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ABSTRACT

We analyze the optimal Taylor rule in a standard New Keynesian model. If the central bank can observe the output gap and the inflation rate without error, then it is typically optimal to respond infinitely strongly to observed deviations from the central bank's targets. If it observes inflation and the output gap with error, the central bank will temper its responses to observed deviations so as not to impart unnecessary volatility to the economy. If the Taylor rule is expressed in terms of estimated output and inflation then it is optimal to respond infinitely strongly to estimated deviations from the targets. Because filtered estimates are based on current and past observations, such Taylor rules appear to have an interest smoothing component. Under such a Taylor rule, if the central bank is behaving optimally, the estimates of inflation and the output gap should be perfectly negatively correlated. In the data, inflation and the output gap are weakly correlated, suggesting that the central bank is systematically underreacting to its estimates of inflation and the output gap.

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1. INTRODUCTION

Taylor rules are simple linear relationships between a central bank's choice of a target interest rate, observed output (or the "output gap") and observed inflation (Taylor, 1993). Suitably parameterized, they are reasonable descriptions of how actual central banks set interest rates. In addition, monetary policy is often discussed by journalists, researchers and central bankers themselves in terms that fit comfortably into a Taylor rule framework. For example, whether the federal funds rate should be increased or decreased is typically discussed in terms of whether inflation is relatively too high or whether GDP (or employment) is too low. A central banker who fights inflation aggressively can be modeled as following a Taylor rule with a large coefficient on inflation. A central banker who reacts more strongly to output would have a relatively higher coefficient on the output gap, and so forth.

While Taylor rules are useful descriptions of actual policy and common components of many prominent New Keynesian models, it is well-known that *optimal* monetary policy is rarely given by a Taylor rule. Instead, optimal policy depends in complicated ways on the underlying state variables and is often history dependent (see Woodford, 1999). By confining attention to current inflation and the current output gap, a Taylor rule is unnecessarily restrictive.¹ Nevertheless, given the underlying model, it is a meaningful question to ask what the optimal parameterization of a Taylor rule is. Addressing this question is the goal of this paper.

We anticipate that much of what we have to say will not come as a surprise to researchers at the frontier of New Keynesian economics. Indeed, we suspect that many of our results exist as folk wisdom among New Keynesian researchers. (If you are Mike Woodford or Jordi Galí, you can stop reading now). Instead, our intended audience consists of consumers of New Keynesian economics – researchers who often use macroeconomic models with sticky prices and want to draw on established results from the New Keynesian literature (we include ourselves in this group).

In this paper, we consider the nature of the optimal Taylor rule in the basic New Keynesian model. That is, we assume the monetary authority is committed to using a Taylor rule, and ask what coefficients maximize the central bank's objective function. When the output gap and inflation are observed without error, it is typically best to adopt infinitely aggressive responses to output and inflation – i.e., the optimal coefficients on inflation and output are arbitrarily large. If the only shocks to the economy are shocks to the efficient rate of output ("demand shocks"), then any Taylor rule which responds infinitely strongly to either output or inflation (or both) will maximize welfare. If the only shocks perturbing the economy are shocks to the New Keynesian Phillips Curve ("cost-push shocks"), then there exists a continuum of optimal coefficients for the Taylor rule. If the output gap and inflation are observed with error, then the

¹ See Svensson (2003) for a more general criticism of Taylor rules.

Taylor coefficients are finite. As the variance of measurement error grows, the optimal coefficients fall and the central bank reacts less to measured inflation and the output gap.

We extend our analysis to the case in which the Taylor rule is expressed in terms of estimated output and estimated inflation. In this extension, the central bank solves a signal extraction problem to estimate the output gap and inflation and sets the interest rate as a function of these estimates. Under such a policy rule, the optimal responses to estimated inflation and the estimated gap are again infinite. Because filtered estimates are typically revised gradually over time, a central bank that responds to estimates of inflation and output will often appear to be adhering to a form of interest rate smoothing. Indeed, there are special cases in which a Taylor rule with interest rate smoothing can duplicate the policy of responding to estimated inflation and the output gap.

Under an optimal Taylor rule, estimated deviations of output and inflation from their targets should be strongly negatively correlated. Intuitively, the optimal Taylor rule eliminates the effects of estimated demand shocks on inflation and the output gap. At the optimum, only variation due to estimated cost-push shocks should remain, implying that estimated inflation and output should move in opposite directions. Actual data on estimates of inflation and the gap are not strongly correlated, suggesting that the central bank is not reacting to demand shocks as aggressively as it should.

Our work is related to a large literature on optimal monetary policy and instrument rules. The contributions of Giannoni and Woodford (2003a, 2003b), Woodford (2003, Ch. 7) and Giannoni (2010, 2012) are closest to our work. Unlike us, Giannoni and Woodford (2003b) assume that the central bank has an explicit preference for smooth interest rates. Our paper highlights the role played by measurement error in tempering the central bank's optimal reaction to changes in the measured output gap and inflation. This type of measurement error has been termed "data uncertainty" by some researchers to distinguish it from other sources of policy uncertainty (such as parameter and model uncertainty, see Dennis, 2005 and the references cited therein). Orphanides (2001, 2003) shows that real-time measures of inflation and the output gap are sufficiently noisy to justify relatively small Taylor rule coefficients. Rudebusch (2001), Smets (2002) and Billi (2012) all conclude that measurement error naturally encourages central banks to adopt less aggressive policy reaction rules.²

Our paper is also closely related to the literature on signal extraction and optimal monetary policy. Aoki (2003) considers optimal monetary policy with signal extraction in an environment similar to ours. There are several differences between Aoki's work and ours. First, unlike our

² Both Rudebusch (2001) and Smets (2002) numerically analyze optimal policy based on estimated New Keynesian systems. Billi (2012) numerically analyzes optimal monetary policy at the zero lower bound on nominal interest rates. Both Billi and Smets assume that the central bank wants to minimize interest rate variation in addition to variation in output and inflation. See Taylor (1999) for additional work on optimal monetary policy rules.

model, Aoki's does not have shocks to the New Keynesian Phillips Curve and thus the so-called "divine coincidence" holds. If the central bank successfully stabilizes prices, it simultaneously eliminates the output gap. Second, while Aoki discusses optimal policy under discretion, we restrict attention to the optimal Taylor rule – a form of commitment. Third, when formulating the signal extraction problem, Aoki assumes that the central bank learns the true values of output and inflation with a one period lag. In our formulation, these values are never revealed. Methodologically, we draw heavily on results in Svensson and Woodford (2003, 2004). Although they do not consider restricted instrument rules such as the Taylor rule, several of their findings continue to hold in our setting.

2. BASELINE MODEL AND POLICY OBJECTIVE

Our analysis is based on simple variations of the standard New Keynesian framework consisting of a Phillips Curve and the New Keynesian IS curve. These log-linear aggregate relationships are typically derived under the assumption that firms infrequently adjust prices according to the Calvo (1983) mechanism.³ (See Gali (2008) or Woodford (2003) for standard treatments of this derivation.)

The Phillips curve relates inflation, π_t , to the output gap, y_t , expected future inflation, and a cost-push shock, u_t ,

$$\pi_t = \kappa y_t + \beta E_t[\pi_{t+1}] + u_t, \quad (1)$$

where $\kappa > 0$ is the macroeconomic rate of price adjustment. The presence of the cost-push shock implies a trade-off between the stabilization of the output gap and inflation (e.g. Clarida, Gali, and Gertler, 1999). Woodford (2003, Ch. 6) shows how the cost-push shock can be motivated from first principles.

The New Keynesian IS curve is given by

$$y_t = E_t[y_{t+1}] - \frac{1}{\sigma} (i_t - \rho - r_t^e - E_t[\pi_{t+1}]). \quad (2)$$

Here $\rho + r_t^e$ is the efficient rate of interest – the interest rate consistent with the level of output that would prevail under perfect price flexibility in the absence of all other distortions. We express this rate as the sum of the rate of time preference ρ and a shock r_t^e which is centered at zero to simplify the exposition of later results. Below we often (somewhat imprecisely) refer to

³ The standard New Keynesian model abstracts from investment in physical capital and durable goods. While this assumption is common, it has important consequences for the analysis of the model and for optimal policy. See Barsky et al. (2003, 2007) and Barsky et al. (2014) for a more detailed discussion of the consequences of this assumption.

r_t^e as the efficient rate. The remaining terms are i_t , the nominal interest rate, and σ , the coefficient of relative risk aversion (equivalently, the inverse of the intertemporal elasticity of substitution).

The efficient rate shock and the cost-push shock are assumed to follow the AR(1) processes

$$r_{t+1}^e = \varrho_r r_t^e + \varepsilon_{t+1}^r, \quad \varrho_r \in [0,1), \quad (3)$$

$$u_{t+1} = \varrho_u u_t + \varepsilon_{t+1}^u. \quad \varrho_u \in [0,1). \quad (4)$$

We close the model by assuming that the monetary authority commits to a Taylor rule,

$$i_t = \rho + \phi_\pi \pi_t^m + \phi_y y_t^m. \quad (\text{TR1})$$

Importantly, we distinguish between the actual output gap and the output gap observed by the monetary authority, and similarly between actual inflation and measured inflation. The central bank can respond only to measured output and inflation, so in (TR1) π_t^m and y_t^m denote *measured* inflation and the *measured* output gap. Using the analogy in Bernanke (2004), measurement error creates a “foggy windshield” through which the monetary authority sees the economy.

Orphanides (2001) suggests that modeling measurement error as additive is a reasonable approximation of reality. We follow Orphanides and assume that measured inflation and the measured output gap are $\pi_t^m = \pi_t + m_t^\pi$ and $y_t^m = y_t + m_t^y$, where m_t^π and m_t^y denote the respective measurement errors. Both types of measurement error follow AR(1) processes

$$m_{t+1}^\pi = \varrho_{m\pi} m_t^\pi + \varepsilon_{t+1}^{m\pi}, \quad \varrho_{m\pi} \in [0,1), \quad (5)$$

$$m_{t+1}^y = \varrho_{my} m_t^y + \varepsilon_{t+1}^{my}. \quad \varrho_{my} \in [0,1). \quad (6)$$

Unless stated otherwise, all error terms in the efficient rate, the cost-push shock and the measurement error processes are assumed to be uncorrelated.

A comprehensive literature has examined equilibrium determinacy in the New Keynesian model when monetary policy follows a Taylor rule. We will not dwell on this issue further and simply note that the well-known condition for a unique equilibrium is

$$\kappa(\phi_\pi - 1) + (1 - \beta)\phi_y > 0 \quad (7)$$

for the baseline model discussed in Section 4.⁴ This condition is assumed to hold at all times unless otherwise stated.

⁴ The argument in Bullard and Mitra (2002) continues to hold in our setting.

We assume that the central bank wishes to minimize an expected discounted sum of weighted squared inflation and the output gap

$$(1 - \beta)E \left[\sum_{t=0}^{\infty} \beta^t (\alpha \pi_t^2 + y_t^2) \right] = E[\alpha \pi_t^2 + y_t^2]. \quad (8)$$

Here, α denotes the relative weight that the central bank places on inflation and $E[\cdot]$ is the unconditional expectations operator. An objective of this form can be derived as a quadratic approximation of the representative consumer's utility function (see Rotemberg and Woodford, 1997 and Woodford, 2003). The optimal policy problem is then to choose ϕ_y and ϕ_π to minimize (8) subject to (1), (2) and (TR1).

3. EQUILIBRIUM IN THE BASELINE MODEL

Our framework is sufficiently simple to solve for the equilibrium analytically. Inflation and the output gap are linear functions of the four exogenous state variables r_t^e, u_t, m_t^π , and m_t^y . Applying the method of undetermined coefficients yields expressions for the behavior of inflation and the output gap. We characterize the model's solution in the following lemma. Proofs of all results are in the appendix.

Lemma 1: *Under the assumptions in section 2, the unique competitive equilibrium of the model is given by the equations*

$$\pi_t = \frac{\kappa}{\Phi_r + \phi_y(1 - \beta \varrho_r) + \kappa \phi_\pi} r_t^e + \frac{\phi_y + (1 - \varrho_u)\sigma}{\Phi_u + \phi_y(1 - \beta \varrho_u) + \kappa \phi_\pi} u_t \quad (9)$$

$$- \frac{\kappa \phi_\pi}{\Phi_{m\pi} + \phi_y(1 - \beta \varrho_{m\pi}) + \kappa \phi_\pi} m_t^\pi - \frac{\kappa \phi_y}{\Phi_{my} + \phi_y(1 - \beta \varrho_{my}) + \kappa \phi_\pi} m_t^y,$$

$$y_t = \frac{1 - \beta \varrho_r}{\Phi_r + \phi_y(1 - \beta \varrho_r) + \kappa \phi_\pi} r_t^e + \frac{\varrho_u - \phi_\pi}{\Phi_u + \phi_y(1 - \beta \varrho_u) + \kappa \phi_\pi} u_t \quad (10)$$

$$- \frac{(1 - \beta \varrho_{m\pi})\phi_\pi}{\Phi_{m\pi} + \phi_y(1 - \beta \varrho_{m\pi}) + \kappa \phi_\pi} m_t^\pi - \frac{(1 - \beta \varrho_{my})\phi_y}{\Phi_{my} + \phi_y(1 - \beta \varrho_{my}) + \kappa \phi_\pi} m_t^y,$$

where $\Phi_j = \sigma(1 - \varrho_j)(1 - \beta \varrho_j) - \kappa \varrho_j$, $j \in \{r, u, m\pi, my\}$, are constants that are independent of monetary policy.

