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UNCERTAINTY SHOCKS, ASSET SUPPLY AND PRICING OVER THE BUSINESS CYCLE

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ABSTRACT

This paper estimates a business cycle model with endogenous financial asset supply and ambiguity averse investors. Firms' shareholders choose not only production and investment, but also capital structure and payout policy subject to financial frictions. An increase in uncertainty about profits lowers stock prices and leads firms to substitute away from debt as well as reduce shareholder payout. This mechanism parsimoniously accounts for postwar comovement in investment, stock prices, leverage and payout, at both business cycle and medium term cycle frequencies. Ambiguity aversion permits a Markov-Switching VAR representation of the model, while preserving the effect of uncertainty shocks on the time variation in the equity premium.

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1 Introduction

Standard business cycle models have trouble reconciling the dynamics of stock prices with optimizing behavior by households and firms. On the household side, standard models assume expected utility and low risk aversion. As a result, they predict that households choose asset positions so as to keep the expected return on equity closely aligned with the riskless interest rate. In the data, however, the conditional expected return on equity is much more volatile; in particular, low stock prices predict returns that are higher than the riskless interest rate.

On the firm side, standard models assume costless access to external finance. They predict that shareholders choose investment so as to equate the return on equity and the return on capital. At the same time, the Modigliani-Miller theorem implies that standard models make no predictions about capital structure and shareholder payout. Quantitative exercises thus typically specify exogenous rules for leverage and payout that are not confronted with the data.

Postwar US data shows distinct patterns of comovement between stock prices, debt issuance and shareholder payout. It is well known that all three variables are procyclical.¹ This paper documents that there is also strong comovement at medium term cycle frequencies.² In particular, compared to the 1960s and more recent decades, the 1970s and early 1980s were not only a time of low stock prices but also saw low debt issuance and shareholder payout.

This paper proposes and estimates a business cycle model that jointly matches the dynamics of stock prices, leverage and shareholder payout in postwar US data. On the household side, we depart from standard models by making the representative household averse to Knightian uncertainty (ambiguity). As in recent work on asset pricing with recursive multiple priors utility, the household chooses asset holdings so as to keep a *worst case* expected return on equity aligned with the riskless interest rate.

On the firm side, we depart from standard models by introducing financing costs, following the literature on dynamic corporate finance. In particular, our representative firm faces adjustment costs that encourage dividend smoothing, as well as an upward sloping marginal cost of debt that is familiar from the "tradeoff theory" of capital structure. In particular, issuing additional debt is cheaper than raising equity at low levels of debt, but becomes eventually more expensive as debt increases. Shareholders maximize the value of the firm by choosing both optimal investment and determinate optimal leverage and payout ratios.

The key mechanism at the heart of the model is the response of households and firms to changes in uncertainty. Other things equal, an increase in uncertainty about future profits would lower the worst case return on equity below the riskless interest rate. Households would then choose to

¹A large asset pricing literature has documented the cyclicity of stock prices (see Lettau and Ludvigson (2010) and Cochrane (2011) for recent surveys). At the same time, recent work on financial flows, such as Covas and Den Haan (2011) and Jermann and Quadrini (2012), emphasizes the procyclicality of debt issuance and shareholder payout.

²Following Comin and Gertler (2006), we define medium frequencies as those between 8 and 50 years.

hold less equity and more bonds. To restore equilibrium, the stock price falls and the conditional expected return on equity – as measured by an observer such as an econometrician – increases.

Since shareholders are ambiguity averse, an increase in uncertainty also changes optimal capital structure and payout. Indeed, an increase in uncertainty about future profits increases the worst case expected cost of debt to firms who are concerned about future financing costs. Firms optimally respond by cutting debt. The scarcity of external funds further leads them to rely more on internal funds and hence reduce shareholder payout. Changes in uncertainty thus endogenously generate the comovement of stock prices, debt and payout observed in the data.

Our estimation allows for two sources of uncertainty: shocks to the marginal product of capital (MPK) and shocks to a fixed operating cost that affects corporate earnings but does not scale with production.³ Uncertainty about either source lowers worst case profits and makes debt less attractive. There is a key difference, however, in the response of investment: while uncertainty about MPK lowers investment because it lowers the worst case return on capital, uncertainty about operating cost increases investment since it encourages precautionary savings.

Formally, our model can be represented by a Markov-switching VAR. The two shocks – MPK and operating cost – follow autoregressive processes. Regime switches allow for occasional large innovations as well as nonlinear dynamics in MPK. They also affect the conditional volatility of future innovations as well as agents’ perceived ambiguity. A convenient technical feature of our setup is that agents’ endogenous response to changes in uncertainty is reflected in the coefficients of the MS-VAR. Indeed, since ambiguity affects the worst case conditional mean, a linear approximation of the model solution works well even though uncertainty changes over time.

We estimate the model with postwar US data on four observables: investment, leverage and the ratios of shareholder payout as well as equity to GDP. We allow for measurement error on all observables, but show that its role is minor: our two shocks alone, together with occasional changes in uncertainty, deliver a good fit for the distribution of the observables. We show in particular that we match key moments of the observables and related statistics like the price-payout ratio at both business cycle and medium term cycle frequencies.

The estimated model provides a three-part account of how uncertainty drove real and financial variables in the postwar era. First, shocks to MPK – as well as ambiguity about MPK – drive the business cycle. Second, a large increase in the volatility of operating cost induced a medium term cycle in the 1970s and 1980s. A similar increase in volatility occurred during the recent financial crisis. Finally, the end of the long booms of the 1960s, 1980s, 1990s and early 2000s saw a decline in ambiguity that was not accompanied by a change in volatility. Those booms ended with a spike in ambiguity that generated a stock market crash.

³The operating cost captures reorganization in the corporate sector that redistributes resources away from shareholders. One example is expenses incurred in “packaging” earnings in the form of individual firm payouts such as the cost of mergers, spin-offs or IPO’s, all of which can vary over time because of changes in financial conditions. Other examples are changes in compensation or the relationship with the government.

We also investigate model implications for the riskless interest rate and excess returns, two series that our estimation does not target. In the data, the riskless rate is stable. Moreover, the excess return on equity relative to the riskless rate is predictable: a regression of excess returns on the price-payout ratio show that low prices reliably predict high excess returns – the excess volatility puzzle. DSGE models driven by uncertainty shocks often struggle to reconcile these facts. Indeed, excess volatility calls for large movements in uncertainty, whereas stable interest rates suggest little variation in precautionary savings.

In our model, there is no tension between excess volatility and stable interest rates. This is because operating cost represents a small share of overall consumption, but a sizable share of shareholder payout. An increase in uncertainty about operating cost thus discourages stock holding and lowers the stock price. At the same time, it affects uncertainty about future consumption only weakly, and the effect on precautionary savings and bond prices is small.

To show further how ambiguity matters for fit, we estimate an alternative model without uncertainty about operating cost. The alternative model is otherwise identical to our baseline specification: there are two shocks and nonlinear dynamics in MPK. There is also time varying uncertainty about MPK, much like in the growing literature on uncertainty shocks in DSGE models. We find that the alternative model does not fit well: without occasional regime switches that alter uncertainty about operating cost, it is not possible to fit our four observables. In particular, the alternative model cannot jointly fit payout and equity value.

The alternative estimation illustrates the issues faced by traditional models once shareholder payout and investment are included as observables. It must somehow account for volatile stock prices. Since investment is observable, it cannot rely on large changes in uncertainty about MPK – this would generate excessive precautionary savings. Since dividends are observable, it cannot rely on large movements in realized operating cost. In the end, what remains is measurement error. Moreover, the price volatility the alternative estimation does generate is largely due to fluctuations in realized operating cost. As a result, the alternative model cannot generate predictability of excess returns.

A number of papers study asset pricing in production economies with aggregate uncertainty shocks. Some consider rational expectations models with time variation in higher moments of the shock distributions. The latter can take the form of time varying disaster risk (Gourio (2012)) or stochastic volatility (Basu and Bundick (2012), Caldara et al. (2012), Malkhozov and Shamloo (2012)). Another line of work investigates uncertainty shocks when agents have a preference for robustness (Cagetti et al. (2002), Bidder and Smith (2012), Jahan-Parvar and Liu (2012)). Most of these papers identify equity with the value of firm capital or introduce leverage exogenously. In contrast, our interest is in how uncertainty shocks drive valuation, payout and leverage when shareholders respond optimally to those shocks.

We rely on financial frictions to obtain predictions for payout and capital structure, following a large literature on dynamic corporate finance (see for example the survey by Strebulaev and

Whited (2011)). In particular, we share the perspective of that literature that a firm is not simply a production technology, but instead makes decisions on how to package cash flows into debt and equity claims. Several recent papers have emphasized shocks that occur at this packaging stage, such as costs to raising different types of external funds (e.g. Bolton et al. (2013), Eisfeldt and Muir (2014) and Belo et al. (2014)).

We also build on a recent literature that tries to jointly understand financial flows and macro quantities. Jermann and Quadrini (2012), Covas and Den Haan (2011, 2012), Khan and Thomas (2013) and Begeau and Salomao (2013) develop evidence on the cyclical behavior of debt and equity flows. The modeling exercises in these papers also point to the importance of shocks to firm profits other than productivity. In our setup, a key shock is the operating cost, which is also independent of the marginal product of capital. We differ from the above papers in emphasizing the effects of uncertainty shocks, and their joint effects on firm decisions and equity risk premia.

There is also recent work that explores whether the interaction of uncertainty shocks and financial frictions can jointly account for credit spreads and investment. Most of this work considers changes in firm-level volatility (Arellano et al. (2010), Gilchrist et al. (2010), Christiano et al. (2014) and Elenev et al. (2016)). Gourio (2013) incorporates time varying aggregate risk and thus allows risk premia to contribute to spreads. In contrast to our paper, this line of work does not focus on the determination of equity prices.

At the macro level, our finding that movements in operating cost and uncertainty are important is also related to Greenwald et al. (2014), who decompose changes in stock market wealth into three components that they label productivity, factor share and risk aversion shocks. The latter are responsible for most of the variation in equity prices in their setting and thus play a role similar to changes in perceived uncertainty shocks in our model. At the same time, our specification allows for stochastic volatility and thus relates high equity premia in part to the occurrence of large shocks.

Glover et al. (2011) and Croce et al. (2012) study the effects of taxation in the presence of uncertainty shocks. Their setups are similar to ours in that they combine a representative household, a trade-off theory of capital structure and aggregate uncertainty shocks (in their case, changes in stochastic volatility under rational expectations). While their interest is in quantifying policy effects, our goal is to assess the overall importance of different uncertainty shocks.

The multiple priors model was introduced by Gilboa and Schmeidler (1989); it was extended to intertemporal choice by Epstein and Wang (1994) and Epstein and Schneider (2003). It has since been used in a number of studies in finance and macroeconomics (see Epstein and Schneider (2010) or Guidolin and Rinaldi (2013) for surveys). In particular, Epstein and Schneider (2008), Leippold et al. (2008), Sbuelz and Trojani (2008) and Drechsler (2013) consider the effect of time varying ambiguity on asset prices. Ilut and Schneider (2014) study a tractable linear business cycle model with ambiguity shocks where all movements in uncertainty come from intangible information. The present paper instead considers a nonlinear model in which stochastic volatility

matters for decisions and focuses on financial decisions and asset prices.

Our estimation strategy follows the literature in using Bayesian techniques for inference of DSGE models, but also incorporates financial variables and particularly asset prices. We build on the Markov switching (MS) model introduced by Hamilton (1989), which is now a popular tool for capturing parameter instability. For example, Sims and Zha (2006) use a MS-VAR to investigate the possibility of structural breaks in the conduct of monetary policy, while Schorfheide (2005), Baele et al. (2011), Liu et al. (2011), Bianchi (2013) and Davig and Doh (2013) estimate MS-DSGE models.⁴ We show that the MS-DSGE setup can also accommodate uncertainty averse agents' responses to volatility shocks. Indeed, since switches in volatilities have first order effects on decision rules, agents' responses are reflected in switches of the constants in the MS-DSGE.

The paper is structured as follows. Section 2 presents the model. Section 3 uses first order conditions for households and firms to explain the effect of uncertainty shocks on firm asset supply and asset prices. Section 4 describes our estimation strategy. Sections 5 and 6 discuss the estimation results by analyzing alternative models where shocks or frictions are shut down. Section 7 investigates the behavior of real rates and excess return predictability.

2 Model

Our model determines investment, production and financing choices of the US nonfinancial corporate sector as well as the pricing of claims on that sector by an infinitely-lived representative household. Firms are owned by the household and maximize shareholder value.

2.1 Technology and accounting

There is a single perishable good that serves as numeraire.

Production

The corporate sector produces numeraire from physical capital K_{t-1} according to the production function

$$Y_t = Z_t K_{t-1}^\alpha \xi^{(1-\alpha)t}, \quad (2.1)$$

where ξ is the trend growth rate of the economy and α is the capital share. The shock Z_t to the marginal product of capital accounts for fluctuations in variable factors including labor.

Capital is produced from numeraire and depreciates at rate δ

$$K_t = (1 - \delta)K_{t-1} + [1 - 0.5\Theta(I_t/I_{t-1} - \xi)^2] I_t. \quad (2.2)$$

⁴An alternative approach models smooth changes in the parameters (see Justiniano and Primiceri (2008), Fernández-Villaverde et al. (2010) and Fernández-Villaverde et al. (2011) for applications in a DSGE model and Primiceri (2005) and Cogley and Sargent (2006) for applications in a VAR).

Capital accumulation is subject to adjustment costs that are convex in the growth rate of investment I_t , as in Christiano et al. (2005). This functional form captures the idea that the investment scale affects the organization of the firm. For example, investing at a scale I_t requires allocating the right share of managerial effort to guide expansion rather than oversee production. Moving to a different scale entails reallocating managerial effort accordingly.

Cost of debt and dividend smoothing

Besides investment, shareholders choose firms' net payout and their debt level. We introduce two frictions that are familiar from dynamic models in corporate finance. The first is an upward sloping marginal cost of debt that gives rise to a tradeoff theory of capital structure.⁵ The representative firm can issue riskless one period noncontingent debt at price Q_t^b . If it issues $Q_{t-1}^b B_{t-1}^f$ worth of bonds at date $t-1$, it repays B_{t-1}^f at date t . It also incurs the cost

$$\kappa \left(B_{t-1}^f \right) = 0.5 \Psi \frac{1}{\xi^t} \left(B_{t-1}^f \right)^2. \quad (2.3)$$

The marginal cost of an extra dollar of debt is thus increasing in the face value of debt.⁶ Reduced form convex cost functions such as (2.3) can be derived for example from deadweight costs of bankruptcy.⁷

The second friction is an adjustment cost that creates an incentive to smooth shareholder payout. Every period, shareholders pay

$$\phi \left(D_t, D_{t-1} \right) = 0.5 \Phi \xi^t \left(D_t / D_{t-1} - \xi \right)^2. \quad (2.4)$$

Shareholders thus respond to shocks to cash flow or asset prices by changing payout gradually. The corporate finance literature has developed various microfounded models to motivate observed dividend smoothing.⁸

Operating cost and cash flow

Consider the firm's cash flow statement at date t . Denoting the corporate income tax rate by

⁵The key idea is that an extra dollar of debt is cheaper than an extra dollar of equity at low debt levels, for example because of tax benefits, but more expensive at high debt levels. The optimal debt level equates the marginal costs of debt and equity. See Frank and Goyal (2011) and Graham and Leary (2011) for surveys.

⁶Binsbergen et al. (2010) estimate the convexity of a reduced form marginal cost function using exogenous variation in tax benefits. We relate the magnitude of our estimated cost to their findings below.

⁷In models with costly bankruptcy, the risk adjusted return on debt is equal to the riskless rate plus compensation for bankruptcy costs. In our context, the representative firm stands in for the entire corporate sector. The cost κ can thus be interpreted as the expected bankruptcy cost on a diversified portfolio of corporate bonds that are each subject to idiosyncratic default risk.

⁸The relative merits of asymmetric information and agency cost explanations are the subject of an active debate based on cross-sectional evidence. See Brav et al. (2005) and Farre-Mensa et al. (2014) for surveys on payout policy.

τ_k , we can write net payout as

$$D_t = \alpha Y_t - I_t - f_t \xi^t - \kappa(B_{t-1}^f) - \phi(D_t, D_{t-1}) - (B_{t-1}^f - Q_t^b B_t^f) \quad (2.5)$$

$$- \tau_k \left[\alpha Y_t - B_{t-1}^f (1 - Q_{t-1}^b) - \delta Q_{t-1}^k K_{t-1} - I_t \right]$$

The first line records cash flow in the absence of taxation: payout equals revenue less investment, a fixed operating cost and financing costs as well as net debt repayment. The second line subtracts the corporate income tax bill: the tax rate τ_k is applied to profits, that is, income less interest, depreciation and investment.

The fixed operating cost is motivated by expenditure that is unrelated to the scale of payout and varies independently from shocks shifting the production technology. Examples include costs from restructuring of the corporate sector (due to mergers and spin-offs, for example), reorganization of compensation, compliance with regulation, or lawsuits. We model f_t as a resource cost measured in units of goods. In practice one would expect it to reflect a drop in the labor productivity of management or other employees not directly related to increasing the scale of production (such as the legal or human resources department).

Household wealth

We denote the price of aggregate corporate sector equity by P_t . In addition to owning the firm, the household receives an endowment of goods $\pi \xi^t$ and government transfers $t_r \xi^t$. We assume a proportional capital income tax. Moreover, capital gains on equity are taxed immediately at the same rate. The household budget constraint is then

$$C_t + P_t \theta_t + Q_t^b B_t^h = (1 - \alpha) Y_t + \pi \xi^t + t_r \xi^t + B_{t-1}^h + (P_t + D_t) \theta_{t-1} \quad (2.6)$$

$$- \tau_l \left[(1 - \alpha) Y_t + (1 - Q_{t-1}^b) B_{t-1}^h + D_t \theta_{t-1} + (P_t - P_{t-1}) \theta_{t-1} + \pi \xi^t \right] - \tau_c C_t.$$

The first line is the budget in the absence of taxation: consumption plus holdings of equity and bonds – equals labor and endowment income plus the (cum dividend) value of assets. The second line subtracts the tax bill. The income tax rate applies to labor income, interest, dividends as well as capital gains. The consumption tax rate is denoted by τ_c .

We do not explicitly model the government, since we do not include observables that identify its behavior in our estimation. To close the model, one may think of a government that collects taxes based on the rates τ_l, τ_c, τ_k and uses lump sum transfers to balance its budget. The asset market clearing conditions then states that bonds issued by the firm equal to bonds held by the household, i.e. that $B_t^f = B_t^h$, and that equity holdings $\theta_t = 1$.⁹

⁹The goods market clearing condition states $C_t + I_t + f_t \xi^t + \kappa(B_{t-1}^f) + \phi(D_t, D_{t-1}) = Y_t + \pi \xi^t$.

2.2 Uncertainty and preferences

We describe the exogenous shocks by a Markov chain $s_t = (Z_t, f_t, \omega_t)$. There are two continuous shocks – to the marginal product of capital Z_t and to the operating cost f_t . In order to capture nonlinearities, s_t also contains the time homogeneous finite-state Markov chain ω_t . We represent the continuous shocks as

$$\log Z_{t+1} = \mu_z(s_t) + \sigma_z(s_t) \varepsilon_{t+1}^z, \quad (2.7a)$$

$$f_{t+1} = \mu_f(s_t) + \sigma_f(s_t) \varepsilon_{t+1}^f, \quad (2.7b)$$

where the random variables ε_{t+1}^z and ε_{t+1}^f are iid and mutually independent with mean zero and variance one. We further assume that s_t has a unique invariant distribution. For our estimated model below, this follows from specific functional forms for the conditional means and volatilities. We denote the unconditional means by $\bar{s} = (\bar{Z}, \bar{f}, \bar{\omega})$.

Ambiguity and changes in perceived ambiguity

Ambiguity averse agents observe the state s_t . However, they are not confident about the distribution of the continuous shocks in (2.7). Instead, they entertain an entire family of conditional distributions with different conditional means:

$$\log Z_{t+1} = \mu_z(s_t) + \sigma_z(s_t) \varepsilon_{t+1}^z + u_t^z, \quad u_t^z \in [-a_t^z, a_t^z], \quad (2.8a)$$

$$f_{t+1} = \mu_f(s_t) + \sigma_f(s_t) \varepsilon_{t+1}^f + u_t^f, \quad u_t^f \in [-a_t^f, a_t^f]. \quad (2.8b)$$

Here the random variable a_t^z bounds an interval of mean shifters u_t^z for the shock $\log Z$ – it captures the degree of ambiguity the agent perceives about Z next period conditional on state s_t at date t . Similarly, the random variable a_t^f captures ambiguity about the shock f .

If ambiguity is low, then agents are relatively confident in forecasting shocks. As a result, their set of conditional distributions is small and each distribution is close to (2.7). In contrast, when ambiguity is high, then agents do not feel confident about forecasting and entertain a large set of distributions. Much like volatility, ambiguity is shock-specific: for example, in a state with high ambiguity about the marginal product of capital (MPK), ambiguity about operating cost can be low and vice versa.

