OPTIMAL CONSUMPTION AND SAVINGS WITH STOCHASTIC INCOME AND
RECURSIVE UTILITY

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ABSTRACT

We develop a tractable incomplete-markets model with an earnings process Y subject to permanent shocks and borrowing constraints. Financial frictions cause the marginal (certainty equivalent) value of wealth W to be greater than unity and decrease with liquidity $w = W/Y$. Additionally, financial frictions cause consumption to decrease with this endogenously determined marginal value of liquidity. Risk aversion and the elasticity of inter-temporal substitution play very different roles on consumption and the dispersion of w. Permanent earnings shocks, especially large discrete stochastic jumps, make consumption smoothing quantitatively difficult to achieve. Borrowing constraints and permanent discrete jump shocks can generate empirically plausible values for marginal propensities to consume in the range of 0.2 to 0.6.

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1   Introduction

A fundamental question in economics is how an agent dynamically chooses consumption when markets offer limited opportunities for her to smooth consumption and manage earnings risk. Since Friedman's permanent-income hypothesis and Modigliani's life-cycle hypothesis developed in 1950s, economists have developed a large body of research on income-fluctuation, self-insurance, and optimal savings problems.[1]

We contribute to this literature by (1) generalizing standard buffer-stock savings models (e.g., Deaton, 1991, and Carroll, 1997) along preferences and earnings specifications, (2) deriving a tractable consumption rule via continuous-time dynamic programming, which sharpens the underlying economic mechanism and develops new economic intuition, and (3) generating new quantitative implications and empirical predictions consistent with data.

Model Generalization and New Results.   For preferences, we choose the non-expected homothetic utility developed by Epstein and Zin (1989) and Weil (1990) to separate risk aversion γ from the elasticity of intertemporal substitution (EIS) ψ. Why do we want to separate these two preference parameters? Conceptually speaking, *a priori,* there is no theoretical reason to impose $\psi = 1/\gamma$, as implied in CRRA-utility-based models as these two parameters reflect different economics. Additionally, this generalization is well motivated by ample (though not yet conclusive) evidence that calls for the separation of these two parameters. For example, Bansal and Yaron (2004) and the follow-up long-run-risk finance literature critically rely on high risk aversion and high EIS (greater than one) to match asset-pricing moments.[2] Additionally, empirical estimates of the EIS reflect significant heterogeneity.[3]

Moreover, we show that separating risk aversion from the EIS generates new economic insights. We find that consumption decreases with risk aversion at all levels of wealth. The

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[1]Early work includes Leland (1968), Levhari and Srinivasan (1969), Sandmo (1970), Dreze and Modigliani (1972), and Schechtman (1976), among others. Hall (1978) formalizes Friedman's permanent-income hypothesis via martingale (random-walk) consumption in a dynamic programming framework. Zeldes (1989), Caballero (1990, 1991), Deaton (1991, 1992), and Carroll (1997) study optimal consumption using discrete-time dynamic programming method. For additional related work, see Chamberlain and Wilson (2000), Gourinchas and Parker (2002), Storesletten, Telmer, and Yaron (2004), Aguiar and Hurst (2005), Guvenen (2006, 2007), Fernández-Villaverde and Krueger (2007), and Kaplan and Violante (2010), among others.

[2]Tallarini (2000) is an early application of Epstien-Zin utility to asset pricing with production. Also for related applications of risk-sensitive preferences to asset pricing, see Hansen, Sargent, and Tallarini (1999). In an equilibrium model with Epstein-Zin utility, Lustig and Van Nieuwerburgh (2008) show that the restrictions on the joint distribution of financial wealth returns, human wealth returns, and consumption depends only on the elasticity of intertemporal substitution, not on the coefficient of risk aversion.

[3]For example, see Vissing-Jorgensen (2002), Attanasio, Banks, and Tanner (2002), and Guvenen (2006).

higher risk aversion, the stronger precautionary savings demand and the lower consumption. In contrast, the effect of the EIS on consumption is ambiguous. The higher the EIS, the more flexible consumption responds to changes in the interest rate. When W is low, the agent with a higher EIS consumes less as she is more willing to reduce her current consumption for higher consumption in the future, *ceteris paribus*. When W is high, the higher the EIS the more she is more willing to substitute her consumption over time. Therefore, the effects of the EIS on consumption at the two ends of wealth W are opposite. By separating risk aversion from the EIS, we identify different effects of these two key parameters.

For earnings shocks, we find that permanent shocks make consumption smoothing quantitatively much more difficult. Additionally, discrete large downward earnings shocks matter much more than small continuous diffusive shocks. Guvenen et al. (2015) show that most individuals experience small earnings shocks, but a small and non-negligible experience very large shocks. Large earnings losses are often due to unemployment and job displacement.[4] By incorporating jumps into diffusion models, we well capture these empirical features for the earnings process Y and also generate empirically plausible predictions for consumption. With jumps, annual marginal propensities to consume (MPCs) typically range from 0.2 to 0.6, which are in line with empirical estimates in Parker (1999) and Souleles (1999). In contrast, MPCs implied by standard buffer-stock models typically range from 0.04 to 0.07.

Why do jumps (even when they rarely occur and are of moderate sizes) generate much higher MPCs than diffusion models? This is because jump risk is much harder to manage than continuous diffusive shocks and hence consumption has to adjust discretely in response under incomplete markets causing the MPC to be large especially when liquidity is low. Additionally, jump risks can potentially increase wealth dispersion as jump risks remain significant for the wealthy and hence the wealthy's savings motives remain strong. We can also incorporate transitory shocks into our tractable framework. Next, we discuss how tractability helps us generate new economic insights.

Tractability and Intuition. We measure the agent's welfare via her certainty equivalent wealth $P(W, Y)$, which is the minimal amount of wealth that she requires in order for her to permanently giving up her *status quo* (with wealth W, labor income process under incomplete markets and subject to borrowing constraints) and living with no labor income thereafter.

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[4]For example, Jacobson, LaLonde, and Sullivan (1993) and Low, Meghir, and Pistaferri (2010) show that wages may fall dramatically at job displacement.

The model's homogeneity property implies that the effective state variable is $w = W/Y$, the liquidity ratio between wealth W and income Y. The scaled certainty equivalent wealth $p(w) = P(W,Y)/Y$ solves a nonlinear ordinary differential equation (ODE).