Examining the coefficients in (9) and (10) provides simple intuition about how monetary policy influences the economy. First, notice that the coefficients on the efficient rate shock (r_t^e) are positive in both (9) and (10) (all of the denominators in (9) and (10) are positive). Thus, an increase in the efficient rate (an "aggregate demand" shock) increases inflation and the output

gap. Moreover, the Taylor rule coefficients ϕ_y and ϕ_π appear only in the denominators of the coefficients on the efficient rate shock, so strong responses by the central bank dampen fluctuations caused by shocks to the efficient rate.

Now consider the coefficients on the cost-push disturbances (u_t). Assuming that ϕ_y is positive and $\phi_\pi > \rho_u$, cost-push shocks reduce output and raise inflation. Strong reactions to output (i.e., large values of ϕ_y) imply that the economy can enjoy reduced output gap volatility only at the expense of higher inflation volatility. Unlike shocks to the efficient rate of interest, cost-push shocks clearly entail a trade-off between inflation and output stabilization.

Finally, measurement error of either type impacts equilibrium inflation and the output gap negatively. The interpretation of this relationship is natural. For example, a positive innovation to m_t^π makes inflation appear higher than actual inflation. In response, the central bank raises interest rates causing both output and inflation to fall. Similar reasoning applies to measurement error in the output gap.

With Lemma 1 in hand, we now turn our attention to the central banks' optimal choices of ϕ_y and ϕ_π .

4. OPTIMAL TAYLOR RULES IN THE BASELINE MODEL

The optimal Taylor rule coefficients, ϕ_y^* and ϕ_π^* , minimize the central bank's objective (8) subject to the equilibrium conditions (9) and (10). To build intuition, we begin by considering four special cases. All of these special cases share the property that both inflation and the output gap are observed without error.

SPECIAL CASE 1: ONLY SHOCKS TO THE EFFICIENT RATE OF INTEREST. Assume that the only shocks to the economy are shocks to the efficient rate of interest, r_t^e . In this case, equations (9) and (10) simplify to

$$\pi_t = \frac{\kappa}{\Phi_r + \phi_y(1 - \beta\rho_r) + \kappa\phi_\pi} r_t^e,$$

$$y_t = \frac{1 - \beta\rho_r}{\Phi_r + \phi_y(1 - \beta\rho_r) + \kappa\phi_\pi} r_t^e.$$

Clearly, it is optimal for the central bank to respond infinitely strongly to deviations in either inflation or the output gap (or both). This is the well-known "divine coincidence" case in which it is possible for the central bank to kill two birds with one stone by simply eliminating inflation variability (see e.g., Blanchard and Galí, 2007).

SPECIAL CASE 2: ONLY COST-PUSH SHOCKS. Optimal monetary policy is not so simple when there are shocks to the Phillips curve. Suppose the *only* shocks to the economy are cost-push shocks. The central bank can no longer eliminate both inflation and output variability and instead must choose whether to endure large swings in inflation to reduce the variation in output or vice versa. The best choice will depend on the underlying parameters of the model, particularly the relative weight α that the central bank attaches to inflation variability in its objective (8).

In this case, (9) and (10) collapse to

$$\pi_t = \frac{\phi_y + (1 - \rho_u)\sigma}{\Phi_u + \phi_y(1 - \beta\rho_u) + \kappa\phi_\pi} u_t,$$

$$y_t = \frac{\rho_u - \phi_\pi}{\Phi_u + \phi_y(1 - \beta\rho_u) + \kappa\phi_\pi} u_t.$$

The optimal Taylor coefficients are given in Proposition 1.

Proposition 1:

(i) If the only shocks to the model are cost-push shocks, then the optimal policy requires that the Taylor rule coefficients ϕ_y^* and ϕ_π^* lie on the affine manifold

$$\phi_\pi^* = \rho_u + \frac{\alpha\kappa\sigma(1 - \rho_u)}{1 - \beta\rho_u} + \frac{\alpha\kappa}{1 - \beta\rho_u} \phi_y^*. \quad (11)$$

(ii) For any ϕ_y^* and ϕ_π^* satisfying (11), the equilibrium satisfies

$$\pi_t = \frac{1 - \beta\rho_u}{\alpha\kappa^2 + (1 - \beta\rho_u)^2} u_t, \quad y_t = -\frac{\alpha\kappa}{\alpha\kappa^2 + (1 - \beta\rho_u)^2} u_t.$$

Any combination of coefficients on line (11) is equally desirable. Moreover, the equilibrium paths for output and inflation do not depend on the exact coefficient values, provided that they are on the manifold.

To understand the intuition for this result, consider a positive innovation to u_t . By itself, the cost-push shock puts upward pressure on inflation and downward pressure on output. The central bank then faces a dilemma: any interest rate response which closes one of the gaps, widens the other. If the bank cuts the interest rate to raise output, inflation rises. If the bank fights inflation instead, it must endure even lower production in the short term.

Suppose the bank decides to raise interest rates. Suppose further that it has chosen a pair of coefficients (ϕ_y^*, ϕ_π^*) which satisfy (11) and imply the desired increase in interest rates.

According to Proposition 1, the same interest rate response can be achieved by many different Taylor rules. If the central bank instead opts for a stronger response to inflation (a higher ϕ_π) the nominal interest rate will increase a bit more. To re-establish the prior interest rate change, the central bank can adopt a stronger response to output. The higher ϕ_y implies a lower interest rate because cost-push shocks lower output whenever they raise inflation. Equation (11) simply gives all pairs of coefficients that result in the same interest rate.

SPECIAL CASE 3: SHOCKS TO BOTH THE IS-CURVE AND THE PHILLIPS CURVE. We next combine the previous two cases and consider (uncorrelated) shocks to both the IS-Curve (r_t^e) and the Phillips Curve (u_t). In Special Case 1, optimal policy required infinitely large Taylor Rule coefficients. In Special Case 2, optimal policy required coefficients on the upward sloping line (11). Not surprisingly, the optimal Taylor rule in this third case calls for arbitrarily large coefficients which also satisfy (11). Equation (11) implies that the optimal ratio of ϕ_π to ϕ_y converges to $\alpha\kappa/(1 - \beta\varrho_u)$. This ratio is higher if the central bank places greater weight on inflation or if the tradeoff between inflation and output (captured by the parameter κ) is more favorable.

This policy will imply the same equilibrium paths as in Proposition 1: all of the disturbances originating from variation in the efficient rate will be eliminated and only the cost-push shocks affect output and inflation. Combining the two equations in Proposition 1 part (ii) yields

$$\pi_t = -\frac{1 - \beta\varrho_u}{\alpha\kappa} y_t.$$

Hence, for the optimal Taylor rule, the observed output gap and inflation are perfectly negatively correlated.⁵

SPECIAL CASE 4: CORRELATED SHOCKS IN THE I.I.D. CASE. For Special Case 3 we assumed uncorrelated cost-push and demand shocks. Here we briefly consider the optimal Taylor rule coefficients for transitory but correlated shocks. This case is relevant for later results and also shows that properties of the optimal Taylor rule can be extended to allow for correlated structural shocks.

Proposition 2: *Suppose r_t^e and u_t are i.i.d. over time and have covariance $Cov[r_t^e, u_t]$. The Taylor rule coefficients satisfying*

$$\phi_\pi^* = \alpha\kappa(\phi_y^* + \sigma) + (1 + \alpha\kappa^2) \frac{Cov[r_t^e, u_t]}{V[u_t]} \quad (11')$$

and $\phi_y^ \rightarrow \infty$ are optimal.*

⁵ Note that under the optimal policy, estimation of a New Keynesian Phillips curve will be particularly problematic. Typically, the structural shock u_t is correlated with the regressors y_t and $E_t[\pi_{t+1}]$ but under optimal policy, both regressors are functions *only* of u_t and are therefore *perfectly* correlated with the error. The optimal Taylor rule eliminates all variation other than variation associated with u_t making the bias particularly pronounced.

As before, the central bank must respond infinitely strongly to inflation and the output gap. Additionally, the relationship between ϕ_π^* and ϕ_y^* barely changes. To see this, notice that (11) simplifies to $\phi_\pi^* = \alpha\kappa(\phi_y^* + \sigma)$ when ϱ_u is set to zero (the first term on the right hand side of (11')). Hence, shock correlation only introduces the term $(1 + \alpha\kappa^2)\text{Cov}[r_t^e, u_t]/V[u_t]$ that was not present in the earlier condition. It is easy to verify that equilibrium outcomes only depend on the limit of ϕ_π^*/ϕ_y^* which is independent of the correlation of shocks. As in special cases 2 and 3, the output gap and inflation are perfectly negatively correlated in equilibrium.⁶

MONETARY POLICY IN THE PRESENCE OF MEASUREMENT ERROR

The simple New Keynesian framework captures many realistic features of monetary policy. The model embodies a tradeoff between inflation and output and suggests that the central bank has a particular advantage in minimizing economic instabilities that arise from “demand shocks” (shocks to the IS-Curve). Despite these attractive features, the model does not entail any costs to excessively strong reactions on the part of the central bank. In stark contrast to the modest empirical estimates of actual Taylor rules (see Judd and Rudebusch, 1998, and more recently Hofmann and Bogdanova, 2012), optimal Taylor rule coefficients are often infinite.

In this section, we consider the model in which the output gap and inflation are measured with error. Measurement error is a natural candidate for why central banks do not respond more to observed changes in GDP and inflation. This concern was emphasized by Friedman (1953) who pointed out that activist policies might be destabilizing if policy actions were not sufficiently correlated with the true targets of policy. From observing equations (9) and (10) it is clear that, for any fixed coefficients ϕ_y and ϕ_π , greater measurement error reduces the correlation between the policy instrument and the targets and thus entails greater unwanted variation in output and inflation. Indeed, it would seem that if measurement error was sufficiently high, it would be optimal not to respond to observed variations in inflation and output at all.⁷

When we allow for arbitrary variation in the efficient rate, the cost-push shock, and both types of measurement error, an analytical solution of the optimal policy problem is generally not feasible and so we instead use numerical methods to characterize the optimal Taylor rule. To build intuition, however, we first consider another special case.

⁶ Even though ϕ_y and ϕ_π are infinite, the equilibrium interest rate will be finite. This can easily be seen easily in the i.i.d. case using (TR1), (9), (10) and (11').

⁷ This might present a problem for equilibrium determinacy. It is well known that determinacy requires that the central bank responds sufficiently strongly to inflation and output. If measurement error is large however, the Fed might be caught between a rock and a hard place. It would be forced to choose between a locally indeterminate equilibrium on the one hand and destabilization resulting from reactions to erroneous signals on the other. Accurate measurement and state estimation are therefore at the heart of monetary stabilization policy. We consider optimal state estimation on the part of the central bank below.

Proposition 3: Suppose all shocks are white noise, that is, $\varrho_r = \varrho_u = \varrho_{m\pi} = \varrho_{my} = 0$. Then the minimization of (8) subject to (9) and (10) yields the following optimal Taylor rule coefficients

$$\phi_y^* = \frac{1}{\sigma} \cdot \frac{V[r_t^e]}{V[m_t^y]}, \quad (12)$$

$$\phi_\pi^* = \frac{\kappa}{\sigma} \cdot \frac{(\alpha\kappa^2 + 1)V[m_t^y] + \alpha V[u_t]}{(\alpha\kappa^2 + 1)V[m_t^\pi] + V[u_t]} \cdot \frac{V[r_t^e]}{V[m_t^y]} + \frac{\alpha\kappa\sigma V[u_t]}{(\alpha\kappa^2 + 1)V[m_t^\pi] + V[u_t]}, \quad (13)$$

where $V[\cdot]$ denotes the unconditional variance operator.

There are several striking features of the optimal Taylor rule in this setting. First, the optimal Taylor coefficients in this model are finite. The central bank avoids aggressive reactions to measured inflation and output because it knows that its actions would cause excessive fluctuations in actual output and inflation. As measurement error decreases, the Fed can adopt more and more aggressive reactions to inflation and output. It is also worth noting that only measurement error in the output gap is necessary for finite Taylor coefficients. Assuming that $V[u_t] > 0$, measurement error in inflation is neither necessary nor sufficient for finite coefficients.

Second, the optimal choice of ϕ_y^* depends neither on α , nor on κ , nor on $V[u_t]$. Instead, ϕ_y^* depends only on the ratio of the variance of shocks to the efficient rate, $V[r_t^e]$, to the variance of measurement error in the output gap, $V[m_t^y]$, together with the coefficient of relative risk aversion, σ . The reader might find the result in (12) somewhat counterintuitive. If risk aversion is relatively high then the household will strongly dislike output variability and presumably prefer a stronger output reaction. In contrast, the actual best reaction is decreasing in σ . The reason for this apparent contradiction is that we chose to specify the IS shocks as shocks to the efficient rate of interest itself (as is common in the literature). If we instead stated the shocks in terms of exogenous changes in the efficient growth rate of output, we would have $r_t^e = \sigma E_t[\Delta y_{t+1}^e] = -\sigma y_t^e$ (the second equality uses the assumption that the autocorrelation of shocks is zero). In this case, the central bank's choice of ϕ_y^* can be written as

$$\phi_y^* = \sigma \frac{V[y_t^e]}{V[m_t^y]},$$

which is increasing in σ .

