THE INFORMATIVENESS PRINCIPLE UNDER LIMITED LIABILITY

Pierre Chaigneau
Alex Edmans
Daniel Gottlieb

The Informativeness Principle Under Limited Liability
Pierre Chaigneau, Alex Edmans, and Daniel Gottlieb
NBER Working Paper No. 20456
September 2014
JEL No. D86,J33

ABSTRACT

This paper shows that the informativeness principle does not automatically extend to settings with limited liability. Even if a signal is informative about effort, it may have no value for contracting. An agent with limited liability is paid zero for certain output realizations. Thus, even if these output realizations are accompanied by an unfavorable signal, the payment cannot fall further and so the principal cannot make use of the signal. Similarly, a principal with limited liability may be unable to increase payments after a favorable signal. We derive necessary and sufficient conditions for signals to have positive value. Under bilateral limited liability and a monotone likelihood ratio, the value of information is non-monotonic in output, and the principal is willing to pay more for information at intermediate output levels.

Pierre Chaigneau
HEC Montreal
3000 chemin de la côte-Sainte-Catherine
Montreal H3T 2A7
Canada
pierre.chaigneau@hec.ca

Alex Edmans
The Wharton School
University of Pennsylvania
2460 Steinberg Hall - Dietrich Hall
3620 Locust Walk
Philadelphia, PA 19104
and NBER
aedmans@wharton.upenn.edu

Daniel Gottlieb
The Wharton School
University of Pennsylvania
3303 Steinberg Hall-Dietrich Hall
3620 Locust Walk
Philadelphia, PA 19104
dgott@wharton.upenn.edu

The informativeness principle (Holmstrom (1979), Shavell (1979), Gjesdal (1982), Grossman and Hart (1983), Kim (1995)) is believed to be one of the most robust results in contract theory. The textbook of Bolton and Dewatripont (2005, Section 4.7) states that the moral hazard literature has produced very few general results – for example, even the intuitive result that a lower cost of effort increases the optimal effort level need not hold – but the informativeness principle is one of the few results that is general. They write: "among the main general predictions of the model is the informativeness principle, which says that the incentive contract should be based on all variables that provide information about the agent's actions." Formally, a signal s is valuable for the principal – i.e. an optimal contract is a function of s – if and only if output q is not a sufficient statistic for effort e given (q, s).

The informativeness principle was derived in settings without constraints on the contract. However, contracting constraints do exist in reality. Perhaps the most important one is limited liability, which applies to almost all contracts between employees and firms. Entrepreneurs raising financing also enjoy limited liability, because their equity cannot fall below zero. This paper studies whether the informativeness principle continues to hold under this important constraint. The textbook of Tirole (2006, p123) conjectured that it does: he writes: "added risk is bad when the limited liability constraint is binding", but we show that this is not always the case.[1]

Section 1 considers a model in which both the agent and principal are risk-neutral and the only contracting constraint is that the agent is protected by limited liability. We show that, if output q is a sufficient statistic for effort e given (q, s), then it remains the case that the signal has zero value. However, the reverse is no longer true: the optimal contract may be independent of s even if output is not a sufficient statistic. Some signals have zero value even though they are informative.

The intuition is as follows. Without contracting constraints, the principal can always make use of an informative signal by adjusting the agent's wage accordingly. If

---

[1]To establish the existence of an optimal mechanism, Holmstrom (1979) – and much of the ensuing literature – assumed that sharing rules have a bounded total variation. As he states, "this restriction is natural from a pragmatic point of view as well, since the agent's wealth puts a lower bound, and the principal's wealth ... an upper bound [on payments]." When deriving the informativeness principle, however, he implicitly assumed that the solution was interior. He conjectured that the informativeness principle may not hold under contracting constraints ("if, for administrative reasons, one has restricted attention a priori to a limited class of contracts (e.g., linear price functions or instruction-like step-functions), then informativeness may not be sufficient for improvements within this class") but does not formally study the implications of limited liability for the informativeness principle.

low output is accompanied by a low signal, the agent is paid even lower than with a high signal. However, with limited liability, the agent is already being paid zero under low output. The principal cannot punish the agent further upon a low signal, and so the signal is of no value even though it is informative about effort. Signals are only useful if they can be used to reduce the wage, which – under limited liability – requires the wage to be strictly positive to begin with. In turn, the agent is only paid a positive wage for the output level that maximizes the likelihood ratio. Thus, signals are only useful if they are informative about effort at this output level.

Section 2 considers additional contracting constraints. In Section 2.1, the principal, as well as the agent, is protected by limited liability. Thus, the contract can no longer involve the principal paying only in the maximum likelihood ratio state, as the required payment would exceed her pledgeable income. As in Innes (1990), the optimal contract is a "live-or-die" contract where the agent receives nothing if output falls below a threshold q^*, and the entire output if it exceeds the threshold. Section 2.2 imposes an additional monotonicity constraint which requires payments to each party to be non-decreasing in output. The optimal contract is an option on firm output: the agent receives nothing if output falls below q^*, and the residual $q - q^*$ if output exceeds it.

In both cases, the contract depends on the signal s if and only if it is informative about effort when $q = q^*$, i.e. at the center of the distribution, because then it can be used to allow the threshold q^* to vary with the signal. Signals that are informative about effort at the tails of the output distribution are of no value. The intuition for $q < q^*$ is as above: the agent is already being paid zero, and cannot be paid less upon a low signal without violating limited liability. Turning to $q > q^*$, the principal cannot receive any less upon a high signal. Without the monotonicity constraint, the principal receives zero and cannot receive less without violating limited liability; with the monotonicity constraint, the principal receives q^* and cannot receive less without violating monotonicity. Appendix B in the Online Appendix shows that the results continue to hold under a continuum of effort levels, holding fixed the target effort level.