Ambiguity varies over time with new information. We allow for two sources of time variation. On the one hand, ambiguity can move together with volatility. It is plausible that in more turbulent times – that is, when larger shocks are expected – agents also find it harder to settle on a forecast of the future. On the other hand, ambiguity can move with intangible information that is not reflected in current fundamentals or volatility. Our estimated model below uses specific functional forms to parsimoniously capture both possibilities. For now, we describe the joint evolution of ambiguity

and other variables by general functions $a_{t+1}^z = \tilde{a}^z(s_{t+1}, s_t, a_t^z)$ and $a_{t+1}^f = \tilde{a}^f(s_{t+1}, s_t, a_t^f)$.¹⁰

Preferences

How do ambiguity averse agents who entertain multiple distributions make decisions? We assume recursive multiple priors utility. A consumption plan is a family of functions $c_t(s_t, a_t)$, where the state (s_t, a_t) contains the exogenous shocks as well as the agent's perceived ambiguity $a_t = (a_t^z, a_t^f)$. Conditional utilities derived from a given consumption plan c are defined by the recursion

$$U(c; s_t, a_t) = \log c_t(s_t, a_t) + \beta \min_{(u^z, u^f) \in \times_i [-a_t^i, a_t^i]} E^{(u^z, u^f)} [U(c; s_{t+1}, a_{t+1})], \quad (2.9)$$

where the conditional distribution of s_{t+1} on the right hand side is (2.8), with the mean shifters (u_t^z, u_t^f) chosen to *minimize* expected continuation utility. Finally, the agent maximizes the recursive multiple priors utility in (2.9) by choosing over available consumption plans.

The "maxmin" utility function (2.9) captures the idea that lack of confidence in probability assessments leads to cautious behavior. If $a_t^z = a_t^f = 0$, the solution to the recursion is the standard separable log utility with conditional beliefs (2.7). More generally, positive a_t^z or a_t^f imply that households behave as if the worst case mean is relevant. Importantly, the worst case mean depends on the consumption plan c that is currently being evaluated.¹¹

With the particular shocks considered in our model, it is easy to solve the minimization step in (2.9) at the equilibrium consumption plan: the worst case expected cash flow is low and the worst case expected operating cost is high. Indeed, consumption depends positively on cash flow and negatively on the operating cost. It follows that agents act throughout as if forecasting under the worst case mean $u_t^f = a_t^f$ and $u_t^z = -a_t^z$. This property pins down the representative household's worst case belief after every history and thereby a worst case belief over entire sequences of data. In what follows, we denote worst case expectations by stars; for example E^*D_{t+k} is the worst case expected dividend k periods in the future.

To illustrate the asset pricing properties of the model, it is helpful to develop notation for the "change of measure" from the true data generating process (2.7) to the agent's equilibrium worst case belief. Since the innovations ε_t^z and ε_t^f are continuous, and the only difference between conditional distributions is the mean shifter, there exists a positive stochastic process X_t adapted to the agent's information set with $E_t[X_{t+1}] = 1$ such that $E_t^*[Y_{t+1}] = E_t[X_{t+1}Y_{t+1}]$ for any

¹⁰Writing ambiguity a_{t+1} as a function of both the current state s_{t+1} and the past state s_t as well as its own lag allows the change in ambiguity to depend on unexpected changes in s_t , such as the innovation ε_{t+1}^z . In our specification below, ambiguity increases when MPK is unexpectedly low.

¹¹In the expected utility case, time t conditional utility can alternatively be represented as $E_t[\sum_{\tau=0}^{\infty} \log c_{t+\tau}]$ where the expectation is taken under a conditional probability measure over sequences that is updated by Bayes' rule from a measure that describes time zero beliefs. An analogous representation exists under ambiguity: time t utility can be written as $\min_{\pi \in \mathcal{P}} E_t^\pi[\sum_{\tau=0}^{\infty} \log c_{t+\tau}]$. Epstein and Schneider (2003) shows how a "rectangular" time zero set of beliefs \mathcal{P} can be derived from a process of one step ahead conditionals such as (2.8) so as to guarantee that conditional preferences are dynamically consistent.

random variable Y_{t+1} . The adjustment X_{t+1} can be thought of as a ratio of densities that shifts relatively more weight to states of the world that offer low consumption to the agent.

3 Uncertainty shocks, firm financing and asset prices

In this section, we describe the key trade-offs faced by investors and firms and explain how our model differs from existing production based asset pricing models. On the investor side, we model perception of uncertainty as ambiguity about means. This approach results in a more flexible theory of how uncertainty – and in particular the realized volatility of shocks – drives asset prices. Moreover its technical properties facilitate computation and hence estimation of our model. On the firm side, we endogenously derive both its real and financial decisions, rather than assume free adjustment of equity and exogenous leverage. This approach results in additional discipline on firm payout, a key ingredient in asset pricing models.

3.1 Asset pricing under ambiguity about means

The prices of assets held by the representative agent – in particular debt and equity – must satisfy the agent’s first order conditions. The information in those conditions can be summarized by the implied prices of one-period-ahead contingent claims. We follow convention and normalize contingent claims prices by conditional probabilities to obtain the stochastic discount factor

$$M_{t+1} = \beta \frac{C_t}{C_{t+1}} X_{t+1}, \quad (3.1)$$

where X_{t+1} reflects the change of measure to the worst case belief.¹² The date t price of any payoff Π_{t+1} is $Q_t^\Pi = E_t[M_{t+1}\Pi_{t+1}]$.

Precautionary savings and the safe interest rate

To compare the effects of ambiguity and risk, we approximate asset prices via a Taylor expansion of marginal utility C_{t+1}^{-1} around expected consumption $E_t C_{t+1}$.¹³ The price Q_t^b of a safe bond with constant payoff $\Pi_{t+1} = 1$ can be written as

$$Q_t^b = \beta E_t^* \left[\frac{C_t}{C_{t+1}} \right] \approx \beta \frac{C_t}{E_t[C_{t+1}]} \left(1 + \frac{\text{var}_t(C_{t+1})}{(E_t C_{t+1})^2} + \frac{E_t[C_{t+1}] - E_t^*[C_{t+1}]}{E_t[C_{t+1}]} \right). \quad (3.2)$$

Here the first factor is the bond price that would obtain under certainty. It reflects intertemporal substitution: if agents expect consumption to fall, then they want to buy assets and the bond

¹²In this section for ease of exposition we set the household tax rate on income, including bond income, to $\tau_l = 0$. The appropriate more general formulas allowing for that taxation are presented in Appendix 9.3.

¹³Formulas are derived in Appendix 9.1. We follow convention and drop terms of third or higher moments.

price is high. The terms in the bracket show the effect of risk and ambiguity on the value of the safe asset.

Both risk and ambiguity create incentives for precautionary savings. Indeed, if agents worry that consumption *might* fall then they want to buy the safe asset and the bond price is higher than under certainty. Risk and ambiguity differ only in how uncertainty about consumption is described. Under risk, uncertainty is measured by the conditional variance of consumption; under ambiguity, it is measured by the difference between the actual and worst case mean of consumption. In either case, a shock that increases uncertainty increases the bond price and lowers the safe interest rate.

Ambiguity, risk and volatility

A key difference between our model and standard models without ambiguity is the role of time varying volatility. Without ambiguity, there is a tight connection between agents' worry about future consumption on the one hand and the typical size of realized consumption surprises on the other. Indeed, without ambiguity, the last term in the bracket in (3.2) is zero; perceived uncertainty is thus perceived risk, which in turn is equal to conditional volatility. As a result, an increase in uncertainty can lower the safe interest rate only if it is also followed by unusually large consumption surprises.

Our model also allows agents' worry to be higher in times when larger consumption surprises are expected. Our specification of ambiguity (2.8b)-(2.8a) allows for this plausible effect because the size of the belief set (2.8) can be correlated with volatility. At the same time, ambiguity can also move in response to intangible information that is uncorrelated with volatility. The relative size of each source of ambiguity is inferred by our estimation.

The bond price formula (3.2) also illustrates a technical difference between our model and standard models with risk. While the risk term is a second order term in the Taylor expansion and contains a second moment of consumption, the ambiguity term is a first order term and involves only means. As a result, the ambiguity term (but not the risk term) appears also in a linearized representative agent Euler equation. Our estimation strategy below exploits this fact: we follow linearization steps to obtain the likelihood. With ambiguity in means, we can do so without sacrificing precautionary savings effects on the interest rate.

Time variation in uncertainty premia

The price of a generic payoff Π_{t+1} can be decomposed into expected present value less an uncertainty premium, which in turn consists of risk and ambiguity premia:

$$Q_t^\Pi \approx Q_t^b E_t \Pi_{t+1} \left(1 - \frac{\text{cov}_t(C_{t+1}, \Pi_{t+1})}{E_t C_{t+1} E_t \Pi_{t+1}} - \frac{E_t[\Pi_{t+1}] - E_t^*[\Pi_{t+1}]}{E_t[\Pi_{t+1}]} \right). \quad (3.3)$$

The first factor is the expected present value of future payoffs. The "volatility puzzle" is that, for many uncertain assets such as equity, the price is much more volatile than the expected

payoff. In other words, the uncertainty premium – the term in brackets – fluctuates over time. In a stationary environment, this is equivalent to saying that the excess return $Q_t^b \Pi_{t+1} / Q_t^\Pi$ is predictable by observables such as prices that comove with uncertainty premia.

Investors require positive uncertainty premia on assets that pay off little in "bad states" when consumption is low. Equation (3.3) shows that this property is shared by both risk and ambiguity premia. Indeed, in a DSGE model, consumption and payoff are both endogenous and "bad states" can be generated by many exogenous innovations. According to (3.3), a particular innovation contributes to a positive uncertainty premium if and only if it affects consumption and payoff in the same direction. It contributes to a positive risk premium because it contributes to a positive covariance between consumption and payoff. It contributes to a positive ambiguity premium because the worst case belief about the innovation puts more weight on realizations that lower consumption and hence also payoff. As a result, the worst case expected payoff is lower than the actual expected payoff.

As with the precautionary savings term, there are two key differences between the risk and ambiguity premium. First, a conceptual difference is that the size of the risk premium depends on the covariance of the payoff with consumption. Higher risk premia must therefore be followed by unusually close comovement of consumption and payoff.¹⁴ In contrast, the ambiguity premium may also move due to intangible information; large consumption volatility is not required to generate large uncertainty premia. The second difference is technical: the ambiguity premium involves means, not second moments, and hence it appears in a linearized Euler equation.

Uncertainty shocks, price volatility and the role of leverage

Comparison of the price formulas (3.2) and (3.3) illustrate the tension that typically exists in asset pricing models between the twin goals of matching a volatile uncertainty premium and a stable interest rate. Suppose there is an increase in uncertainty (either risk or ambiguity) about an innovation that affects both consumption and shareholder payout. Investors then require a higher uncertainty premium: the stock price falls relative to the expected present value of payout. At the same time, however, investors attempt to buy safe bonds and push up the bond price and – via discounting – also the expected present value of payout. The effect on the stock price is generally ambiguous – a successful model must ensure that the former effect is larger than the latter.

As an extreme example, suppose equity is a claim to consumption next period only. In this case, uncertainty shocks do not affect the stock price at all. Indeed, if $\Pi_{t+1} = C_{t+1}$, then the precautionary savings terms in (3.2) are *identical* to the uncertainty premium terms in (3.3), for both risk and ambiguity. In other words, whatever the nature of uncertainty, an increase in uncertainty lowers the interest rate (and increases the present value of payoff) by the same amount that it raises the uncertainty premium. While the uncertainty premium is volatile, and

¹⁴Analogous implications hold for risk preferences that are more complicated than the time separable log utility case considered here. For example, with Epstein-Zin utility, what matters for risk premia is covariance of asset payoffs with continuation utility, and hence covariance with state variable that continuation utility depends on.

hence excess returns are predictable, the interest rate is as volatile as the premium and the stock price is constant.

When do uncertainty shocks generate volatile stock prices and stable safe interest rates? A sufficient condition is that consumption contains a sizable component that is not correlated with the payoff to equity. To see this, write without loss of generality

$$C_{t+1} = \hat{\alpha}_t + \hat{\beta}_t \Pi_{t+1} + \gamma_{t+1}, \quad (3.4)$$

where $\hat{\alpha}_t$ and $\hat{\beta}_t$ are based on date t information and γ_{t+1} has mean zero and is orthogonal to Π_{t+1} given date t information. An increase in uncertainty could arrive either as an increase in $\text{var}_t(\Pi_{t+1})$ or as an increase in ambiguity $E_t[\Pi_{t+1}] - E_t^*[\Pi_{t+1}]$. In either case, the ratio of the change in the uncertainty premium in the stock price relative to the precautionary savings term in the bond price is $1 + \hat{\alpha}_t/\hat{\beta}_t E_t[\Pi_{t+1}]$. Large $\hat{\alpha}_t$ and small $\hat{\beta}_t$ thus deliver the desired result.

Uncertainty shocks thus deliver the "right kind" of price volatility if uncertainty about payout moves relatively more than uncertainty about consumption. This property underscores the importance of modeling the dynamics of payout. Existing models with risk typically assume exogenous leverage: either payout is defined as a power of consumption, or the amount of debt is fixed as proportional to capital. While this approach mechanically generates a payout series with the desired property, it is not disciplined by actual data on payout, except perhaps average leverage. In contrast, our model explicitly derives optimal leverage and payout which are matched directly to data. We thus impose additional restrictions on a key mechanism for price volatility.

3.2 Payout and capital structure choice

To write shareholder value, we define t -period ahead contingent claims prices $M_0^t = \Pi_{\tau=1}^t M_\tau$. The firm thus maximizes

$$E \sum_{t=1}^{\infty} M_0^t D_t$$

subject to the budget constraint (2.5). Since M_t contains the change of belief X_t , shareholder value is computed under shareholders' worst case expectation of payout. Ambiguity (as well as risk) perceived by shareholders affects the valuation of the firm as well as the actions (such as investment) shareholders choose for the firm.

Optimal debt issuance

Firms' optimal choice of debt choice follows classic "tradeoff theory": the firm equates the tax benefit of debt (relative to equity) to the extra financing cost of debt. Let λ_t denote the multiplier on the firm's date t budget constraint (2.5), normalized by the contingent claims price M_0^t . The

first order condition for debt is

$$Q_t^b \lambda_t + E_t [M_{t+1} \lambda_{t+1} \tau_k (1 - Q_t^b)] = E_t [M_{t+1} \lambda_{t+1} (1 + \kappa'(B_t^f))]. \quad (3.5)$$

The left hand side is the marginal benefit of issuing an additional dollar of debt. Per unit of debt the firm raises Q_t^b dollars of funds today, which is worth $Q_t^b \lambda_t$ dollars inside the firm. Moreover, the firm expects a tax benefit proportional to the interest payment $1 - Q_t^b$ next period. The right hand side is the marginal cost of issuing debt which includes not only the dollar to be repaid but also the marginal financing cost.

Without equity adjustment costs, the shadow value of funds is always $\lambda_t = 1$. Since in equilibrium $Q_t^b = E_t M_{t+1}$, expectations in (3.5) cancel and we get an essentially static tradeoff between the tax benefit and the cost of debt. At low debt levels, a positive tax benefit implies that marginal benefit is always higher than marginal cost and some debt is always issued. As debt rises, marginal cost rises as well until the benefit and cost are equated at the optimal debt level.

In the presence of equity adjustment costs, the basic tradeoff remains the same. However, future benefits and costs are discounted not with shareholders' stochastic discount factor M_{t+1} , but with the firm's internal stochastic discount factor $M_{t+1} \lambda_{t+1} / \lambda_t$. Since all of the cost but only part of the benefit of debt accrue in the future, optimal debt is lower if funds inside the firm are relatively scarce in the future, that is, if λ_{t+1} is expected to be relatively high. In fact, what matters is shareholders' uncertainty adjusted expectation: it is optimal to lower debt if shareholders worry about future scarcity because the conditional volatility or ambiguity about λ_{t+1} is relatively high.

Optimal investment

Firms' choice of capital reflects familiar principles of investment with adjustment costs. First, the marginal value of installed capital, Q_t^k say, reflects the discounted present value of the marginal benefit of capital. The first order conditions for investment and capital imply

$$\lambda_t Q_t^k = E_t [M_{t+1} \lambda_{t+1} ((1 - \tau_k) \alpha (k_t)^{\alpha-1} L^{1-\alpha} + (1 - \delta) Q_{t+1}^k + \delta \tau_k)]. \quad (3.6)$$

Without adjustments costs to equity ($\lambda_t = 1$), we have a standard Q-theory equation. The difference here is that financial frictions again matter for how costs and benefits of capital are calculated. Since capital is held inside the firm – as opposed to directly by households – future marginal benefits are discounted by the internal stochastic discount factor of the firm. For example, a temporary scarcity of funds inside the firm (that is, high λ_t) lowers the value of capital.

The second principle is that the difference between the marginal values of installed and uninstalled capital reflects the marginal adjustment cost. The first order conditions further imply

$$\lambda_t (Q_t^k - (1 - \tau_k)) = \lambda_t Q_t^k \psi_1(i_t, i_{t-1}) + E_t [M_{t+1} \lambda_{t+1} Q_{t+1}^k \psi_2(i_{t+1}, i_t)], \quad (3.7)$$

where for our functional form the adjustment cost function is defined as $\psi(i_t, i_{t-1}) = 0.5\Theta i_t(i_t/i_{t-1} - \xi)^2$. In steady state, marginal adjustment cost on the right hand side is zero: installed and uninstalled capital are worth the same, that is, Q_t^k equals the after tax value of a dollar. More generally, temporarily higher Q_t^k encourages higher investment. Indeed, ψ_1 is positive and increasing in i_t if $i_t > i_{t-1}$, whereas ψ_2 is positive and increasing in i_t if $i_t > i_{t+1}$. Optimal investment thus moves with the value of installed capital Q_t^k , but any response is limited by the firm's desire to stabilize the growth rate of investment.

Optimal payout

To understand the optimal choice of payout, consider first what happens without adjustment costs to equity. If $\lambda_t = 1$, then debt, investment and capital are determined by (3.5)-(3.7) and (2.2). In particular, investment decisions proceed as if households were direct holders of capital. Optimal payout follows as a residual from (2.5). Shareholders freely transfer funds into and out of the firm in response to investment and tax incentives as well as changes in the operating cost. The result is a volatile payout series: net changes in equity always quickly satisfy any funding need. The role of adjustment costs is to add a motive for payout smoothing that allows the model to match the smoothness of payout in the data.

When deciding payout subject to adjustment costs, shareholders face a similar tradeoff as for investment. They compare the difference $1 - \lambda_t$ between the value of a dollar outside versus inside the firm to the cost of paying out the extra dollar. Formally, the firm's first order condition for payout is

$$1 - \lambda_t = \lambda_t \phi_1(D_t, D_{t-1}) + E_t[M_{t+1} \lambda_{t+1} \phi_2(D_{t+1}, D_t)]. \quad (3.8)$$

In steady state, the marginal adjustment cost on the right hand side is zero so funds inside and outside the firm are worth the same: we have $\lambda_t = 1$. More generally, temporarily lower λ_t (that is, an abundance of funds inside the firm) encourages higher payout.¹⁵ Optimal payout thus moves against the value of internal funds λ_t , but any response is limited by the firm's desire to stabilize the growth rate of payout.

Firms' response to shocks

Consider now the firm's response to shocks. We start with shocks that change current cash flow, that is actual changes in MPK or operating cost. A negative shock to cash flow temporarily lowers dividends and makes current funds more scarce relative to funds in the future. From (3.5) the firm should then borrow temporarily so as to cover the shortfall in funds. Cash flow shocks thus tend to move payout and debt in opposite directions. If cash flow falls because of a low MPK, persistence of the shock further lowers the return on investment and hence investment itself. In contrast, changes in operating cost are unrelated to the return on investment and affect investment

¹⁵Analogously to the case of investment, ϕ_1 is positive and increasing in D_t if $D_t > D_{t-1}$, whereas ϕ_2 is positive and increasing in D_t if $D_t > D_{t+1}$

only weakly through the cost of funds.

Consider now the firm’s response to an increase in uncertainty. Under the worst case belief, future dividends are now low so that funds are scarce, that is, the relative shadow value of funds $E_t [M_{t+1}\lambda_{t+1}] / \lambda_t$ increases. From (3.5), holding fixed the riskless rate, the marginal cost of debt increases and the firm responds by cutting current debt B_t^f . At the same time, (3.8) suggests that the firm will decrease payout already at date t in order to smooth the drop in the growth rate of payout. As a result, uncertainty shocks make payout and debt move together.¹⁶

The nature of the uncertainty shock matters for how the valuation of the firm responds. As discussed above, uncertainty shocks generate excess volatility in stock prices if uncertainty about payout moves more than uncertainty about consumption. Since an increase in uncertainty about MPK implies that future output and consumption become more uncertain as well, it is not suitable as a driver of the stock market. In contrast, operating cost is a small share of output, but a sizable share of payout. Shocks to uncertainty about operating cost thus generate the ”right” kind of volatility. At the same time, firms’ endogenous responses help identify such shocks – if they matter, large drops in price should coincide with a joint decline in payout and debt. This comovement is key to inference in our estimation exercise below.