The optimal response to measured inflation (equation (13)) is somewhat more complex. We start with its more intuitive properties. First, as with ϕ_y^* , a larger variance of the efficient rate $V[r_t^e]$ implies a stronger response to inflation. Second, greater measurement error in inflation requires more attenuated responses. Third, one can show that $\partial\phi_\pi^*/\partial\alpha > 0$ for any choice of

model parameters so a stronger preference for price stability always implies stronger reactions to measured inflation.

The relationship of ϕ_π^* with the remaining parameters is less clear. To see how this coefficient depends on the shock variances consider the following limiting cases. Suppose first that cost-push shocks are dominant – that is, consider the behavior of ϕ_π^* as $V[u_t] \rightarrow \infty$. In this case, the optimal reaction to inflation approaches

$$\phi_\pi^* = \frac{\kappa\alpha}{\sigma} \cdot \frac{V[r_t^e]}{V[m_t^y]} + \alpha\kappa\sigma = \kappa\alpha\sigma \left(\frac{V[y_t^e] + V[m_t^y]}{V[m_t^y]} \right),$$

where we have again used the relationship $r_t^e = -\sigma y_t^e$. The inflation response is increasing in the signal-to-noise ratio for the output gap, the weight the central bank places on inflation stability, the slope of the IS curve and the macroeconomic rate of price adjustment. Notice also that for large $V[u_t]$ neither ϕ_y^* nor ϕ_π^* depend on measurement error in inflation.

Alternatively, consider the opposite extreme – suppose there are no cost-push shocks at all. In this case,

$$\phi_\pi^* = \frac{\kappa}{\sigma} \cdot \frac{V[r_t^e]}{V[m_t^\pi]} = \kappa\sigma \frac{V[y_t^e]}{V[m_t^\pi]}.$$

Analogous to equation (12), the ratio of the variance of the efficient rate of output to that of measured inflation governs the strength of the reaction. In particular, as $V[u_t]$ approaches zero, ϕ_π^* becomes independent of measurement error in the output gap. Finally, a larger macroeconomic rate of price adjustment, κ , raises the policy response.

Much of the optimal monetary policy literature assumes a quadratic objective function and linear constraints. In these settings the globally optimal policy exhibits certainty equivalence. That is, the presence and nature of additive stochastic disturbances does not affect optimal policy (see, e.g. Sargent and Ljungqvist, 2004, Ch. 5). However, when optimal policy is restricted to a Taylor rule of the form (TR1), equations (9) and (10) show that the constraints are no longer linear in the choice variables. Hence, it is not surprising that certainty equivalence breaks down in our setting. This result is consistent with earlier findings (see, e.g. Smets, 2002).

QUANTITATIVE ANALYSIS

We now analyze a calibrated version of the model to illustrate how the optimal Taylor rule changes with plausible variations in the model parameters. The length of one time period is a quarter. We assume logarithmic utility ($\sigma = 1$). The discount factor β is set to 0.99 and we set

the macroeconomic rate of price adjustment to $\kappa = 0.34$.⁸ We choose an autoregressive parameter of the efficient rate ρ_r equal to 0.9. This value is similar to calibrations of trend stationary productivity shocks used in the real business cycle literature. Existing literature provides less guidance for the persistence of the cost-push shocks ρ_u . Admitting that this choice is somewhat *ad hoc* we select a value of 0.5 as in one of the cases considered in Galí (2008, Ch. 5). The variances of the innovations to the efficient rate and the cost-push shocks are chosen so that the annual unconditional variances of r_t^e and u_t equal unity.

Orphanides (2003) provides estimates for the measurement error processes. Based on his calculations, measurement error in inflation is best approximated by a white noise process with a quarterly standard deviation of roughly 0.50. In contrast, measurement error in the output gap is highly persistent with a quarterly autoregressive coefficient of about 0.95 and an innovation standard deviation of 0.66 (also quarterly). Finally, we assume that the monetary authority dislikes inflation and the output gap equally so $\alpha = 1$. Table 1 summarizes the baseline calibration.

The optimal policy coefficients, ϕ_π^* and ϕ_y^* , maximize objective (8) subject to the Phillips curve (1), the dynamic IS equation (2), and the Taylor rule (TR1). For the numerical solutions, we additionally impose condition (7) to ensure that the equilibrium is locally determinate. (This constraint never binds for the parameter values considered below.) For the benchmark calibration, the central bank's optimal Taylor rule has coefficients $\phi_\pi^* = 2.00$ and $\phi_y^* = 0.61$.⁹

We now examine how optimal policy depends on the model parameters. Figure 1 shows the optimal Taylor Rule coefficients as we vary the persistences of the shock processes in the neighborhood of the baseline calibration. To isolate the effect of shock persistence on policy, we adjust the variance of the innovations to ensure that the unconditional variance of the shocks is unchanged. In Figures 1 to 3, baseline parameter values are indicated by dotted vertical lines. All other parameters are held fixed at the level of the baseline calibration.

As the persistence of the efficient rate increases, the optimal reactions to inflation and the output gap both rise. In line with Rudebusch (2001), we interpret this finding as justifying intervention in the case of more persistent shocks while less intervention is necessary if the economy automatically and quickly reverts to its efficient allocation. However, this intuition does not always hold. For instance, the persistence of cost-push shocks has little influence on the optimal Taylor rule coefficients. Interestingly, persistent measurement error in inflation increases the optimal Taylor rule coefficients while persistent measurement error in the output

⁸ This value of κ can be derived from a Calvo model with a probability of price rigidity of 2/3 per quarter, together with our calibrated value of β , a Frisch labor supply elasticity of 1.00 and a linear production function.

⁹ Whenever we report numerical values for the coefficient on the output gap, we annualize it by multiplying the quarterly value by four.

gap reduces the optimal response to the output gap (and leaves the inflation coefficient almost unchanged).

Figure 2 shows how the optimal Taylor rule depends on the standard deviation of shock innovations. The upper left panel shows the Taylor rule coefficients as we change the volatility of the IS shocks. Not surprisingly (and incidentally, consistent with Proposition 3), a larger standard deviation of the efficient rate shocks raises ϕ_π^* and ϕ_y^* . The upper right panel considers changes in the standard deviation of cost-push shocks. Given our baseline calibration, as the cost-push shocks become more volatile the reaction to the output gap increases while the reaction to inflation falls. Intuitively, as the variance of cost-push shocks increases, the model behaves more and more like the special case in which there are only cost-push shocks. In that case, the optimal coefficients were restricted to the manifold (11). At the same time, the logic of Proposition 3 suggests finite Taylor rule coefficients as long as the output gap is observed with error. A combination of these two arguments seems to imply that as the variance of cost-push shocks becomes greater, the coefficients will approach a specific finite point on the line (11).

The bottom panels in Figure 2 show how the optimal Taylor rule depends on measurement error. Not surprisingly, for both types of measurement error, greater data uncertainty implies less active policy. It is worth noting that, at least for our baseline calibration, measurement error in the output gap is much more influential than measurement error in inflation. This is again consistent with the analytical result in Proposition 3.

Figure 3 depicts the optimal Taylor rule for alternative values of the weight on inflation in the objective function (α) and the macroeconomic rate of price adjustment (κ). The left panel illustrates the coefficients when we change the weight on inflation. As the preference for price stability increases (a greater α), the central bank adopts a Taylor rule with a higher reaction coefficient on inflation and a lower reaction coefficient on the output gap. The panel on the right shows how the coefficients vary with the macroeconomic rate of price adjustment. While the relationship is not monotone over the range considered in the figure, for relatively high values of κ , the central bank again chooses a stronger reaction to inflation and a weaker reaction to the output gap.

5. SIGNAL EXTRACTION AND OPTIMAL TAYLOR RULES

To this point, we have assumed that the central bank directly responds to *measured* inflation π_t^m and the *measured* gap y_t^m . Alternatively, we could consider a modified Taylor rule which stipulates that the central bank sets the interest rate as a function of *estimated* inflation and the output gap. Among others, Orphanides (2001) has advocated this specification. Drawing on the

results of Svensson and Woodford (2003, 2004), we consider signal extraction and optimal Taylor rules in this section.¹⁰

In our model, agents in the private sector have full information. Their information set includes all variables dated t or earlier and all model parameters. In contrast, the central bank observes only measured inflation and the measured output gap. We assume that the central bank uses the Kalman filter to construct estimates of the true values of the output gap and the inflation rate. For a generic variable x_t , we let $x_{t|t}$ denote the central bank's estimate of the variable given all of the information available at date t .¹¹ Having solved the signal extraction problem, the central bank sets the interest rate according to the modified Taylor rule

$$i_t = \rho + \psi_\pi \pi_{t|t} + \psi_y y_{t|t} . \quad (\text{TR2})$$

ψ_π and ψ_y are the Taylor rule coefficients which operate on the estimates $\pi_{t|t}$ and $y_{t|t}$. We list this Taylor rule as (TR2) to distinguish it from the more conventional Taylor rule (TR1). The remaining model equations are unchanged.

OPTIMAL POLICY

We begin by characterizing the central bank's estimates of inflation and the output gap as a function of the estimated disturbances. Lemma 2 is analogous to Lemma 1 for the model without noisy observations.

Lemma 2: *The central bank's estimates of inflation and the output gap satisfy*

$$\pi_{t|t} = \frac{\kappa}{\Phi_r + \psi_y(1 - \beta \varrho_r) + \kappa \psi_\pi} r_{t|t}^e + \frac{\psi_y + (1 - \varrho_u)\sigma}{\Phi_u + \psi_y(1 - \beta \varrho_u) + \kappa \psi_\pi} u_{t|t}, \quad (14)$$

$$y_{t|t} = \frac{1 - \beta \varrho_r}{\Phi_r + \psi_y(1 - \beta \varrho_r) + \kappa \psi_\pi} r_{t|t}^e + \frac{\varrho_u - \psi_\pi}{\Phi_u + \psi_y(1 - \beta \varrho_u) + \kappa \psi_\pi} u_{t|t}, \quad (15)$$

where Φ_r and Φ_u are defined as in Lemma 1.

The lemma shows that, with a suitable reinterpretation of the shocks, the equilibrium paths of the filtered variables $y_{t|t}$ and $\pi_{t|t}$ obey the same equilibrium conditions as the actual underlying variables y_t and π_t (see equations (9) and (10)).

¹⁰ A number of researchers have examined signal extraction problems of central banks. Further references include Swansson (2000), Aoki (2003), and Smets (2002).

¹¹ Formally, the central bank's information set is $I_t^{CB} = \{\Theta, \pi_{t-j}^m, y_{t-j}^m : j \geq 0\}$ where Θ is a vector of all model parameters. Then, for any variable x_t , the central bank's date t estimate is $x_{t|t} = E[x_t | I_t^{CB}]$. The observation equations are $\pi_t^m = \pi_t + m_t^\pi$ and $y_t^m = y_t + m_t^y$.

We are now interested in finding the coefficients $\{\psi_y^*, \psi_\pi^*\}$ that minimize (8) subject to (1) - (6), (TR2), and the central bank's informational constraints. The following proposition presents a result for the i.i.d. case in which we can obtain a closed-form solution.

Proposition 4: *Suppose all shocks are contemporaneously uncorrelated with each other and i.i.d over time. Then the coefficients $\{\psi_y^*, \psi_\pi^*\}$ satisfying*

$$\psi_\pi^* = \alpha\kappa(\psi_y^* + \sigma) + (1 + \alpha\kappa^2) \frac{\text{Cov}[r_{t|t}^e, u_{t|t}]}{V[u_{t|t}]} \quad (11'')$$

and $\psi_y^* \rightarrow \infty$ are optimal.

Proposition 4 shows that, there is an optimal Taylor rule in terms of filtered output $y_{t|t}$ and inflation $\pi_{t|t}$ which embodies the same properties as Proposition 2 (Special Case 4). As in Special Case 4, the optimal Taylor rule coefficients are again infinitely large and ψ_π^*/ψ_y^* converges to $\alpha\kappa$. Notice the following subtle difference: in Proposition 2, we assumed an exogenous correlation between r_t^e and u_t . In contrast, the correlation between the estimates $r_{t|t}^e$ and $u_{t|t}$ in Proposition 4 arises endogenously even though the underlying shocks r_t^e and u_t are uncorrelated.

The correlation between $r_{t|t}^e$ and $u_{t|t}$ comes from the central bank's effort to infer the true shocks. For example, suppose the central bank observes positive inflation and a negative output gap. A negative supply shock ($u_t > 0$) is a natural candidate for such observations. Another possibility is the occurrence of a positive demand shock together with a negative innovation to measurement error in the output gap. Because the central bank attaches positive probability to many potential combinations of shocks, its estimates $r_{t|t}^e$ and $u_{t|t}$ will typically be correlated. For the optimal Taylor rule, the induced correlation is immaterial. Equations (14) and (15) with $q_u = q_r = 0$ imply that only the limit of ψ_π^*/ψ_y^* affects the central bank's estimates of inflation and the output gap.