Our model's tractability allows us to conveniently demonstrate the following important implications of financial frictions (incomplete markets and/or borrowing constraints): 1) $p(w)$ is increasing and bounded from the above by the linear CM benchmark $p^*(w)$; 2) the marginal value of liquidity $p'(w)$ decreases with w and approaches unity, the CM benchmark value, as $w \to \infty$; 3) the consumption-income ratio $c(w)$ is increasing and bounded from the above by the classic Ramsey-Friedman's linear consumption rule; 4) the MPC $c'(w)$ decreases with w and approaches the CM MPC, m^*, the CM benchmark value, as $w \to \infty$.

These results are consistent with our intuition, as financial frictions cause the agent to value earnings less and to consume less than under the CM benchmark. Additionally, as liquidity w increases, self insurance becomes more effective hence both $p'(w)$ and $c'(w)$ should intuitively decrease and approach their respective CM benchmarks as $w \to \infty$. Moreover, under CM, risk aversion has no role on consumption as a risk-averse agent optimally chooses to bear no risk if not being compensated by risk premium, risk aversion has very important effects on consumption under incomplete markets and/or borrowing constraints.

The tight connection between analytical results and key features of the model makes economic intuition transparent: (1) the left-end boundary condition $c(0) \leq 1$ maps to the economics of the borrowing constraint; (2) the right-end boundary describes the CM solution as the limit of our self-insurance model's solution; and (3) the deviations of solutions for $p(w)$ and $c(w)$ from their respective CM benchmark levels reflect the impact of financial frictions. Next, we discuss our model's quantitative implications and empirical predictions.

Quantitative Implications and Empirical Predictions. While self insurance against transitory shocks is generally effective, we find that self insurance against permanent shocks is much less effective and a large savings buffer is generally necessary to manage permanent shocks. For example, with standard coefficient of relative risk aversion, e.g., $\gamma = 3$, the steady-state savings target is around six times of her earnings. Additionally, savings are sensitive to risk aversion. For example, by increasing γ from three to six, steady-state savings increases by about three folds from six times of her earnings to almost eighteen times.

Empirically, Parker (1999) and Souleles (1999) document that the MPCs range from 0.2

to 0.6. We find that it is generally difficult to generate MPCs in this range if we only rely on incomplete-markets-induced precautionary savings demand. We show that the MPCs for borrowing-constrained agents tends to be much higher at low levels of w. The intuition is as follows. An agent whose savings demand is sufficiently large optimally avoids binding borrowing constraints and hence her MPC out of wealth cannot be high as the additional value of liquidity for an unconstrained agent cannot be that high as she could have chosen to increase her consumption from current income.

Finally, we generate predictions that are consistent with empirical findings, e.g., *excess sensitivity* (Flavin, 1981) and *excess smoothness* (Campbell and Deaton, 1989). We show analytically how the drift and volatility of the endogenous consumption process naturally arise in a model with buffer-stock savings demand giving rise to "excess sensitivity" and "excess smoothness" documented in the empirical consumption literature.

2   Model

We consider a continuous-time setting where an infinitely-lived agent receives an exogenously given perpetual stream of stochastic earnings. The agent saves via a risk-free asset that pays interest at a constant rate $r > 0$. There exist no other traded financial assets. Additionally, the agent is not allowed to borrow implying that wealth is non-negative at all times.

Labor-income/earnings process.   A commonly used labor-income process involves both permanent and transitory shocks that our framework is able to fully capture.[5] For expositional convenience, in our baseline model, we specify Y as one with permanent diffusion shocks only and leave the generalization to allow for large jump shocks to Section **??** and the generalization to allow for both permanent and transitory shocks to Appendix **??**.

Consider the following widely used labor-income process specification:

$$dY_t = \mu Y_t dt + \sigma Y_t d\mathcal{B}_t, \ Y_0 > 0, \tag{1}$$

where $\mathcal{B}$ is a standard Brownian motion, μ is the expected income growth rate, and σ measures income growth volatility. The labor-income process (**??**) implies that the growth rate of income, dY_t/Y_t, is independently, and identically distributed (i.i.d.)

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[5]See MaCurdy (1982), Abowd and Card (1989), and Meghir and Pistaferri (2004), for example. Given our focus, we also ignore life-cycle variations and various fixed effects such as education and gender.

Empirical labor-income processes are often specified in terms of logarithmic income $\ln Y$. We may also write the dynamics for logarithmic income, $\ln Y$ as follows by using Ito's formula:

$$d \ln Y_t = \alpha \, dt + \sigma d\mathcal{B}_t \,, \tag{2}$$

where the expected change of the logarithmic income, i.e., the drift in (**??**), equals

$$\alpha = \mu - \frac{\sigma^2}{2} \,. \tag{3}$$

Here, $\sigma^2/2$ is the Jensen's inequality correction term. Equation (**??**) is an arithmetic Brownian motion which implies the following discrete-time specification:

$$\ln Y_{t+1} - \ln Y_t = \alpha + \sigma \epsilon_{t+1} \,, \tag{4}$$

where the time-t conditional distribution of ϵ_{t+1} is the standard normal.

In our baseline model, $\ln Y$ is a unit-root process and its first difference is independently and normally distributed with mean α and volatility σ. That is, our labor-income model (**??**) is the same as the simplest form among commonly used empirical specifications.

Note that due to the Jensen's inequality, the growth rate of income in logarithm, α, can differ significantly from μ, the growth rate of labor income Y in levels. For example, at the annual frequency, with $\mu = 1.5\%$ and $\sigma = 10\%$, we have $\alpha = 1\%$, which is one third lower than the growth rate $\mu = 1.5\%$ due to the Jensen's term, $\sigma^2/2 = 0.5\%$. While income growth shocks are i.i.d., shocks are permanent in *levels* of Y.