3.3 Linearization and asset prices

In this section, we explain how we compute equilibria. Rather than work through all the equations, we first sketch the general approach and then focus on the household first order conditions for assets; the full model solution is described in the appendix. The approximate asset pricing equations are of interest in their own right since they illustrate how ambiguity delivers simple linear asset pricing formulas that allow for an equity premium in steady state as well as excess volatility of stock prices.

Linearization around the worst case steady state

Let w_t denote the vector of endogenous state variables. As usual in DSGE models, we can characterize the law of motion for w_t as the solution to a stochastic difference equation. It consists of (i) first order conditions for the representative agent under the worst case belief as well as for the firm, (ii) budget constraints and market clearing conditions and (iii) the evolution of the exogenous state s_t under the worst case belief. The worst case belief is used in (i) and (iii) because we are computing the law of motion of the endogenous variables, which depends on agents’ actions and hence their perception of the future.

The Markov property of the exogenous state process and regularity conditions on the economy imply that the stochastic difference equation given by (i) – (iii) has a stationary solution with a recursive representation $w_t = W(w_{t-1}, s_t)$. Since it is based on the worst case belief, the law of

¹⁶Uncertainty shocks are transmitted through state prices. The effect we highlight will thus occur more generally whenever there is a change in state prices that does not affect current cash flow directly.

motion W describes the effect of ambiguity on agents' decision rules. Let $s^* = (Z^*, f^*, \bar{w})$ denote the worst case mean of the exogenous state; since the finite state chain ω_t is not ambiguous, its worst case mean is equal to the true mean \bar{w} . The worst case steady state w^* is defined by $w^* = W(w^*, s^*)$.

From the perspective of an outside observer, the joint distribution of the exogenous and endogenous variables (s_t, w_t) is described by the function W together with the true evolution of the exogenous state s_t that has mean $\bar{s} = (\bar{Z}, \bar{f}, \bar{w})$. The steady state \bar{w} satisfies $\bar{w} = W(\bar{w}, \bar{s})$. It is typically different from the worst case steady state. Indeed, at the state \bar{s} , agents observe the true mean shocks \bar{Z} and \bar{f} , but behave as if those shocks revert to their worst case means Z^* and f^* , respectively, in the future. Mechanically, agents thus behave *as if* they are on an impulse response following an unanticipated shock that took the economy from s^* to \bar{s} . Intuitively, agents' actions reflect precautionary behavior in the face of ambiguity about future shocks.

Computation of the model proceeds in two steps. First, we linearize the stochastic difference equation around the worst case steady state w^* . We thus find coefficient matrices ε_{ww} and ε_{ws} such that the log deviation of w_t from the worst case steady state is represented as

$$\log w_t - \log w^* = \varepsilon_{ww} (\log w_{t-1} - \log w^*) + \varepsilon_{ws} (\log s_t - \log s^*). \quad (3.9)$$

Second, given an initial condition for w_0 , the distribution w_t follows (3.9) together with the true distribution of s_t . In particular, the true steady state is the vector \bar{w} that satisfies (3.9) for $s_t = \bar{s}$. As the exogenous state fluctuates around \bar{s} , the approximate solution fluctuates around \bar{w} .¹⁷

We evaluate the accuracy of this approximation using the Euler equations errors approach of Judd (1992). The objective is to evaluate the typical optimization error incurred by agents from using the proposed linearized policy functions. There are two basic properties of the economy that matter for this evaluation: first, the endogenous state variables w_{t-1} are determined by equation (3.9) together with true distribution of s_t . Second, at each time t in that sequence of shocks, agents inside the model form beliefs according to the worst-case conditional probability distribution E_t^* . Appendix 9.5 presents general computational details and specific quantitative results for the estimated model discussed in Section 4.

Euler equations for assets

We now illustrate the role of ambiguity for asset prices in steady state as well as in the linear approximation. From the Euler equations (3.2) and (3.3), the bond price at the worst case steady state is equal to β and the price payout ratio is equal to $\beta/(1-\beta)$ – the same values as in a steady

¹⁷The steady state of the model is different from both the worst case steady state and the deterministic steady state. As in the case of risk, averages under the true law of motion are different from the deterministic steady state; some authors thus label the steady state as the "stochastic" or "ergodic" steady state.

In contrast to the risk case, the steady state here is easy to find since it solves a linear equation such as (3.9) (see also equation (9.6) in Appendix 9.2.2). With risk, the steady state instead solves a non-linear equation, which typically requires dealing with a pruned state space, since otherwise unconditional moments may not exist (see Andreasen et al. (2013)).

state with perfect foresight. We indicate log deviations from the worst case steady state by hats, for example we write $\hat{c}_t = \log C_t - \log C^*$. The loglinearized pricing kernel and the household Euler equation for bonds and equity are then

$$\begin{aligned}\hat{q}_t &= E_t^*[\hat{m}_{t+1}], \\ \hat{p}_t &= E_t^* \left[\hat{m}_{t+1} + \beta \hat{p}_{t+1} + (1 - \beta) \hat{d}_{t+1} \right],\end{aligned}\tag{3.10}$$

where $\hat{m}_{t+1} = \hat{c}_t - \hat{c}_{t+1}$ and the short term interest rate is $\hat{r}_t = -\hat{q}_t = -E_t^* \hat{m}_{t+1}$.

Stock price and interest rate volatilities

We can use the loglinearized Euler equations to understand the relative volatility of stock prices and interest rates with ambiguity shocks. Substituting into the Euler equation for stocks, the price payout ratio can be written as

$$\hat{p}_t - \hat{d}_t = -\hat{r}_t + E_t^* \left[\beta(\hat{p}_{t+1} - \hat{d}_{t+1}) + \hat{d}_{t+1} - \hat{d}_t \right].\tag{3.11}$$

The price payout ratio is the worst case expected payoff relative to dividends, discounted at the riskless interest rate. As in (3.3), an increase in uncertainty can move both the payoff term (if cash flow becomes more uncertain), and the interest rate (if consumption becomes more uncertain). To see how these opposing forces shape price volatility, we solve forward to express the price payout ratio as the present value of future growth rates in the payout-consumption ratio

$$\hat{p}_t - \hat{d}_t = E_t^* \sum_{\tau=1}^{\infty} \beta^{\tau-1} \left((\hat{d}_{t+\tau} - \hat{c}_{t+\tau}) - (\hat{d}_t - \hat{c}_t) \right).\tag{3.12}$$

If dividends are proportional to consumption, then the price dividend ratio is constant – with log utility, income and substitution effects cancel. In contrast, if dividends are a small share of consumption, then uncertainty about dividends will tend to dominate and an increase in uncertainty can decrease the price dividend ratio.

The steady state and unconditional premia

Average asset premia predicted by the model depend on the average amount of ambiguity reflected in decisions. In contrast to models with risk, compensation for uncertainty thus appears already in formulas for steady state prices. To show this, we use the fact that in the absence of shocks, the equilibrium laws of motion for consumption and payout satisfy the Euler equations. Let (\bar{C}, \bar{D}) denote steady state consumption and payout. Both are generally above their worst case counterparts (C^*, D^*) . We denote the log difference in consumption by $\bar{c} = \log \bar{C} - \log C^*$ and similarly use the notation \bar{d} and \bar{q} .

In steady state, agents behave as if they are on an impulse response from (\bar{C}, \bar{D}) to the worst case steady state (C^*, D^*) . We know that along this (linearized) impulse response, the Euler

equations (3.10) hold deterministically. The steady state log bond price therefore satisfies

$$\bar{Q} = \beta \exp(\bar{q}) = \beta \exp(\bar{c} - \hat{c}_1),$$

where \hat{c}_1 is the first value along the impulse response. If there is ambiguity about consumption, we would expect the impulse response to decline towards the worst case. In this case, the bond price is higher than β , the worst case (and the rational expectations) steady state bond price. In other words, ambiguity about consumption lowers the interest rate – a precautionary savings effect.

Consider now the steady state price dividend ratio. The log deviation of \bar{P}/\bar{D} from the worst case value $\beta/(1-\beta)$ is given by (3.12), where the sum is over the consumption and dividend path along the linearized impulse response. For example, if the dividend-consumption ratio declines along the impulse response – say because there is a lot of average ambiguity about dividends and dividends are a small share of consumption – then $\bar{p} - \bar{d}$ is negative, that is, the steady state price dividend ratio \bar{P}/\bar{D} is below $\beta/(1-\beta)$. The presence of ambiguity thus induces a price discount.

Combining the bond and stock price calculations, the steady state equity premium is¹⁸

$$\log(\bar{P} + \bar{D}) - \log \bar{P} + \log \bar{Q} = (1 - \beta)(\bar{d} - \bar{p}) - (\bar{c} - \hat{c}_1).$$

There are two reasons why ambiguity can generate a positive steady state equity premium. First, the average stock return can be higher than under rational expectations because the price dividend ratio is lower. This is the first term. Second, the interest rate can be lower. The second effect is small if dividends are a small share of consumption and ambiguity is largely about dividends. We emphasize the role of the first effect: it says that average equity returns themselves are higher than in the rational expectations steady state. Ambiguity thus does not simply work through low real interest rates.

4 Estimation

In this section we describe the data on real and financial variables, the specific functional forms for the exogenous shocks and ambiguity as well as our estimation approach.

4.1 Data

Our estimation uses data on investment growth, leverage, the ratios of shareholder payout to GDP and equity value to GDP. The time period is 1959Q1 to 2011Q3. All firm variables are for the US

¹⁸The log stock return at the steady state is

$$\log(\bar{P} + \bar{D}) - \log \bar{P} \approx (1 - \beta)(\bar{d} - \bar{p}) - \log \beta$$

where we are using the fact that all asset returns are equal to $-\log \beta$ at the worst case steady state.

nonfinancial corporate sector. We thus include all nonfinancial firms that are corporations for tax purposes. In terms of value, however, most fluctuations in financial variables are driven by the largest firms, which are also publicly traded (see for example, Covas and Den Haan (2011)).

Investment and GDP numbers come from the National Income and Product Accounts (NIPA), published by the Bureau of Economic Analysis. The nonfinancial corporate sector accounts for about one half of total US GDP. The NIPA accounts are integrated with the Flow of Funds Accounts (FFA), published by the Federal Reserve Board, from which we take financial variables.

We define net debt issuance as debt issuance less increases in bond holdings. Here we add up over all fixed income instruments listed in the flow of funds accounts. The idea is that all types of bonds are close substitutes, at least compared to equity. To quantify a model that delivers a choice between debt and equity, it thus makes sense to lump all types of bonds together. The market leverage ratio of the nonfinancial corporate sector is defined as outstanding net debt divided by the market value of equity. We define shareholder payout as dividends plus share repurchases less issuance of equity. Since asset flows in raw FFA data are highly seasonal, we compute four quarter trailing moving averages.

4.2 Dynamics of shocks, volatility, and ambiguity

We need functional forms for the exogenous shocks to MPK and operating cost introduced in (2.7), as well as for uncertainty about those shocks as in (2.8). We are guided by two considerations. First, the dynamics of our observables are not easily captured by standard linear Gaussian dynamics – for example, booms are longer than recessions and return distributions are heteroskedastic and skewed. This fact motivates the introduction of a finite state Markov chain to model shock distributions.

The finite state Markov chain $\omega_t = (\eta_t^f, \eta_t^z, v_t)$ consists of three independent "regime processes". Here v_t captures joint movements in volatility and ambiguity, whereas η_t^f and η_t^z capture movements in intangible ambiguity (that is orthogonal to volatility) about operating cost and MPK, respectively. Each regime process can take one of two values. Conditional on the realized sequence of regimes, the shocks are persistent AR(1) processes with Gaussian innovations. We introduce heteroskedasticity by allowing conditional volatilities to depend on the regime process v_t .

The second consideration is that MPK in our model captures not only TFP, but also the response of (unmodeled) variable inputs such as labor. Previous work has shown that uncertainty can be an important driver of variable inputs.¹⁹ Those findings motivate correlation between uncertainty about Z_t and Z_t itself. We assume in particular that a one-unit-innovation to ambiguity about Z_t is associated with a negative \varkappa -unit-innovation to Z_t itself. Movements in

¹⁹Ilut and Schneider (2014) study the response of hours worked to an increase in ambiguity about TFP. They show that more ambiguity makes firms and households act more cautiously, so hours worked and economic activity contract even if current labor productivity does not change. Time variation in ambiguity induces a time varying "labor wedge" that accounts for the bulk of variation in hours over the business cycle.

Z_t (and ambiguity about Z_t) are then subject to both "small and frequent" Gaussian innovations and "large and rare" movements due to regime switches in η_t^z and v_t .

Our parsimonious setup is not the only way to accommodate regime changes as well as correlation between MPK and ambiguity. We have conducted preliminary specification analysis for several more complicated setups – in particular more regimes for volatility, three rather than two states for each regime process, additional Gaussian forcing processes for ambiguity and a more general correlation structure between Z_t and uncertainty about Z_t . We have concluded that the potential for better fit from those alternatives does not outweigh the additional technical burden.

We now proceed to a formal description of the shock distributions.

Operating cost

It is helpful to think of an innovation to operating cost as drawn in two steps. Nature first draws a new volatility regime v_{t+1} for date $t + 1$, conditional on the current volatility regime v_t . It then draws from a normal distribution with mean zero and volatility $\tilde{\sigma}_f(v_{t+1})$. The distribution of the innovation ε_{t+1}^f is thus a mixture of two normals with different variances. In terms of the general notation in (2.7), the conditional mean and variance of f_{t+1} are now given by

$$\mu_f(s_t) = \bar{f} + \rho_f(f_t - \bar{f}), \quad (4.1a)$$

$$\sigma_f^2(s_t) = E[\tilde{\sigma}_f^2(v_{t+1}) | v_t], \quad (4.1b)$$

where the conditional expectation in the second line is computed from the transition matrix of the Markov chain v_t .

Ambiguity about operating cost depends on intangible ambiguity η_t^f as well as on the expected volatility of innovations to operating cost. The idea here is that agents are less confident about forecasting the future if they anticipate more turbulence. The function \tilde{a}^f that describes ambiguity depends only on the current state and takes the multiplicative form

$$\tilde{a}^f(s_t, s_{t-1}, a_{t-1}) = \eta_t^f E[\tilde{\sigma}_f(v_{t+1}) | v_t]. \quad (4.2)$$

The regime η_t^f thus measures the role of intangible information per unit of expected volatility. Since each chain takes on two values, there are four possible values for ambiguity about operating cost.

Marginal product of capital

Consider now an innovation to ambiguity about Z_t , or equivalently a negative \varkappa -unit-innovation to Z_t itself. It is again helpful to think of nature proceeding in two steps. It first draws new regimes $\omega_{t+1} = (\eta_{t+1}^f, \eta_{t+1}^z, v_{t+1})$ that affect ambiguity analogously to (4.2). It then draws a Gaussian random variable with mean zero and volatility $\tilde{\sigma}_z(v_{t+1})$ that moves Z_t one for one. In contrast to the case of the operating cost, however, our correlation assumption means that regime switches also affect Z_t itself (\varkappa -for-one with ambiguity), and that the Gaussian random variable has a

\varkappa^{-1} -for-one effect on ambiguity.

Some additional notation is helpful in order to describe the distribution of the innovation. We denote the contribution of regimes to ambiguity about Z_t by \tilde{a}_ω^z and the contribution of regimes to Z_t itself by \tilde{z}_ω :

$$\begin{aligned}\tilde{a}_\omega^z(\omega_t) &:= \eta_t^z E[\tilde{\sigma}_z(v_{t+1}) | v_t] \\ \tilde{z}_\omega(\omega_t) &:= -\varkappa \tilde{a}_\omega^z(\omega_t).\end{aligned}\tag{4.3}$$

Similarly to the specification above, the Markov chain η_t^z measures the role of intangible information per unit of expected volatility.

We can now write the conditional mean and variance of MPK in terms of the general notation in equation (2.7):

$$\mu_Z(s_t) = \log \bar{Z} + \rho_z (\log Z_t - \log \bar{Z}) + E[\tilde{z}_\omega(\omega_{t+1}) | \omega_t]\tag{4.4a}$$

$$\sigma_z^2(s_t) = E[\tilde{\sigma}_z^2(v_{t+1}) + (\tilde{z}_\omega(\omega_{t+1}) - E[\tilde{z}_\omega(\omega_{t+1}) | \omega_t])^2 | \omega_t].\tag{4.4b}$$

The conditional mean consists not only of a standard AR(1) mean reversion term, but also adds the conditional mean of the regime component $\tilde{z}(\omega_t)$. Similarly, the conditional variance takes into account that movements in Z_t can occur either because of a Gaussian innovation with volatility $\tilde{\sigma}_z(v_{t+1})$ drawn at date $t + 1$ or because of a regime switch between at dates t and $t + 1$.

It remains to specify conditional ambiguity about Z_t which can also change because of either a Gaussian surprise or a regime switch. We allow the impulse response of a Gaussian surprise on ambiguity to differ from that on Z_t itself. Intuitively, uncertainty caused by intangible information might subside more quickly than the response to MPK since the former captures information dynamics whereas the latter depends on technology. We define

$$\tilde{a}^z(s_{t+1}, s_t, a_t) = \rho_a (a_t - \tilde{a}_\omega^z(\omega_t)) - \varkappa^{-1} (\log Z_{t+1} - (\log \bar{Z} + \rho_z (\log Z_t - \log \bar{Z}))),\tag{4.5}$$

where ρ_a is the AR(1) coefficient for ambiguity. The second term represents the innovation to MPK, multiplied by \varkappa^{-1} to respect the negative correlation with ambiguity.

The parameter \varkappa measures the strength of the negative response of variable inputs to ambiguity, which is estimated from the data. If it is close to zero, then there is essentially no response of variable inputs to ambiguity. Of course, firms may still respond to ambiguity for example by their choice of investment. At the other extreme, if \varkappa is very large then movements in Z_t are driven by either Gaussian innovations or regime changes, but are accompanied by negligible movements in ambiguity. Our specification thus effectively nests a nonlinear process for Z_t itself that is not perceived to be ambiguous.

4.3 Likelihood-based estimation

Our model has two continuously distributed shocks but four observables. To avoid stochastic singularity, we introduce measurement error on all observables. Appendix 9.2.2 shows that under the assumptions of the previous section, the model solution can be represented as a Markov-switching (MS) VAR with conditional moments that depend on the regime. A key technical feature of our setup is that agents' endogenous response to changes in uncertainty is reflected in the time-varying constants of the MS-VAR. Indeed, as illustrated in section 3.3, since ambiguity affects the worst case conditional mean, a linear approximation of the model solution works well even though uncertainty changes over time. We combine this model solution with a set of observation equations to obtain a state space system that can be estimated with likelihood based techniques developed in Bianchi (2013).

We divide parameters into two subsets. The parameters reported in Table 3 are calibrated up front to match key ratios from the NIPA accounts; further details are in Appendix 9.4. The parameters reported in Tables 4 and 5 are estimated. We choose priors such that unconditional means of key variables are centered around values that are *a-priori* economically sensible; further details are in Appendix 9.6.

Table 4 contains several parameters for which the estimated modes are similar to values typically found in the literature: the trend growth rate, the discount factor, the depreciation rate and the capital share. We now discuss the reported magnitudes of four sets of parameters that are either nonstandard or for which our estimates are very different from other studies. Those parameters are displayed in Tables 4 and 5. They govern adjustments costs on investment and dividends, borrowing costs, operating cost and ambiguity. Identification of the parameters is discussed in more detail in Sections 5 and 6 below.

Adjustment costs

The parameter Θ scales investment adjustment costs. We estimate a posterior mode of .93, significantly lower than typical values in the literature that range between 3 and 10.²⁰ The parameter Φ scales dividend adjustment costs. Since the adjustment cost is quadratic, it determines both the level of adjustment costs and how the cost changes with payout. The estimated level of adjustment costs is negligible. It is zero at steady state and amounts to only 1.3 basis points of payout even when dividend growth is as high as three model-implied unconditional standard deviations.

To put the sensitivity of cost to payout in perspective, consider a one time change in payout away from the balanced growth path. Our estimate implies that the adjustment cost increases linearly by one cent for every additional dollar of payout. This is again lower than effects found in the literature. For the cost function in Jermann and Quadrini (2012, JQ), the same experiment

²⁰See for example Christiano et al. (2005), Del Negro and Schorfheide (2008), Justiniano and Primiceri (2008), Bianchi (2013) and Ilut and Schneider (2014).

implies that the adjustment cost increases linearly by 15 cents for every additional dollar of payout.