We next turn to the general case in which all shocks have arbitrary autocorrelation. The analytical solution of the optimal policy problem is difficult so we use numerical methods to characterize the optimal Taylor rule.

Figure 4 shows level curves of the central bank's objective function when all parameters are set to the values of our baseline calibration summarized in Table 1. As we have seen in the i.i.d. case (Proposition 4), stronger responses to both expected inflation and the expected output gap reduce the central bank's loss function (8). This continues to be the case when there are persistent innovations. Again, the optimal Taylor rule coefficients are infinitely large.¹²

¹² In all numerical cases we consider, the optimal Taylor rule coefficients approach infinity.

As the Taylor rule coefficients approach infinity, (14) and (15) imply that expected inflation and the expected output gap depend only on the estimated supply shock. In particular, $\pi_{t|t} = Au_{t|t}$ and $y_{t|t} = -Bu_{t|t}$ for some strictly positive constants A and B . Eliminating $u_{t|t}$ immediately yields

$$y_{t|t} = -BA^{-1}\pi_{t|t}.$$

The central bank acts to neutralize the effect of estimated demand shocks on $\pi_{t|t}$ and $y_{t|t}$ so that under the optimal Taylor rule, only the effects of cost-push shocks (i.e., “supply shocks”) remain. As a consequence, the central bank’s best estimates of current inflation and output are perfectly negatively correlated.¹³

DOES THE FEDERAL RESERVE FOLLOW AN OPTIMAL TAYLOR RULE?

If the central bank followed an optimal Taylor rule (TR2) and the model were correct, then expected inflation and the expected output gap should be perfectly negatively correlated. To see whether this prediction holds in the data, we examine the Fed’s real time forecasts of current quarter inflation and the output gap. These forecasts are produced by the Federal Reserve staff before every meeting of the Federal Open Market Committee and made publically available with a five year lag by Federal Reserve Bank of Philadelphia. We take the Fed’s contemporaneous estimate of Core CPI inflation as the empirical analogue of $\pi_{t|t}$.

Figure 5 plots the current-quarter forecasts of Core CPI inflation and the output gap from 1987Q3 - 2007Q4. In the figure, ‘initial’ refers to the forecast of the output gap produced for the first quarterly meeting and ‘revised’ refers to the forecast for the second meeting. Panel A shows the unfiltered series. There is no obvious relationship between the two series in the figure. The sample correlation between the initial output gap estimate and the estimate of Core CPI inflation is -0.09 (the correlation changes to -0.1 if we use the revised estimate). In contrast, under the optimal Taylor rule, the correlation between the two series should be -1.00.

One concern with using the raw series to compute this correlation is that much of the variation in inflation is due to a steady declining trend since the mid 1980’s. In Panel B, we show HP-filtered time series for both the estimated output gap and estimated inflation. (We use the standard quarterly smoothing parameter of 1600). In this figure, the estimates of inflation and the output gap have a modest positive correlation. The correlation coefficient is 0.35 for the initial output gap estimate and 0.29 for the revised estimate.

¹³ This result technically requires that the Taylor rule coefficients approach infinity at asymptotically the same rate so that their ratio converges to a strictly positive and finite value at the optimum.

In neither case is the correlation close to the prediction of negative one. When interpreted through the lense of our model it appears that the Fed is underreacting to its own estimates of inflation and the gap.

SIGNAL EXTRACTION AND INTEREST RATE SMOOTHING

There is a close connection between the Taylor rule given by (TR2) and interest rate smoothing. Interest rate smoothing can be captured by a policy rule of the form

$$i_t = \rho + \phi_\pi \pi_t^m + \phi_y y_t^m + \nu i_{t-1}. \quad (\text{TR3})$$

In this specification the central bank sets the interest rate as a function of measured inflation, measured output, and the lagged interest rate.¹⁴ The parameter ν governs the extent to which the central bank anchors its current policy with the interest rate from the previous quarter.

Taylor rules with interest rate smoothing are similar to Taylor rules based on filtered output and inflation. In fact, there are special cases in which the two rules exactly coincide. To see this, consider the model in the previous section given by equations (1) to (6) and (TR2), in which the central bank observes $\pi_t^m = \pi_t + m_t^\pi$ and $y_t^m = y_t + m_t^y$ and uses the Kalman filter to estimate the output gap and inflation. Suppose further that there are no cost-push shocks ($V[u_t] = 0$) and that the persistence of the measurement error shocks is zero ($\varrho_{my} = \varrho_{m\pi} = 0$). For this special case, we have the following result.

Proposition 5: *Given the assumptions above, for any Taylor rule (TR2) with coefficients ψ_y and ψ_π , there exist coefficients $\{\tilde{\phi}_y, \tilde{\phi}_\pi, \tilde{\nu}\}$ such that the policy rule (TR3) generates the same equilibrium paths for all variables.*

Proposition 5 states that in the special case where there are no cost-push shocks and measurement error is i.i.d., the Taylor rule in which the central bank responds to the estimates $y_{t|t}$ and $\pi_{t|t}$ can be implemented exactly by a Taylor rule with interest rate smoothing of the form (TR3).¹⁵

While exact equivalence between interest rate smoothing and Taylor rules of the form (TR2) requires strong assumptions, the spirit of Proposition 5 extends to very general settings. As long as the central bank follows Taylor rule (TR2) and forms its estimates $y_{t|t}$ and $\pi_{t|t}$ using the Kalman filter, the interest rate will change gradually because the estimates of the output gap and inflation change gradually. In such a setting, an econometrician would continue to conclude that lagged interest rates play an important role in shaping policy. To demonstrate this claim quantitatively, we again consider the model consisting of equations (1) to (6), the central bank's

¹⁴ See, for example, Sack and Wieland (2000), and Rudebusch (2006) for discussions of interest rate smoothing.

¹⁵ Aoki (2003) reaches a similar conclusion in a setting where the central bank optimizes under discretion.

informational constraints described above and Taylor rule (TR2). When we estimate the interest rate smoothing rule (TR3) on data simulated from this model using the benchmark calibration summarized in Table 1, we obtain a smoothing parameter ν of about 0.7.

6. NUMERICAL EVALUATION

Here we use a calibrated numerical model to quantitatively compare the policy rules (TR1), (TR2), and (TR3). All numerical results are based on the calibration discussed in Section 4 and summarized in Table 1. For each rule, we compute the optimal coefficients and evaluate the value of the objective function as well as the variance of inflation and output.

Table 2 reports the value of the objective function (equation (8)), as well as the associated variance of inflation and the output gap for all three policy rules evaluated at their optimal coefficients. For rules (TR1) and (TR3) the optimal coefficients are shown in the bottom rows of the table. Not surprisingly, the simple Taylor rule in which the central bank responds to measured inflation and measured output (equation (TR1)) is inferior to both alternatives. For our baseline calibration, Taylor rule (TR2) outperforms the interest rate smoothing rule (TR3) though we note that it is possible to construct cases in which optimal interest rate smoothing outperforms Taylor rule (TR2).¹⁶

Figures 6 and 7 report impulse response functions for the four structural shocks $(r_t^e, u_t, m_t^y, m_t^\pi)$. For each case, we consider a Taylor rule which is optimal given the specification (TR1), (TR2) or (TR3) as indicated. The top panels of Figure 6 show the impulse response functions of the output gap, inflation, and the interest rate to a shock to the efficient rate r_t^e (a ‘demand’ shock). When the central bank follows the simple Taylor rule (TR1), the output gap is remarkably close to zero. However, inflation jumps up substantially and only slowly converges back to its steady state value of zero. The history dependent rules (TR2) and (TR3) are superior in this regard.

Impulse response functions for the cost-push shock are shown in the bottom panels of Figure 6. On impact, the output gap is largest for the simple Taylor rule (TR1). No further striking differences of the paths of output and inflation are revealed for the different policy rules.

¹⁶ Consider a calibration in which all of the structural shocks have no persistence. In this case, the filtered estimates $\pi_{t|t}$ and $y_{t|t}$ are functions of only current observations π_t^m and y_t^m . As a consequence, the optimized value of the objective is the same for the simple Taylor rule (TR1) and for the Taylor rule with filtered data (TR2). However, since a central bank following (TR3) can choose an additional parameter it must be able to weakly improve on the restricted rules. The interest rate smoothing parameter allows the central bank some ability to commit to future actions which is often a feature of the globally optimal policy. See, e.g., Clarida, Galí and Gertler (1999), Woodford (1999) and Rotemberg and Woodford (1999).

As in our baseline calibration, Orphanides (2003) estimated that measurement error in the output gap is highly persistent. The top panels of Figure 7 show that this persistence is reflected in the dynamic reaction to a noise shock. When the central bank solves a signal extraction problem and sets the interest rate according to Taylor rule (TR2), it learns gradually that the shock must have been measurement error. Although the output gap and inflation drop immediately after the shock, optimal state estimation reveals relatively quickly that the shock is likely noise and the central bank quickly guides the economy back to steady state. When the central bank follows rules (TR1) or (TR3) instead, the persistence of the measurement error shock causes the economy to go through a protracted period of low inflation.

Not surprisingly, signal extraction is particularly important when measurement error is persistent. Because our baseline calibration features no persistence for measurement error in inflation, when the economy experiences such a shock, there are only slight differences in output and inflation across the policies. This is shown in the bottom panels of Figure 7. While the impact response of the output gap is largest for the simple Taylor rule (TR1), the lack of history dependence implies a return to the steady state in a single period. Rules (TR2) and (TR3) show a somewhat slower convergence rate.

6. CONCLUSION

One of the Taylor rule's most appealing features is its simplicity: the central bank's behavior is characterized by two parameters only. In this paper we analyze optimal Taylor rules in standard New Keynesian models. In the absence of measurement error, activist monetary policy is costless and the optimal Taylor rule coefficients are often infinite. When inflation and output are measured with error, the optimal Taylor rule coefficients are finite. If the central bank instead sets the interest rate as a function of *estimated* output and inflation then the optimal coefficients on estimated inflation and the estimated gap are again infinite.

Optimal monetary policy in the model with signal extraction implies a strong negative correlation between estimated inflation and estimated output. In contrast, data on the Federal Reserve's estimates of current inflation and the output gap exhibit either zero correlation or a modest positive correlation depending on whether or not they are filtered. This suggests that the Fed is insufficiently aggressive in responding to deviations from its targets.

Signal extraction on the part of the central bank also introduces behavior which mimics history dependence in monetary policy. Because the central bank's beliefs about the state of the economy are updated gradually, the interest rate also changes gradually and observed policy actions will resemble interest rate smoothing even if the central bank is reacting only to current estimates of inflation and the output gap.

The results in this paper are not meant to be the final word on optimal Taylor rules in New Keynesian models but rather to provide a simple benchmark for optimal Taylor policy rules in more realistic and complex economic models. Such models would likely include wage rigidity in addition to price rigidity and would almost surely include durable consumer and investment goods. While we anticipate that optimal Taylor rules will differ in more articulated models, we also suspect that many of the properties we derived for the basic New Keynesian model will carry over to more realistic environments.

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Appendix: Proofs of the Propositions

Lemma 1 *Under the assumptions in section 2, the unique competitive equilibrium of the model is characterized by equations (9) and (10), where $\Phi_j = \sigma(1 - \varrho_j)(1 - \beta\varrho_j) - \kappa\varrho_j$ for $j = \{r, u, my, m\pi\}$ are constants that are independent of monetary policy.*

Proof: Consider the system (1), (2) and (TR1) with the exogenous processes (3) to (6). Using (TR1) to eliminate i_t in (2) gives the system

$$\pi_t = \kappa y_t + \beta E_t[\pi_{t+1}] + u_t, \quad (\text{A1})$$

$$y_t = E_t[y_{t+1}] - \frac{1}{\sigma} (\phi_\pi (\pi_t + m_t^\pi) + \phi_y (y_t + m_t^y) - r_t^e - E_t[\pi_{t+1}]). \quad (\text{A2})$$

We conjecture an equilibrium solution

$$\pi_t = s_{\pi r} r_t^e + s_{\pi m^\pi} m_t^\pi + s_{\pi m^y} m_t^y + s_{\pi u} u_t$$

$$y_t = s_{y r} r_t^e + s_{y m^\pi} m_t^\pi + s_{y m^y} m_t^y + s_{y u} u_t$$

and solve for the eight unknown coefficients $s_{h,j}$ for $h = \pi, y$ and $j = r, u, my, m\pi$. Substituting the conjectured relationships into (A1) and (A2) and using (3) to (6) to evaluate the expectations gives

$$0 = (s_{\pi r} - \kappa s_{y r} - \beta s_{\pi r} \varrho_r) r_t^e + (s_{\pi m^\pi} - \kappa s_{y m^\pi} - \beta s_{\pi m^\pi} \varrho_{m^\pi}) m_t^\pi + (s_{\pi m^y} - \kappa s_{y m^y} - \beta s_{\pi m^y} \varrho_{m^y}) m_t^y + (s_{\pi u} - \kappa s_{y u} - 1 - \beta s_{\pi u} \varrho_u) u_t$$

and

$$0 = [(\varrho_r - 1)\sigma s_{y r} - \phi_\pi s_{\pi r} + 1 - \phi_y s_{y r} + s_{\pi r} \varrho_r] r_t^e + [(\varrho_{m^\pi} - 1)\sigma s_{y m^\pi} - \phi_\pi s_{\pi m^\pi} - \phi_\pi - \phi_y s_{y m^\pi} + s_{\pi m^\pi} \varrho_{m^\pi}] m_t^\pi + [(\varrho_{m^y} - 1)\sigma s_{y m^y} - \phi_\pi s_{\pi m^y} - \phi_y - \phi_y s_{y m^y} + s_{\pi m^y} \varrho_{m^y}] m_t^y + [(\varrho_u - 1)\sigma s_{y u} - \phi_\pi s_{\pi u} - \phi_y s_{y u} + s_{\pi u} \varrho_u] u_t.$$

Each coefficient in these expressions must be zero. This gives a system of eight equations in eight unknowns. Solving for these unknowns gives the coefficients in (9) and (10). (By collecting terms for each of the shocks r_t^e , m_t^π , m_t^y , and u_t one can split the system into four sub-systems, each with two equations and two unknowns. The four subsystems can then be solved separately.)