Non-expected recursive utility. The widely-used standard preference in the consumption/savings literature is the expected utility with constant relative risk aversion (CRRA). While this utility has the homogeneity property, it unfortunately ties the elasticity of intertemporal substitution (EIS) to the inverse of the coefficient of relative risk aversion. Conceptually, risk aversion and the EIS are fundamentally different and have very different effects on consumption-savings decisions, as we will show. By using the non-expected recursive utility developed by Epstein and Zin (1989) and Weil (1990), who build on the work by Kreps and Porteous (1978), we are able to disentangle the effect of risk aversion from that of EIS. Specifically, we use the continuous-time formulation of this non-expected utility developed by Duffie and Epstein (1992a), and write the recursive utility process V as,

$$V_t = \mathbb{E}_t \left[\int_t^\infty f(C_u, V_u) du \right], \tag{5}$$

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where $f(C,V)$ is known as the normalized aggregator for consumption C and utility V. Duffie and Epstein (1992a) show that $f(C,V)$ for the Epstein-Zin utility is given by

$$f(C,V) = \frac{\rho}{1-\psi^{-1}} \frac{C^{1-\psi^{-1}} - ((1-\gamma)V)^{\theta}}{((1-\gamma)V)^{\theta-1}}, \tag{6}$$

and

$$\theta = \frac{1-\psi^{-1}}{1-\gamma}. \tag{7}$$

Here, ψ is the EIS, γ is the coefficient of relative risk aversion, and ρ is the subjective discount rate. The widely used time-additive separable CRRA utility is a special case of the recursive utility where the coefficient of relative risk aversion γ equals the inverse of the EIS, $\gamma = \psi^{-1}$ implying $\theta = 1$. For the expected utility special case, we thus have $f(C,V) = U(C) - \rho V$, which is additively separable in C and V, with $U(C) = \rho C^{1-\gamma}/(1-\gamma)$. For the general specification of the recursive utility, $\theta \neq 1$ and $f(C,V)$ is non-separable in C and V.

Wealth dynamics and borrowing constraints. The agent's wealth W evolves as:

$$dW_t = (rW_t + Y_t - C_t)dt, \quad t \geq 0. \tag{8}$$

The agent cannot borrow against her future incomes, i.e. wealth is non-negative at all t,

$$W_t \geq 0 , \tag{9}$$

which implies $C_t \leq Y_t$ when $W_t = 0$. Therefore, we have two mutually exclusive cases:

Type A: The agent's savings motive is sufficiently strong such that (**??**) never binds;

Type B: The agent may run out of savings at stochastic time τ and then will choose $C_\tau = Y_\tau$ thereafter permanently living from paycheck to paycheck. For this case, whenever $W_t > 0$, the agent is unconstrained in the short term. But the fact that she may be constrained in the future influences her current decisions.

In summary, the agent maximizes the non-expected recursive utility given in (**??**)-(**??**) subject to the labor-income process (**??**), the wealth accumulation process (**??**), and the borrowing constraint (**??**). Before analyzing the incomplete-markets model, we first present the complete-markets (CM) solution and introduce the certainty-equivalence-based PIH.

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3   CM and PIH

We first develop the CM benchmark and then compare it with Friedman's PIH.

3.1   CM Solution

To make the model dynamically complete, we introduce a traded asset that is perfectly correlated with earnings shocks. By applying the standard dynamic replicating portfolio argument as in Black and Scholes (1973), we know that CM can be achieved by *dynamically and frictionlessly trading* the risk-free asset and the newly introduced financial asset. As all labor income risks are assumed to be idiosyncratic in self-insurance models, the newly introduced financial asset must demand *no risk premium.* Appendix **??** provides details.

Next, following Friedman (1957) and Hall (1978), we define "human" wealth, denoted by H, as the expected present value of labor incomes, discounted at the interest rate r:

$$H_t = \mathbb{E}_t \left(\int_t^\infty e^{-r(u-t)} Y_u du \right) . \tag{10}$$

With CM, (**??**) gives the market value of labor income because labor income risk is fully diversifiable earning no risk premium and thus can be discounted at the risk-free rate.[6]

To ensure that human wealth H is finite, which is necessary for convergence under CM, we assume that the expected income growth rate μ is lower than the interest rate r, in that

$$Condition\ 1:\ r > \mu\,. \tag{11}$$

Then human wealth H is thus proportional to contemporaneous income, $H_t = hY_t$, where

$$h = \frac{1}{r - \mu}\,. \tag{12}$$

Our models (both CM and incomplete-markets settings) have the homogeneity property (in wealth W and income Y).[7] We use the lower case to denote the corresponding variable in the upper case scaled by contemporaneous earnings Y. For example, $w_t = W_t/Y_t$ and $c_t = C_t/Y_t$ denote the wealth-income ratio and the consumption-income ratio, respectively.

To ensure that the problem is well posed under CM, we require the following:

$$Condition\ 2:\ \rho > (1 - \psi^{-1})\,r\,. \tag{13}$$

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[6]Obviously, in general under incomplete markets, with borrowing constraints, or other frictions, H defined in (**??**) does not provide true economic valuation of labor incomes.

[7]Deaton (1991) and Carroll (1997) show the homogeneity property for the CRRA utility case and numerically solves for the optimal consumption rule in the discrete-time setting.

The subjective discount rate ρ needs to be sufficiently large so that the MPC is sufficiently large ensuring that the optimization problem is well defined. We next summarize the main results for our CM benchmark.

Proposition 1 *Under Condition 1 and Condition 2 given by (??) and (??) that ensure convergence, the agent's value function under CM, denoted by $V^*(W,Y)$, is given by*

$$V^*(W,Y) = \frac{(bP^*(W,Y))^{1-\gamma}}{1-\gamma}, \tag{14}$$

where the coefficient b is given by

$$b = \rho \left[\frac{r + \psi(\rho - r)}{\rho} \right]^{\frac{1}{1-\psi}}, \tag{15}$$

and the "total" wealth $P^*(W,Y) = W + H = p^*(w)Y$ with

$$p^*(w) = w + h = w + \frac{1}{r-\mu}. \tag{16}$$

The optimal consumption-income ratio $c_t = C_t/Y_t = c(w_t)$ is given by

$$c^*(w) = m^* p^*(w) = m^* (w + h), \tag{17}$$

and m^* is the marginal propensity to consume (MPC) under CM and is given by

$$m^* = r + \psi(\rho - r). \tag{18}$$

For the case with $\psi = 1$, we have $b = \rho e^{(r-\rho)/\rho}$ and $c^*(w) = \rho(w + h)$.

Next, we discuss key intuition for the CM solution.

3.2 Intuition

First, the "total" wealth $P^* = W + H$ measures the agent's economic value. Under CM with no frictions, total wealth P^* is simply additive in wealth W and "human" wealth H.

Second, consumption follows the linear Ramsey rule given by (??). The agent consumes a fixed fraction m^* of her total wealth $P^* = W + H$. Note that m^* is the CM MPC given by the sum of the interest rate r and $\psi(\rho - r)$, the latter being the product of the EIS ψ and the wedge $(\rho - r)$. If and only if $\rho > r$, the agent behaves relatively impatiently by consuming

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more than $rP^*(W, Y)$, the annuity value of her total wealth. Additionally, the higher the EIS ψ, the more elastic/responsive her consumption is to changes in the wedge $(\rho - r)$.