In sum, we find lower adjustment costs for both payout and investment. This is even though we follow other studies in inferring adjustment costs from the smoothness of firm actions, that is, investment in many business cycle studies and the ratio of equity payout to GDP in JQ. Likely reasons are that – in contrast to the above papers – we combine both physical adjustment costs and financial frictions, and that our model relies to a greater extent on uncertainty shocks that generate different responses than other shocks.

Borrowing costs

The estimated parameter Ψ_y is a normalized version of the parameter Ψ that controls the marginal cost of debt in (2.3). Using the posterior mode estimate, the level of the borrowing cost along the balanced growth path $0.5\Psi (b^f/\xi)^2$ is only 2 bp of GDP or 2.5% of payout. For the model dynamics, the key statistic is how much the marginal cost of debt grows with an additional dollar of debt, that is $\Psi b^f/\xi^2 = 0.0011$.

Our estimated sensitivity of borrowing costs is reasonable in light of micro evidence. Indeed, Binsbergen et al. (2010) use changes in tax law to identify marginal cost curves for debt. They assume that the marginal cost of interest expense grows linearly with debt, in line with our functional form. They find that the average firm paying one dollar more in annualized interest expense (relative to the book value of debt) incurs an additional cost of 14 cents.²¹ In our model, annualized interest expense relative to book value is $4(1 - Q^b) = 0.0093$, so that the additional cost per dollar of interest expense is 0.0011 divided by 0.0093, which equals 12 cents.

Operating cost

For operating cost, we estimate the AR(1) process with heteroskedastic innovations as in (4.1).²² The posterior mode implies an average operating cost of .8% of GDP with a standard deviation of .25% of GDP. Much like the other frictional costs, operating cost therefore is not an important item in the aggregate resource constraint. It does however matter for shareholders: on average it amounts to 9.2% of payout. It is also very persistent with an AR(1) coefficient of .992.

Operating cost displays considerable heteroskedasticity. Volatility of operating cost takes on two values, selected by the regime process v_t . The high value is estimated to be 2.5 times the low value. Given our specification (4.1)-(4.4), the estimation allows the process v_t to also select different volatilities for MPK innovations. However those estimated values are virtually identical, so we can interpret v_t as simply a shifter of volatility in operating cost.

How much ambiguity?

For each innovation, the magnitude of agents' perceived ambiguity is parametrized by the interval of possible worst case means in (2.8). Ilut and Schneider (2014) discuss what sets of

²¹In their Table III the estimated slope on interest expense is 4.81 and in their Table II, Sample B, the sample mean for the interest expense variable is 0.029. This results in an additional average cost of $4.81 \cdot 0.029 = 0.14$ dollars.

²²Appendix 9.4 discusses how to map the estimated parameters f_y, ς^f and η_r^f into values for \bar{f}, f^*, η_L^f and η_H^f .

models are consistent with a sample of iid innovations measured by an econometrician. They propose a bound on worst case means given by twice the standard deviation of the innovation measured by an econometrician. The basic idea is that agents should not entertain forecasts that perform badly along the average long sample. If an econometrician estimates a more volatile process, there is more room for agents' concern about ambiguity and hence the interval of means can be wider.

With this bound in mind, consider ambiguity about operating cost. On average, the worst case mean shifter a_t^f – the extra operating cost agents are concerned about – is equal to 2.3% of the actual average operating cost \bar{f} , or 2 basis points of GDP. The worst case shifter is thus small relative to the typical negative innovation observed by investors – it amounts to only 7% of the average conditional standard deviation.

Perceived ambiguity about operating cost differs significantly across regimes. The worst case extra operating cost ranges from a low of .9% of average cost when both volatility and intangible ambiguity η_t^f are low, to a high of 8.8% of average cost when both regime processes are high. Changes in intangible ambiguity play a key role here: the low value is $\eta_t^f = .024$ standard deviations whereas the high value is $\eta_t^f = .1$ standard deviations, a fourfold increase. It follows that even at its peak, ambiguity about operating cost is well inside the bound of two standard deviations.

Consider now ambiguity about MPK. On average, the worst case mean shifter a_t^z – the extra decrease in MPK agents worry about – equals 1.25% of the actual average value $\bar{Z} = 1$. The shifter is about as large as a typical observed innovation, at 87% of the average conditional standard deviation. Perceived ambiguity about MPK also differs across regimes, but its variation is much less pronounced than for operating cost and comes almost entirely from changes in intangible information. The low value is $\eta_t^z = .8$ standard deviations whereas the high value is $\eta_t^z = .945$ standard deviations. Again we find that estimated ambiguity is well inside the reasonable bound.

Finally, consider the parameter \varkappa in (4.3) that governs the relative size of the perfectly correlated movements in MPK and ambiguity about MPK. We estimate $\varkappa = 5.7$, that is, a one unit change in actual MPK goes along with a $1/5.7 = .18$ unit change in ambiguity. This finding of sizable comovement of output with ambiguity about MPK is consistent with the estimation in Ilut and Schneider (2014) who explicitly model the response of variable factors to ambiguity.

4.4 Model fit

Figure 1 and Table 1 summarize the data that the estimation must account for. In particular, the four raw series used in the estimation consist of investment growth, given by the log difference of investment, the payout-GDP ratio and the equity value-GDP ratio in logarithms, and the leverage ratio (debt divided by equity value) in levels.²³ Figure 1 shows our observables as solid lines as

²³Following standard practice in the DSGE literature, we work with raw data and do not demean or detrend any series prior to estimation. Stationarity of observables is achieved by taking the growth rate of investment and ratios for the financial variables. See Fernández-Villaverde et al. (2016) for a recent survey.

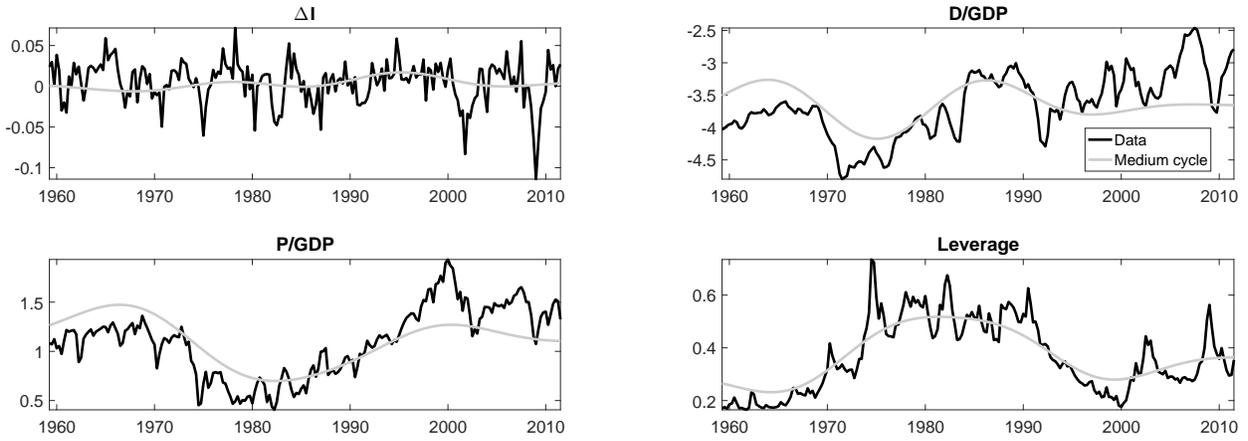


Figure 1: The figure reports the raw series used in the estimation (black line) and the corresponding medium cycle (gray line). The medium term cycle considers the frequency range between 8 and 50 years and is extracted from each series by using a bandpass filter.

well as their medium term movements as lighter gray lines. We define the medium term cycle by a frequency range between 8 and 50 years and extract the medium term cycle component from each series using a bandpass filter. Table 1 reports summary statistics for the series and their medium term cycle components. We also report the correlation between investment growth and the log dividend yield, as well as between the log payout-GDP ratio and the ratio of net debt issuance ΔB to GDP.

Stylized facts

Four stylized facts stand out about our sample. First, investment moves primarily with the business cycle. Second, stock prices are not cyclical: the dividend yield, and hence the price-dividend ratio, is essentially uncorrelated with investment growth. Third, financial variables have large medium term cycle components that are more important than their business cycle movements. Table 1 shows that medium term cycle components explain more than half of the variation in every financial series (in particular, 78% of the equity-GDP ratio), but less than one quarter of investment movements. Moreover, the positive correlation between net debt issuance and payout relative to GDP is much stronger at medium term frequencies than it is unconditionally.

Fourth, comovement of financial variables with real activity – that is, investment – is very different at medium term and business cycle frequencies. Over the medium term, payout and equity value decline relative to GDP as investment growth and leverage increase. This pattern is especially strong in the 1970s and early 1980s. In contrast, at business cycle frequencies, the payout-GDP and equity-GDP ratios are procyclical and move with investment, whereas leverage is countercyclical.

Figure 2 shows the role of observation error in the estimation. The figure reports the data (dashed line) and the filtered series (solid line) based on the posterior mode estimates. The difference between the two lines is accounted for by the observation error. The shaded areas

indicate 99% error bands. To allow an assessment of fit in terms of stylized facts, Table 1 reports smoothed model implied counterparts for each of the data statistics described above.

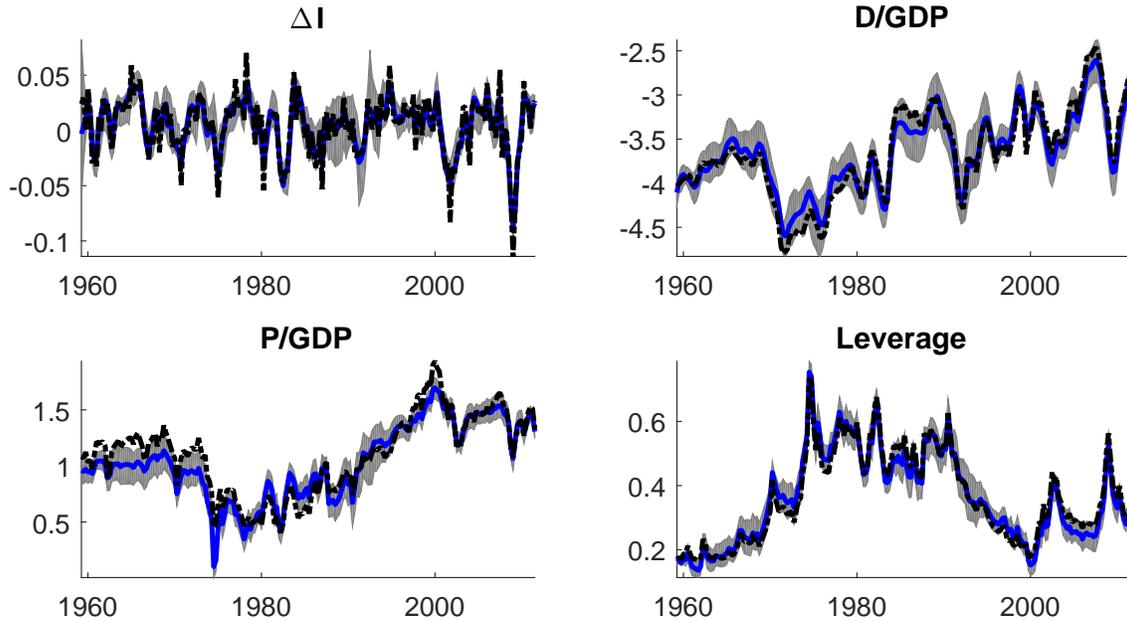


Figure 2: Variables used for estimation and filtered series. The figure reports the data (dashed line) and the filtered series (solid line) based on the posterior mode estimates. The difference between the two lines is accounted for by the observation error. The gray shaded areas correspond to 99% error bands.

We cannot expect a model with two shocks to fit four variables perfectly. It is remarkable therefore how small the estimated measurement errors are. Two notable discrepancies between data and model are that the model somewhat understates the volatility of payout and that it allocates a somewhat too large share of variation in equity value to high frequencies. As a result, the weak overall correlation between investment growth and payout as well as equity value is even weaker in the model. Nevertheless, the model captures all four stylized facts described above, not only qualitatively but also quantitatively.

5 Identification of shocks and the role of time varying uncertainty

How does the model accomplish the fit reported in Figure 2 and Table 1? Our estimation results point to a three part narrative. First, shocks to MPK – as well as ambiguity about MPK – drive the business cycle. Second, a large increase in the volatility of operating cost induced a medium term cycle in the 1970s and 1980s. A similar increase in volatility occurred during the recent financial crisis. Finally, intangible ambiguity about operating cost increased during the long booms of the 1960s, 1980s, 1990s and early 2000s, all of which ended with a spike in ambiguity. Movements in

Table 1: Second moments for data, baseline model and counterfactuals

	100* $\Delta \log I$			Std deviation			Correlation with $\Delta \log I$						Corr $\Delta B/Y$ with $\log(D/Y)$										
	All	MC		All	MC		$\log(D/Y)$	All	MC	All	MC	B/P		$\log(D/P)$									
Data	2.58	.62		.51	.25		.35	.24	.14	.10	-.01	-.31	.09	-.28	-.27	.13	-.07	-.04	All	MC		.27	.55
Model	2.58	.62		.44	.21		.34	.19	.14	.09	.07	-.41	.04	-.26	-.23	.19	.05	-.18	All	MC		.29	.41
<i>Lower uncertainty about operating cost</i>																							
Only volatility	2.57	.72		.42	.20		.30	.17	.13	.08	.10	-.47	.03	-.54	-.23	.38	-.03	.38	All	MC		.33	.59
No ambiguity	2.61	.58		.31	.14		.11	.03	.04	.02	.33	-.29	-.10	-.51	-.22	.16	.38	-.20	All	MC		.60	.59
Reestimated	2.58	.62		.51	.33		.29	.11	.13	.08	.01	-.69	.03	.12	-.27	.09	.01	-.69	All	MC		.23	.83
<i>Lower uncertainty about marginal product of capital</i>																							
No ambiguity	3.25	1.10		.58	.48		.45	.26	.21	.13	-.62	-.85	-.06	-.70	.01	.61	-.57	-.62	All	MC		.59	.96
<i>Lower frictions</i>																							
Const. leverage	2.54	.86		.67	.10		.43	.20	.24	.09	-.12	-.22	-.03	-.48	.03	.57	-.17	.31	All	MC		.08	.07
No inv adj costs	4.00	1.76		.46	.18		.34	.20	.14	.09	.26	-.07	.07	-.44	-.18	.27	.24	.30	All	MC		.23	.56
No div adj costs	2.51	.98		5.34	2.77		.35	.20	.15	.10	.35	-.24	-.04	-.48	.07	.50	.35	-.04	All	MC		.99	.98

Note: This table reports some key summary statistics for the data, the benchmark model, and a series of alternative models. The summary statistics are: (1) the standard deviation for the observables used in the estimation; (2) the correlation of $\log(D/Y)$, $\log(P/Y)$, B/P , and $\log(D/P)$ with the growth rate of investment, $\Delta \log I$; (3) and the correlation between $\log(D/Y)$ and $\Delta B/Y$. Each statistic is computed for the raw series (All) and for the corresponding filtered medium term cycle (MC). The first panel reports the summary statistics for the data and the benchmark model. The second panel reports the results for alternative models obtained by reducing uncertainty about the operating cost. The third panel focuses on a model with lower uncertainty about marginal product of capital. Finally, the fourth panel considers models with lower frictions.

uncertainty give rise to precautionary behavior by firms and investors which alters the comovement of real and financial variables and generates larger changes in equity prices, leverage and payout.

To establish this narrative in detail, we report the estimation results in three steps. Sections 5.1 and 5.2 explain the separate roles of uncertainty about operating cost and MPK in our baseline estimation. Section 5.3 then shows that uncertainty about operating cost – a novel ingredient of our story – is essential in the sense that a reestimated model without it cannot account for the joint dynamics of macro and financial variables.

5.1 Uncertainty about operating cost

Our estimation infers shifts in uncertainty about operating cost from the comovement of prices and quantities. To clarify this inference, we first summarize the estimated realization of the uncertainty regimes. We then discuss comovements induced by a one time increase in uncertainty, thus quantifying the theoretical responses discussed in Section 3.

Historical evolution of uncertainty

The left panel of Figure 3 displays, for each sample date, the smoothed probability at the posterior mode that the volatility of operating cost was high. The estimation clearly establishes a high volatility regime from the mid 1970s to the early 1990s as well as in the wake of the financial crisis. Ambiguity about operating cost also changes when volatility does not move. The right panel shows the probability that the process η_t^f selects high ambiguity due to intangible information. Here the estimation draws an interesting parallel between the 1960s and the recent three long booms in the 1980s, the 1990s and the 2000s. Toward the end of each of those booms, there was a marked increase in confidence (that is, a drop in ambiguity) about operating cost that resembled the sustained confidence of the 1960s.

Figure 4 isolates the contribution of uncertainty about operating cost to the estimated realizations of investment growth and the ratio of equity to GDP. To derive it, we start the economy at the estimated initial state and shut down all innovations except regime switches about the operating cost. These are drawn based on the estimated regime probabilities at the posterior mode. The resulting lines display jumps around the regime switches shown in both panels of Figure 3.

The figure shows that uncertainty about operating cost is an important driver of the equity-GDP ratio. Changes in volatility had a larger impact than changes in intangible ambiguity – in particular they played a key role over the medium term cycle as well as during the stock market correction of the late 2000s. The effect of uncertainty on investment growth is comparatively smaller. Nevertheless, the increase in volatility in the 1970s did elevate the level of investment for a number of years, while higher confidence held back investment at the end of long booms.

The middle panel of Table 1 shows that changes in uncertainty are critical for the model to match stylized facts on both prices and quantities. The row labeled "No ambiguity" shuts down any movements in uncertainty about operating cost. As a result, volatilities of all variables

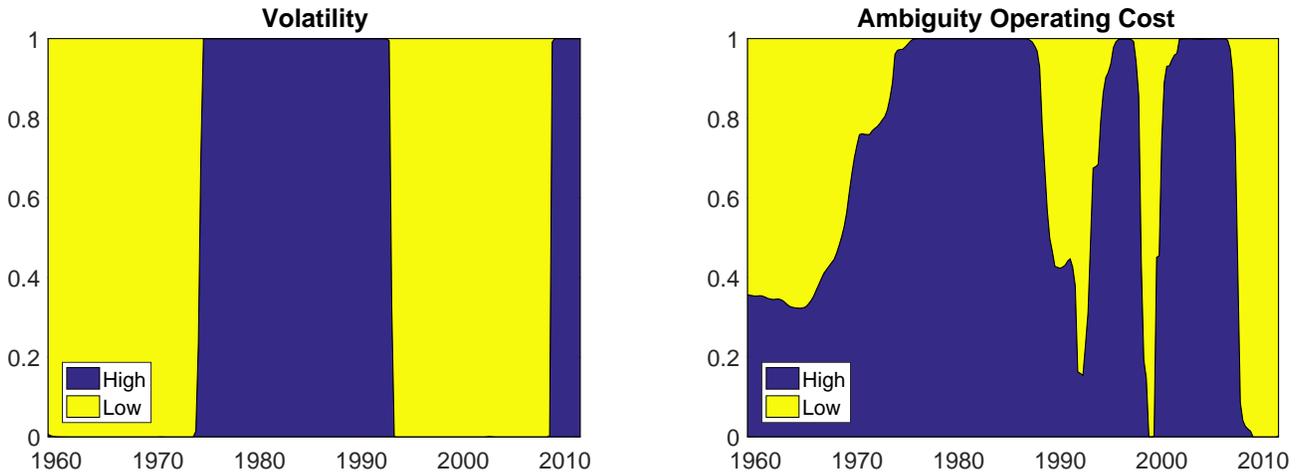


Figure 3: Smoothed regime probabilities. The figure reports the smoothed probabilities at the posterior mode for the volatility regimes (left panel) and for the regimes capturing ambiguity about the operating cost (right panel). The smoothed probabilities for the ambiguity regimes about the marginal product of capital are reported in Figure 6.

except investment growth drop substantially, both overall and at medium term cycle frequencies. Moreover, without changes in uncertainty the model overstates the cyclicity of the stock market and the payout-GDP ratio. It then also fails to match the key medium term correlation between debt issuance and payout relative to GDP.²⁴

The response to an increase in uncertainty

Why does the estimation infer particularly high uncertainty about operating cost in the 1970s and 1980s, but particularly low uncertainty at the end of major booms? The common denominator of these episodes is a large common movement in stock prices, payout and debt, but not in investment. We now show that this joint movement in financial and macro variables closely resembles the response to a change in uncertainty about operating cost.

Figure 5 displays responses to a one time increase in intangible ambiguity, that is a jump in η_t^f from its low to its high value that is not followed by further regime switches. Up to a scale factor, the figure also illustrates the response to an increase in volatility.²⁵ The six panels in the figure show our four observables as well as debt to GDP and the real interest rate. In each panel, the model is initially in the conditional steady state for the low ambiguity regime. It then experiences a regime shift to the high ambiguity regime in period 20.