Uniqueness follows from the determinacy condition (7). ■

Proposition 1

(i) *If the only shocks to the model are cost-push shocks, then the optimal policy requires that the Taylor rule coefficients lie on the affine manifold (11).*

(ii) *For any ϕ_π^* and ϕ_y^* satisfying (11), the equilibrium is*

$$\pi_t = \frac{1 - \beta\varrho_u}{\kappa^2 + \alpha(1 - \beta\varrho_u)^2} u_t, \quad y_t = \frac{\alpha\kappa}{\alpha\kappa^2 + (1 - \beta\varrho_u)^2} u_t. \quad (\text{A3})$$

Proof: Substituting (9) and (10) (and using $r_t^e = m_t^y = m_t^\pi = 0$) into (8) gives

$$V[u_t] \left(\alpha \left(\frac{\phi_y + (1 - \varrho_u)\sigma}{\Phi_u + \phi_y(1 - \beta\varrho_u) + \kappa\phi_\pi} \right)^2 + \left(\frac{\varrho_u - \phi_\pi}{\Phi_u + \phi_y(1 - \beta\varrho_u) + \kappa\phi_\pi} \right)^2 \right)$$

The first order condition for ϕ_π requires

$$0 = \alpha\kappa(\phi_y + (1 - \varrho_u)\sigma) + (\varrho_u - \phi_\pi)(1 - \beta\varrho_u)$$

which implies (11). The first order condition for ϕ_y requires

$$0 = \alpha\kappa\phi_y(\phi_\pi - \varrho_u) + \alpha\kappa\sigma(1 - \varrho_u)(\phi_\pi - \varrho_u) - (\varrho_u - \phi_\pi)^2(1 - \beta\varrho_u).$$

This condition can be satisfied either by setting $\phi_\pi = \varrho_u$ or by (11). This establishes (i).

To establish (ii), use $\Phi_u = (1 - \varrho_u)(1 - \beta\varrho_u)\sigma - \kappa\varrho_u$ to write the equilibrium conditions as

$$\pi_t = \frac{\phi_y + (1 - \varrho_u)\sigma}{(1 - \varrho_u)(1 - \beta\varrho_u)\sigma + \kappa(\phi_\pi - \varrho_u) + \phi_y(1 - \beta\varrho_u)} u_t,$$

$$y_t = \frac{\varrho_u - \phi_\pi}{(1 - \varrho_u)(1 - \beta\varrho_u)\sigma + \kappa(\phi_\pi - \varrho_u) + \phi_y(1 - \beta\varrho_u)} u_t.$$

Substituting (11) gives (A3). This establishes (ii). \blacksquare

Proposition 2 *Suppose r_t^e and u_t are i.i.d. over time and have covariance $Cov[r_t^e, u_t]$. The Taylor rule coefficients satisfying (11') and $\phi_y \rightarrow \infty$ are optimal.*

Proof: When r_t^e and u_t are i.i.d. equations (9) and (10) simplify to

$$\begin{aligned}\pi_t &= \frac{\kappa}{\sigma + \phi_y + \kappa\phi_\pi} r_t^e + \frac{\phi_y + \sigma}{\sigma + \phi_y + \kappa\phi_\pi} u_t \\ y_t &= \frac{1}{\sigma + \phi_y + \kappa\phi_\pi} r_t^e - \frac{\phi_\pi}{\sigma + \phi_y + \kappa\phi_\pi} u_t\end{aligned}$$

Plugging them into objective (8), and simplifying, shows that the optimal ϕ_y and ϕ_π minimize

$$(\sigma + \phi_y + \kappa\phi_\pi)^{-2} \left[(1 + \alpha\kappa^2) V[r^e] + \left(\alpha(\phi_y + \sigma)^2 + (\phi_\pi)^2 \right) V[u] + 2(\alpha\kappa(\phi_y + \sigma) - \phi_\pi) Cov[r_t^e, u_t] \right]$$

First we fix the overall strength of the policy response by setting $\sigma + \phi_y + \kappa\phi_\pi = A$ and minimizing

$$A^{-2} \left\{ (1 + \alpha\kappa^2) V[r^e] + \left(\alpha(\phi_y + \sigma)^2 + (\phi_\pi)^2 \right) V[u] + 2(\alpha\kappa(\phi_y + \sigma) - \phi_\pi) Cov[r_t^e, u_t] \right\}$$

subject to $\sigma + \phi_y + \kappa\phi_\pi = A$.

The Lagrangian is

$$\mathcal{L} = A^{-2} \left\{ (1 + \alpha\kappa^2) V[r^e] + \left(\alpha(\phi_y + \sigma)^2 + (\phi_\pi)^2 \right) V[u] + 2(\alpha\kappa(\phi_y + \sigma) - \phi_\pi) Cov[r_t^e, u_t] \right\} + \lambda [A - \sigma - \phi_y - \kappa\phi_\pi]$$

and the first order conditions w.r.t. ϕ_y and ϕ_π , respectively, are

$$\begin{aligned}A^{-2} \{ 2\alpha(\phi_y + \sigma) V[u] + 2\alpha\kappa Cov[r_t^e, u_t] \} &= \lambda \\ A^{-2} \{ 2(\phi_\pi) V[u] - 2Cov[r_t^e, u_t] \} &= \kappa\lambda\end{aligned}$$

Combining them yields (11').

Since the objective is decreasing in A , it is optimal to let ϕ_y approach infinity. \blacksquare

Proposition 3 *Suppose all shocks are white noise, that is, $\varrho_r = \varrho_u = \varrho_{m_y} = \varrho_{m_\pi} = 0$. Then the minimization of (8) subject to (9) and (10) yields the optimal Taylor rule coefficients given by (12) and (13).*

Proof: Setting $\varrho_r = \varrho_u = \varrho_{m_y} = \varrho_{m_\pi} = 0$ in (9) and (10) and substituting into the objective (8) and using the fact that the shocks are (by assumption) uncorrelated implies that the central bank wants to choose parameters ϕ_y and ϕ_π to minimize

$$(\sigma + \phi_y + \kappa\phi_\pi)^{-2} \left([\alpha\kappa^2 + 1] V[r_t^n] + [\alpha\kappa^2 + 1] (\phi_\pi)^2 V[m_t^\pi] + [\alpha\kappa^2 + 1] (\phi_y)^2 V[m_t^y] + \left[\alpha(\phi_y + \sigma)^2 + (\phi_\pi)^2 \right] V[u_t] \right)$$

The first order condition for ϕ_y is

$$\begin{aligned}& \{ [\alpha\kappa^2 + 1] V[m_t^y] \phi_y + \alpha(\phi_y + \sigma) V[u_t] \} (\sigma + \phi_y + \kappa\phi_\pi) \\ &= [\alpha\kappa^2 + 1] V[r_t^n] + [\alpha\kappa^2 + 1] V[m_t^\pi] (\phi_\pi)^2 + [\alpha\kappa^2 + 1] V[m_t^y] (\phi_y)^2 + \left[\alpha(\phi_y + \sigma)^2 + (\phi_\pi)^2 \right] V[u_t]\end{aligned} \tag{A4}$$

The first order condition w.r.t. ϕ_π is

$$\begin{aligned}& \frac{\phi_\pi}{\kappa} \{ [\alpha\kappa^2 + 1] V[m_t^\pi] + V[u_t] \} (\sigma + \phi_y + \kappa\phi_\pi) \\ &= [\alpha\kappa^2 + 1] V[r_t^n] + [\alpha\kappa^2 + 1] V[m_t^\pi] (\phi_\pi)^2 + [\alpha\kappa^2 + 1] V[m_t^y] (\phi_y)^2 + \left[\alpha(\phi_y + \sigma)^2 + (\phi_\pi)^2 \right] V[u_t]\end{aligned} \tag{A5}$$

It is immediate to see that this implies

$$\phi_\pi = \kappa \frac{[\alpha\kappa^2 + 1] V[m_t^y] + \alpha V[u_t]}{[\alpha\kappa^2 + 1] V[m_t^\pi] + V[u_t]} \phi_y + \frac{\alpha\kappa\sigma V[u_t]}{[\alpha\kappa^2 + 1] V[m_t^\pi] + V[u_t]} \tag{A6}$$

We can rewrite equation (A5) as

$$\begin{aligned} & \kappa [\alpha\kappa^2 + 1] V [r_t^n] + \kappa [\alpha\kappa^2 + 1] V [m_t^y] (\phi_y)^2 \\ = & \phi_\pi [\alpha\kappa^2 + 1] (\sigma + \phi_y + \kappa\phi_\pi) V [m_t^\pi] - \kappa [\alpha\kappa^2 + 1] V [m_t^\pi] (\phi_\pi)^2 + \\ & \phi_\pi (\sigma + \phi_y + \kappa\phi_\pi) V [u_t] - \kappa [\alpha (\phi_y + \sigma)^2 + (\phi_\pi)^2] V [u_t] \end{aligned}$$

Cancelling like terms and simplifying gives

$$\begin{aligned} & \frac{\kappa}{\sigma + \phi_y} [\alpha\kappa^2 + 1] V [r_t^n] + \frac{\kappa}{\sigma + \phi_y} [\alpha\kappa^2 + 1] V [m_t^y] (\phi_y)^2 \\ = & \phi_\pi \{ [\alpha\kappa^2 + 1] V [m_t^\pi] + V [u_t] \} - \kappa\alpha\phi_y V [u_t] - \kappa\alpha\sigma V [u_t] \end{aligned}$$

Using condition (A6) we have

$$\phi_\pi ([\alpha\kappa^2 + 1] V [m_t^\pi] + V [u_t]) - \kappa\alpha\sigma V [u_t] = \kappa ([\alpha\kappa^2 + 1] V [m_t^y] + \alpha V [u_t]) \phi_y$$

Eliminating this term gives

$$\begin{aligned} & \frac{\kappa}{\sigma + \phi_y} [\alpha\kappa^2 + 1] V [r_t^n] + \frac{\kappa}{\sigma + \phi_y} [\alpha\kappa^2 + 1] V [m_t^y] (\phi_y)^2 \\ = & \kappa ([\alpha\kappa^2 + 1] V [m_t^y] + \alpha V [u_t]) \phi_y - \kappa\alpha\phi_y V [u_t]. \end{aligned}$$

Finally, we cancel terms to get (12). To find (13) use condition (A6) and rearrange terms. ■

Model with signal extraction To solve the model with signal extraction, we closely follow the setup in Svensson and Woodford (2003, 2004). The proofs of the results use the following notation and calculations. The model can be written as

$$\begin{pmatrix} X_{t+1} \\ \tilde{E}\mathbb{E}_t[x_{t+1}] \end{pmatrix} = A \begin{pmatrix} X_t \\ x_t \end{pmatrix} + B(i_t - \rho) + \begin{pmatrix} s_{t+1} \\ 0 \end{pmatrix} \quad (\text{A7})$$

where $X_t = (r_t^e, u_t, m_t^y, m_t^\pi)'$; $s_{t+1} = (\varepsilon_{t+1}^r, \varepsilon_{t+1}^u, \varepsilon_{t+1}^y, \varepsilon_{t+1}^\pi)'$; $x_t = (y_t, \pi_t)'$; and $\tilde{E} = \begin{pmatrix} 0 & 1 \\ \sigma & 1 \end{pmatrix}$. Partition the matrices A and B as follows

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

and let

$$\begin{aligned} A_{11} &= \begin{pmatrix} \varrho_r & 0 & 0 & 0 \\ 0 & \varrho_u & 0 & 0 \\ 0 & 0 & \varrho_{m^y} & 0 \\ 0 & 0 & 0 & \varrho_{m^\pi} \end{pmatrix}, A_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ A_{21} &= \begin{pmatrix} 0 & -\frac{1}{\beta} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, A_{22} = \begin{pmatrix} -\frac{\kappa}{\beta} & \frac{1}{\beta} \\ \sigma & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

The flow objective is

$$y_t^2 + \alpha\pi_t^2 = x_t' W x_t, W = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \quad (\text{A8})$$

and the observation equations are

$$Z_t = \begin{pmatrix} y_t^m \\ \pi_t^m \end{pmatrix} = D \begin{pmatrix} X_t \\ x_t \end{pmatrix}, \text{ with } D = (D_1, D_2), D_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (\text{A9})$$

Notice that the measurement errors are included in X_t . This allows for arbitrary persistence of both types of measurement error.