The results have a number of economic implications. First, the presence of limited liability requires us to refine our notion of informativeness. When only the agent has limited liability, a signal is valuable if and only if it is informative about effort for output levels that maximize the likelihood ratio. When the principal also has limited liability, a signal is valuable if and only if it is informative about effort at the center of the distribution. In particular, a signal could be informative about effort almost every-

3

where, and still have no value in contracting. Second, the common practice of paying agents for luck, i.e. not filtering out industry shocks, is not necessarily suboptimal. If a firm suffers a catastrophe, the manager is typically paid zero, regardless of whether it was down to bad luck (e.g. industry performance was also poor) or shirking. In reality, instances of "pay for luck" typically concern very good or very bad outcomes – for example, Bertrand and Mullainathan (2001) consider how CEO pay varies with spikes and troughs in the oil price – but additional signals are only valuable for moderate outcomes. Third, the value of information is non-monotonic in output. Our results suggest that the principal should only invest in signals (e.g. through costly monitoring) at moderate output realizations.

1 Limited Liability on Agent

We consider a model with a principal (firm) and an agent (worker). Both parties are risk-neutral and the agent is protected by limited liability. The agent exerts unobservable effort of $e \in \{0, 1\}$, where $e = 0$ ("low effort") costs the agent 0, and $e = 1$ ("high effort") costs $C > 0$.

Effort improves the probability distribution of output $q \in \{q_1, q_2, ..., q_Q\}$ in the sense of first-order stochastic dominance. It also affects the probability distribution of an additional signal $s \in \{s_1, s_2, ..., s_S\}$. Let $\pi_{q,s}$ ($p_{q,s}$) denote the joint probability of (q, s) conditional on high (low) effort. Both output and the signal are contractible. We refer to each realization of an output/signal pair (q, s) as a "state".

The principal offers a vector of payments $\{w_{q,s}\}$ to the agent conditional on the state. She has full bargaining power. The agent accepts the contract if it satisfies his individual rationality constraint ("IR"):

$$\sum_{q,s} \pi_{q,s} w_{q,s} - C \geq 0. \tag{1}$$

The agent exerts high effort if the following incentive compatibility constraint ("IC") is satisfied:

$$\sum_{q,s} (\pi_{q,s} - p_{q,s}) w_{q,s} \geq C. \tag{2}$$

4

Finally, the agent's limited liability constraint ("LL") implies:

$$w_{q,s} \geq 0 \quad \forall q, s. \tag{3}$$

The IC (2) and LL (3) imply that the IR (1) is automatically satisfied, and so we ignore it in the analysis that follows.

We assume that the cost of effort C is sufficiently low that the principal wishes to implement high effort, else the optimal contract trivially involves a zero wage. The principal's problem is to find the contract $\{w_{q,s}\}$ that minimizes the expected wage:

$$\min_{w_{q,s} \geq 0} \pi_{H,1} w_{H,1} + \pi_{H,0} w_{H,0} + \pi_{L,1} w_{L,1} + \pi_{L,0} w_{L,0} \tag{4}$$

subject to IC (2).

Lemma 1 below states that a signal is valuable if and only if it is informative about effort in the states where the wage is strictly positive. Thus, the informativeness principle – that the contract does not depend on s if and only if the likelihood ratio is independent of s – continues to hold under limited liability when considering states in which the wage is strictly positive. (All proofs are in Appendix A.)

Lemma 1 *Let $(w_{q,s})$ be an optimal contract for implementing $e = 1$ with $w_{q,s} > 0$ and $w_{q,s'} > 0$ for some q, s, and s'. Then, $w_{q,s} = w_{q,s'}$ if and only if $\frac{\pi_{q,s}}{p_{q,s}} = \frac{\pi_{q,s'}}{p_{q,s'}}$.*

Lemma 2 states that the wage is strictly positive only in states that maximize the likelihood ratio.

Lemma 2 *Let the vector of payments $(w_{q,s})$ be an optimal contract for implementing $e = 1$. If $\frac{\pi_{q,s}}{p_{q,s}} < \max_{(q',s')} \left\{ \frac{\pi_{q',s'}}{p_{q',s'}} \right\}$, then $w_{q,s} = 0$.*

Combining these results, a signal is valuable if and only if it is informative about effort in states with the highest likelihood ratio. This result is stated in Proposition 1 below:

Proposition 1 *A signal has positive value if and only if, for all $(q, s) \in \arg\max_{(q',s')} \frac{\pi_{q',s'}}{p_{q',s'}}$, there exists $\hat{s}$ such that $\frac{\pi_{q,s}}{p_{q,s}} \neq \frac{\pi_{q,\hat{s}}}{p_{q,\hat{s}}}$.*

When the agent exhibits limited liability, a signal has positive value if and only if it affects the likelihood ratio at the output level at which the likelihood ratio is maximized without the signal, as only then is the wage associated with this output level positive. In this case, the principal can improve on the contract by making the wage at this output level contingent upon the signal – increase it at the signal where (q, s) has the highest likelihood ratio and decrease it to zero at other signal realizations. In contrast, a signal is not useful if it changes the likelihood ratio only for output levels at which the likelihood ratio is not maximized. Since the wage is zero to begin with, the principal cannot decrease it upon a low signal.

In sum, if output q is a sufficient statistic for effort e given (q, s), the signal s has zero value to the principal. However, even if q is not a sufficient statistic, s still has zero value if it is informative about effort only for output levels at which the likelihood ratio is not maximized. The presence of limited liability requires us to refine our notion of informativeness – what matters is whether signals are informative about effort in *states with the maximum likelihood ratio*, rather than in general.