Why is consumption independent of risk aversion under CM? Because labor-income risk is purely idiosyncratic and can be fully diversified away at no risk premium. A risk-averse agent optimally chooses zero net risk exposure for her total wealth P_t^* at all time t, which is achievable via dynamic hedging. As a result, the consumption problem under CM is converted into a deterministic one and P_t^* evolves deterministically as:

$$dP_t^* = (r - m^*)P_t^* dt = -\psi(\rho - r)P_t^* dt. \tag{19}$$

If $\rho = r$, we have $P_t^* = W_0 + hY_0$, constant at all time t. In general, P_t^* evolves exponentially at the rate of $-\psi(\rho - r)$, consistent with the intuition as that in the Ramsey's model. Neither total wealth $P^*(W_t, Y_t)$ nor the MPC m^* depends on risk aversion γ. Indeed, risk aversion cannot be identified because there is no net risk exposure under optimality. As the EIS and risk aversion are two independent parameters in our model, we can thus clearly show that it is the EIS ψ rather than risk aversion γ that matters under CM.

Third, the value function $V^*(W, Y)$ is increasing and homothetic with degree $(1 - \gamma)$ in "total" wealth $P^*(W, Y)$ due to the geometric earnings process and homothetic preferences. The only effect that risk aversion γ has for the CM solution is to make the value function $V^*(W, Y)$ homogenous in $P^*(W, Y)$ with degree $(1 - \gamma)$.

Our intuition for the homogeneity of the value function $V^*(W, Y)$ builds on Samuelson-Merton's insight and closed-form solution that the demand for risky assets decreases in $-WV_{WW}/V_W$, which equals γ in the original Merton (1971) formulation. By conjecturing that $V^*(W, Y)$ is homogeneous with degree $(1 - \gamma)$ in W and Y in our model, we link the curvature of the value function to risk aversion, not the EIS.[8]

Certainty Equivalence and PIH. Our CM model has the certainty-equivalence property and by further assuming $\rho = r$, we uncover Friedman-Hall's certainty-equivalence rule:

$$C_t = rP^*(W_t, Y_t) = r(W_t + H_t). \tag{20}$$

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[8]Conjecturing that $V^*(W, Y)$ is homogeneous in W and Y with degree $(1 - \psi^{-1})$ would have generated counter-intuitive and wrong portfolio predictions in our model with Epstein-Zin utility. This is consistent with the result that the demand for risky assets only depends on γ not the EIS ψ even in a generalized Merton's (1971) problem with Epstein-Zin utility.

However, importantly, consumption is a martingale under the PIH, but is deterministic in our CM setting because financial market structures differ. The only financial claim in the PIH framework is the risk-free asset. Next, we turn to the incomplete-markets problem.

4   Incomplete Markets

We first characterize our model's analytical solution and then provide economic intuition.

4.1   Solution

The continuous-time methodology allows us to analytically solve the model in closed form up to an ordinary differential equation (ODE) with economically intuitive boundary conditions.

We proceed in three steps. First, we consider the agent's decision problem in the interior region. Second, we analyze the boundary behaviors (e.g., $W = 0$.) Third, we explore the homogeneity property to convert the two-dimensional optimization problem into a one-dimensional one. Let $V(W, Y)$ denote the value function.

The Interior Region.   Using the standard dynamic programming method, we know that optimal consumption solves the Hamilton-Jacobi-Bellman (HJB) equation,[9]

$$0 = \max_{C>0} \ f(C, V) + (rW + Y - C)V_W(W, Y) + \mu Y V_Y(W, Y) + \frac{\sigma^2 Y^2}{2} V_{YY}(W, Y). \quad (21)$$

Equation (??) reflects incomplete markets as labor income has the volatility V_{YY} term but there is no traded risky financial asset to hedge earnings shocks.[10] The first-order condition (FOC) for consumption is given by

$$f_C(C, V) = V_W(W, Y), \quad (22)$$

which equates the marginal benefit of consumption $f_C(C, V)$ with the marginal utility of savings $V_W(W, Y)$. Note that $f_C(C, V)$ is non-separable in C and V, which is in sharp contrast with the standard FOC for expect utility where the marginal utility of consumption does not depend on V. Before characterizing the solution for the general case, it is useful to first summarize the solution for the case with no labor income, $Y = 0$.

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[9]Duffie and Epstein (1992b) generalize the standard HJB equation for the expected-utility case to allow for non-expected recursive utility such as the Epstein-Weil-Zin utility used here.

[10]Hence, technically speaking, in the HJB equation (??), we do not have a term involving V_{WW} nor the hedging term involving V_{WY}.

No Labor Income: $Y = 0$. Because Y is assumed to follow a GBM process, both the drift and volatility of Y are zero and hence $Y = 0$ is an absorbing state, in that $Y_t = 0$ for all $t \geq \tau$ if $Y_\tau = 0$. Therefore, the last two terms in the HJB equation (??) involving $V_Y(W, 0)$ and $V_{YY}(W, 0)$ are zero. In Appendix **??**, we show that the value function $V(\cdot, 0)$ for the case with no labor income has the following closed-form solution:

$$V(W, 0) = \frac{(bW)^{1-\gamma}}{1 - \gamma}, \tag{23}$$

where b is given by (??). Because b in (??) is the same as the value of b in the CM value function $V^*(W, Y)$ given in (??), we thus have for any given level of wealth W,

$$V(W, 0) = V^*(W, 0). \tag{24}$$

Consumption and utils are the same path by path for the deterministic case (with no labor income) and the CM case as all risks are idiosyncratic and are completely diversified away.

Next, we analyze the general case where $W > 0$ and $Y > 0$.

Value Function $V(W, Y)$ and Certainty Equivalent Wealth $P(W, Y)$. We show that the value function $V(W, Y)$ is given by

$$V(W, Y) = \frac{(bP(W, Y))^{1-\gamma}}{1 - \gamma}, \tag{25}$$

where we set b to the value given by (??). By doing so, we may interpret $P(W_t, Y_t)$ as the certainty equivalent wealth, which is the time-t total wealth that makes the agent indifferent between (i) the *status quo* (with wealth W and the earnings process Y) and (ii) the alternative of living with no earnings permanently thereafter. Therefore, $P(W, Y)$ solves:

$$V(W, Y) = V(P(W, Y), 0) = V^*(P(W, Y), 0), \tag{26}$$

where the second equality follows from (??).