The central bank's information set is $I_t^{CB} = \{\Theta, y_{t-j}^m, \pi_{t-j}^m : j \geq 0\}$ where Θ is a vector of all model parameters.

We write the Taylor rule as

$$i_t - \rho = F x_{t|t} \quad (\text{A10})$$

where $F = (\psi_y, \psi_\pi)$. Notice the difference to Svensson and Woodford who write the policy as a linear function of the estimates of the state variables X_t .

Next, we conjecture that

$$x_{t|t} = GX_{t|t} \quad (\text{A11})$$

Then the upper block of (A7) gives

$$X_{t+1} = A_{11}X_t + s_{t+1}$$

and taking expectations (based on I_t^{CB}) yields

$$X_{t+1|t} = A_{11}X_{t|t}.$$

Taking expectations of the lower block of (A7) gives

$$\tilde{E}x_{t+1|t} = A_{21}X_{t|t} + A_{22}x_{t|t} + B_2(i_t - \rho).$$

Combining these equations with (A10) and (A11), we arrive at

$$x_{t|t} = [A_{22}]^{-1} \left(-A_{21} + \tilde{E}GA_{11} - B_2FG \right) X_{t|t}.$$

Hence, G must satisfy

$$G = [A_{22}]^{-1} \left(-A_{21} + \tilde{E}GA_{11} - B_2FG \right). \quad (\text{A12})$$

Next we conjecture that

$$x_t = G^1 X_t + (G - G^1) X_{t|t} \quad (\text{A13})$$

and rewrite the observation equation (A9) as

$$\begin{aligned} Z_t &= D_1 X_t + D_2 x_t \\ &= (D_1 + D_2 G^1) X_t + D_2 (G - G^1) X_{t|t} \end{aligned}$$

Following Svensson and Woodford,

$$Z_t = LX_t + MX_{t|t} \quad (\text{A14})$$

where

$$L = (D_1 + D_2 G^1) \quad (\text{A15})$$

and

$$M = D_2 (G - G^1).$$

The state equation of the Kalman filter is

$$X_{t+1} = A_{11}X_t + s_{t+1}$$

and the observation equation is (A14). Svensson and Woodford (2004) show that the Kalman filter updating equation takes the form

$$X_{t|t} = X_{t|t-1} + KL (X_t - X_{t|t-1}) \quad (\text{A16})$$

(their equation 26 with $v_t = 0$) and that it is possible to write

$$X_{t+1|t+1} = (I + KM)^{-1} [(I - KL) A_{11}X_{t|t} + KZ_{t+1}] \quad (\text{A17})$$

(their equation 30) where

$$K = PL' (LPL')^{-1}. \quad (\text{A18})$$

Furthermore,

$$P = \mathbb{E} \left[(X_t - X_{t|t-1}) (X_t - X_{t|t-1})' \right] = A_{11} \left[P - PL' (LPL')^{-1} LP \right] A_{11}' + \Sigma_s \quad (\text{A19})$$

where Σ_s is the covariance matrix of the errors s_{t+1} .

Finally, one needs to find G^1 . Again, following Svensson and Woodford (2003, 2004) we obtain

$$G^1 = [A_{22}]^{-1} \left(-A_{21} + \tilde{E}GKLA_{11} + \tilde{E}G^1 (I - KL) A_{11} \right). \quad (\text{A20})$$

Lemma 2 *The central bank's estimates of inflation and the output gap satisfy (14) and (15) where Φ_r and Φ_u are defined as in Lemma 1.*

Proof: Take conditional expectations based on the central bank's information set I_t^{CB} of equations (1) to (4) to obtain

$$\begin{aligned}\pi_t &= \kappa y_t + \beta \pi_{t+1|t} + u_{t|t} \\ y_t &= y_{t+1|t} - \frac{1}{\sigma} \left(i_t - \rho - r_{t|t}^e - \pi_{t+1|t} \right) \\ r_{t+1|t}^e &= \varrho_r r_{t|t}^e, \text{ and } u_{t+1|t} = \varrho_u u_{t|t}\end{aligned}$$

Imposing the Taylor rule (TR2) and repeating steps in the proof of Lemma 1 yields the desired result. ■

Lemma A1 *The welfare objective (8) can be decomposed as follows*

$$(1 - \beta) \mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t (y_t^2 + \alpha \pi_t^2) \right] = \left(\alpha \mathbb{E} \left[\pi_{t|t}^2 \right] + \mathbb{E} \left[y_{t|t}^2 \right] \right) + \left(\alpha \mathbb{E} \left[(\pi_t - \pi_{t|t})^2 \right] + \mathbb{E} \left[(y_t - y_{t|t})^2 \right] \right)$$

Proof: See Svensson and Woodford (2003, 2004). ■

Lemma A2 $\alpha \mathbb{E} \left[(\pi_t - \pi_{t|t})^2 \right] + \mathbb{E} \left[(y_t - y_{t|t})^2 \right] = \text{tr} \left[P (G^1 (I - KL))' W G^1 (I - KL) \right]$ where I is the identity matrix.

Proof: Define $T = \alpha \mathbb{E} \left[(\pi_t - \pi_{t|t})^2 \right] + \mathbb{E} \left[(y_t - y_{t|t})^2 \right]$ and notice that $T = \mathbb{E} \left[(x_t - x_{t|t})' W (x_t - x_{t|t}) \right]$ where W is defined in (A8) above. Equation (A13) implies that

$$x_t - x_{t|t} = G^1 X_t + (G - G^1) X_{t|t} - G X_{t|t} = G^1 (X_t - X_{t|t})$$

so we obtain

$$T = \mathbb{E} \left[(X_t - X_{t|t})' (G^1)' W G^1 (X_t - X_{t|t}) \right].$$

Next subtract $X_{t|t}$ from X_t and use (A16) to get

$$\begin{aligned}X_t - X_{t|t} &= X_t - (X_{t|t-1} + KL (X_t - X_{t|t-1})) \\ &= (I - KL) (X_t - X_{t|t-1})\end{aligned}$$

Then

$$\begin{aligned}T &= \mathbb{E} \left[(X_t - X_{t|t-1})' (I - KL)' (G^1)' W G^1 (I - KL) (X_t - X_{t|t-1}) \right] \\ &= \mathbb{E} \left[\text{tr} \left[(X_t - X_{t|t-1}) (X_t - X_{t|t-1})' (I - KL)' (G^1)' W G^1 (I - KL) \right] \right] \\ &= \text{tr} \left[\mathbb{E} \left[(X_t - X_{t|t-1}) (X_t - X_{t|t-1})' \right] (G^1 (I - KL))' W G^1 (I - KL) \right]\end{aligned}$$

where $\text{tr}(\cdot)$ is the trace operator. Using (A19) yields the desired result. ■

Lemma A3 *Suppose all shocks are contemporaneously uncorrelated with each other and i.i.d over time. Then the term $\alpha \mathbb{E} \left[(\pi_t - \pi_{t|t})^2 \right] + \mathbb{E} \left[(y_t - y_{t|t})^2 \right]$ is independent of the Taylor rule coefficients.*

Proof: Given Lemma A2, it is sufficient to show that T is independent of the Taylor rule coefficients. If all shocks are i.i.d. over time then $A_{11} = 0_{4 \times 4}$ and G^1 as defined in (A20) reduces to

$$G^1 = -[A_{22}]^{-1} A_{21}$$

which is independent of policy. Furthermore, $P = \Sigma_s$ (see A19) and because G^1 is independent of policy, so is L (defined in A15). Then $K = PL' (LPL')^{-1}$ is independent of policy (equation A18). As a result, T is independent of policy. ■

Lemma A4 *Suppose all shocks are contemporaneously uncorrelated with each other and i.i.d over time. Then $r_{t|t}^e$ and $u_{t|t}$ are independent of the Taylor rule coefficients.*

Proof: In the iid case with $A_{11} = 0_{4 \times 4}$, equation (A17) reduces to

$$X_{t+1|t+1} = (I + KM)^{-1} K Z_{t+1}$$

Combining this with (A14) gives

$$X_{t|t} = (I + KM)^{-1} K Z_t = (I + KM)^{-1} K (LX_t + MX_{t|t}).$$

Rearranging yields

$$X_{t|t} = KLX_t.$$

In the proof of Lemma A3 we showed that K and L are independent of policy when shocks are i.i.d. Hence $r_{t|t}^e$ and $u_{t|t}$ are independent of the Taylor rule coefficients. ■

Proposition 4 *Suppose all shocks are contemporaneously uncorrelated with each other and i.i.d over time. Then the coefficients $\{\psi_y^*, \psi_\pi^*\}$ satisfying (11'') and $\psi_y \rightarrow \infty$ are optimal.*

Proof: Lemmas A1 and A3 imply that minimizing objective (8) is equivalent to minimizing $\alpha \mathbb{E} \left[\pi_{t|t}^2 \right] + \mathbb{E} \left[y_{t|t}^2 \right]$. Furthermore, equations (14) and (15) simplify to

$$\begin{aligned} \pi_{t|t} &= \frac{\kappa}{\sigma + \psi_y + \kappa\psi_\pi} r_{t|t}^e + \frac{\psi_y + \sigma}{\sigma + \psi_y + \kappa\psi_\pi} u_{t|t} \\ y_{t|t} &= \frac{1}{\sigma + \psi_y + \kappa\psi_\pi} r_{t|t}^e - \frac{\psi_\pi}{\sigma + \psi_y + \kappa\psi_\pi} u_{t|t} \end{aligned}$$

in the i.i.d. case. Lemma A4 shows that $r_{t|t}^e$ and $u_{t|t}$ are independent of the Taylor rule coefficients, but importantly, their covariance need not to be zero. The remainder of the proof is identical to the proof of Proposition 2. ■

Proposition 5 *Suppose the model is given by equations (1) to (6), (TR2), and the observation equations $\pi_t^m = \pi_t + m_t^\pi$ and $y_t^m = y_t + m_t^y$. The central bank uses the Kalman filter with information set I_t^{CB} to estimate the true state of the economy. Suppose further that $V[u_t] = 0$ and that $\varrho_{m_y} = \varrho_{m_\pi} = 0$. Then, for any Taylor rule (TR2), with coefficients ψ_y and ψ_π , there exist coefficients $\{\tilde{\phi}_y, \tilde{\phi}_\pi, \tilde{\nu}\}$ such that the policy rule (TR3) generates the same equilibrium paths for all variables.*

Proof: Under the assumption that $V[u_t] = 0$ and $\varrho_{m_y} = \varrho_{m_\pi} = 0$, it is possible to write the model as follows

$$\begin{aligned} X_t = r_t^e, s_{t+1} = \varepsilon_{t+1}^r, x_t = \begin{pmatrix} y_t \\ \pi_t \end{pmatrix}, \tilde{E} = \begin{pmatrix} 0 & 1 \\ \sigma & 1 \end{pmatrix}, A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \\ A_{11} = \varrho_r, A_{12} = \begin{pmatrix} 0 & 0 \end{pmatrix}, B_1 = 0, A_{21} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, A_{22} = \begin{pmatrix} -\frac{\kappa}{\beta} & \frac{1}{\beta} \\ \sigma & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

The observation equations are

$$Z_t = \begin{pmatrix} y_t^m \\ \pi_t^m \end{pmatrix} + v_t = D \begin{bmatrix} X_t \\ x_t \end{bmatrix} + v_t, D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $v_t = (m_t^y, m_t^\pi)'$. Furthermore, $D_1 = (0, 0)'$ and $D_2 = I_2$. Notice that A_{11} is a scalar. Because v_t is nonzero, the equations (and matrices) associated with the Kalman filter change somewhat (see Svensson and Woodford 2004) though (A17) continues to hold. Since A_{11} is a scalar, KL and KM are also scalars.