An increase in uncertainty leads to a joint reduction in payout and debt. At the same time, the

²⁴This counterfactual illustrates an interesting comparison to Tallarini (2000). There, a change in the risk aversion parameter of the Epstein-Zin representative agent in a model that has only TFP shocks, leads only to a change in the market price of risk, while quantities are not affected. Moreover, price volatility and even the size of premia also change little. Our results are similar in that increased concern with one source of uncertainty – about operating cost – has a larger effect on asset pricing moments than on real moments. However, the details are quite different and in particular price volatility itself is directly affected.

²⁵Due to nonlinearity, the response to a decrease in ambiguity is not exactly the mirror image of the response to an increase. However, it is qualitatively very similar and hence not shown separately.

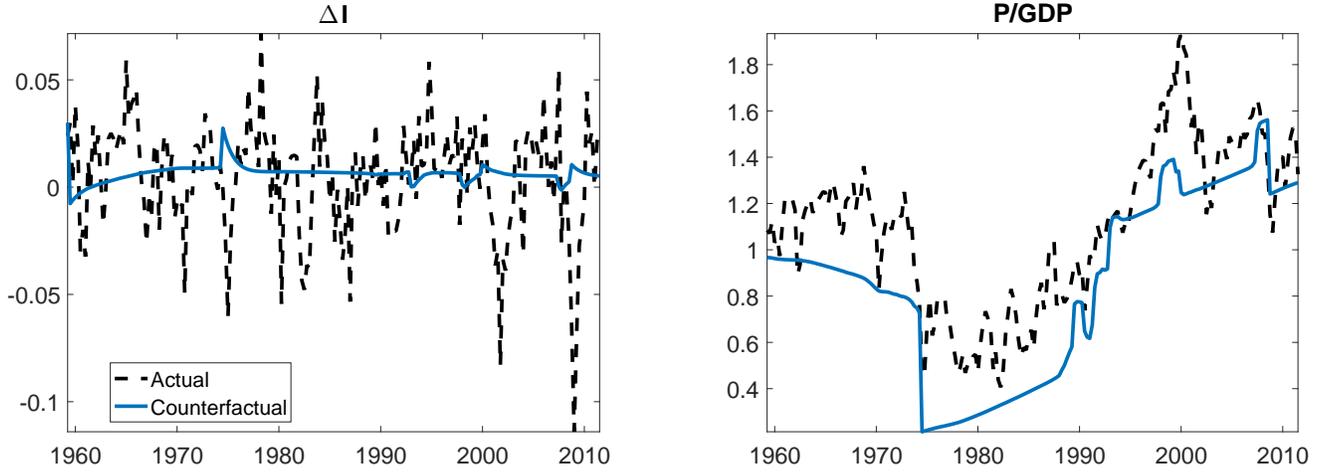


Figure 4: Evolution induced by draws for the regimes sequences of volatility and intangible information about the operating cost, based on the smooth probabilities of Figure 3.

stock market plummets. The price effect is strong enough that leverage of the corporate sector rises even as debt shrinks. The stock market crash is not due to an increase in interest rates – in fact the real rate even declines slightly along with investor confidence. In contrast to the decline in all financial ratios, investment growth increases on impact. All responses are persistent, with the stock price, interest rate and debt/GDP ratio even exhibiting large permanent changes. Permanent effects are to be expected since the economy converges to a new regime-dependent steady state after the initial regime switch.

The intuition for these responses follows from the firm and household optimality conditions in Section 3. An increase in uncertainty makes agents fear scarcity of resources in the future. In particular, households fear lower consumption and increase precautionary savings: their discount factor M_{t+1} increases and the safe interest rate falls. This force would be present even in the absence of financial frictions. In addition, uncertainty about operating cost makes shareholders fear scarcity of resources *within the firm*. The firm’s internal discount factor $M_{t+1}\lambda_{t+1}/\lambda_t$ thus increases in part because shareholders fear a higher cost of capital.

Precautionary savings generates higher investment. At the same, financial frictions imply that extra investment is not funded by more leverage, but by reducing payout. Indeed, holding fixed the interest rate, an increase in the firm’s discount factor increases the worst case marginal cost of repaying debt. On top of that, a lower interest rate lowers the tax benefit of debt. With higher investment and lower debt, a decrease in payout follows from the firm budget constraint. In the presence of adjustment costs, the firm draws out the decrease to smooth the growth of payout.

The drop in the stock price is due to an increase in the uncertainty premium. In the notation of Section 3.1, the future payoff of equity is $\Pi_{t+1} \equiv P_{t+1} + D_{t+1}$. An innovation to uncertainty about operating cost then triggers the two opposing forces present in equation (3.3): a higher uncertainty premium $E_t[\Pi_{t+1}] - E_t^*[\Pi_{t+1}]$ lowers the price whereas a lower interest rate raises

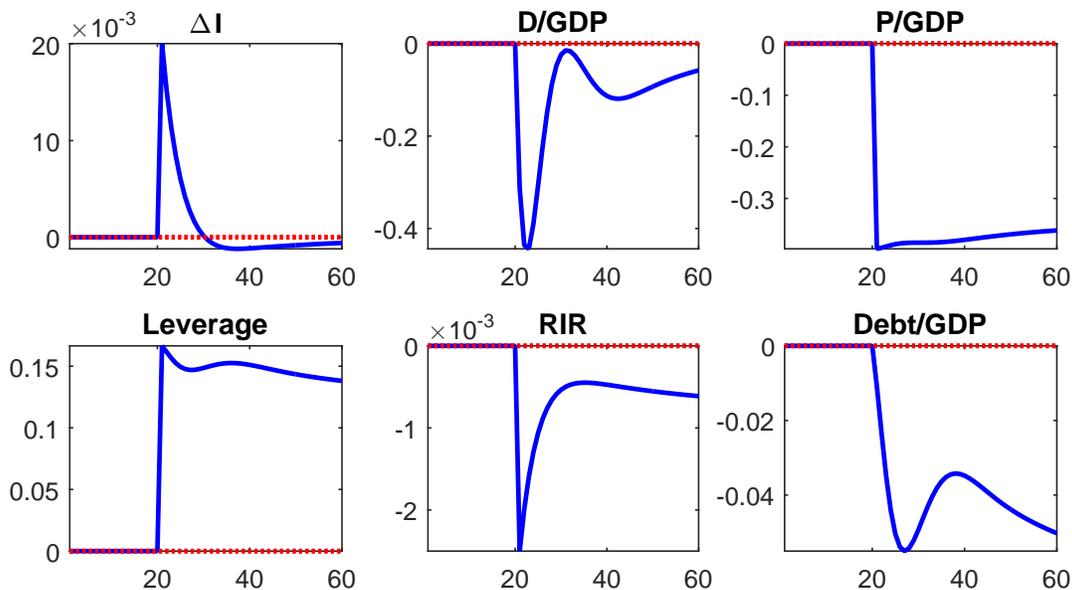


Figure 5: Impulse response for a switch from a regime of low ambiguity to a regime of high ambiguity about the operating cost. The switch occurs in period 20.

the price as the present value of cash flow increases. The uncertainty premium effect dominates because payout D_t is a small fraction of overall consumption C_t .²⁶ The drop in the stock price thus goes along with an increase in the excess return measured by an econometrician while interest rates are stable.

What flavor of uncertainty?

So far we have emphasized the similarity between changes in volatility and changes in intangible ambiguity. Both types of uncertainty generate precautionary behavior and measured risk premia that lower stock prices. How does the estimation decide that the 1970s and 1980s as well as the Great Recession were characterized by volatile operating cost? In the raw data in Figure 2, both episodes exhibit sharp variations in leverage and payout relative to GDP. The estimation uses the high volatility regime to account for these movements.

To isolate further the role of each type of uncertainty, the row labeled "volatility only" in Table 1 shows moments when intangible ambiguity is shut down but movements in volatility are retained. We can compare it to the baseline model to see the impact of intangible ambiguity alone as well as to the row "no ambiguity" to see the effect of volatility.

Time variation in volatility is overall more important for dynamics than intangible ambiguity. Indeed, when intangible ambiguity is shut off, the standard deviations of all observables are close

²⁶The asset market clearing condition $\theta_{t+1} = 1$ implies that the price P_{t+1} disappears from the determination of equilibrium consumption C_{t+1} in the household budget constraint. Instead, the uncertain payoff Π_{t+1} matters for C_{t+1} only from its effect through the payout D_{t+1} . In the notation of equation (3.4), this means that there is a large $\hat{\alpha}_t$, not directly affected by the uncertainty about cash flow, and a small $\hat{\beta}_t$.

to those in the baseline model. At the same time, intangible ambiguity plays an important role in fitting correlations. In particular, the "volatility only" specification strongly overstates the negative correlation between the stock market and investment growth at medium term cycle frequencies. While both sources of uncertainty induce a negative correlation, shutting down the intangible information channel shifts the weight towards the stronger medium cycle correlation effects coming through the volatility regimes.

Our estimated high volatility regime is quite different from regimes found in work on the "Great Moderation". The latter work is concerned with drops in the volatility of macro variables between the early 1980s and the Great Recession.²⁷ In contrast, our high volatility regime is identified by changes in the volatility of financial variables together with low asset prices. While it also starts in the mid 1970s, it also includes the financial market turbulence of the late 1980s which saw the savings and loan crisis and the 1987 stock market crash.

5.2 Uncertainty about the marginal product of capital

The left panel of Figure 6 shows the estimated joint evolution of MPK and ambiguity about MPK. As a backdrop, the dashed black line is investment growth, the observable most closely associated with MPK. The blue solid line is the contribution of MPK to investment growth. Since our model solution is linear, that contribution represents a scaled version of the shock Z_t itself.

Comparing the two lines we conclude that shifts in MPK account for the bulk of movements in investment. In contrast, the solid blue line in the right panel of Figure 6 shows the minor contribution of realized operating cost to investment growth. Movements in uncertainty about operating cost also account for a minor variation, as shown in Figure 4 and discussed earlier.

According to the shock specification (4.4)-(4.5), movements in MPK (and ambiguity about MPK) have both a Gaussian component and a discrete component due to regime switches. The shading in the left panel of Figure 6 indicates the posterior probability that the regime process η_t^z takes on its high value, so ambiguity about MPK is high and MPK itself is low. Comparison with Figure 3 shows that η_t^z is much less persistent than uncertainty about operating cost. Instead, its role is to generate relatively frequent non-Gaussian movements.

The response to MPK shocks

How does the estimation identify joint movements in MPK and ambiguity about MPK as drivers of business cycle fluctuations? Whatever is the driving force behind those fluctuations must account for business cycle movements in investment as well as procyclical comovement in payout and debt. At the same time, however, it must not generate large fluctuations in the stock market and in leverage. We now show that our correlated shock to MPK and ambiguity about MPK has these properties.

²⁷See, for example, Justiniano and Primiceri (2008), Fernández-Villaverde et al. (2010) and Bianchi (2013).

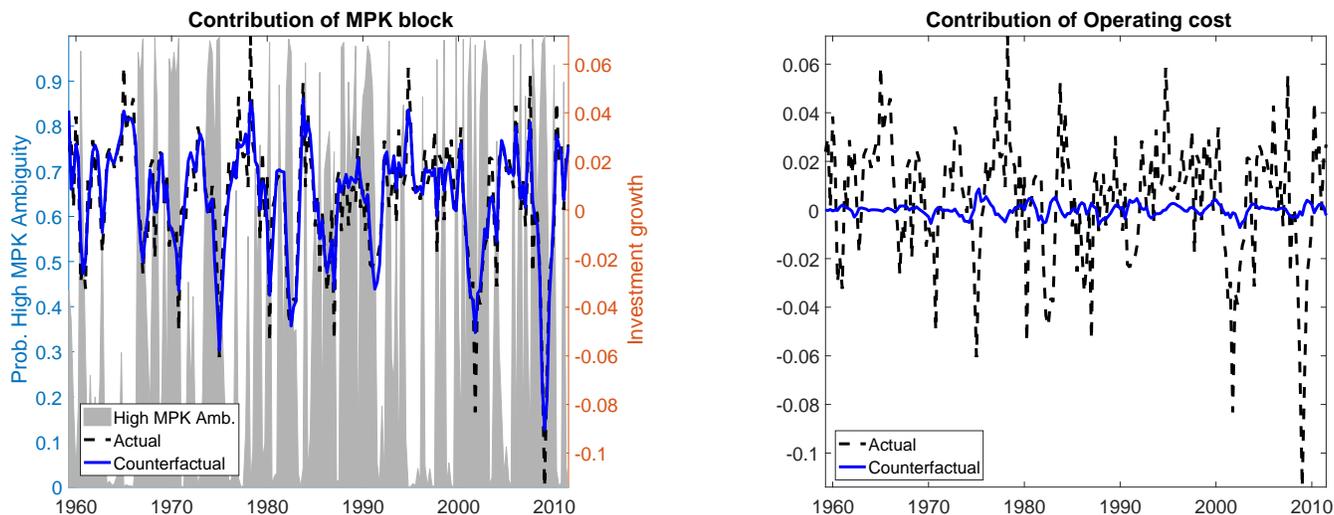


Figure 6: Shock decomposition for investment growth. In both panels, the solid line refers to the counterfactual simulation, whereas the dashed line corresponds to the actual data. In the left panel, the counterfactual shuts down all disturbances except for those affecting MPK: Shocks to MPK and regime changes in the level of ambiguity about MPK (right scale). This plot also reports the smoothed probability at the posterior mode of being in the high MPK ambiguity regime (left scale). In the right plot, the counterfactual shuts down all disturbances except for the shocks to the operating cost.

A decrease in MPK and an increase in ambiguity about MPK have distinct and partially offsetting effects on investment and the real interest rate. On the one hand, a decrease in the persistent shock Z_t lowers both the current and the expected MPK. Much like in a standard real business cycle model, it lowers investment and it increases the real interest rate. On the other hand, an increase in ambiguity about MPK lowers the worst case expected return on capital while current MPK is unaffected. The response is dominated by a precautionary savings effect that increases investment and decreases the real interest rate. However, the effect of investment is dampened by the lower worst case return on investment.

There are also offsetting effects on financial variables. A time of low MPK is a bad time for a firm to invest, but a good time to issue debt: future funds within the firm are relatively more abundant and the higher interest rate raises the tax benefit of debt. The increase in debt and decrease in investment are large enough to allow higher payout, even though current cash flow is lower. In contrast, higher ambiguity about future MPK makes future funds uncertain and lowers the real interest rate, thus decreasing debt issuance. Since investment is also higher and current cash flow is unchanged, the budget constraint implies that payout must decrease.

To quantify the two opposing forces, the sixth row in Table 1, labeled "lower uncertainty about marginal product of capital: no ambiguity", shuts off ambiguity to leave a shock to MPK only. Relative to the baseline model, all volatilities are higher, as one would expect if one of two opposing forces is eliminated. Moreover, the counterfactual produces stronger correlation between payout on the one hand and investment and debt on the other. Ambiguity about MPK thus helps the

model to simultaneously account for the cyclical fluctuations in these three quantity variables – shocks to MPK alone would create too strong of an association.

In contrast to ambiguity about operating cost, ambiguity about MPK does not help generate stock price volatility: the latter is higher under the counterfactual that eliminates ambiguity about MPK. The reason is the precautionary savings effect discussed in Section 3.1. While higher ambiguity about future payoff induces an uncertainty premium that decreases the equity price, the precautionary savings motive lowers the interest rate and increases the present value of payoffs. At the estimated parameter values, the second effect is sufficiently strong so as to dominate.

5.3 Is uncertainty about operating cost essential?

The results in Section 5.1 show that if we shut down variation in uncertainty about operating cost – keeping other estimated parameters the same – then the model performance worsens dramatically, especially for stock market volatility. It follows that variation in uncertainty about operating cost plays a key role *within* the narrative of the data provided by our model. We cannot yet conclude, however, whether a more parsimonious model without variation in uncertainty about operating cost could also get close to the data and baseline model, thus supplying a quite different narrative.

To answer the latter question, the row labeled "reestimated" in Table 1 reports results from an alternative estimation without any uncertainty about operating cost; in particular the possibility of regime switches as in Figure 3 are left out. Otherwise, this alternative model is driven by the same two shocks as before and allows for variation in uncertainty about MPK. As for the baseline estimation, we allow for measurement error on all observables.

While one would expect the more parsimonious alternative model to perform worse than the baseline model, its lack of fit is striking. Measurement error is now needed to account for 25% of overall variation in the equity-GDP ratio, and for 80% of the variation of that ratio over the medium term cycle. This is although the model overstates variation in payout over the medium term cycle by 60% relative to the data. The need for measurement error to account for low frequency movements is a clear sign of misspecification. Moreover, the alternative model is unable to account for the correlation patterns over the medium term cycle; in particular, it makes the equity-GDP ratio and the dividend yield counterfactually strongly countercyclical.

Since the alternative model does not allow for uncertainty about operating cost, it might rely on movements in MPK and uncertainty about MPK to generate asset price volatility. The discussion in Section 3.1 suggests that this is an uphill battle: while a decrease in MPK (as well as an increase in ambiguity in MPK) will tend to lower expected payoff, it also makes investors discount those payoffs at a lower real interest rate, leaving the stock price relatively stable. The estimation deals with this dilemma by introducing a counterfactually large medium term component in payout, driven largely by realized operating cost. As we show in Section 7 below, this feature implies that the model cannot generate excess return predictability, in contrast to the baseline model.

The results here have broader significance for production based asset pricing models driven by "business cycle shocks" such as TFP together with time varying volatility of those shocks. While it may be possible for such models to generate a lot of price volatility, especially if the payout series is not fit to the data, it is important to check whether the volatility in payout and prices indeed arises at the right frequencies. Comparison of our two estimations suggests that while business cycle shocks – in our case, to MPK and ambiguity about MPK – can generate sensible movements in investment that are only weakly connected to the stock market, they cannot simultaneously generate the observed comovement over the medium term cycle.

6 The role of frictions

Throughout our discussion, firms' optimal choice of leverage and payout has been central to our interpretation. To quantify the importance of this margin, we now contrast our model to one with constant leverage. We also discuss the role of different adjustment costs for firm behavior.

6.1 Endogenous leverage

Our paper differs from existing production based asset pricing exercises in that leverage and payout are determined endogenously in the model and matched to their counterparts in the data. Existing papers typically consider frictionless models of the firm and choose constant book leverage, that is, debt is proportional to the capital stock:

$$B_t = \vartheta K_t. \tag{6.1}$$

Payout is then determined as a residual from the firm budget constraint. In a frictionless model, this (or any other budget feasible) exogenous specification can be justified by the Modigliani-Miller theorem. However, it is not disciplined by data except that ϑ is selected to match average leverage.

In this section we show that the endogenous choice of leverage plays a key role for the performance of our model. The seventh row in Table 1, labeled "constant leverage", considers a counterfactual that sets debt as in equation (6.1), with payout determined as a residual. All other parameters and shock realizations are the same as in the baseline case. A key property of the reduced form policy function for debt is that the change in debt is now closely tied to investment; debt responds to shocks only to the extent that those shocks move investment. In contrast, we have shown that under the optimal policy function firms respond to shocks that move the perceived future cost of funds – such as uncertainty about operating cost – even if the effect of such shocks on investment is relatively small.

In the constant leverage counterfactual, volatilities of the financial variables increase relative to the baseline model, but the share of variation at medium term cycle frequencies declines to a

level that is much lower than in the data. This is what one would expect since financial decisions are now mechanically tied to investment, a variable that fluctuates mostly over the business cycle.²⁸ For the same reason, the correlation between investment growth and (market) leverage becomes counterfactually high and positive. Finally, the procyclical comovement of debt issuance and payout virtually disappears. In the baseline model, the latter key fact is driven by the response of optimal leverage choice to uncertainty shocks that make investment and debt issuance move in opposite directions, in sharp contrast to (6.1).²⁹

6.2 The role of adjustment costs

Our baseline model assumes adjustment costs to the rate of change in investment and payout. The last two rows in Table 1 report moments from counterfactuals that shut down those costs, again everything else equal.

Investment adjustment cost

The main effect of dropping investment adjustment costs (that is, $\Theta = 0$) is that investment becomes more volatile. This is not surprising – adjustment costs play the usual role of smoothing investment. At the same time, the volatilities of payout and leverage change little – the additional movements in investment track internal cash flow, but do not generate movement in external finance. What also stands out in the table is that adjustment costs are key to matching the comovement of stock prices and investment over the medium term cycle. In particular, an increase in uncertainty about operating cost in the counterfactual model generates a stronger precautionary increase in investment together with lower stock prices as before. As a result, negative comovement between the equity-GDP ratio and investment over the medium term cycle becomes too strong.

The fact that adjustment costs are not crucial for price volatility distinguishes our model from many existing production based asset pricing models. Indeed, the traditional approach is to work with TFP shocks only and to identify the value of an unlevered firm with the capital stock. In the absence of adjustment cost the stock price is then constant. With higher adjustment costs, the stock price becomes more volatile, at the cost of making investment more smooth. This tension makes it hard to match price and quantity volatility jointly. Our model start from a different view of the firm – based on packaging of payout subject to financial frictions – together with uncertainty shocks, so that this tension does not arise.