Why do we work with certainty equivalent wealth $P(W, Y)$ rather than directly with the value function $V(W, Y)$? First, certainty equivalent wealth is an intuitive concept and is measured in units of consumption goods, while the unit for value function $V(W, Y)$ is utils, which cannot be directly measured. Second, $P(W, Y)$ is analytically convenient to work with. For example, by using $P(W, Y)$ and its derivative, the marginal (certainty equivalent) value of wealth $P_W(W, Y)$, we can clearly explain how consumption depends on $P(W, Y)$ and $P_W(W, Y)$. Third, $P_W(W, Y)$ is a natural measure for the impact of financial frictions.

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Let $c(w) = C(W,Y)/Y$ and $p(w) = P(W,Y)/Y$. By using the homogeneity property, the "effective" state variable is $w = W/Y$ and $P_W(W,Y) = p'(w)$. The following proposition summarizes the main results on $c(w)$ and $p(w)$.

Proposition 2 *The optimal consumption-income ratio $c(w)$ is given by*

$$c(w) = m^* p(w)(p'(w))^{-\psi}, \tag{27}$$

where m^* is the CM MPC given in (??) and the certainty equivalent wealth $p(w)$ solves:

$$
\begin{aligned}
0 =& \left(\frac{m^*(p'(w))^{1-\psi} - \psi\rho}{\psi - 1} + \mu - \frac{\gamma\sigma^2}{2} \right) p(w) + p'(w) + (r - \mu + \gamma\sigma^2)wp'(w) \\
&+ \frac{\sigma^2 w^2}{2} \left(p''(w) - \gamma \frac{(p'(w))^2}{p(w)} \right).
\end{aligned}
\tag{28}
$$

The above ODE for $p(w)$ is solved subject to the following condition:

$$\lim_{w \to \infty} p(w) = p^*(w) = w + h = w + \frac{1}{r - \mu}. \tag{29}$$

Additionally, the ODE (??) for $p(w)$ satisfies the borrowing constraint at the origin,

$$0 < c(0) \leq 1. \tag{30}$$

Equation (??) state that $c(w)$ is given by the CM MPC m^* multiplied by $p(w)$ and also importantly $(p'(w))^{-\psi}$. Unlike in the CM setting, where $p'(w) = 1$ and consumption is linear in w, with financial frictions, both $p(w)$ and $p'(w)$ directly influence $c(w)$. The ODE (??) reflects the nonlinearity of $p(w)$. The condition (??) describes the convergence result that self insurance against income shocks becomes as effective as CM risk sharing as $w \to \infty$. Finally, whether the constraint binds or not is captured by the boundary condition (??). We will focus on the implications of binding borrowing constraints in Subsection ??.

Next, we discuss key qualitative economic insights for the main results in Proposition ??. It is useful to graphically sketch out the shapes and highlight the key properties of the certainty equivalent wealth $p(w)$ and the optimal consumption-income ratio $c(w)$.

4.2 Graphic Illustrations of Key Insights

We plot $p(w)$ and the marginal value of liquidity, $p'(w)$, in Panels A and B of Figure ??, respectively. To aid our discussions, we also plot the dotted straight lines for the CM
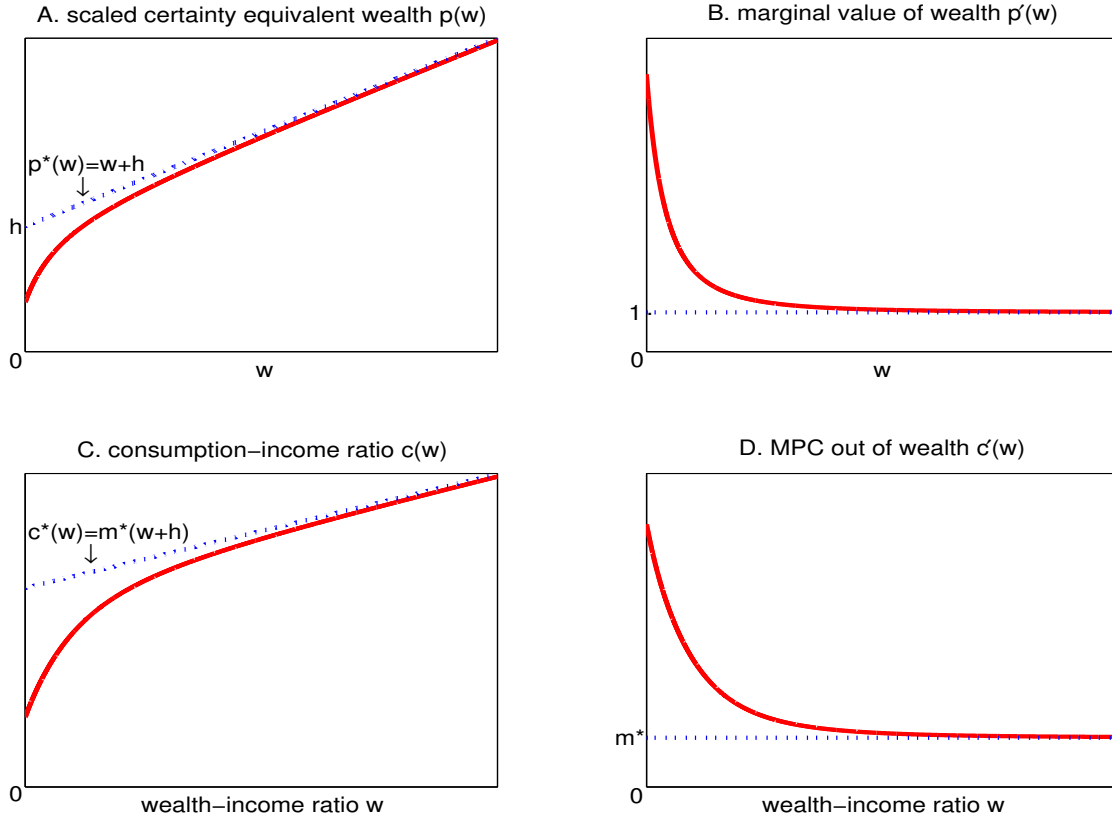
Figure 1: **Graphical Illustrations: the scaled certainty equivalent wealth** $p(w)$**, marginal (certainty equivalent) value of wealth** $p'(w)$**, consumption-income ratio** $c(w)$**, and the MPC out of wealth** $c'(w)$**.** The dashed straight lines in all panels correspond to the CM benchmark results.

results. We see that $p(w)$ is increasing and concave, lies below the CM line $p^*(w) = w + h$, i.e., $p(w) \leq p^*(w)$. The monotonicity of $p(w)$ is immediate. The intuition for $p(w) \leq p^*(w)$ and the concavity of $p(w)$ is as follows. Financial frictions make the agent worse off, i.e., $p(w) \leq p^*(w)$ and cause her to value wealth more than its accounting value, i.e., $p'(w) \geq 1$. Additionally, the higher the liquidity w, the more effective her self insurance against financial frictions and the less she values her wealth at the margin, i.e., a lower $p'(w)$. As $w \to \infty$, self insurance is very effective achieving the CM benchmark, hence $\lim_{w\to\infty} p(w) = p^*(w)$ and $\lim_{w\to\infty} p'(w) = 1$. Next we discuss intuition for optimal consumption and the MPC.