We conclude this section with two examples. The first one illustrates the result from Proposition 1:

Example 1 *Consider a setting with binary outputs and binary signals with the following conditional probabilities:*

	$e = 1$		$e = 0$		Likelihood Ratio	
	$q = q_H$	$q = q_L$	$q = q_H$	$q = q_L$	$q = q_H$	$q = q_L$
$s = 1$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{8}{3}$	$\frac{1}{3}$
$s = 0$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{8}{3}$	$\frac{2}{3}$
Marginal	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{3}{4}$		

Note that q is not a sufficient statistic for e given (q, s) since the likelihood ratio conditional on $q = q_L$ is not constant:

$$\frac{p_{L,1}}{\pi_{L,1}} = \frac{1/6}{1/2} = \frac{1}{3}, \quad \frac{p_{L,0}}{\pi_{L,0}} = \frac{1/6}{1/4} = \frac{2}{3}.$$

By Lemma 2, the optimal contract entails paying only in states $(q_H, 1)$ and $(q_H, 0)$ where the likelihood ratio is maximized at $\frac{8}{3}$. Since the likelihood ratio is equal in both states, any share of payments across these states generates the same payoff to the principal as long as they add up to $\frac{24}{5}$ (which is the amount required to satisfy incentive

compatibility). One solution is $w_{H,1} = w_{H,0} = \frac{12}{5}$, $w_{L,1} = w_{L,0} = 0$. The wage does not depend on s, because it is only informative at output levels for which the wage is already zero. Since an optimal contract is not a function of the signal realization, the signal has zero value.

The second example shows that the key driver of our results is limited liability and not risk neutrality. With risk aversion, paying different amounts in states with equal likelihood ratios not only fails to improve incentives, but also forces the principal to compensate the agent for the additional risk. Thus, risk aversion removes the multiplicity of optimal contracts from the previous example.

Example 2 *Let the agent's utility be $u(w, e) = \sqrt{w} - e$, $e \in \{0, 1\}$. Consider the following conditional probabilities:*

	$e = 1$		$e = 0$	
	$q = q_H$	$q = q_L$	$q = q_H$	$q = q_L$
$s = 1$	$\frac{\alpha}{2}$	$\frac{\beta}{2}$	0	γ
$s = 0$	$\frac{1-\alpha}{2}$	$\frac{1-\beta}{2}$	0	$1 - \gamma$
Marginal	$\frac{1}{2}$	$\frac{1}{2}$	0	1

Let $\frac{\beta}{2} \leq \gamma \leq \frac{1+\beta}{2}$. The likelihood ratios conditional on $q = q_L$ are not constant whenever $\beta \neq \gamma$:

$$\frac{p_{L,1}}{\pi_{L,1}} = 2\frac{\gamma}{\beta}, \quad \frac{p_{L,0}}{\pi_{L,0}} = 2\frac{1-\gamma}{1-\beta}.$$

In Appendix A, we show that the unique solution is $w_{H,1} = w_{H,0} = 4$ and $w_{L,1} = w_{L,0} = 0$. Thus, the optimal contract is again independent of the signal even though the signal is informative about effort.

2   Additional Contracting Constraints

When the only constraint is limited liability on the agent, generically the wage is positive in a single state only. In a tradition initiated by Innes (1990), we now introduce additional contracting constraints that yield more realistic contracts. For tractability, we retain the assumption of binary effort levels; Appendix B studies the continuous effort case.

Although the results can be easily replicated for the discrete output case of Section 1, we expand the model to a continuum of outputs since it simplifies notation. Formally, output is now distributed over an interval $q \in [0, \bar{q}]$, where $\bar{q}$ may be $+\infty$. Effort $e \in \{0, 1\}$ and the signal $s \in \{s_1, ..., s_S\}$ are specified as before.[2]

Since the model combines discrete and continuous variables, it is convenient to specify the distributions in terms of the conditional $f(q|e, s)$ and the marginal $\phi_{\hat{e}}^{\hat{s}} \equiv \Pr(s = \hat{s}|e = \hat{e})$. The joint distribution of (q, s) conditional on effort is determined by their product. Conditional on effort e and signal s, output q is distributed according to the probability density function ("PDF"):

$$f(q|e, s) \equiv \begin{cases} \pi_s(q) & \text{if } e = 1 \\ p_s(q) & \text{if } e = 0 \end{cases}.$$

The marginal distribution of q is:

$$f(q|e = 1) = \sum_s \phi_1^s \pi_s(q), \text{ and } f(q|e = 0) = \sum_s \phi_0^s p_s(q). \tag{5}$$

Let

$$LR_s(q) \equiv \frac{\phi_1^s \pi_s(q)}{\phi_0^s p_s(q)} \tag{6}$$

denote the likelihood ratio associated with signal s at output q, and $w_s(q)$ denote the agent's payment conditional on output q and signal s.

The IC is:

$$\sum_s \int_0^{\bar{q}} w_s(q) [\phi_1^s \pi_s(q) - \phi_0^s p_s(q)] \, dq \geq C. \tag{7}$$

2.1 Bilateral Limited Liability

In this subsection, we assume that both parties are subject to limited liability:

$$0 \leq w_s(q) \leq q. \tag{8}$$

As in Section 1, IC and the agent's LL guarantee that IR holds.

---

[2] Although the assumption of a discrete signal space is unimportant for our results, it allows us to avoid unnecessary measurability issues. Since s is discrete and q is continuous, we use the notation $\pi_s(q)$ rather than $\pi_{q,s}$ as in the previous section.

The principal offers a contract that minimizes the expected wage:

$$\min_{\{w_s(q)\}} \sum_s \int_0^{\bar{q}} w_s(q)\, \phi_1^s \pi_s(q)\, dq,$$

subject to (7) and (8). Recall that in Section 1, where only the agent is subject to limited liability, the principal pays only in the maximum likelihood ratio state. This requires her to make a very large payment, which would violate her LL.[3] Thus, she must spread the payments out across more states. As in Innes (1990), the solution involves paying the minimum amount possible – zero – when the likelihood ratio is below a certain threshold κ, and the maximum amount possible – the whole output – when it exceeds that threshold. The threshold level is chosen so that IC holds with equality. If more than one such threshold exists, the optimal contract involves the largest one:[4]

$$\kappa \equiv \sup\left\{ \hat{\kappa} : \sum_s \int_{LR_s(q)>\hat{\kappa}} w_s(q)\left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q) \right] dq = C \right\}. \tag{9}$$

As we show in Appendix A, κ exists.

Lemma 3 *The optimal contract is* $w_s(q) = \mathbf{1}_{LR_s(q)>\kappa} \cdot q$.