Using (A10) and (A11), write the interest rate as

$$i_t - \rho = (\psi_y, \psi_\pi) (y_{t|t}, \pi_{t|t})' = Fx_{t|t} = FGX_{t|t}.$$

Using (A17) and substituting backwards yields

$$i_t = \sum_{j=0}^{\infty} FG(I + KM)^{-1} \left[(I - KL) A_{11} (I + KM)^{-1} \right]^j KZ_{t-j}$$

Since KL , and KM are scalars, set $\tilde{\nu} = (I - KL) A_{11} (I + KM)^{-1}$. Then write

$$i_t = \sum_{j=0}^{\infty} FG(I + KM)^{-1} \tilde{\nu}^j KZ_{t-j} = FG(I + KM)^{-1} KZ_t + \tilde{\nu} i_{t-1} = \tilde{\phi}_y y_t^m + \tilde{\phi}_\pi \pi_t^m + \tilde{\nu} i_{t-1}$$

where $(\tilde{\phi}_y, \tilde{\phi}_\pi) = FG(I + KM)^{-1} K$. ■

TABLE 1: BASELINE CALIBRATION

Parameter	Value	Description
β	0.99	Discount factor
σ	1	Coefficient of relative risk aversion
θ	2/3	Poisson rate of price stickiness
η	1	Labor supply elasticity
$(\varrho_r, \varrho_u, \varrho_{m\pi}, \varrho_{my})$	(0.9, 0.5, 0, 0.95)	Persistence of shock processes
$sd[\varepsilon^r]$	0.22	Standard deviation (SD) of innovation to efficient rate
$sd[u]$	0.43	SD of innovation to the cost-push shock
$sd[\varepsilon^{m\pi}]$	0.50	SD of innovation to measurement error in inflation
$sd[\varepsilon^{my}]$	0.66	SD of innovation to measurement error in the gap
α	1	Weight on inflation in the policy objective function

TABLE 2: EVALUATION OF POLICY RULES

	Standard Taylor Rule (TR1)	Taylor Rule with signal extraction (TR2)	Interest Rate Smoothing (TR3)
Criterion (8)	1.56	1.03	1.16
$V[y]$	1.11	0.69	0.86
$V[\pi]$	0.45	0.34	0.30
ϕ_y^*	0.61	-	0.27
ϕ_π^*	2.00	-	0.42
ν^*	-	-	1.08

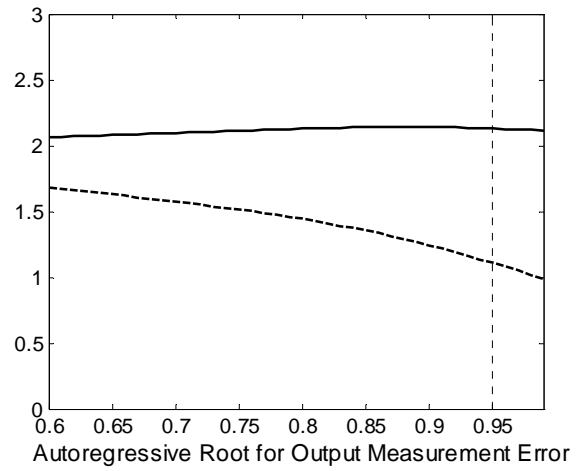
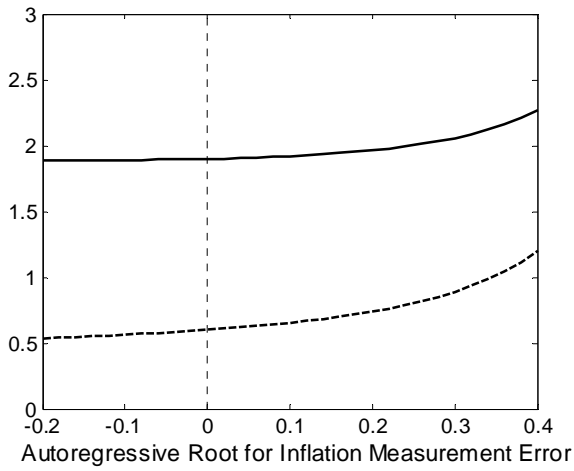
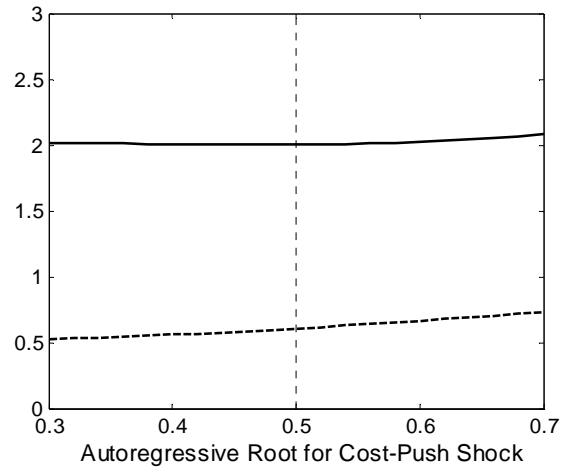
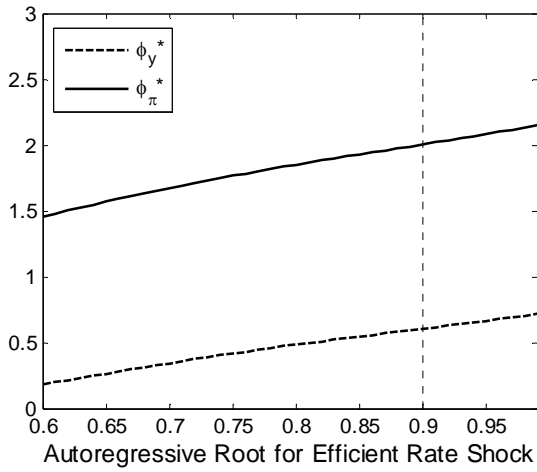


Figure 1: Parametric Variations in Shock Persistence.

Notes: Each panel plots the optimal Taylor rule coefficients for inflation (heavy solid line) and the output gap (heavy dashed line). The vertical dashed line indicates the parameter value in the baseline calibration. For all cases, as the autoregressive coefficient changes, the innovation variance is adjusted so as to hold constant the variance of the overall process. All other parameters are held constant at the baseline level.

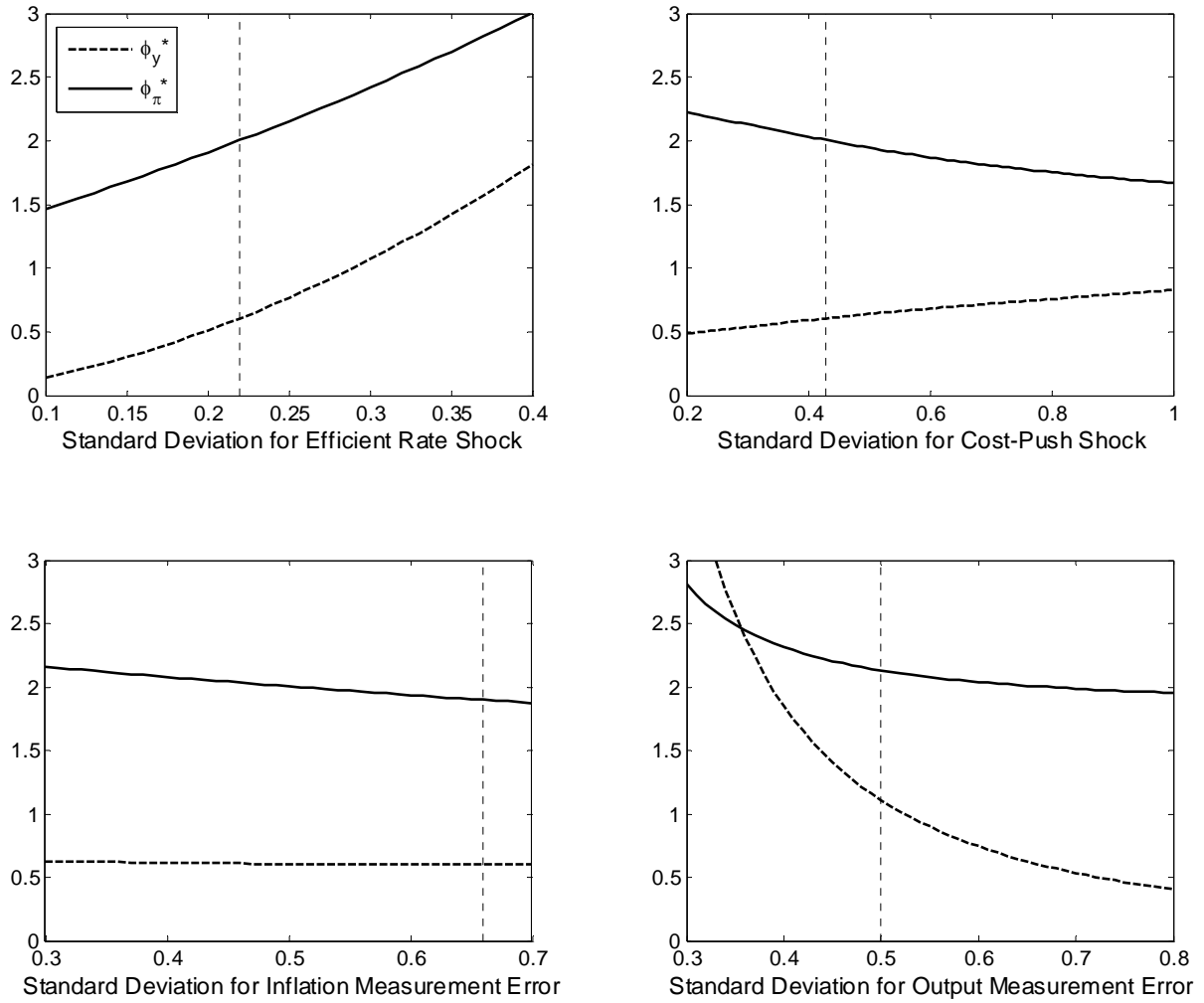


Figure 2: Parametric Variations in Shock Variance.

Notes: Each panel plots the optimal Taylor rule coefficients for inflation (heavy solid line) and the output gap (heavy dashed line). The vertical dashed line indicates the parameter value in the baseline calibration. All other parameters are held constant at the baseline level.

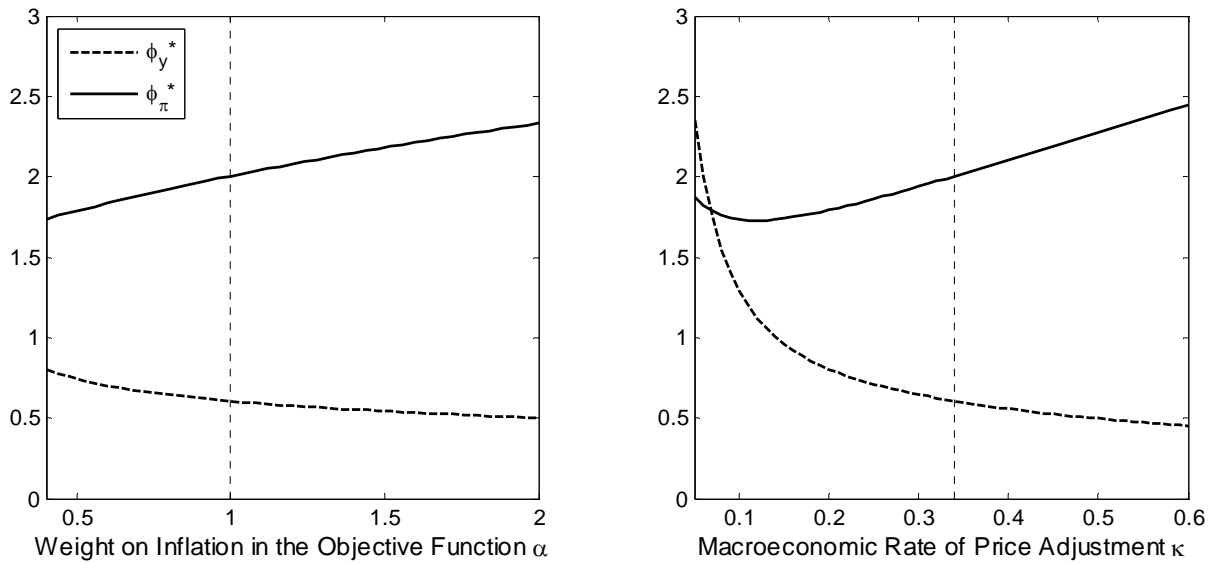


Figure 3: Parametric Variations in Inflation Weight and Rate of Price Adjustment.

Notes: Each panel plots the optimal Taylor rule coefficients for inflation (heavy solid line) and the output gap (heavy dashed line). The vertical dashed line indicates the parameter value in the baseline calibration. All other parameters are held constant at the baseline level.

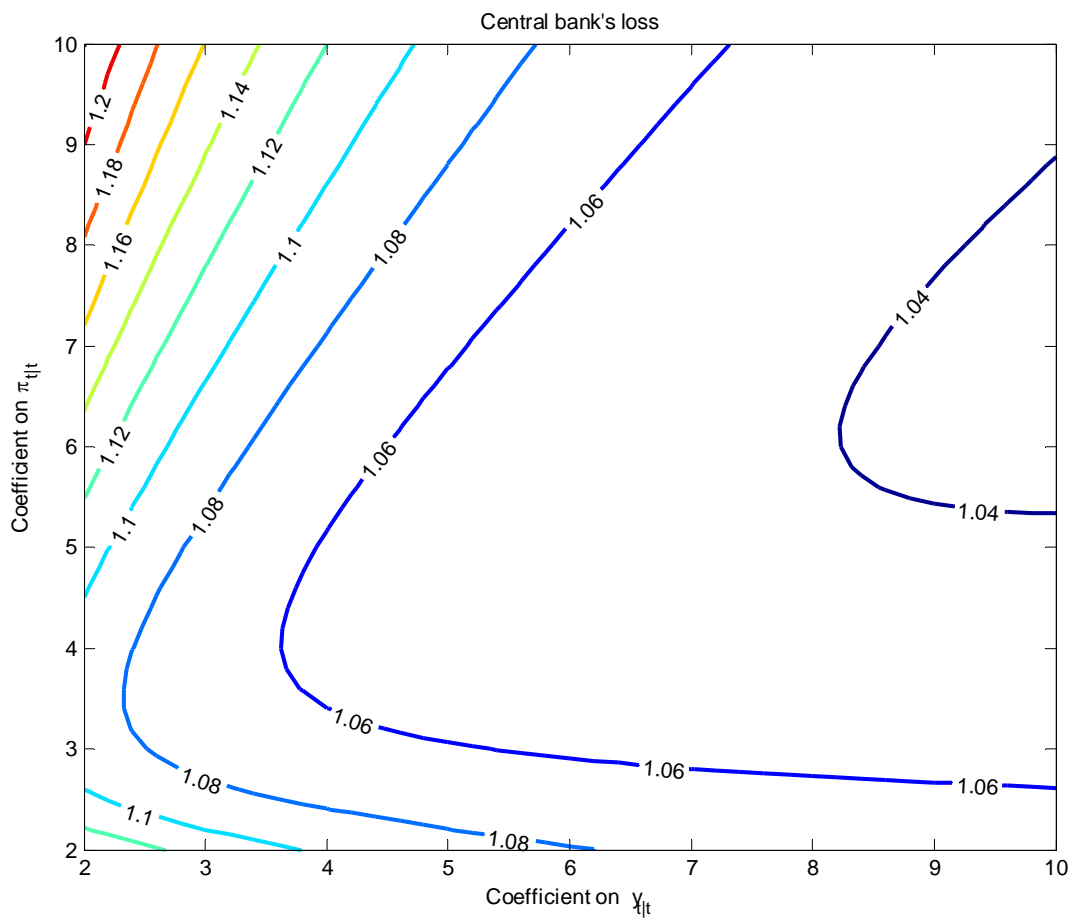


Figure 4: Level curves of the central bank's loss function.

