H(t'|t)$ . Such CDF solves:

$$H(t'|t) = e^{-\lambda t} J(\hat{\zeta}(t'); t | \hat{\zeta}(t)) + \int_0^t \lambda e^{-\lambda s} J(\hat{\zeta}(t'); s | \hat{\zeta}(t)) ds \quad \text{for all } t, t'. \quad (46)$$

These equations trace the ways in which firms with last planned observation of length  $t$  will plan a time to the next observation of length  $t'$ . The current observation can occur either at the planned observation date ( $t$  periods after the last observation), or following a substitution taking place any time  $s \in [0, t]$  since the last observation. The first term on the right of [equation \(46\)](#) takes into account of the case in which no substitution has occurred, which happens with probability  $e^{-\lambda t}$ . The second term on the right of [equation \(46\)](#) takes into account of the case in which a substitution has occurred at some time  $s \in [0, t]$ , and weights each of these event with the appropriate density  $\lambda e^{-\lambda s}$ .

We next apply [equation \(46\)](#) to two cases of interest. First, consider the case of i.i.d. observations of [Section 5](#). In this case, the distribution  $J(\zeta'; s | \zeta)$  becomes  $J(\zeta'; s | \zeta) = 1 - \hat{G}(\zeta')$  for all  $s$ ,  $\zeta'$  and  $\zeta$ . Substituting the latter into [equation \(46\)](#) we obtain  $H(t'|t) \equiv \hat{H}(t') = 1 - \hat{G}(\hat{\zeta}(t'))$ .

Next, consider the case of Markov observation times in [Section 4](#). In this case the distribution  $J(\zeta'; s | \zeta)$  becomes  $J(\zeta'; s | \zeta) = J(\theta'; s | \theta) = 1 - F(\theta'; s | \theta)$ . Using the specification for  $F$  in [equation \(23\)](#) in the latter, and substituting into [equation \(46\)](#) we obtain:

$$H(t'|t) = \begin{cases} 1 - \hat{F}(\tau^{-1}(t')) \frac{\kappa^{-1}}{\kappa^{-1} + \lambda} \left( 1 - e^{-(\lambda + \kappa^{-1})t} \right) - \frac{\lambda}{\kappa^{-1} + \lambda} - \frac{\kappa^{-1}}{\kappa^{-1} + \lambda} e^{-(\lambda + \kappa^{-1})t} & \text{if } t' > t \\ 1 - \hat{F}(\tau^{-1}(t')) \frac{\kappa^{-1}}{\kappa^{-1} + \lambda} \left( 1 - e^{-(\lambda + \kappa^{-1})t} \right) & \text{if } t' \leq t \end{cases}. \quad (47)$$

Finally, we can then use the results of [Lemma 2](#) to obtain the cross-sectional right CDF of planned observation times  $K(T) = -\int_0^{\infty} H(T|t) dK(t)$  for given CDF  $H$ , and from that

obtain the cross-sectional right CDF of times between consecutive observations including substitutions,  $\hat{K}(T) \equiv e^{-\lambda T} K(T)$ .

## C Calibration

We divide the parameters of the model in two categories. In the first category, we have parameters determining the benefit of price observations,  $(\rho, \lambda, \eta, \sigma, \phi, \mu)$ , which we set ex-ante. In particular, the annual discount rate is set equal to  $\rho = 0.02$ . Consistently with estimates in by Nakamura and Steinsson (2008), the frequency of product substitutions is assumed to be equal to  $\lambda = 0.25$ . We assume a steady state markup of one fourth implying  $\eta = 5$ . Using Lemma 3, together with the frequency and variance of price changes estimated by Nakamura and Steinsson (2008) for the U.S. CPI, i.e.  $Var(\Delta \log(p)) = 0.01$  and  $R = 1.3$ , we obtain that  $\sigma = 0.114$ . We assume that that marginal cost is a martingale, implying  $\mu - \phi + \sigma^2/2 = 0$ , so that prices change infrequently as it is well documented. We note in fact that if there is a non-zero drift in the level of the nominal marginal cost, i.e.  $\mu - \phi + \sigma^2/2 \neq 0$ , then the optimal price plan implies a continuous price change. On the other hand, if the level of the nominal marginal cost is a martingale, i.e.  $\mu - \phi + \sigma^2/2 = 0$ , then there is no incentive to use price plans, i.e. the optimal price plan features a constant nominal price between observations.