In the remainder of this section, we assume that the distributions of outputs conditional on (q,s) satisfy the monotone likelihood ratio property ("MLRP"): $\frac{\pi_s(q)}{p_s(q)}$ is increasing in q $\forall$ s. Then, Lemma 3 implies that there exist unique thresholds $q_{s_1}^*, ..., q_{s_S}^*$, such that the principal keeps (pays) the entire output when it is below (above) that threshold. For a given s, the threshold levels solve $LR_s(q_s^*) = \kappa$ and

$$\sum_s \int_{q>q_s^*} q\left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q) \right] dq = C. \tag{10}$$

---

[3]Intuitively, with a continuum of outputs, the principal wishes to concentrate the payment as much as possible in a neighborhood around the maximum likelihood ratio state. Without limited liability on the principal, existence of an optimal contract is typically an issue. The contract cannot involve the principal paying only in the state with the highest likelihood ratio (as with discrete outputs) since this is a set of measure zero, so it must involve her paying in a neighborhood around that state. Without limited liability, the principal can generically improve on the contract by concentrating the payment in a smaller neighborhood, in which case an optimal contract fails to exist. Accordingly, Section 1 assumed discrete outputs.

[4]In general, there may be an interval of optimal thresholds which happen with probability zero. Under full support and continuity of the PDFs, the optimal threshold κ is unique.

In general, these thresholds will be contingent on the signal s, and so the contract is contingent upon both output and the signal. Proposition 2 gives a necessary and sufficient condition under which the thresholds are independent of the signal ($q^* = LR_s^{-1}(\kappa) \ \forall \ s$). In this case, there exists an optimal contract that is contingent only upon output and the signal has zero value.

Proposition 2 *Suppose the conditional distributions satisfy MLRP. The optimal contract is independent of the signal if and only if, $\forall \ s$ and s'*

$$\frac{\phi_1^s}{\phi_0^s}\frac{\pi_s(q^*)}{p_s(q^*)} = \frac{\phi_1^{s'}}{\phi_0^{s'}}\frac{\pi_{s'}(q^*)}{p_{s'}(q^*)}$$

for q^* given by $\int_{q^*}^{\bar{q}} q\left[f\left(q|e=1\right) - f\left(q|e=0\right)\right]dq = C$.

Thus, any signal that does not affect the likelihood ratio at the threshold output q^* does not influence the optimal contract and has zero value. Information about effort at output levels in the tails of the distribution is not valuable. The signal has no value for $q < q^*$ because the principal is already paying the lowest amount possible (zero). It has no value for $q > q^*$ because the principal is already paying the full output. A signal may affect the likelihood ratio almost everywhere and still have zero value: all that matters is the likelihood ratio at the threshold q^*, which has zero measure.

As a result, signals are only useful for moderate output realizations. It may not be optimal to investigate whether very good or very bad performance was down to luck (e.g. industry conditions) or effort. For example, if the firm suffers a catastrophe, the manager is fired and paid zero. Investigating whether the catastrophe was due to shirking or bad luck is not useful, since the agent will be paid zero in either case. In practice, situations in which executives are rewarded for luck typically involve extreme realizations; our model suggests that these contracts might indeed be optimal.

2.2 Monotonicity Constraint

We now introduce a monotonicity constraint as in Innes (1990):

$$0 \leq w\left(q + \epsilon\right) - w\left(q\right) \leq \epsilon, \ \forall \ \epsilon > 0. \tag{11}$$

Innes (1990) justifies these constraints on two grounds. If the constraint on the right did not hold, the agent could borrow on his own account to artificially increase output,

raising his payoff. If the constraint on the left did not hold, the principal would exercise her control rights to "burn" output, raising her payoff.

For simplicity, we assume that the likelihood ratio is unbounded from above:

$$\lim_{q \to \bar{q}} LR_s(q) = +\infty, \ \forall s. \tag{12}$$

This assumption is not needed for our main results. However, it allows us to rule out corner solutions, thereby ensuring that the thresholds in the optimal contract are lower than $\bar{q}$. The optimal contract is then given by Lemma 4:

Lemma 4 *Suppose the conditional distributions satisfy MLRP. The optimal contract is $w_s(q) = \max\{q, z_s\}$, where $(z_{s_1}, ..., z_{s_S})$ satisfy the IC (7) with equality and*

$$\frac{\phi_1^s \int_{z_s}^{\bar{q}} \pi_s(q) \, dq}{\phi_0^s \int_{z_s}^{\bar{q}} p_s(q) \, dq} = \frac{\phi_1^{s'} \int_{z_{s'}}^{\bar{q}} \pi_{s'}(q) \, dq}{\phi_0^{s'} \int_{z_{s'}}^{\bar{q}} p_{s'}(q) \, dq} \ \forall s, s',$$

Lemma 4 yields a standard option contract, where the threshold z_s is the strike price.[5] If output exceeds z_s, the agent receives the residual $q - z_s$, rather than the entire output as in Section 2.1. In general, z_s may depend on the realized signal s. Proposition 3 gives the conditions under which the strike price is independent of the signal (i.e. $z_s = z^* \ \forall \ s$), and so the signal has no value for the contract.

Proposition 3 *Suppose the conditional distributions satisfy MLRP. The optimal contract is independent of the signal if and only if, $\forall \ s, s'$,*

$$\frac{\Pr(q > z^*, s | e = 1)}{\Pr(q > z^*, s | e = 0)} = \frac{\Pr(q > z^*, s' | e = 1)}{\Pr(q > z^*, s' | e = 0)},$$

where z^* is given by $\int_{z^*}^{\bar{q}} (q - z^*) [f(q | e = 1) - f(q | e = 0)] \, dq = C$.