²⁸The counterfactual here simply shuts down optimal leverage and payout choice given our estimated parameters. It thus serves to establish the importance of optimal leverage choice in the context of our model. However, the stylized facts we have emphasized point to a more general problem for models that impose the mechanical link in equation (6.1): any model that matches the cyclical dynamics of investment will have trouble matching the medium term cycle dynamics of debt.

²⁹Our constant leverage counterfactual also exhibits countercyclical payout. This property is familiar from existing production based asset pricing models with volatility shocks. In those models, investment and hence debt tends to respond negatively to volatility shocks; dividends work as a residual that has to compensate by moving in the opposite direction. Along this dimension, our model makes uncertainty shocks work because it allows for an endogenous leverage.

Dividend adjustment cost

The main effect of dropping dividend adjustment costs (that is, $\Phi = 0$) is that dividends become more volatile. In addition, there is a notable increase in the volatility of investment growth at medium term cycle frequencies, whereas volatilities of leverage and the equity-GDP ratio change little. Without dividend adjustment costs, the firm first order conditions (3.5)-(3.7) simplify: the investment decision works as if households were direct holders of capital, while debt remains determined by a tradeoff theory of capital structure. Payout is determined as a residual from the budget constraint of the levered firm and hence becomes excessively volatile.

Dividend adjustment costs are also important to generate the right type of comovement over the medium term cycle. As in the baseline model, an increase in uncertainty about operating cost – the key driver of the medium term cycle – jointly lowers payout, debt and the stock price and increases investment. The absence of dividend adjustment costs results in a stronger reduction in payout that in turn allows a stronger increase in investment. At the same time, the stock price – which reflects worst case present values – is less sensitive to the stronger impact response and reacts much like in the baseline model. The upshot is a medium term cycle with excessive comovement in payout and debt, investment and the equity-GDP ratio as well as investment and leverage.

7 The real interest rate and excess volatility

The results reported so far show that our model can match the volatility of the payout yield – or equivalently, the price-payout ratio. While this is promising, it does not yet demonstrate that it generates the right kind of volatility. As discussed in Section 3.1, stock prices can move because of (i) changes in the expected present value of cash flows, or (ii) changes in premia for uncertainty. The literature has established that the bulk of volatility is due to (ii) – the excess volatility puzzle. This section shows that our model also has this property.

In particular, we need to make sure that we do not obtain volatile stock prices because there are counterfactually large movements in the real interest rate. Our estimation did not target the real interest rate, so it might in principle use rate movements to shift the present value of payout. It would thereby generate the wrong type of stock price volatility (i). The power of discounting implies that our model may not be a good model of the stock market even though it does a good job matching the joint distribution of prices and payout.

The volatility of the model implied real interest rate is 1% per year. It has an autocorrelation coefficient of .5 and is mildly procyclical – the correlation coefficient with investment is .2. As a concrete data counterpart, we rely on a state space model of interest rate and inflation dynamics

estimated by Piazzesi et al. (2013).³⁰ The resulting series has a volatility of 2%, an autocorrelation coefficient of .8 and its correlation with investment is .1. Both the model series and its data counterpart thus share standard properties of real interest rate series in the literature: they are stable, persistent and not particularly cyclical.

The real return on short bonds thus looks very different from the real return on equity. In our data, the average equity premium is 5.2% per year and the volatility of equity returns is 17.6%. The model also produces an equity premium of 5.2% as well as a return volatility of 19.8%. In both the model and the data, the autocorrelation of stock returns is small at .1. We conclude that the model replicates the unconditional high risk and high return in stocks relative to bonds.

Predictability of excess returns

We use a standard diagnostic for time varying premia: we regress excess return on stocks on the dividend yield. As shown in Section 3.1, if price were equal to expected present value, then we should see no predictive power – both the slope and the R^2 should be equal to zero. We define the one period log excess return on stocks relative to the safe interest rate by

$$x_{t+1}^e = \log(P_{t+1} + D_{t+1}) - \log P_t + \log(Q_t^b) \quad (7.1)$$

and define h -period excess returns by summing up one period excess returns.

Table 2 reports results for predictive regressions with horizon one, two and five years, comparing the data and the baseline model. To be consistent with the estimation approach, our construction of returns is also based on measures of payout and equity value from the Flow of Funds Accounts. The predictive regressions exhibit familiar patterns and magnitudes from the asset pricing literature: the regression slopes are significantly positive and both slopes and R^2 s increase with the forecast horizon.³¹ The model reproduces both patterns; in fact, it slightly overstates the predictive power of the dividend yield.

Ambiguity and predictability

How does the model generate excess volatility of stock prices and a stable real interest rate? The key mechanism is that uncertainty about operating cost has persistent effects on the dividend yield but not on the real interest rate. Indeed, an increase in uncertainty lowers prices and increases the dividend yield as investors require more compensation to hold stocks. However, since operating cost is a small share of output, worst case expectations of future consumption and hence the real rate remain stable. As the shock mean reverts, so does the stock price. An econometrician thus finds that low prices are followed by high excess returns. Investors do not exploit this predictability because they do not want to bear the ambiguity associated with high premia.

³⁰Since indexed bonds were not traded in the US before 1997, estimates of real interest rate always involve assumptions on inflation expectations. The advantage of a state space approach is that it separates slow moving and transitory components of the inflation process, thus yielding a better fit of inflation dynamics.

³¹Since we are using the statistic only as a diagnostic tool, we report OLS standard errors in both the data and model regressions.

We confirm that ambiguity about operating cost is critical for predictability in our model by rerunning the predictive regressions for the alternative model introduced in Section 5.3. The bottom rows of Table 2 shows that the alternative model is not successful on that front: while the slopes have the right sign and the results look good for short horizon, there is essentially no predictability at the five year horizon, in contrast to the data and the baseline model. It follows that ambiguity about MPK is not enough to generate the right type of stock market volatility.

Table 2: Predictability of future excess returns by current dividend yield

	Statistic	Horizon		
		1 year	2 years	5 years
Data	<i>Slope</i>	0.0986	0.1611	0.3669
	(<i>s.e.</i>)	0.0262	0.0330	0.0406
	R^2	0.0653	0.1068	0.3040
Baseline model	<i>Slope</i>	0.1504	0.2503	0.4280
	(<i>s.e.</i>)	0.0307	0.0381	0.0518
	R^2	0.1077	0.1801	0.2643
Alternative model	<i>Slope</i>	0.0997	0.1551	0.1061
	(<i>s.e.</i>)	0.0229	0.0293	0.0379
	R^2	0.0865	0.1241	0.0392

Note: This table reports results for regressing future excess returns on stocks on a constant and the dividend yield $\log(D_t/P_t)$. The horizon columns report results based on summing up the one period returns x_{t+1}^e , defined in equation (7.1), for one, two or five years ahead. *Slope* refers to the coefficient on the dividend yield, (*s.e.*) refers to the OLS standard errors and R^2 to the regression R-squared. The alternative model, introduced in Section 5.3, is a re-estimated model without any uncertainty about operating cost.

8 Conclusion

How to reconcile the behavior of real activity and equity prices is a long standing question in macro finance. To provide a solid foundation for understanding aggregate fluctuations and the effects of policy, it would be desirable to work with DSGE models that can jointly match firm cash flows and their valuation. This paper contributes to that agenda by proposing one such model. Introducing time varying ambiguity on the household side and financial frictions on the firm side delivers a parsimonious account of postwar US data. More broadly, we emphasize four takeaways for the agenda as a whole.

First, it is worthwhile to look beyond business cycle frequencies. To understand postwar asset price dynamics, it is not enough to simply generate "high risk premia in recessions", a common theme in the literature. A narrow focus on the business cycle ignores the large medium term cycle component in stock prices. We show that this point is even more salient once we care

about matching corporate cash flows and leverage. Our results suggest that shocks that drive the business cycle are unlikely to account for the right type of price volatility, even if a mechanism that generates time varying risk premia is added to the model.

Second, time varying uncertainty is useful not only to relate asset prices and investor behavior, but also to understand firm financial decisions. Previous work has shown that time varying uncertainty – whether risk or ambiguity – can reconcile excessively volatile prices with moderate volatility in consumption and cash flow. This works because uncertainty shocks add extra “discount rate” volatility to the stochastic discount factor. This paper describes a mechanism by which discount rate volatility – and hence uncertainty shocks – affect shareholder value maximization so as to generate observed comovement in prices, investment, payout and leverage at different frequencies.

Third, our model assigns an important role to uncertainty about shocks that affect corporate earnings but do not scale with production. In our view such shocks are particularly plausible once one thinks about the corporate sector not simply as a production technology, but instead an organization that expends resources in order to package cash flows into financial claims. Operating cost can then then represent a sizable share of shareholder payout but still remain a small share of overall consumption.

Compared to standard uncertainty shocks about productivity, uncertainty about operating cost is promising for explaining asset prices. In particular, our results show that it reduces the standard tension between excess volatility of stock prices and stable interest rates. Moreover, it changes the comovement of stock prices and shareholder payout with investment by encouraging precautionary savings when stock prices are low.

While we have provided some motivating examples for operating cost, it would be valuable to develop more direct evidence on the nature of these costs, their variability as well as uncertainty perceived about them by investors. A particularly interesting question is to what extent uncertainty arises at the level of the individual firm or only at the level of the corporate sector as a whole (where it might reflect, for example, reorganization via mergers and acquisitions). In the former case, the first order effects of ambiguity about operating cost as we have modeled them could also help understand also the cross section of stock returns.

Finally, our model demonstrates by example that recursive multiple priors utility with ambiguity about means offers a tractable way to analyze DSGE models with time varying uncertainty. This tractability allows us to represent the model as an MS-VAR and estimate it by likelihood methods. Adding ambiguity about means is thus a straightforward way for DSGE modelers to augment their models with quantitatively sensible asset pricing implications.

References

- ANDREASEN, M. M., J. FERNÁNDEZ-VILLAYERDE, AND J. RUBIO-RAMÍREZ (2013): “The pruned state-space system for non-linear DSGE models: Theory and empirical applications,” NBER Working Paper 18983.
- ARELLANO, C., Y. BAI, AND P. KEHOE (2010): “Financial Markets and Fluctuations in Uncertainty,” FRB of Minneapolis, Research Department Staff Report.
- ARUOBA, S. B., J. FERNANDEZ-VILLAYERDE, AND J. F. RUBIO-RAMIREZ (2006): “Comparing Solution Methods for Dynamic Equilibrium Economies,” *Journal of Economic Dynamics and Control*, 30, 2477–2508.
- BAELE, L., G. BEKAERT, S. CHO, K. INGHELBRECHT, AND A. MORENO (2011): “Macroeconomic Regimes,” Columbia University working paper.
- BASU, S. AND B. BUNDICK (2012): “Uncertainty shocks in a model of effective demand,” NBER Working Paper 18420.
- BEGENAU, J. AND J. SALOMAO (2013): “Firm financing over the business cycle,” Mimeo.
- BELO, F., X. LIN, AND F. YANG (2014): “External equity financing shocks, financial flows, and asset prices,” NBER Working Paper 20210.
- BIANCHI, F. (2013): “Regime Switches, Agents’ Beliefs, and Post-World War II U.S. Macroeconomic Dynamics,” *Review of Economic Studies*, 80, 463–490.
- BIDDER, R. AND M. SMITH (2012): “Robust Animal Spirits,” *Journal of Monetary Economics*, 59, 738–750.
- BINSBERGEN, V., H. JULES, J. R. GRAHAM, AND J. YANG (2010): “The cost of debt,” *The Journal of Finance*, 65, 2089–2136.
- BOLTON, P., H. CHEN, AND N. WANG (2013): “Market timing, investment, and risk management,” *Journal of Financial Economics*, 109, 40–62.
- BRAV, A., J. R. GRAHAM, C. R. HARVEY, AND R. MICHAELY (2005): “Payout policy in the 21st century,” *Journal of Financial Economics*, 77, 483–527.
- CAGETTI, M., L. P. HANSEN, T. SARGENT, AND N. WILLIAMS (2002): “Robustness and Pricing with Uncertain Growth,” *Review of Financial Studies*, 15, 363–404.
- CALDARA, D., J. FERNÁNDEZ-VILLAYERDE, J. F. RUBIO-RAMÍREZ, AND W. YAO (2012): “Computing DSGE models with recursive preferences and stochastic volatility,” *Review of Economic Dynamics*, 15, 188–206.

- CHRISTIANO, L. J., M. EICHENBAUM, AND C. L. EVANS (2005): “Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy,” *Journal of political Economy*, 113, 1–45.
- CHRISTIANO, L. J., R. MOTTO, AND M. ROSTAGNO (2014): “Risk shocks,” *The American Economic Review*, 104, 27–65.
- COCHRANE, J. H. (2011): “Presidential address: Discount rates,” *The Journal of Finance*, 66, 1047–1108.
- COGLEY, T. AND T. J. SARGENT (2006): “Drifts and Volatilities: Monetary Policies and Outcomes in the Post WWII U.S.” *Review of Economic Dynamics*, 8, 262–302.
- COMIN, D. AND M. GERTLER (2006): “Medium-term business cycles,” *The American Economic Review*, 96, 523–551.
- COVAS, F. AND W. J. DEN HAAN (2011): “The Cyclical Behavior of Debt and Equity Finance,” *The American Economic Review*, 101, 877–899.
- (2012): “The Role of Debt and Equity Finance Over the Business Cycle,” *The Economic Journal*, 122, 1262–1286.
- CROCE, M. M., H. KUNG, T. T. NGUYEN, AND L. SCHMID (2012): “Fiscal policies and asset prices,” *Review of Financial Studies*, 25, 2635–2672.
- DAVIG, T. AND T. DOH (2013): “Monetary Policy Regime Shifts and Inflation Persistence,” *The Review of Economics and Statistics*, forthcoming.
- DEL NEGRO, M. AND F. SCHORFHEIDE (2008): “Forming Priors for DSGE Models (And How it Affects the Assessment of Nominal Rigidities),” *Journal of Monetary Economics*, 55, 1191–1208.
- DRECHSLER, I. (2013): “Uncertainty, Time-Varying Fear, and Asset Prices,” *The Journal of Finance*, 68, 1843–1889.
- EISFELDT, A. L. AND T. MUIR (2014): “Aggregate Issuance and Savings Waves,” NBER Working Paper 20442.
- ELENEV, V., T. LANDVOIGT, AND S. VAN NIEUWERBURGH (2016): “A Macroeconomic Model with Financially Constrained Producers and Intermediaries,” Working Paper.
- EPSTEIN, L. AND T. WANG (1994): “Intertemporal Asset Pricing under Knightian Uncertainty,” *Econometrica*, 62, 283–322.
- EPSTEIN, L. G. AND M. SCHNEIDER (2003): “Recursive Multiple-Priors,” *Journal of Economic Theory*, 113, 1–31.

- (2008): “Ambiguity, information quality, and asset pricing,” *The Journal of Finance*, 63, 197–228.
- (2010): “Ambiguity and Asset Markets,” *Annual Review of Financial Economics*, forthcoming.
- FARRE-MENSA, J., R. MICHAELY, AND M. SCHMALZ (2014): “Payout Policy,” *Annual Review of Financial Economics*, 6, 75–134.
- FERNÁNDEZ-VILLAVÉRDE, J., P. GUERRÓN-QUINTANA, J. RUBIO-RAMÍREZ, AND M. URIBE (2011): “Risk Matters. The Real Effects of Volatility Shocks,” *American Economic Review*, 101, 2530–2561.
- FERNÁNDEZ-VILLAVÉRDE, J., P. GUERRÓN-QUINTANA, AND J. F. RUBIO-RAMÍREZ (2010): “Fortune or Virtue: Time-Variant Volatilities Versus Parameter Drifting in U.S. Data,” NBER Working Papers 15928.
- FERNÁNDEZ-VILLAVÉRDE, J., J. F. RUBIO-RAMÍREZ, AND F. SCHORFHEIDE (2016): “Solution and estimation methods for DSGE models,” *Handbook of Macroeconomics*, 2, 527–724.
- FOERSTER, A., J. RUBIO-RAMÍREZ, D. F. WAGGONER, AND T. ZHA (2015): “Perturbation Methods for Markov-switching DSGE Models,” *Quantitative Economics*, forthcoming.
- FRANK, M. AND V. GOYAL (2011): “Trade-off and Pecking Order Theories of Debt,” *Handbook of Empirical Corporate Finance: Empirical Corporate Finance*, 2, 135.
- GILBOA, I. AND D. SCHMEIDLER (1989): “Maxmin Expected Utility with Non-unique Prior,” *Journal of Mathematical Economics*, 18, 141–153.
- GILCHRIST, S., J. W. SIM, AND E. ZAKRAJSEK (2010): “Uncertainty, financial frictions, and investment dynamics,” Unpublished working Paper.
- GLOVER, B., J. F. GOMES, AND A. YARON (2011): “Corporate taxes, leverage, and business cycles,” Unpublished working Paper.
- GOURIO, F. (2012): “Disaster Risk and Business Cycles,” *The American Economic Review*, 102, 2734–2766.
- (2013): “Credit risk and disaster risk,” *American Economic Journal: Macroeconomics*, 5, 1–34.
- GRAHAM, J. R. AND M. T. LEARY (2011): “A review of empirical capital structure research and directions for the future,” *Annual Review of Financial Economics*, 3.

- GREENWALD, D. L., M. LETTAU, AND S. C. LUDVIGSON (2014): “The Origins of Stock Market Fluctuations,” NBER Working Papers 19818.
- GUIDOLIN, M. AND F. RINALDI (2013): “Ambiguity in Asset Pricing and Portfolio Choice: A Review of the Literature,” *Theory and Decision*, 74, 183–217.
- HAMILTON, J. D. (1989): “A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle,” *Econometrica*, 57, 357–384.
- (1994): *Time Series Analysis*, Princeton: Princeton University Press.
- ILUT, C. AND M. SCHNEIDER (2014): “Ambiguous Business Cycles,” *American Economic Review*, 104, 2368–2399.
- JAHAN-PARVAR, M. AND H. LIU (2012): “Ambiguity Aversion and Asset Prices in Production Economies,” Unpublished working Paper.
- JERMANN, U. AND V. QUADRINI (2012): “Macroeconomic Effects of Financial Shocks,” *American Economic Review*, 102, 238–271.
- JUDD, K. L. (1992): “Projection Methods for Solving Aggregate Growth Models,” *Journal of Economic Theory*, 58, 410–452.
- JUDD, K. L. AND S.-M. GUU (1997): “Asymptotic Methods for Aggregate Growth Models,” *Journal of Economic Dynamics and Control*, 21, 1025–1042.
- JUSTINIANO, A. AND G. PRIMICERI (2008): “The Time Varying Volatility of Macroeconomic Fluctuations,” *American Economic Review*, 98, 604–41.
- KHAN, A. AND J. K. THOMAS (2013): “Credit shocks and aggregate fluctuations in an economy with production heterogeneity,” *Journal of Political Economy*, 121, 1055–1107.
- LEIPPOLD, M., F. TROJANI, AND P. VANINI (2008): “Learning and asset prices under ambiguous information,” *Review of Financial Studies*, 21, 2565–2597.
- LETTAU, M. AND S. LUDVIGSON (2010): “Measuring and Modeling Variation in the Risk-Return Tradeoff,” in *Handbook of Financial Econometrics*, ed. by Y. Ait-Sahalia and L. Hansen, Elsevier Science B.V., North Holland, Amsterdam, 617–690.
- LIU, Z., D. WAGGONER, AND T. ZHA (2011): “Sources of the Great Moderation: A Regime-Switching DSGE Approach,” *Quantitative Economics*, 2, 251–301.
- LUBIK, T. A. AND F. SCHORFHEIDE (2004): “Testing for indeterminacy: an application to US monetary policy,” *The American Economic Review*, 94, 190–217.

- MALKHOZOV, A. AND M. SHAMLOO (2012): “News Shocks, Volatility Shocks, and Asset Prices,” Unpublished working paper.
- PIAZZESI, M., J. SALOMAO, AND M. SCHNEIDER (2013): “Trend and Cycle in Bond Premia,” Mimeo.
- PRIMICERI, G. (2005): “Time Varying Structural Vector Autoregressions and Monetary Policy,” *Review of Economic Studies*, 72, 821–852.
- SANTOS, M. S. (2000): “Accuracy of Numerical Solutions using the Euler Equation Residuals,” *Econometrica*, 1377–1402.
- SBUELZ, A. AND F. TROJANI (2008): “Asset prices with locally constrained-entropy recursive multiple-priors utility,” *Journal of Economic Dynamics and Control*, 32, 3695–3717.
- SCHORFHEIDE, F. (2005): “Learning and Monetary Policy Shifts,” *Review of Economic Dynamics*, 8, 392–419.
- SIMS, C. A. (2002): “Solving Linear Rational Expectations Models,” *Computational Economics*, 20, 1–20.
- SIMS, C. A. AND T. ZHA (2006): “Were There Regime Switches in US Monetary Policy?” *American Economic Review*, 91, 54–81.
- STREBULAEV, I. A. AND T. M. WHITED (2011): “Dynamic models and structural estimation in corporate finance,” *Foundations and Trends in Finance*, 6, (1-2).
- TALLARINI, T. D. (2000): “Risk-sensitive real business cycles,” *Journal of Monetary Economics*, 45, 507–532.