Notes: The figure uses the baseline calibration summarized in Table 1.

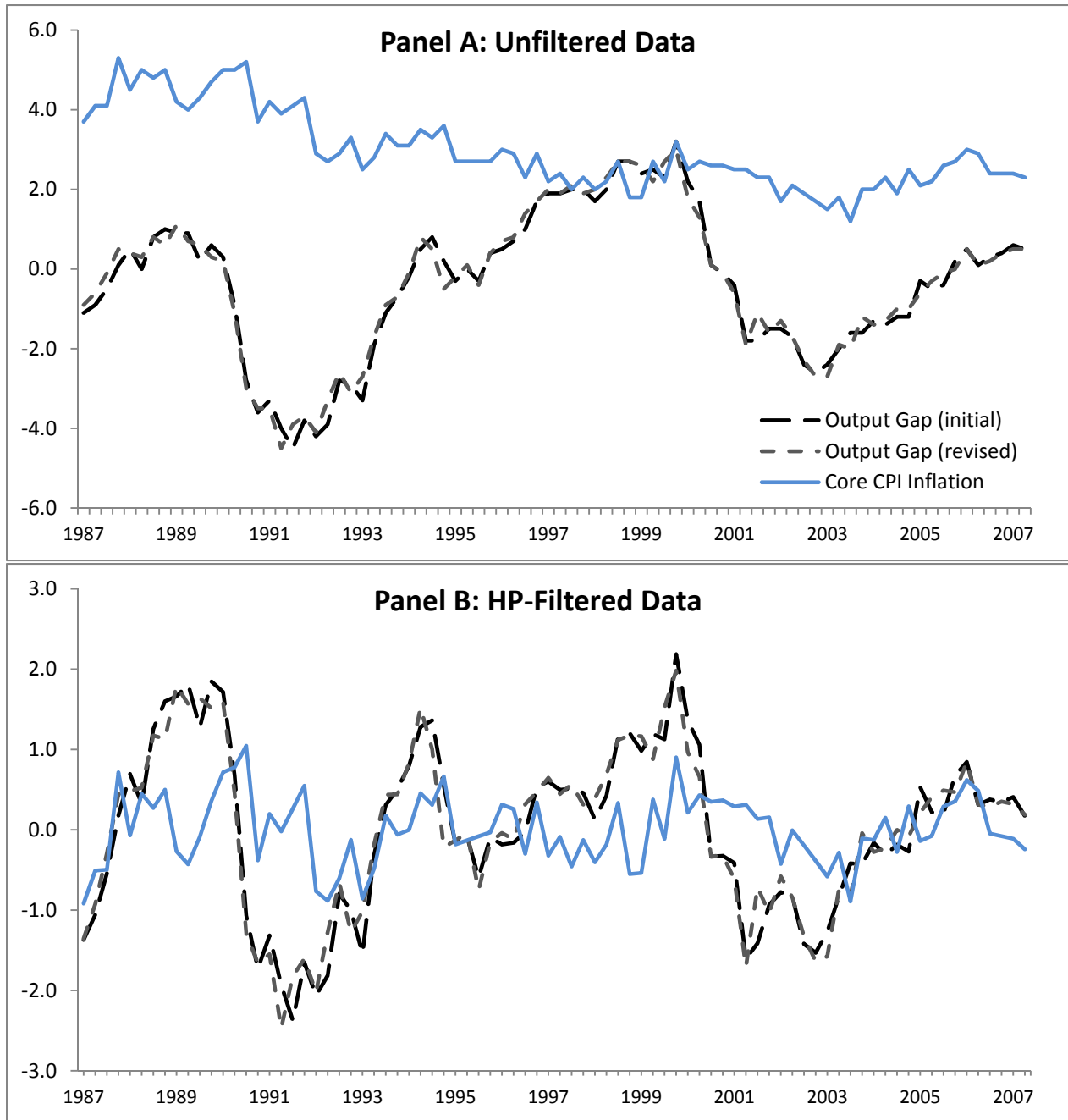


Figure 5: Inflation and output gap estimates of the Fed.

Notes: The graphs show the current-quarter estimates of Core CPI inflation and the output gap. All series in Panel B are HP-filtered with a smoothing parameter of 1600. The Federal Open Market Committee meets twice per quarter and two estimates for the output gap are available. “Initial” refers to the estimate for the first and “revised” to the estimate for the second meeting.

Source: Federal Reserve Bank of Philadelphia.

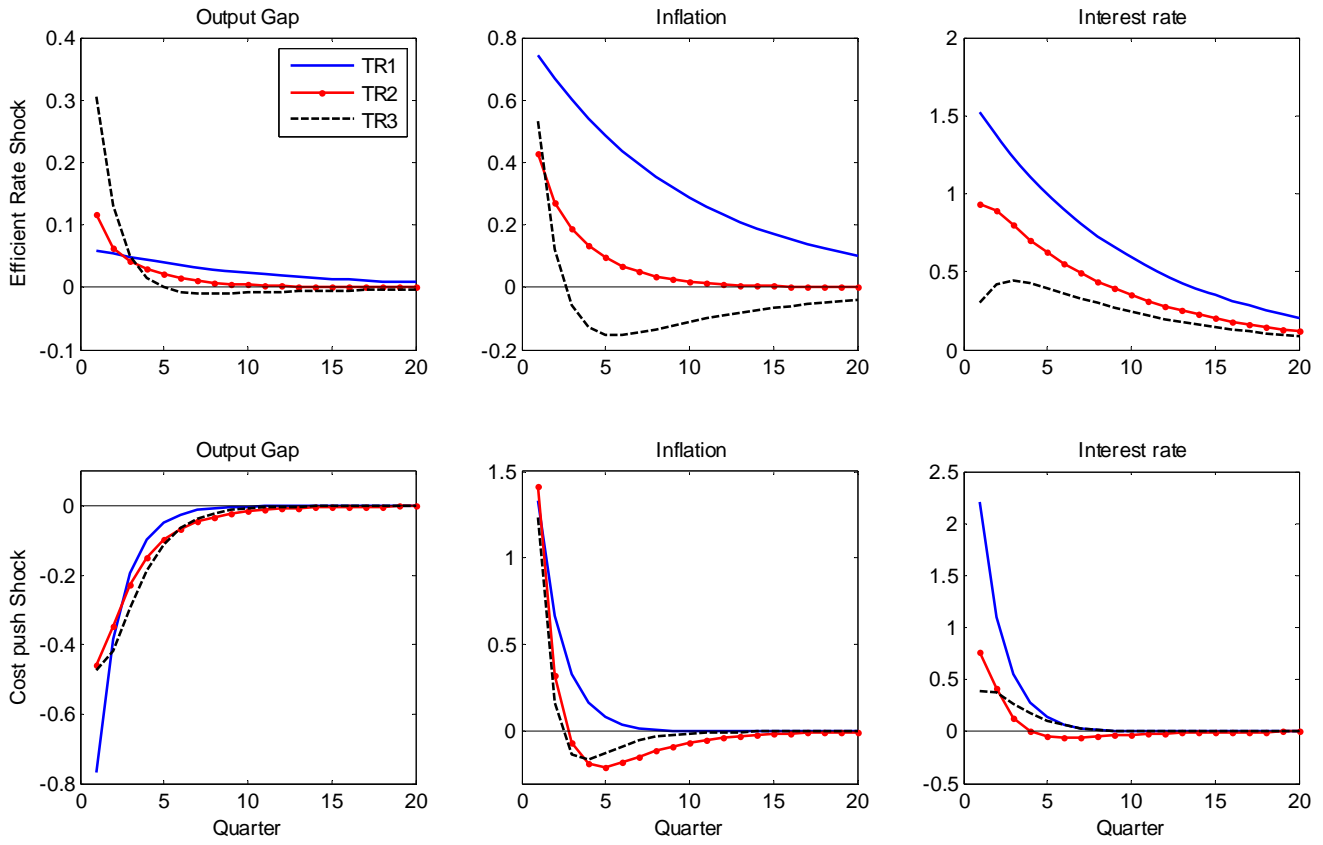


Figure 6: Impulse Response Functions.

Notes: All shocks are one standard deviation in size. The unit of the output gap is percentage deviations from its steady state value of zero. The units of inflation and the interest rate are annualized percentage points.

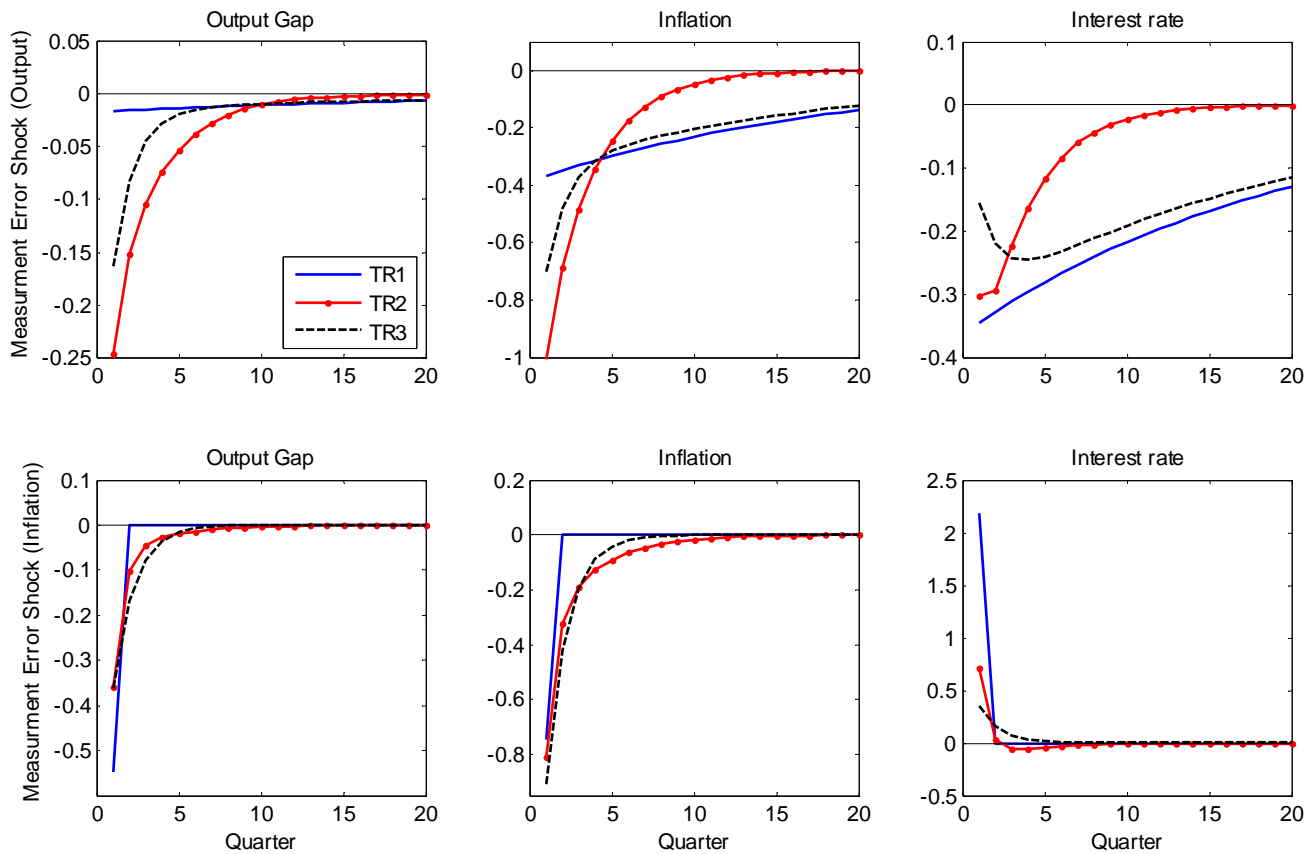


Figure 7: Impulse Response Functions.

Notes: All shocks are one standard deviation in size. The unit of the output gap is percentage deviations from its steady state value of zero. The units of inflation and the interest rate are annualized percentage points.