Proposition 3 shows that the intuition behind Section 2.1 continues to apply when a monotonicity constraint is introduced. Regardless of whether the monotonicity constraint is imposed, the agent receives zero for low output levels and the maximum

---

[5]Innes (1990) considers a financing model, where the contract stipulates the payment from the agent to the principal. The optimal contract is debt, and so the agent has equity – an option on output. Here, we consider a contracting model, where the contract stipulates the payment from the principal to the agent, in line with the literature on the informativeness principle.

possible for high output levels – without monotonicity he receives the maximum possible without violating the principal's limited liability constraint; with monotonicity he receives the maximum possible without violating the monotonicity constraint. The principal's only degree of freedom is on the threshold z_s that separates "low" from "high" output levels. Thus, the signal is only valuable if it leads to the principal optimally setting different strike prices – i.e. it affects the likelihood ratio of the event that the agent's option is in-the-money.

3   Conclusion

This paper studies whether the informativeness principle holds under limited liability, an important constraint in most contracting environments. The limited liability (and, where imposed, monotonicity) constraints bind at almost all output levels under the optimal contract. As a result, signals may be informative about effort at almost all outputs and still have zero value – since the wage already lies at the boundary of the contracting space, the principal cannot use the signal to modify the wage. When the agent's limited liability is the only contractual constraint, a signal is valuable if and only if it is informative about effort at the state with the highest likelihood ratio, which has zero measure. With limited liability on the principal or monotonicity constraints, a signal is valuable if and only if it is informative about effort at a single intermediate output, which also has zero measure. Thus, the principal's willingness to invest in signals is greatest for intermediate output realizations.

References

[1] Bertrand, Marianne, and Sendhil Mullainathan (2001): "Are CEOs rewarded for luck? The ones without principals are." *Quarterly Journal of Economics* 116, 901–932.

[2] Bolton, Patrick, and Mathias Dewatripont (2005): *Contract Theory* (MIT Press, Cambridge).

[3] Gjesdal, Froystein (1982): "Information and incentives: the agency information problem." *Review of Economic Studies* 49, 373–390.

[4] Grossman, Sanford J., and Oliver D. Hart (1983): "An analysis of the principal-agent problem." *Econometrica* 51, 7–45.

[5] Holmstrom, Bengt (1979): "Moral hazard and observability." *Bell Journal of Economics* 10, 74–91.

[6] Innes, Robert D. (1990): "Limited liability and incentive contracting with ex-ante action choices." *Journal of Economic Theory* 52, 45–67.

[7] Kim, Son Ku (1995): "Efficiency of an information system in an agency model." *Econometrica* 63, 89–102.

[8] Poblete, Joaquín and Daniel Spulber (2012): "The form of incentive contracts: Agency with moral hazard, risk neutrality, and limited liability." *Rand Journal of Economics* 43, 215–234.

[9] Shavell, Steven (1979): "Risk sharing and incentives in the principal and agent relationship." *Bell Journal of Economics* 10, 55–73.

[10] Tirole, Jean (2006): *The Theory of Corporate Finance* (Princeton University Press, Princeton).

A Proofs

Proof of Lemma 1

Fix a vector of wages that satisfy IC, and consider the following perturbation:

$$w'_{q,s} = w_{q,s} + \frac{\epsilon}{\pi_{q,s} - p_{q,s}}, \quad \text{and} \quad w'_{q,s'} = w_{q,s'} - \frac{\epsilon}{\pi_{q,s'} - p_{q,s'}}.$$

This perturbation keeps the incremental benefit from effort constant and therefore preserves IC. LL continues to hold for $\epsilon > 0$ if $w_{q,s'} > 0$, and for $\epsilon < 0$ if $w_{q,s} > 0$. The expected wage (4) increases by:

$$\left(\frac{\pi_{q,s}}{\pi_{q,s} - p_{q,s}} - \frac{\pi_{q,s'}}{\pi_{q,s'} - p_{q,s'}} \right) \epsilon. \tag{13}$$

If the original contract entails $w_{q,s} = w_{q,s'} > 0$ (i.e., a strictly positive wage for output q that does not depend on whether the signal is s or s'), then such a perturbation would satisfy both IC and LL. Thus, for this contract to be optimal, such a perturbation cannot reduce the expected wage. The term in (13) must be non-positive $\forall \; \epsilon$:

$$\frac{\pi_{q,s}}{\pi_{q,s} - p_{q,s}} = \frac{\pi_{q,s'}}{\pi_{q,s'} - p_{q,s'}},$$

which yields $\frac{\pi_{q,s}}{p_{q,s}} = \frac{\pi_{q,s'}}{p_{q,s'}}$.

Proof of Lemma 2

Consider the following perturbation, which, as before, keeps the incremental benefit from effort constant, thereby preserving IC:

$$w'_{q,s} = w_{q,s} + \frac{\epsilon}{\pi_{q,s} - p_{q,s}}, \quad \text{and} \quad w'_{q',s'} = w_{q',s'} - \frac{\epsilon}{\pi_{q',s'} - p_{q',s'}}.$$

LL continues to hold for $\epsilon > 0$ if $w_{q',s'} > 0$, and for $\epsilon < 0$ if $w_{q,s} > 0$. The expected wage (4) increases by:

$$\left(\frac{\pi_{q,s}}{\pi_{q,s} - p_{q,s}} - \frac{\pi_{q',s'}}{\pi_{q',s'} - p_{q',s'}} \right) \epsilon. \tag{14}$$

Let $(q, s) \in \arg\max_{(q'',s'')} \left\{ \frac{\pi_{q'',s''}}{p_{q'',s''}} \right\}$ denote a state with the highest likelihood ratio

14

and consider a state (q', s') that does not have the highest likelihood ratio:

$$\frac{\pi_{q',s'}}{p_{q',s'}} < \frac{\pi_{q,s}}{p_{q,s}}. \tag{15}$$

From (15), the term inside the parentheses in (14) is strictly negative. Thus, the principal can reduce the expected wage by selecting $\epsilon > 0$, which is feasible without violating LL when $w_{q',s'} > 0$. As a result, the solution entails zero payments in all states that do not maximize the likelihood ratio.