In Panels C and D of Figure **??**, we plot $c(w)$ and the MPC out of wealth, $c'(w)$, respectively. Again, we plot the CM results via dotted straight lines to aid our discussions. First, $c(w)$ is increasing and concave in w. Additionally, $c(w)$ lies below the CM line $c^*(w)$,

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i.e., $c(w) \leq c^*(w)$. The monotonicity of $c(w)$ is immediate. But why is $c(w)$ concave in w? This is due to the agent's optimal response to buffer shocks and mitigate borrowing constraints. The higher the value of w, the more effective her self insurance against financial frictions and the less her consumption responds to wealth explaining that $c'(w)$ decreases with w or equivalently consumption is concave in w, as conjectured by Keynes (1935).[11] Finally, as $w \to \infty$, self insurance achieves perfect risk sharing as under CM, in that $\lim_{w \to \infty} c(w) = c^*(w) = m^*(w + h)$ and $\lim_{w \to \infty} c'(w) = m^* = r + \psi(\rho - r)$.

Our analysis and intuition essentially go throughout regardless of whether the borrowing constraint binds at $w = 0$ or not. Having qualitatively characterized the key features of our model solution, we next turn to the model's quantitative implications in the next section.

5   Buffer-Stock Savings and Stationary Distribution

Parameter value choices. Despite being parsimonious, our model generates sensible quantitative predictions. The entire model only has six parameters including three (γ, ψ, and ρ) for preferences, two for labor income (μ and σ), and the interest rate r. Parameters μ, σ, ρ, and r are annualized and continuously compounded. We set the annual expected growth rate and volatility of the earnings process to $\mu = 1.5\%$ and $\sigma = 10\%$, respectively, in line with values used in the consumption literature, e.g., Deaton (1991) and Carroll (1997). The implied logarithmic annual income growth rate $\alpha = 1\%$. The Jensen's inequality term $\sigma^2/2 = 0.5\%$ is quantitatively significant, which is one third of the growth rate $\mu = 1.5\%$.

We choose the annual interest rate $r = 3.5\%$ and set the annual discount rate $\rho = 4\%$, commonly used values in the literature.[12] The agent is relatively impatient (compared with the market), with a wedge of $\rho - r = 0.5\%$, as typically done in the buffer-stock savings literature so that the rich has incentives to dis-save when w is sufficiently high.

There are significant disagreements about a reasonable value for the EIS. For example, Hall (1988) obtains an estimate of EIS near zero by using aggregate consumption data. Vissing-Jorgensen (2002), Attanasio, Banks, and Tanner (2002), and Guvenen (2006) choose a low EIS for non-stock holders but a higher value for stock holders. Bansal and Yaron

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[11]Carroll and Kimball (1996) show that the consumption function is concave under certain conditions for the expected-utility case.

[12]For example, Caballero (1990), Krusell and Smith (1998), and Guvenen (2007) choose an annual interest rate around 4%. In order to meet *Condition 2* for convergence purposes, we need to choose r satisfying $r > \mu$. A choice of 3.5% with $\mu = 1.5\%$ gives a "human wealth" multiple of $h = 1/(r - \mu) = 50$ when converting labor income to "human" wealth. A lower interest rate will give an even higher multiple h.

Table 1: PARAMETER CHOICES.

This table summarizes the parameter values for our baseline quantitative analysis. Parameter values are annualized and continuously compounded when applicable.

Parameters	Symbol	Value
Risk-free rate	r	3.5%
Subjective discount rate	ρ	4%
Coefficient of relative risk aversion	γ	3
Elasticity of intertemporal substitution (EIS)	ψ	1
Expected income growth rate in levels	μ	1.5%
Income growth volatility	σ	10%

(2004) choose a high estimate of EIS in the range of 1.5 to 2 in order to fit long-run risk asset pricing models with aggregate evidence. While an EIS parameter larger than one has become a common practice in the macro-finance literature, however, there is no consensus on what the sensible value of EIS should be. The Appendix in Hall (2009) provides a brief survey of estimates in the literature. For the baseline analysis, we set $\psi = 1$, which is in the mid-range of the EIS values and is the estimated EIS value for stockholders by Vissing-Jorgensen (2002), Attanasio, Banks, and Tanner (2002), and Guvenen (2006).[13] As in Guvenen (2006), we set risk aversion at $\gamma = 3$, which is within the range of plausible estimates viewed by many economists. Note that our non-expected utility features $\gamma = 3 > 1 = 1/\psi$.

5.1 $p(w)$, $p'(w)$, Consumption $c(w)$, and MPC $c'(w)$

To highlight the impact of risk aversion, we consider two levels of risk aversion, $\gamma = 3, 6$. Panel A and B of Figure ?? plot $p(w)$ and $p'(w)$, respectively. Quantitatively, frictions significantly lower $p(w)$ from its CM benchmark value $p^*(w)$ and the marginal value of liquidity $p'(w)$ is much higher than unity, the value under CM. For example, with $\gamma = 3$, we have $p(0) = 28.7$, which is only 57.4% of $p^*(0) = h = 50$ under CM, and the marginal value of liquidity $p'(0) = 1.34$ implying a 34% premium over the face value of liquidity.