The second category of parameters refers to the modelling of the cost of the observations, and in particular to the distributions  $F$  and  $G$ . We consider two specifications of  $F$  and  $G$ . The first specification refers to the case of i.i.d. observations of Section 5. We assume that  $F$  is given by equation (26) and that  $G(\zeta' | \theta) \equiv \hat{G}(\zeta')$  for all  $\zeta'$  and  $\theta$ , with  $\hat{F} = \hat{G}$ ;  $\hat{G}$  is given by a  $Gamma(\alpha_1, \alpha_2)$  distribution where  $\alpha_1$  controls the coefficient of variation of observation cost and  $\alpha_2$  its scale. The Gamma distribution is appealing as it encompasses many shapes with only two parameters. We consider alternative parametrizations of  $\gamma$  and  $\alpha_1$ . For each combination of  $\gamma$  and  $\alpha_1$ , we set  $\alpha_2$  so that the average (yearly) frequency of observations is  $R = 1.3$ . The second specification of  $F$  and  $G$  refers to the case of persistent Markov observations of Section 4, where  $F$  is given by equation (23) and  $G(\zeta | \theta) = \mathbf{1}_{\zeta \geq \theta}$  for all  $\theta'$  and  $\theta$ . Similarly to the case of i.i.d. observations, we parametrize the distribution  $\hat{F}$  by a  $Gamma(\alpha_1, \alpha_2)$  distribution. We consider alternative parametrizations of  $\kappa$  and  $\alpha_1$ . For each combination of  $\kappa$  and  $\alpha_1$ , we set  $\alpha_2$  so that the average (yearly) frequency of observations is  $R = 1.3$ .

## D The real effects of a monetary shock

We describe how the distribution of times between consecutive observations relate to the real effects of monetary shocks. We assume that at time  $t = 0$  there is an unanticipated permanent increase in the level of the money supply by  $\delta$  log points. We use the result that, for small shocks  $\delta$  and small average duration of observations, the general equilibrium feedback effect on firms' decision rule is negligible, so that ignoring the effect of the shock on the distribution of times between observations,  $H(t|t)$ , gives a good approximation of the firm's behavior during the convergence to the steady state.<sup>28</sup>

We study the effect of an aggregate monetary shock of size  $\delta$  on the deviations of the aggregate price level  $P(t)$  at  $t \geq 0$  from the initial level  $P(0)$ , denoted by  $\mathcal{P}(\delta, t) = P(t)/P(0)$ , and on the deviations of the aggregate output  $c_t$  at  $t \geq 0$  from the initial level  $c_0$ , denoted by  $\mathcal{C}(\delta, t) = c_t/c_0$ . Consider the general equilibrium model described in the appendix of [Alvarez and Lippi \(2014\)](#), where equilibrium conditions in the money market (i.e.  $c_t^\epsilon = M_t/P_t$ ) imply that the equilibrium dynamics of output is given by:  $\log(\mathcal{C}(\delta, t)) = (1/\epsilon)(\delta + \mu t - \log(\mathcal{P}(\delta, t)))$ . Let  $\mathcal{P}(\delta, t; r \geq t)$  and  $\mathcal{P}(\delta, t; r < t)$  be the average price conditional on the subset of firms that made the last observation, respectively, more and less than  $r$  periods ago, divided by  $P(0)$ . It follows from the definition of  $\mathcal{P}(\delta, t)$  and  $Q(t)$  that  $\mathcal{P}(\delta, t) = Q(t) \mathcal{P}(\delta, t; r \geq t) + (1 - Q(t)) \mathcal{P}(\delta, t; r < t)$ . The last expression implies  $\log(\mathcal{P}(\delta, t)) = \mu t + (1 - Q(t)) \delta + o(\delta)$ . It immediately follows that output impulse response  $t$  periods after the shock is proportional to the fraction of firms that are still unaware of the monetary shock, i.e.  $\log(\mathcal{C}(\delta, t)) \approx (\delta/\epsilon) Q(t)$ .

We introduce a measure of the real effects of the monetary shocks, defined by the area under the impulse response of output for a monetary shock of size  $\delta$ , i.e.

$$\int_{t_0}^{\infty} \log(\mathcal{C}(\delta, t)) dt = \frac{1}{\epsilon} \int_{t_0}^{\infty} \log(M_t/P_t) dt \approx \frac{\delta}{\epsilon} \int_0^{\infty} Q(t) dt \equiv \frac{\delta}{\epsilon} \mathcal{M} . \quad (48)$$

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<sup>28</sup> In [Alvarez, Lippi, and Paciello \(2012\)](#) we solve numerically various general equilibrium versions of this model (including menu costs or observation costs or both) and verify the quality of the approximation when the feedback effect is ignored.