Proof of Example 2

It can be shown that the principal would like to implement high effort. Writing in "utils", the principal's program becomes:

$$\min_{(u_{i,j})} \frac{\alpha}{2} u_{H,1}^2 + \frac{(1-\alpha)}{2} u_{H,0}^2 + \frac{\beta}{2} u_{L,1}^2 + \frac{(1-\beta)}{2} u_{L,0}^2$$

$$\text{subject to} \quad \frac{\alpha}{2} u_{H,1} + \frac{(1-\alpha)}{2} u_{H,0} + \frac{\beta}{2} u_{L,1} + \frac{(1-\beta)}{2} u_{L,0} - 1 \geq 0 \tag{IR}$$

$$\frac{\alpha}{2} u_{H,1} + \frac{(1-\alpha)}{2} u_{H,0} - 1 \geq \left(\gamma - \frac{\beta}{2}\right) u_{L,1} + \left(\frac{1+\beta}{2} - \gamma\right) u_{L,0}, \tag{IC}$$

$$u_{i,j} \geq 0, \quad \forall i, j. \tag{LL}$$

As in the core model, IC and LL imply IR, and so IR can be ignored. It is straight-forward to verify that $\gamma > \frac{\beta}{2}$ implies $u_{L,1} = 0$ whereas $\frac{1+\beta}{2} > \gamma$ implies $u_{L,0} = 0$. Substituting back in the program, the principal solves:

$$\min_{u_{H,j} \geq 0} \frac{\alpha}{2} u_{H,1}^2 + \frac{(1-\alpha)}{2} u_{H,0}^2$$

subject to

$$\frac{\alpha}{2} u_{H,1} + \frac{(1-\alpha)}{2} u_{H,0} - 1 \geq 0.$$

The unique solution is $u_{H,1} = u_{H,0} = 2$. Converting to dollar units, we obtain $w_{H,1} = w_{H,0} = 4$ and $w_{L,1} = w_{L,0} = 0$.

Proof of Lemma 3

The principal's program is

$$\min_{\{w_s(q)\}} \sum_s \int_0^{\bar{q}} w_s(q) \, \phi_1^s \pi_s(q) \, dq$$

subject to

$$0 \leq w_s(q) \leq q$$

$$\sum_s \int_0^{\bar{q}} w_s(q) \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q) \right] dq \geq C.$$

While it is possible to obtain a proof along the same lines of Section 1, we follow a more direct approach here. The infinite-dimensional Lagrangian is:

$$\mathcal{L} = \sum_s \int_0^{\bar{q}} w_s(q) \, \phi_1^s \pi_s(q) \, dq - \mu \left\{ \sum_s \int_0^{\bar{q}} w_s(q) \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q) \right] dq - C \right\}.$$

The first-order conditions are:

$$w_s(q) = \left\{ \begin{matrix} q \\ 0 \end{matrix} \right\} \text{ if } \phi_1^s \pi_s(q) - \mu \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q) \right] \left\{ \begin{matrix} > \\ < \end{matrix} \right\} 0, \qquad (16)$$

as well as IC, which must bind:

$$\sum_s \int_{LR_s(q) \geq \frac{\mu}{\mu-1}} q \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q) \right] dq = C. \qquad (17)$$

Rearranging (16), it follows that $w_s(q) = q$ if $LR_s(q) > \frac{\mu}{\mu-1}$, and $w_s(q) = 0$ if $LR_s(q) < \frac{\mu}{\mu-1}$. It remains to be verified that the Lagrange multiplier μ exists. Note that the LHS of (17) converges to zero as $\mu \nearrow +\infty$ and to $E[q|e=1] - E[q|e=0]$ as $\mu \searrow 1$. By assumption, C is small enough that the principal wishes to implement high effort. In particular, high effort must be optimal in the first-best when effort is observable:

$$E[q|e=1] - C > E[q|e=0].$$

Thus, by the Intermediate Value Theorem, there exists a Lagrange multiplier $\mu \in (1, +\infty)$ for which (17) holds.

Proof of Proposition 2

The thresholds q_s^* are independent of s if the likelihood ratios $LR_s(\cdot)$ evaluated at q^* are the same for any s, where q^* is the unique threshold that solves the IC in (10). For all signals s and s' such that the likelihood ratios (6) evaluated at q^* are the same, we have:

$$LR_s(q^*) = LR_{s'}(q^*) \iff \frac{\phi_1^s \pi_s(q^*)}{\phi_0^s p_s(q^*)} = \frac{\phi_1^{s'} \pi_{s'}(q^*)}{\phi_0^{s'} p_{s'}(q^*)}. \tag{18}$$

Using the definition of the marginal distribution (5), when (18) holds, we can write the IC in (10) as

$$\int_{q^*}^{\bar{q}} q \left[f(q|e=1) - f(q|e=0) \right] dq = C. \tag{19}$$

where the threshold q^* is independent of the realization of s.

Finally, we show that q^* is unique. Any contract such that $w(q,s) > 0$ for $q < \hat{q}_s$, where $\hat{q}_s$ is implicitly defined by $f(\hat{q}_s|e=1,s) = f(\hat{q}_s|e=0,s)$ and is unique due to MLRP, is dominated. Indeed, setting $w(q,s) = 0$ for $q < \hat{q}_s$ would not affect payments on other intervals, would increase incentives relative to the initial contract, and would reduce the expected wage. Therefore, restricting attention to contracts described in Lemma 3 such that $w(q,s) > 0$ only if $q \geq \hat{q}_s$, the expression under the integral sign in (19) is positive, which implies that q^* is unique.

Proof of Lemma 4

By the monotonicity constraint (11), $w_s(\cdot)$ is Lipschitz continuous and, therefore, differentiable almost everywhere. Hence, without loss of generality, we can assume that $w_s(q)$ is a cadlag function satisfying $0 \leq w_s'(q) \leq 1$ at all points of differentiability.[6]

We will adopt a two-step approach. First, we solve for the optimal contract for a fixed minimum wage $w_s(0) = Z_s \geq 0$. As we will show, the solution involves a contract where the agent receives a fixed minimum wage of Z_s if output is below a threshold, and is the residual claimant above the threshold. Then, we will show that the minimum is zero, so the solution is a standard option contract.