Panels C of Figure ?? plots $c(w)$. For $\gamma = 3$, the agent consumes 85% of her labor income at $w = 0$, in that $c(0) = 0.85$, which is only 43% of the CM benchmark consumption level $c^*(0) = 2.00$. Even with a very high level of liquidity, e.g., $w = 20$, $c(20) = 1.86$, which is

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[13]We are not using any special property for the case with $\psi = 1$ as results are continuous in ψ.
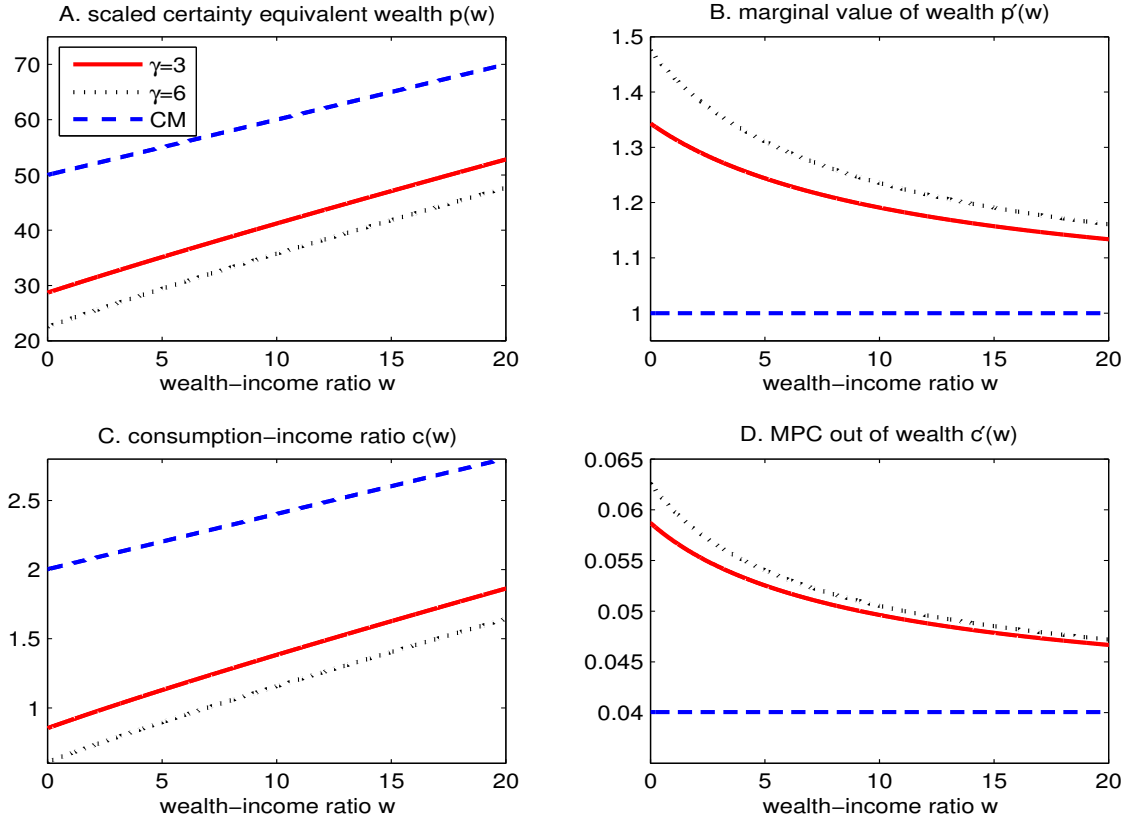
Figure 2: **Certainty equivalent wealth** $p(w)$**, marginal value of liquidity** $p'(w)$**, consumption-income ratio** $c(w)$**, and the MPC** $c'(w)$: *Type A where borrowing constraints do not bind, i.e.,* $c(0) < 1$. We plot for two levels of risk aversion: $\gamma = 3$ and $\gamma = 6$. Other parameter values are: $r = 3.5\%$, $\rho = 4\%$, $\sigma = 10\%$, $\mu = 1.5\%$, and $\psi = 1$.

still only 67% of the CM benchmark level, i.e. $c(20)/c^*(20) = 67\%$.

Panel D of Figure **??** plots the MPC $c'(w)$, which decreases in w and is greater than the CM MPC is $m^* = 4\%$. In this case, the agent's savings motive is sufficiently strong so that she voluntarily chooses not to be in debt even if she can borrow at the risk-free rate. As a result, the variation in MPC is somewhat limited and only ranges from 4% to 6% for $w \geq 0$.

Now consider the impact of increasing risk aversion. At the origin $w = 0$, the marginal value of liquidity increases from $p'(0) = 1.34$ to $p'(0) = 1.48$ and consumption decreases substantially from $c(0) = 0.85$ to $c(0) = 0.61$, as we increase γ from 3 to 6. The more risk averse, the lower her consumption. With $\gamma = 6$, even with plenty liquidity, e.g., $w = 20$, the agent only consumes 59% of the CM benchmark level, i.e. $c(20)/c^*(20) = 59\%$.

It is clear that the quantitative effects of risk aversion γ on consumption are very large

under incomplete markets even for high levels of w because earnings shocks are permanent and self insurance is generally not very effective. This is in sharp contrast with the prediction that risk aversion plays no role at all on consumption for any level of w in the CM benchmark.

5.2 Buffer-Stock Savings and Stationary Distributions

We next turn to the model's implications on buffer stock savings. By using the Ito's formula, we express the dynamics for scaled liquidity w_t as:

$$dw_t = w_t \left[\mu_w(w_t) \, dt - \sigma \, d\mathcal{B}_t \right] , \tag{31}$$

where the drift process $\mu_w(w)$ is given by

$$\mu_w(w_t) = \frac{1}{w_t} \left(n(w_t) - c(w_t) \right) , \tag{32}$$

and

$$n(w) \quad = \quad 1 + \left(r - \mu + \sigma^2 \right) w . \tag{33}$$

Next, we write down the discrete-time implication that we later use for simulation. Over a small time interval $(t, t + \Delta t)$, we may write the implied dynamics as:

$$w_{t+\Delta t} - w_t = w_t \left(\mu_w(w_t) \, \Delta t - \sigma \sqrt{\Delta t} \, \epsilon_{t+\Delta t} \right) , \tag{34}$$

where $\epsilon_{t+\Delta t}$ is a standard normal random variable with mean of zero and variance of one.