9 Appendix

9.1 Approximations for bond and stock prices

In this appendix we derive the approximate formulas for bond and stock prices used in Section 3.1. As in the text, the change of measure from the true distribution to the worst case belief is described by a stochastic process X_{t+1} adapted to the agent's information set and with $E_t X_{t+1} = 1$ such that $E_t^* Y_{t+1} = E_t Y_{t+1} X_{t+1}$ for any random variable Y_{t+1} .

We use a second order Taylor approximation of marginal utility $C_{t+1}^{-\gamma}$ around expected consumption $E_t C_{t+1}$:

$$\begin{aligned} C_{t+1}^{-\gamma} &= (E_t C_{t+1})^{-\gamma} - \gamma (E_t C_{t+1})^{-\gamma-1} (C_{t+1} - E_t C_{t+1}) + \frac{\gamma(\gamma+1)}{2} (E_t C_{t+1})^{-\gamma-2} (C_{t+1} - E_t C_{t+1})^2 \\ &= (E_t C_{t+1})^{-\gamma} \left(1 - \gamma \frac{C_{t+1} - E_t C_{t+1}}{E_t C_{t+1}} + \frac{\gamma(\gamma+1)}{2} \frac{(C_{t+1} - E_t C_{t+1})^2}{(E_t C_{t+1})^2} \right) \end{aligned} \quad (9.1)$$

Consider the bond price. Substituting for $C_{t+1}^{-\gamma}$ from (9.1), we obtain:

$$Q_t^b = \beta E_t^* \left[\frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \right] \approx \beta C_t^\gamma E_t \left[(E_t C_{t+1})^{-\gamma} \left(1 - \gamma \frac{C_{t+1} - E_t C_{t+1}}{E_t C_{t+1}} + \frac{\gamma(\gamma+1)}{2} \frac{(C_{t+1} - E_t C_{t+1})^2}{(E_t C_{t+1})^2} \right) X_{t+1} \right]$$

Omitting terms that contain third or higher moments, we arrive at equation (3.2) in the text, where we use the case $\gamma = 1$:

$$Q_t^b \approx \beta \frac{C_t^\gamma}{(E_t C_{t+1})^\gamma} \left[1 + \frac{\gamma(\gamma+1)}{2} \frac{\text{var}_t(C_{t+1})}{(E_t C_{t+1})^2} + \gamma \frac{E_t C_{t+1} - E_t^* C_{t+1}}{E_t C_{t+1}} \right]$$

Consider now the price of a generic payoff Π_{t+1} :

$$Q_t^\Pi = \beta E_t \left[\frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \Pi_{t+1} X_{t+1} \right]$$

which can be written as

$$Q_t^\Pi = Q_t^b E_t \Pi_{t+1} + \beta E_t \left[\frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} (\Pi_{t+1} - E_t \Pi_{t+1}) X_{t+1} \right]$$

Substituting for C_t from the bond price equation, we obtain:

$$Q_t^\Pi = Q_t^b \left(E_t \Pi_{t+1} + \frac{E_t [C_{t+1}^{-\gamma} (\Pi_{t+1} - E_t \Pi_{t+1}) X_{t+1}]}{E_t [C_{t+1}^{-\gamma} X_{t+1}]} \right)$$

Using $C_{t+1}^{-\gamma}$ from (9.1) and after omitting terms containing third or higher moments, the price Q_t^{Π} becomes

$$Q_t^{\Pi} = Q_t^b E_t \Pi_{t+1} \left(1 + \frac{E_t^* \Pi_{t+1} - E_t \Pi_{t+1} - (E_t C_{t+1})^{-1} \gamma \text{cov}_t(C_{t+1}, \Pi_{t+1})}{E_t \Pi_{t+1} \left(1 - \frac{E_t^* C_{t+1} - E_t C_{t+1}}{E_t C_{t+1}} \right)} \right) \quad (9.2)$$

In applications such as ours, the expected excess return $Q_t^b E_t \Pi_{t+1} / Q_t^{\Pi} - 1$ is on the order of a few percentage points. As a result the big fraction in the bracket on the right hand side is also of that order of magnitude. At the same time, the fraction in the denominator on the right hand side – the percentage difference between the true and worst case one step ahead expected consumption levels – is also on the order of a few percentage points. We thus drop the fraction in the denominator and rearrange to obtain the approximate formula (3.3) in the text, where we use $\gamma = 1$:

$$Q_t^{\Pi} = Q_t^b E_t \Pi_{t+1} \left(1 - \frac{\gamma \text{cov}_t(C_{t+1}, \Pi_{t+1})}{E_t \Pi_{t+1} E_t C_{t+1}} - \frac{E_t \Pi_{t+1} - E_t^* \Pi_{t+1}}{E_t \Pi_{t+1}} \right)$$

9.2 Markov-switching VAR representation of the equilibrium

9.2.1 Regime processes

We can write the equilibrium representation of our model as a Markov-switching VAR (MS-VAR) in the DSGE state vector containing all the variables of the model. The interval of one-step ahead conditional means given by $[-a_t^i, a_t^i]$ for each shock i has a component that is affected by the uncertainty chains ω_t , as shown by formulas (4.2) and (4.5). The chains are stationary and ergodic and their dynamics are the same under the true and worst case dynamics.

The worst case steady state depends on the long run averages of intangible information and standard deviation, given by $\bar{\eta}^i$ and $\bar{\sigma}^i$. As intangible information η_t^i or the standard deviation σ_t^i fluctuate around their respective long run means, there are "shocks" to ambiguity a_t^i and therefore shifts in the constants of the MS-VAR representation.

Formally, for each shock $i = z, f$, we define the vector of linear deviations of the product $\eta_t^i E[\tilde{\sigma}_i(v_{t+1}) | v_t]$ from its ergodic values of $\bar{\eta}^i \bar{\sigma}_i$. We then introduce a vector of dummy variables to load the appropriate volatility-ambiguity combination. The vector follows a four-state Markov chain, which is obtained by mixing the two independent chains η_t^i and v_t . Following Hamilton (1994), we can write the VAR representation of this composite Markov chain as

$$\begin{bmatrix} e_{1,1,t}^{\eta^i \sigma} \\ e_{1,2,t}^{\eta^i \sigma} \\ e_{2,1,t}^{\eta^i \sigma} \\ e_{2,2,t}^{\eta^i \sigma} \end{bmatrix} = H^{\eta^i \sigma} \begin{bmatrix} e_{1,1,t-1}^{\eta^i \sigma} \\ e_{1,2,t-1}^{\eta^i \sigma} \\ e_{2,1,t-1}^{\eta^i \sigma} \\ e_{2,2,t-1}^{\eta^i \sigma} \end{bmatrix} + \begin{bmatrix} v_{1,t}^{\eta^i \sigma} \\ v_{2,t}^{\eta^i \sigma} \\ v_{3,t}^{\eta^i \sigma} \\ v_{4,t}^{\eta^i \sigma} \end{bmatrix} \quad (9.3)$$

where $e_{m,n,t}^{\eta^i\sigma} = 1_{\eta_t^i=m, v_t=n}$ is an indicator operator if at time t the intangible ambiguity regime m and the volatility regime n are in place, where $m, n \in \{1, 2\}$. The realizations of the shock $v_t^{\eta^i\sigma}$ are such that $E_{t-1} \left[v_t^{\eta^i\sigma} \right] = 0$. The transition matrix is $H^{\eta^i\sigma} = H^{\eta^i} \otimes H^\sigma$.

For example, consider the case when the intangible information about the operating cost is in regime 1, normalized to be the higher value η_H^f of intangible ambiguity, and the volatility regime 1 is in place, corresponding to a value σ_1^f for the standard deviation of the operating cost shock. Then the indicator $e_{1,1,t}^{\eta^f\sigma} = 1$ and the rest of the three $e_{m,m,t}^{\eta^f\sigma} = 0$. This means that our system of equations will load the amount of ambiguity about the operating cost to be equal to $a^f(1, 1) = \eta_H^f \left[\sigma_1^f H_{11}^\sigma + \sigma_2^f (1 - H_{11}^\sigma) \right]$, where H_{11}^σ is the persistence parameter giving the probability of the volatility chain to remain in regime 1. When the economy is in this regime combination, the equations get zero weight on the other three ambiguity realizations $a^f(m, n)$. In this case, the realization of the $v_t^{\eta^i\sigma}$ shock such that $e_{1,1,t}^{\eta^i\sigma} = 1$ makes the linear deviation $a^f(1, 1) - \bar{\eta}^f \bar{\sigma}^f$ hit the economy as a discrete shock. Generally, by augmenting the DSGE state vector with the vectors $e_t^{\eta^i\sigma}$ we control for the first order effects of the shifts in intangible ambiguity and volatility.

Given these first-order shifts, we proceed to linearize the rest of the equilibrium conditions of the model. We use an observational equivalence result according to which our economy behaves *as if* the agent maximizes expected utility under the worst-case belief. Given this equivalence, we use standard perturbation techniques that are a good approximation of the nonlinear decision rules under expected utility. This allows us to solve the model using standard solution algorithms, such as *gensys* by Sims (2002). The model solution assumes the form of a Markov-switching VAR that allows for changes in the volatility and in the constants:

$$\widehat{w}_t = C(\omega_t) + T\widehat{w}_{t-1} + R\tilde{\sigma}(v_t)\varepsilon_t \quad (9.4)$$

where \widehat{w}_t is a vector containing the DSGE states of the model. Notice that changes in uncertainty have first order effects captured by changes in the constants of the MS-VAR. Not all variables of the vector \widehat{w}_t are observable. We then combine the solution with a set of observation equations, obtaining a model in state space form that can be estimated with likelihood based techniques.

9.2.2 Solution method

Here we describe further details on the representation and approach to solving the model with regime switching ambiguity and volatility. The objective is to obtain the equilibrium matrices C, T and R in the law of motion of equation (9.4). The steps of the solution are the following:

1. Guess and verify the worst-case scenario. As discussed in detail in Ilut and Schneider (2014), the solution to the equilibrium dynamics of the model can be found through a guess-and-verify approach. To solve for the worst-case belief that minimizes expected continuation

utility over the i sets in (2.9), we propose the following procedure:

- (a) guess the worst case belief (u^z, u^f)
- (b) solve the model assuming that agents have expected utility and beliefs (u^z, u^f) .
- (c) compute the agent's value function V
- (d) verify that the guess (u^z, u^f) indeed achieves the minima.

The following steps detail the point 1.b) above. Here we use an observational equivalence result that makes use of the fact that our economy can be solved *as if* the agent maximizes expected utility under the belief (u^z, u^f) . Given this equivalence, we can use standard perturbation techniques that are a good approximation of the nonlinear decision rules under expected utility. In particular, we will use linearization. When we refer to the guess below, we use $(u^z, u^f) = (-a^z, a^f)$.

2. Compute worst-case steady states

- (a) Compute the ergodic values $\bar{\eta}^i$ and $\bar{\sigma}^i$ for the two shocks $i = z, f$.
- (b) Based on the guess above compute the worst-case steady states for the shocks Z^* and f^* . Denote by $s^* = (Z^*, f^*, \bar{\omega})$ the worst case mean of the exogenous state where, since there is no ambiguity about the finite chain ω_t , $\bar{\omega}$ equals its true mean.
- (c) Compute the worst-case steady state w^* of the endogenous variables. For this, use the FOCs of the economy based on their deterministic version in which the one step ahead expectations are computed under the guessed worst-case belief.

3. Dynamics:

- (a) Linearize around (w^*, s^*) by finding the coefficient matrices from linearizing the FOCs. To accommodate the presence of discrete shocks, we introduce a series of dummy variables corresponding to the different regime combinations. These dummy variables provide the VAR representation of the MS chains as in equation (9.3) and will multiply the corresponding deviation of $\eta_t^i E[\tilde{\sigma}_i(v_{t+1})|v_t]$ from its own steady state $\bar{\eta}^i \bar{\sigma}_i$. The linearized FOCs can be written in the canonical form used for solving rational expectations models:

$$\Gamma_0 \widehat{w}_{r,t} = \Gamma_1 \widehat{w}_{r,t-1} + \Psi \left[\varepsilon_t', v_t^{\eta^f \sigma'}, v_t^{\eta^z \sigma'} \right]' + \Pi \psi_t$$

where $\widehat{w}_{r,t}$ includes the DSGE state vector w_t as well as the dummy variables controlling the regime in place. This vector represents deviations around the worst-case steady state (w^*, s^*) .

- (b) Given that the shock v_t is defined such that $E_{t-1} \left[v_t^{\eta^i \sigma} \right] = 0$, a standard solution method to solve rational expectations general equilibrium models can be employed. The solution can then be rewritten as a VAR with stochastic volatility

$$\widehat{w}_{r,t} = T^* \widehat{w}_{r,t-1} + R\tilde{\sigma}(v_t) \left[\varepsilon'_t, v_t^{\eta^f \sigma'}, v_t^{\eta^z \sigma'} \right]' \quad (9.5)$$

- (c) Verify that the guess (u^z, u^f) indeed achieves the minima of the time t expected continuation utility over the sets in (2.9).

4. Equilibrium dynamics under the true data generating process (DGP). The above equilibrium was derived under the worst-case beliefs. We need to characterize the economy under the econometrician's law of motion. There are two objects of interest: the steady state of our economy and the dynamics around that steady state.

- (a) The steady state, denoted by \bar{w} . This is characterized by shocks, including the volatility regimes, being set to their ergodic values under the true DGP. The steady state \bar{w} is found by taking the linearized solution, adding $R_z \bar{\eta}^z \bar{\sigma}^z$ and subtracting $R_f \bar{\eta}^f \bar{\sigma}^f$:

$$\bar{w} - w^* = T^* (\bar{w} - w^*) + R_z \bar{\eta}^z \bar{\sigma}^z - R_f \bar{\eta}^f \bar{\sigma}^f \quad (9.6)$$

where R_z and R_f are the equilibrium response to positive innovations to \widehat{z}_t and \widehat{f}_t , respectively.

- (b) Dynamics. The law of motion in (9.5) needs to take into account that expectations are under the worst-case beliefs which differ from the true DGP. Then, we define $\widehat{\bar{w}}_{r,t} \equiv w_{r,t} - \bar{w}$ and use (9.5) together with (9.6) to obtain:

$$\widehat{\bar{w}}_{r,t} = T^* \widehat{\bar{w}}_{r,t-1} + R\tilde{\sigma}(v_t) \left[\varepsilon'_t, v_t^{\eta^f \sigma'}, v_t^{\eta^z \sigma'} \right]' + R_z (a_{t-1}^z - \bar{\eta}^z \bar{\sigma}^z) - R_f (a_{t-1}^f - \bar{\eta}^f \bar{\sigma}^f) \quad (9.7)$$

- (c) Define the matrix T accordingly to represent the law of motion in (9.7) as

$$\widehat{\bar{w}}_{r,t} = T \widehat{\bar{w}}_{r,t-1} + R\tilde{\sigma}(v_t) \left[\varepsilon'_t, v_t^{\eta^f \sigma'}, v_t^{\eta^z \sigma'} \right]' \quad (9.8)$$

Intuitively, under the econometrician's DGP the economy responds to the exogenous state variables controlling uncertainty differently since the implied worst-case expectations are not materialized in the current shock processes.

Finally, we can partition out the state vector $\widehat{\bar{w}}_{r,t}$ in a way that the Markov-switching (MS) regimes show up as time-varying constants in the law of motion described by

(9.8). In particular, this allows us to rewrite (9.8) as equation (9.4):

$$\widehat{w}_t = C(\omega_t) + T\widehat{w}_{t-1} + R\tilde{\sigma}(v_t)\varepsilon_t$$

where \widehat{w}_t contains the same state variables of $\widehat{w}_{r,t}$, except the regime dummies $e_{1,1,t}^{\eta^i\sigma}$, $e_{1,2,t}^{\eta^i\sigma}$, $e_{2,1,t}^{\eta^i\sigma}$ and $e_{2,2,t}^{\eta^i\sigma}$ for $i = z, f$, which have been replaced with the MS constant $C(\omega_t)$. To avoid further notation, the matrices T and R in (9.4) are the appropriate partitions of the corresponding objects in (9.8) so that they only capture the effect of \widehat{w}_t . Thus, the DSGE model is represented as a Markov-switching VAR (MS-VAR), where the changes in the constant arise from the first order effects of the composite regimes of stochastic volatility and intangible information that jointly affect ambiguity.

9.3 Equilibrium conditions for the estimated model

Here we describe the equations that characterize the equilibrium of the estimated model in Section 4. To solve the model, we first scale the variables in order to induce stationarity. The variables are scaled as follows:

$$c_t = \frac{C_t}{\xi^t}, y_t = \frac{Y_t}{\xi^t}, k_t = \frac{K_t}{\xi^t}, i_t = \frac{I_t}{\xi^t}$$

Financial variables:

$$p_t = \frac{P_t}{\xi^t}, d_t = \frac{D_t}{\xi^t}, b_t^i = \frac{B_t^i}{\xi^t}; i = f, h$$

The borrowing costs:

$$\frac{\kappa \left(B_{t-1}^f \right)}{\xi^t} = \frac{\Psi}{2} \frac{1}{\xi^2} \left(b_{t-1}^f \right)^2; \quad \frac{\phi(D_t, D_{t-1})}{\xi^t} = \frac{\Phi \xi^2}{2} \left(\frac{d_t}{d_{t-1}} - 1 \right)^2$$

We now present the nonlinear equilibrium conditions characterizing the model, in scaled form. The expectation operator in these equations, denoted by E_t^* , is the one-step ahead conditional expectation under the worst case belief μ^0 . According to our model, the worst case is that future z_{t+1} is low, and that the financing cost f_{t+1} is high.

The firm problem is

$$\max E_0^* \sum M_{0,t}^f D_t$$

subject to the budget constraint

$$\begin{aligned} d_t = (1 - \tau_k) \left[\alpha y_t - \frac{b_{t-1}^f}{\xi} (1 - Q_{t-1}^b) - i_t \right] - f_t - \frac{\Phi \xi^2}{2} \left(\frac{d_t}{d_{t-1}} - 1 \right)^2 \\ - \frac{\Psi}{2} \frac{1}{\xi^2} \left(b_{t-1}^f \right)^2 + \delta \tau_k q_{t-1}^k \frac{k_{t-1}}{\xi} - \frac{b_{t-1}^f}{\xi} Q_{t-1}^b + b_t^f Q_t^b \end{aligned} \quad (9.9)$$

and the capital accumulation equation

$$k_t = \frac{(1-\delta)k_{t-1}}{\xi} + \left[1 - \frac{\Theta}{2}\xi^2 \left(\frac{i_t}{i_{t-1}} - 1 \right)^2 \right] i_t \quad (9.10)$$

Let the LM on the budget constraint be $\lambda_t M_{0,t}^f \epsilon_t$ and on the capital accumulation be $\lambda_t^k M_{0,t}^k \epsilon_t$. Then the scaled pricing kernel is

$$m_{t+1}^f \equiv M_{t+1} \frac{\xi^{t+1}}{\xi^t} = \beta \frac{c_t}{c_{t+1}} \frac{1 - \tau_l}{1 - \tau_l \beta E_t^* [c_t / (c_{t+1} \xi)]}. \quad (9.11)$$

The FOCs associated with the firm problem are then:

1. Dividends:

$$1 = \lambda_t \left[1 + \Phi \xi^2 \frac{1}{d_{t-1}} \left(\frac{d_t}{d_{t-1}} - 1 \right) \right] - E_t m_{t+1}^f \lambda_{t+1} \Phi \xi^2 \frac{d_{t+1}}{d_t^2} \left(\frac{d_{t+1}}{d_t} - 1 \right) \quad (9.12)$$

2. Bonds:

$$Q_t^b \lambda_t = E_t^* m_{t+1}^f \lambda_{t+1} \frac{1}{\xi} \left[1 - \tau_k (1 - Q_t^b) + \frac{\Psi}{\xi} b_t^f \right] \quad (9.13)$$

3. Investment:

$$1 - \tau_k = Q_t^k \left[1 - \frac{\Theta}{2}\xi^2 \left(\frac{i_t}{i_{t-1}} - 1 \right)^2 - \Theta \xi^2 \left(\frac{i_t}{i_{t-1}} - 1 \right) \frac{i_t}{i_{t-1}} \right] + E_t^* m_{t+1}^f \frac{\lambda_{t+1}}{\lambda_t} Q_{t+1}^k \Theta \xi^2 \frac{i_{t+1}^2}{i_t^2} \left(\frac{i_{t+1}}{i_t} - 1 \right) \quad (9.14)$$

where $Q_t^k \equiv \frac{\lambda_t^k}{\lambda_t}$.