Formally, pick constants $Z_s \geq 0$ and consider the following "relaxed" program:

$$\min_{w_s(\cdot)} \sum_s \int_0^{\bar{q}} w_s(q) \phi_1^s \pi_s(q) \, dq \tag{20}$$

---

[6] A cadlag function is everywhere right-continuous and has left limits everywhere.

subject to

$$\sum_s \int_0^{\bar{q}} w_s(q) \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q)\right] dq \geq C,$$

$$0 \leq \dot{w}_s(q) \leq 1, \text{ and } w_s(0) = Z_s \text{ fixed.}$$

Note that monotonicity $\dot{w}_s(q) \geq 0$ implies that $Z_s \geq 0$ is both necessary and sufficient for the agent's LL to hold. We will ignore the principal's LL and verify that it is satisfied later.

Introduce the auxiliary variables $y_s(q) \equiv \dot{w}_s(q)$ and set up the Hamiltonian:

$$H(w, y, \lambda, \mu, q) \equiv \sum_s \left\{-w_s \phi_1^s \pi_s(q) + \lambda \left[w_s \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q)\right] - C\right] + \mu_s(q) y_s\right\},$$

where w_s are state variables, y_s are control variables, μ_s are co-state variables, and λ is a (state-independent) Lagrange multiplier. The necessary optimality conditions are:

$$y_s(q) \in \arg\max_{0 \leq y \leq 1} \mu_s(q) y \therefore y_s(q) = \left\{\begin{matrix} 0 \\ 1 \end{matrix}\right\} \text{ if } \mu_s(q) \left\{\begin{matrix} < \\ > \end{matrix}\right\} 0; \qquad (21)$$

$$\frac{\partial H}{\partial w_s} = -\dot{\mu}_s \therefore \phi_1^s \pi_s(q) - \lambda \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q)\right] = \dot{\mu}_s(q); \qquad (22)$$

and the transversality condition $\mu_s(\bar{q}) = 0$.

Condition (22) yields:

$$\dot{\mu}_s(q) > 0 \iff \frac{1}{\lambda} > 1 - \frac{\phi_0^s p_s(q)}{\phi_1^s \pi_s(q)} \iff \frac{1}{LR_s(q)} > \frac{\lambda - 1}{\lambda},$$

where $LR_s(q) \equiv \frac{\phi_1^s \pi_s(q)}{\phi_0^s p_s(q)}$ is the likelihood ratio, which we assumed to be increasing. Thus, the LHS of the last inequality above is decreasing in q while the RHS is constant. Hence, there exists a threshold $q_s^* \in [0, \bar{q}]$ such that $\dot{\mu}_s(q) > 0$ for $q < q_s^*$ and $\dot{\mu}_s(q) < 0$ for $q > q_s^*$. (Note that if $q_s^* = 0$ or $q_s^* = \bar{q}$, one of these intervals vanishes). Therefore, μ_s is bell-shaped, with a unique maximum at q_s^* and (at most) two local minima – one at 0 and and another at $\bar{q}$.

We claim that $q_s^* < \bar{q}$, i.e. μ_s is increasing $\forall q$ if $q_s^* = \bar{q}$. Then, the transversality condition $\mu_s(\bar{q}) = 0$ and (21) imply that $w_s(q)$ is constant ($w_s(q) = w_s(0) = Z_s \forall q$), which violates IC. Thus, it cannot be a solution.

There are two cases to consider. First, $\mu_s(0) \geq 0$. In this case, we must have $\mu_s(q) > 0 \; \forall \; q \in (0, \bar{q})$, since the only candidates for global minima are 0 and $\bar{q}$ and $\mu_s(0) \geq 0 = \mu_s(\bar{q})$. Second, $\mu_s(0) < 0$, and so there exists a threshold $q_s^{**} \in (0, q_s^*)$ such that $\mu_s(q) < 0$ if $q < q_s^{**}$ and $\mu_s(q) > 0$ if $q > q_s^{**}$. We can combine both cases by letting $q_s^{**} \in [0, q_s^*)$ denote the threshold below which $\mu_s(q) < 0$. The solution is then

$$w_s^*(q) = Z_s + \max\{q - q_s^{**}, \; 0\} \tag{23}$$

for some $q_s^{**} \in [0, \bar{q})$. This concludes the first part of the proof.

We now show that the solution entails $Z_s = 0 \; \forall \; s$. We substitute the agent's wage from (23) into the principal's objective function to yield:

$$\sum_s \left\{ \int_0^{\bar{q}} w_s(q) \phi_1^s \pi_s(q) \, dq \right\} = \sum_s \phi_1^s \left\{ Z_s + \int_{q_s^{**}}^{\bar{q}} (q - q_s^{**}) \pi_s(q) \, dq \right\}, \tag{24}$$

and into the IC constraint to yield:

$$\sum_s \left\{ \int_{q_s^{**}}^{\bar{q}} (q - q_s^{**}) \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q) \right] dq \right\} \geq C. \tag{25}$$

Monotonicity is automatically satisfied by (23). As before, the agent's LL holds if and only if $Z_s \geq 0$. Since Z_s increases the objective function in (24) but does not affect the IC constraint in (25), the solution entails $Z_s = 0 \; \forall \; s$. Thus, the solution of the relaxed program is

$$w_s^*(q) = \max\{q - q_s^{**}, \; 0\}, \tag{26}$$

where the thresholds q_s^{**} are such that the IC constraint holds with equality:

$$\sum_s \left\{ \int_{q_s^{**}}^{\bar{q}} (q - q_s^{**}) \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q) \right] dq \right\} = C.$$

We finally must verify that the principal's LL is satisfied. This is true since $w_s^*(q) = \max\{q - q_s^{**}, \; 0\} \leq q \; \forall \; q$. Thus, we have established that the optimal contract is an option where the strike prices $\{q_s^{**}\}$ are such that the IC holds with equality. Next, we characterize the strike prices.