4. Capital:

$$1 = E_t^* m_{t+1}^f \frac{\lambda_{t+1}}{\xi \lambda_t} R_{t+1}^K \quad (9.15)$$

$$R_{t+1}^k \equiv \frac{(1 - \tau_k) \alpha \left(\frac{k_t}{\xi} \right)^{\alpha-1} L^{1-\alpha} + (1 - \delta) Q_{t+1}^k}{Q_t^k} + \delta \tau_k$$

The household problem is as follows:

$$\begin{aligned} & \max E_0^* \sum \beta^t \log c_t \\ (1 + \tau_c) c_t + p_t \theta_t &= (1 - \tau_l) \left[(1 - \alpha) y_t + \pi + d_t \theta_{t-1} + \frac{b_{t-1}^h}{\xi} (1 - Q_{t-1}^b) \right] + \\ & + p_t \theta_{t-1} - \tau_l (p_t - \frac{1}{\xi} p_{t-1}) \theta_{t-1} + \frac{b_{t-1}^h}{\xi} Q_{t-1}^b - b_t^h Q_t^b + t_r \end{aligned} \quad (9.16)$$

Thus, the FOCs associated to the household problem are:

1. Bond demand:

$$Q_t^b = \beta E_t^* \frac{c_t}{\xi c_{t+1}} [1 - \tau_l (1 - Q_t^b)] \quad (9.17)$$

2. Equity holding:

$$p_t = \beta E_t^* \frac{c_t}{c_{t+1}} \left[(1 - \tau_l) (p_{t+1} + d_{t+1}) + \frac{\tau_l}{\xi} p_t \right] \quad (9.18)$$

The market clearing conditions characterizing this economy are:

$$b_t^h = b_t^f \quad (9.19)$$

$$c_t + i_t + \frac{\Phi \xi^2}{2} \left(\frac{d_t}{d_{t-1}} - 1 \right)^2 + f_t + \frac{\Psi}{2} \frac{1}{\xi^2} \left(b_{t-1}^f \right)^2 = y_t + \pi \quad (9.20)$$

$$\theta_t = 1$$

corresponding to the market for bonds, goods and equity shares, respectively.

Thus, we have the following 11 unknowns:

$$k_t, i_t, b_t^f, b_t^h, Q_t^b, p_t, c_t, d_t, Q_t^k, \lambda_t, m_t^f$$

The equations (9.10), (9.11), (9.12), (9.13), (9.14), (9.15), (9.17), (9.18), (9.19), (9.20) give us 10 equations. By Walras' law, we can use one of the two budget constraints in (9.9) and (9.16) (using $\theta_t = 1$). This gives us a total of 11 equations.

9.4 Rescaling parameters

We estimate a subset of the parameters, with values reported in Tables 4 and 5. The other parameters are fixed, as reported in Table 3, to match some key ratios from the NIPA accounts.

Table 3: Calibrated parameters

τ_l	τ_k	τ_c	π_y	t_y
0.189	0.193	0.09	0.3	0.21

For the steady state calculation of the model it is helpful to rescale some parameters. Specifically, denote by y_{GDP}^* the worst-case steady state measured GDP, where measured GDP equals total goods $y + \pi$ minus financing costs. Then, define the following ratios:

$$\pi_y = \frac{\pi}{y_{GDP}^*}, t_y = \frac{t_r}{y_{GDP}^*}; \Psi_y = \frac{\Psi}{y_{GDP}^*} \quad (9.21)$$

$$f_y = \frac{\bar{f}}{y_{GDP}^*}; 1 + \varsigma^f = \frac{f^*}{\bar{f}}; \eta_r^f = \frac{\eta_L^f}{\bar{\eta}^f} \quad (9.22)$$

Let us first discuss the normalizations in (9.21). Total measured GDP in our model corresponds to the non-financial corporate sector (NFB) output plus goods produced by the other productive sectors- financial, non-corporate and household. We associate the firm in our model with the NFB sector and thus π_y is set to match goods produced by other productive sectors divided by measured GDP. The ratio t_y equals government transfers (including social security and medicare) plus after-tax government wages divided by measured GDP. Finally the parameter Ψ_y is estimated and after computing the level of the economy it implies a value of Ψ .

The tax parameters are computed as follows: τ_l equals total personal taxes and social security contributions divided by total income, where the latter is defined as total wages plus dividends. τ_k equals NFB taxes divided by NFB profits and τ_c equals NFB sales taxes divided by NFB output.

The parameters in (9.22) are estimated and control the size of the operating cost. First, f_y determines \bar{f} , the value of the steady state cost under the true DGP. The parameter ζ^f controls by how much higher is f^* , the worst-case steady state operating cost, compared to \bar{f} . This normalization is helpful because it allows us to use meaningful priors on this ratio in terms of the type and the size of prior mean. Since

$$f^* = \bar{f} + \frac{\bar{\eta}^f \bar{\sigma}^f}{(1 - \rho_f)}$$

then based on ζ^f we compute:

$$\bar{\eta}^f = (1 - \rho_f) \zeta_f \bar{f} / \bar{\sigma}^f$$

where \bar{f} is obtained after solving for the level of the economy. Finally, the parameter η_r^f , which has a beta prior, determines the value of η^f in the regime of low ambiguity about the operating cost, as compared to its ergodic value of $\bar{\eta}^f$:

$$\eta_L^f = \eta_r^f \bar{\eta}^f$$

Based on the estimated transition matrix for the ambiguity regimes, the value of η^f in the regime of high ambiguity about the operating cost is easily computed as:

$$\eta_H^f = \frac{\bar{\eta}^f - p_L^{\eta,f} \eta_L^f}{1 - p_L^{\eta,f}}$$

where $p_L^{\eta,f}$ is the ergodic probability of the regime of low ambiguity about the operating cost computed using the estimated transition matrix $H^{\eta,f}$.

Finally, the effect of a unit increase in ambiguity about the marginal product of capital on the realized Z_t is given by the parameter \varkappa , which we normalize as

$$\varkappa = \zeta \frac{3}{\eta_L^z \sqrt{1 - \rho_a^2}}.$$

We use a Beta prior on the estimated parameter ζ , which allows for a simple interpretation of the size of the typical Gaussian innovation to ambiguity. Indeed, this parametrization implies that even in the low intangible information regime, a negative Gaussian innovation equal to 3ζ times its unconditional standard deviation, given by $\left(\varkappa\sqrt{1-\rho_a^2}\right)^{-1}$, still keeps the a_t^z value non-negative.

9.5 Accuracy of approximation

We evaluate the accuracy of the approximation using the approach in Judd (1992), which is to characterize the normalized Euler Equation Errors (EEE). To describe the approach, it is useful for exposition to consider the household's optimality condition for bonds, given by equation (9.17) as

$$\frac{Q^b(w_{t-1}, \epsilon_t)}{c(w_{t-1}, \epsilon_t) [1 - \tau_l (1 - Q^b(w_{t-1}, \epsilon_t))]} = \beta E_t^* \frac{1}{\xi c(w_t, \epsilon_{t+1})}, \quad (9.23)$$

where c and Q^b represent the household's consumption and bond price functions, respectively. Date t model variables are determined in equilibrium as policy functions of the state variables w_{t-1} and innovations ϵ_t . Date $t + 1$ variables then respond to w_t and ϵ_{t+1} . The E_t^* operator indicates that the conditional expectation is formed under the equilibrium worst-case belief.

Since the solution method used is only an approximation, equation (9.23) will not hold exactly when evaluated using the computed decision rule \hat{c} and price function \hat{Q}^b , say. We define the Euler equation error function as

$$r(w_{t-1}, \epsilon_t) = 1 - \frac{\beta \hat{c}(w_{t-1}, \epsilon_t) \left[1 - \tau_l \left(1 - \hat{Q}^b(w_{t-1}, \epsilon_t) \right) \right]}{\hat{Q}^b(w_{t-1}, \epsilon_t)} E_t^* \frac{1}{\hat{c}(w_t, \epsilon_{t+1})} \quad (9.24)$$

We can evaluate the error function at any any point (w_{t-1}, ϵ_t) . Such evaluation involves integration under the worst case density. To evaluate its magnitude, Judd and Guu (1997) interpret this error as the relative optimization error incurred by the use of the approximated policy rule. For example if $r(w_{t-1}, \epsilon_t) = 10^{-3}$, then the agent is making a \$1 mistake for each \$1000 spent.

As discussed in detail in the solution methods literature (see Aruoba et al. (2006)), the EEE is also important because, under certain conditions, the approximation error of the policy function is of the same order of magnitude as the size of the EEE. Correspondingly, the change in welfare is of the square order of the EEE (Santos (2000)). Compared to most of the literature, which typically deals with Gaussian shocks, our disturbances also have discrete jumps, in the form of Markov-switching (MS) regimes. A recent exception is Foerster et al. (2015), where they also compute EEE for simple MS regime models.

It is first useful to recall that the solution method described in section 9.2.2 produces the linear law of motion under the worst-case beliefs described by equation (9.5). To simulate the economy under the true DGP we have shown how we obtain the law of motion in (9.8). The objective of the accuracy exercise is to evaluate the EEE in (9.24) for the economy that the econometrician

analyzes and takes to the data. There are two basic properties of this economy: (1) the realized shocks are drawn under the econometrician's DGP, and (2) for each moment in that sequence of shocks, agents inside the model form beliefs according to the worst-case conditional probability distribution.

To compute the error in (9.24), we perform the following specific steps:

1. We simulate a path for MS regimes by drawing the innovations $v_t^{\eta^i \sigma}$. Besides the intangible information component, the other part of a regime is the volatility of the Gaussian innovations. Conditional on this path, we simulate a path for the Gaussian shocks ε_t . Notice that the path for regimes matters at this step because it affects the volatility in place at time t . The two types of innovations constitute the ϵ_t innovation in (9.24).
2. For a given path of innovations, we use the law of motion in (9.8) to compute the dynamics of the economy. This generates a simulated path for the state vector w_{t-1} . In particular, we also obtain $\hat{c}(w_{t-1}, \epsilon_t)$ and $\hat{Q}^b(w_{t-1}, \epsilon_t)$ in (9.24) by using the law of motion in (9.8). Notice that these policy functions are based on: (i) the state of the economy given by the path of innovations under the econometrician's DGP, and (ii) worst-case beliefs about the conditional probability distribution of future states.
3. At this stage we are interested in evaluating the conditional distribution of the object $\hat{c}(w_t, \epsilon_{t+1})$ in (9.24). An efficient method to integrate over next period's state is to use optimally selected Gaussian-Hermite (GH) quadrature points and weights. In selecting these points we take into account the possible regimes for volatility for next period, given the volatility state today.
4. We evaluate next period's policy functions at the optimally selected GH points and given the current state w_t that was obtained in step 2. To construct the future values of variables, we use the law of motion in (9.5), which is the behavior of the economy that the agent expects to see under the worst-case belief. In particular, we evaluate the function $\hat{c}(w_t, \epsilon_{t+1})$ under this law of motion.
5. Conditional on a particular volatility regime next period, we integrate over the resulting values of $\hat{c}(w_t, \epsilon_{t+1})$ evaluated at the different GH points in step 4. Then we integrate over the next period's possible volatility regimes by using the probabilities to transition out of the current volatility regime. This double integral gives us the conditional average value $E_t^* \hat{c}(w_t, \epsilon_{t+1})$.
6. At each point t in the simulated path we have now the necessary ingredients to compute the Euler error in (9.24). We obtained $\hat{c}(w_{t-1}, \epsilon_t)$ and $\hat{Q}^b(w_{t-1}, \epsilon_t)$ from step 2 and the conditional expectation $E_t^* \hat{c}(w_t, \epsilon_{t+1})$ from step 5. We simulate the economy for a large number of times

and drop a large initial fraction of the samples to insure the error is computed for the ergodic distribution.

So far we have discussed only one Euler equation. Our model has six Euler equations. These equations are the firm’s optimality for dividends, bonds, investment and capital, given by equations (9.12), (9.13), (9.14) and (9.15), as well as the household’s optimality for bonds and equity, given by equations (9.17) and (9.18). For each of these equations, we evaluate the unit-free EEE following the general logic of the six steps outlined above. In particular, we simulate our economy evaluated at the posterior mode, for 100 samples, each of 10,000 periods with the first 1,000 being dropped. We report the absolute EEE in base 10 logarithms to ease interpretation. The average absolute errors are, in the order of the six equations stated above, as follows: $(-3.2, -2.1, -2, -2.4, -2.1, -2.7)$.

Foerster et al. (2015) compare accuracy for different solution methods of MS models. There they analyze a standard RBC model with one endogenous state variable and MS productivity shocks for which they report the same order of magnitude of the EEE as we find in our model. We interpret this as evidence that the numerical accuracy of our approximation is reasonable, especially in the context of a larger model like ours, with six endogenous state variables.

9.6 Priors and posteriors for model parameters

In DSGE models, there is often a set of free parameters that control the steady state values of key variables. For example, in papers that are focused on monetary policy the steady-state output growth, inflation rate, and real interest rate, are controlled by three different parameters (for example Lubik and Schorfheide (2004)). This property makes it easy to specify priors such that the unconditional means of the variables are centered around values that are *a-priori* economically sensible.

Instead, in our model, the steady state is a function of several model parameters because it depends on the actual solution of the model. Therefore, the priors for the estimated parameters are chosen as follows. We first specify loose independent priors for the different parameters of the model. These priors are described in Tables 4 and 5. We then specify a prior on the steady state of the observables, inducing a joint prior on the model parameters. This second set of priors is described in Table 6. Our approach is in line with the methods developed by Del Negro and Schorfheide (2008) and allows us to take into account that the steady state depends on the actual solution of the model.

Here we describe in detail how the joint prior is implemented. For each draw of the parameters θ , the posterior $p(\theta|Y^T)$ is given by the product of the likelihood $\ell(\theta|Y^T)$ and the prior $p(\theta)$ (up to a constant that is common across all parameter values):

$$p(\theta|Y^T) = \ell(\theta|Y^T) p(\theta)$$

In turn, our approach implies that the prior on the parameter vector $p(\theta)$ is the product of two components. The first component depends on the independent priors for the different parameters of the model. The second component depends on the steady state. This can in turn be expressed as a function $g(\cdot)$ of the parameter vector θ . Therefore, we have:

$$p(\theta) = \underbrace{\left[\prod_{i=1}^n p_i(\theta_i)\right]}_{\text{Independent priors}} * \underbrace{p_g(g(\theta))}_{\text{Joint prior}}$$

Combining the two expressions, we get $p(\theta|Y^T) = \ell(\theta|Y^T) \left[\prod_{i=1}^n p_i(\theta_i)\right] g(\theta)$. Intuitively, our approach restricts the parameter space by giving more weight to parameter combinations that imply economically sensible values for the steady state. Computationally, the extra cost is negligible and consists of evaluating the density function $p_g(g(\theta))$. The steady state $g(\theta)$ is always computed when solving the model, therefore it does not represent an additional computational cost.

Table 4: Priors and posteriors for structural parameters

Description	Param	Posterior				Prior		
		<i>Mode</i>	<i>Mean</i>	5%	95%	<i>Type</i>	<i>Mean</i>	<i>StDev.</i>
<i>Standard</i>								
Growth rate	$100(\xi - 1)$	0.4543	0.4562	0.4281	0.4926	<i>G</i>	0.5	0.05
Discount factor	$100(\beta^{-1} - 1)$	0.6875	0.6762	0.6480	0.7098	<i>G</i>	0.3	0.2
Capital depreciation	δ	0.0067	0.0063	0.0051	0.0075	<i>B</i>	0.01	0.005
Capital share	α	0.2451	0.2451	0.2369	0.2545	<i>B</i>	0.3	0.1
<i>Frictions</i>								
Dividend adjustment	100Φ	0.0199	0.0194	0.0159	0.0232	<i>G</i>	0.4	0.3
Investment adjustment	Θ	0.9273	0.9067	0.8504	0.9657	<i>G</i>	1	0.5
Cost of borrowing	$100\Psi_y$	0.2109	0.2034	0.1937	0.2144	<i>G</i>	0.5	0.4

Note: *Type* refers to the prior parameter distribution, where *B* refers to Beta, *G* to Gamma, *IG* to Inverse-gamma and *D* to the Dirichlet distribution. Posterior percentiles are obtained from nine chains of 5,000,000 draws generated using a Random walk Metropolis algorithm. We retain one out of every 1,000 draws. Appendix 9.4 discusses the normalization implied by the parameter Ψ_y .

Table 5: Priors and posteriors for shock parameters

Description	Param	Posterior				Prior		
		<i>Mode</i>	<i>Mean</i>	5%	95%	<i>Type</i>	<i>Mean</i>	<i>St.dev.</i>
<i>Intangible Z ambig. regimes</i>								
High ambiguity	η_H^z	0.9449	0.9423	0.9057	0.9698	<i>B</i>	0.5	0.25
Low ambiguity	η_L^z	0.8021	0.7987	0.7699	0.8199	<i>B</i>	0.5	0.25
Persistence High	$H_{11}^{\eta,z}$	0.6019	0.5971	0.5867	0.6084	<i>D</i>	0.8333	0.1034
Persistence Low	$H_{22}^{\eta,z}$	0.6879	0.6916	0.6694	0.7179	<i>D</i>	0.8333	0.1034
Negative effect on <i>Z</i>	ζ	0.6996	0.6859	0.6514	0.7094	<i>B</i>	0.5	0.25
<i>Intangible F ambig. regimes</i>								
Low/high ambiguity	η_r^f	0.3379	0.3430	0.3158	0.3773	<i>B</i>	0.5	0.2
Worst-case/True <i>F</i>	ς^f	2.9070	2.9069	2.8575	2.9471	<i>G</i>	3	1
True <i>F</i> /Worst-case GDP	$100f_y$	1.2861	1.2720	1.2300	1.3263	<i>G</i>	2	1.5
Persistence High	$H_{11}^{\eta,f}$	0.9933	0.9925	0.9881	0.9960	<i>D</i>	0.8333	0.1034
Persistence Low	$H_{22}^{\eta,f}$	0.9901	0.9898	0.9882	0.9913	<i>D</i>	0.8333	0.1034
<i>High volatility regime</i>								
<i>Z</i>	$100\tilde{\sigma}_H^z$	1.4271	1.4343	1.3837	1.4870	<i>IG</i>	0.6	0.3
<i>F</i>	$100\tilde{\sigma}_H^f$	0.7338	0.7404	0.6487	0.8388	<i>IG</i>	1	0.5
Persistence of regime	H_{11}^{σ}	0.9974	0.9976	0.9963	0.9988	<i>D</i>	0.8333	0.1034
<i>Low volatility regime</i>								
<i>Z</i>	$100\tilde{\sigma}_L^z$	1.4535	1.4601	1.4097	1.5156	<i>IG</i>	0.6	0.3
<i>F</i>	$100\tilde{\sigma}_L^f$	0.2999	0.3121	0.2779	0.3516	<i>IG</i>	1.0	0.5
Persistence of regime	H_{22}^{σ}	0.9991	0.9993	0.9988	0.9997	<i>D</i>	0.8333	0.1034
<i>Persistence Gaussian</i>								
<i>Z</i>	ρ_z	0.9762	0.9754	0.9727	0.9779	<i>B</i>	0.7	0.2
<i>F</i>	ρ_f	0.9923	0.9922	0.9914	0.9927	<i>B</i>	0.7	0.2
Intangible <i>Z</i> ambiguity	ρ_a	0.8876	0.8823	0.8714	0.8939	<i>B</i>	0.7	0.2

Note: *Type* refers to the prior parameter distribution, where *B* refers to Beta, *G* to Gamma, *IG* to Inverse-gamma and *D* to the Dirichlet distribution. See note to Table 4 on obtaining the posterior percentiles. Appendix 9.4 discusses the normalization of the estimated ζ into \varkappa and how to map the estimated parameter η_r^f , ς^f and f_y into values for \bar{f} , f^* , η_L^f and η_H^f .

Table 6: Priors and posteriors for steady state of observables

Variable	Posterior				<i>Type</i>	Prior	
	<i>Mode</i>	<i>Mean</i>	5%	95%		<i>Mean</i>	<i>st.dev.</i>
$100\Delta I$	0.4533	0.4551	0.4272	0.4913	<i>N</i>	0.47	0.03
B/P	0.0442	0.0445	0.0407	0.0487	<i>IG</i>	0.1	0.06
D/Y	0.0872	0.0879	0.0847	0.0911	<i>N</i>	0.08	0.02
P/Y	7.4723	7.6752	7.3521	7.9894	<i>N</i>	7	1

Note: *Type* refers to the prior parameter distribution, where *N* refers to Normal and *IG* to Inverse-gamma distribution. The observables are: investment growth (ΔI), firm debt to equity value ratio (B/P), dividend to GDP ratio (D/Y) and equity price to GDP ratio (P/Y). The priors for the latter two variables are expressed here in levels. See note to Table 4 on obtaining the posterior percentiles.