Substituting the contract derived in (26) into the objective function (24), the latter

becomes

$$\min_{\{q_s^{**}\}_{s=1,\dots,S}} \sum_s \int_{q_s^{**}}^{\bar{q}} (q - q_s^{**}) \, \phi_1^s \pi_s (q) \, dq. \tag{27}$$

Denoting the Lagrange multiplier associated with the incentive constraint by λ, the necessary first-order condition with respect to q_s^{**} is

$$-\phi_1^s \int_{q_s^{**}}^{\bar{q}} \pi_s (q) \, dq + \lambda \int_{q_s^{**}}^{\bar{q}} [\phi_1^s \pi_s (q) - \phi_0^s p_s (q)] \, dq = 0 \tag{28}$$

$$\frac{\phi_0^s \int_{q_s^{**}}^{\bar{q}} p_s (q) \, dq}{\phi_1^s \int_{q_s^{**}}^{\bar{q}} \pi_s (q) \, dq} = 1 - \frac{1}{\lambda} < 1, \tag{29}$$

where λ is independent of s.

We must verify that the principal's LL holds in the optimal contract. This is the case if the optimal q_s^{**} is nonnegative $\forall \ s$. For $q_s^{**} \leq 0$, using $\pi_s (q) = 0$ for $q < 0$, we can rewrite the principal's objective function (27) as

$$\sum_s \int_{q_s^{**}}^{\bar{q}} (q - q_s^{**}) \, \phi_1^s \pi_s (q) \, dq = \sum_s \int_0^{\bar{q}} (q - q_s^{**}) \, \phi_1^s \pi_s (q) \, dq \tag{30}$$

For $q_s^{**} \leq 0$, using $\pi_s (q) = p_s (q) = 0$ for $q < 0$, $\int_0^{\bar{q}} \pi_s (q) \, dq = \int_0^{\bar{q}} p_s (q) \, dq = 1$, and $\sum_s \phi_1^s = \sum_s \phi_0^s$, we can rewrite the LHS of the incentive constraint in (25) as

$$\sum_s \int_0^{\bar{q}} q \, [\phi_1^s \pi_s (q) - \phi_0^s p_s (q)] \, dq - q_s^{**} \sum_s \int_0^{\bar{q}} [\phi_1^s \pi_s (q) - \phi_0^s p_s (q)] \, dq$$

$$= \sum_s \int_0^{\bar{q}} q \, [\phi_1^s \pi_s (q) - \phi_0^s p_s (q)] \, dq \tag{31}$$

Thus, for $q_s^{**} \leq 0$, the Lagrangian $\mathcal{L}$ is:

$$\mathcal{L}(q_s^{**}) = \sum_s \int_0^{\bar{q}} (q - q_s^{**}) \, \phi_1^s \pi_s (q) \, dq - \lambda \sum_s \int_0^{\bar{q}} q \, [\phi_1^s \pi_s (q) - \phi_0^s p_s (q)] \, dq. \tag{32}$$

Its first derivative with respect to q_s^{**} is $\int_0^{\bar{q}} -\phi_1^s \pi_s (q) \, dq < 0$, and so $q_s^{**} < 0$ cannot be an optimum for any s.

As the necessary first-order condition for an optimum only applies to interior solu-

tions, we need to rule out corner solutions by establishing that $q_s^{**} < \bar{q} \; \forall \; s$. First, note that an option with $q_s^{**} \geq \bar{q}$ is equivalent to an option with $q_s^{**} = \bar{q}$. Thus, we need to check that $\lim_{q_s^{**} \to \bar{q}^-} \mathcal{L}'(q_s^{**}) > 0$. We have

$$\lim_{q_s^{**} \to \bar{q}^-} \mathcal{L}'(q_s^{**}) = \lim_{y \to \bar{q}^-} \left\{ -\phi_1^s \pi_s \left(y \right) \left(\bar{q} - y \right) + \lambda \left[\phi_1^s \pi_s \left(y \right) - \phi_0^s p_s \left(y \right) \right] \left(\bar{q} - y \right) \right\}, \qquad (33)$$

where the expression in brackets has the same sign as $\lambda - 1 - \lambda \frac{\phi_0^s p_s(y)}{\phi_1^s \pi_s(y)}$. Since assumption (12) implies $\frac{\phi_0^s p_s(y)}{\phi_1^s \pi_s(y)} \longrightarrow_{y \to \bar{q}} 0$, we indeed have $\lim_{q_s^{**} \to \bar{q}^-} \mathcal{L}'(q_s^{**}) > 0$ if $\lambda > 1$. Finally, we establish that $\lambda > 1$ by contradiction. If $\lambda \leq 1$, the LHS of (28) is strictly negative $\forall$ s. This implies that the optimum is $q_s^{**} = \bar{q} \; \forall \; s$, which violates the incentive constraint in (25).

Proof of Proposition 3

If the solution entails $z_s = z^*$, IC becomes:

$$\int_{z^*}^{\bar{q}} (q - z^*) \left[f \left(q | e = 1 \right) - f \left(q | e = 0 \right) \right] dq = C,$$

which is not a function of the signal distribution (for fixed marginal distributions $f(q|e = 1)$ and $f(q|e = 0)$).

The condition from Lemma 4 is

$$\begin{aligned}
&\frac{\phi_1^s \int_{z^*}^{\bar{q}} \pi_s \left(q \right) dq}{\phi_0^s \int_{z^*}^{\bar{q}} p_s \left(q \right) dq} = \frac{\phi_1^{s'} \int_{z^*}^{\bar{q}} \pi_{s'}(q) dq}{\phi_0^{s'} \int_{z^*}^{\bar{q}} p_{s'} \left(q \right) dq} \\
&\Rightarrow \frac{\Pr \left(q > z^*, s | e = 1 \right)}{\Pr \left(q > z^*, s | e = 0 \right)} = \frac{\Pr \left(q > z^*, s' | e = 1 \right)}{\Pr \left(q > z^*, s' | e = 0 \right)},
\end{aligned}$$

which states that the likelihood ratios of the event that the agent's option is in-the-money is not a function of the signal realization.