Let w^{ss} denote the value of w at which the drift of w satisfies $\mu_w(w) = 0$ under optimality. The implied consumption-income ratio, denoted by c^{ss}, is then given by

$$c^{ss} = c(w^{ss}) = 1 + \left(r - \mu + \sigma^2 \right) w^{ss} . \tag{35}$$

In Figure **??**, we graphically determine the steady state (scaled) liquidity and consumption (w^{ss}, c^{ss}) as the coordinates for the point of intersection between (i) the optimal concave consumption rule $c(w)$ given in (**??**) and (ii) the linear function $n(w)$ given in (**??**). For $\gamma = 3$, $(w^{ss}, c^{ss}) = (5.96, 1.18)$. That is, at the steady state, the agent targets her wealth to be 5.96 times of and consumes 1.18 times of her current earnings. Consumption demand that is not covered by her current earnings is financed from the interest income rW. Risk aversion has very large effects on steady-state savings. As we increase γ from 3 to 6, (w^{ss}, c^{ss}) increases significantly from $(5.96, 1.18)$ to $(17.6, 1.53)$ demonstrating strong savings motives for risk-averse agents in sharp contrast with the CM or PIH benchmark results where risk aversion has no impact on (w^{ss}, c^{ss}) at all.
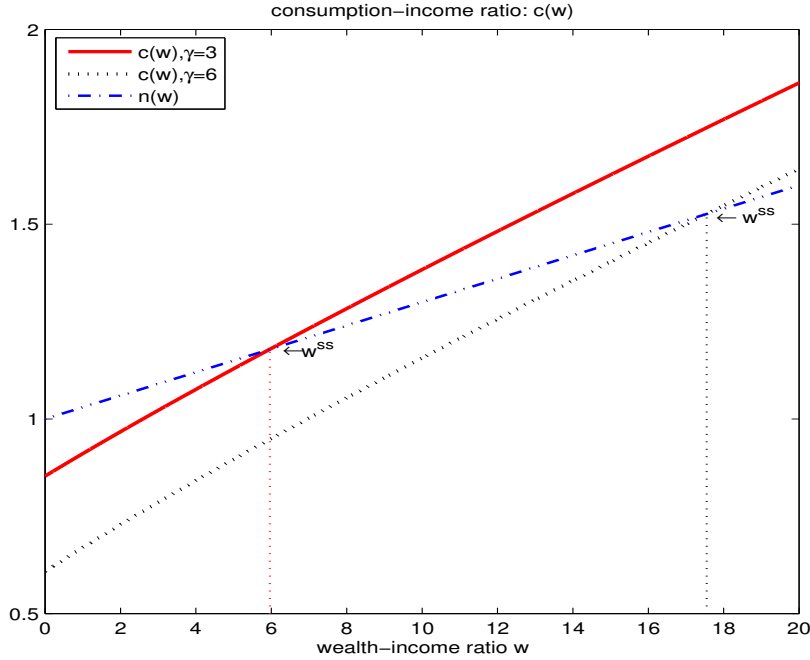
Figure 3: **Determining the steady-state scaled wealth and consumption** (w^{ss}, c^{ss}). The steady-state (w^{ss}, c^{ss}) is determined as the coordinates for the point of intersection between the optimal consumption rule $c(w)$ and the linear function $n(w) = 1 + (r - \mu + \sigma^2)\, w$. For $\gamma = 3$ and $\gamma = 6$, we have $(w^{ss}, c^{ss}) = (5.96, 1.18)$ and $(w^{ss}, c^{ss}) = (17.6, 1.53)$, respectively. Parameter values are: $r = 3.5\%$, $\rho = 4\%$, $\sigma = 10\%$, $\mu = 1.5\%$, and $\psi = 1$.

Stationary Distributions of w and c. Next, we report the stationary distributions of w and c. We simulate our model by starting from the steady state, i.e., $(w_0, c_0) = (w^{ss}, c^{ss})$, and generating four thousand sample paths via the dynamics of w given in (**??**) and the optimal consumption rule $c(w)$ given in (**??**). Each sample path is 5,000-year long with a time increment of $\Delta t = 0.05$ year implying $5,000 \times 20 = 100,000$ observations per path.

In Table **??**, we report the Gini coefficient, steady-state values, mean, standard deviation, and various quantiles for the stationary distribution of w, and that of c in Panels A and B, respectively. In the long run and/or the steady state, risk aversion has first-order effects on wealth and consumption. While the level and dispersion (e.g., measured by Gini coefficients) for both w and c increase with risk aversion, (scaled) wealth responds more than consumption does as risk aversion changes. For example, as we increase γ from 3 to 6, the steady-state wealth-consumption ratio w^{ss}/c^{ss} increases from 5.05 to about 11.5. Also the dispersion of (scaled) consumption is significantly smaller than that of (scaled) wealth. For example, with

Table 2: STATIONARY DISTRIBUTIONS OF w AND c FOR $\gamma = 3$ AND $\gamma = 6$.

This table reports the Gini coefficient, steady-state target w^{ss}, mean, standard deviation, and various quantiles for the stationary distributions of w and c for $\gamma = 3, 6$.

γ	Gini	w^{ss}	mean	std dev	1%	5%	25%	50%	75%	95%	99%
				A. Scaled liquidity w							
3	48.1%	5.96	6.15	3.65	2.19	2.75	3.95	5.25	7.22	12.35	19.40
6	49.9%	17.55	18.39	12.99	5.98	7.59	11.15	15.16	21.47	39.17	65.61
				B. Consumption-income ratio c							
3	35.4%	1.18	1.17	0.18	0.97	1.00	1.06	1.13	1.23	1.49	1.83
6	40.9%	1.53	1.52	0.59	0.93	0.99	1.17	1.38	1.68	2.49	3.63

$\gamma = 3$, the Gini coefficient for c is 35.4%, which is 25% lower than 48.1%, the Gini coefficient for w. These results are consistent with our intuition that the agent builds her buffer stock to smooth consumption over time.

We have so far focused on *Type A* where the constraint $W_t \geq 0$ given in (**??**) does not bind, and hence $c(0) < 1$. For this case, voluntary savings demand is sufficiently strong so that $W_t > 0$ for all t, and therefore, relaxing the borrowing constraint (e.g., by offering a credit line even at the risk-free rate r) has no effect on consumption and thus we can just ignore the borrowing constraint when solving for $p(w)$ for *Type A*. The only friction in this case is the inability to perfectly hedge risk under incomplete markets. Next, we turn to *Type B*, where the constraint $W_t \geq 0$ given in (**??**) eventually binds as the agent runs out of wealth.

5.3   Binding Borrowing Constraints at $W = 0$ (Type B)

Campbell and Mankiw (1989) document that about 50% of households in their sample do not save. We show that this behavior can be consistent with optimality. Consider at time t an agent with no wealth, i.e., $W_t = 0$, and if it is optimal for her to consume all her earnings, in that $C_t = Y_t$, then she will remain permanently constrained at $W_s = 0$ for all time $s \geq t$ living from paycheck to paycheck, or "hand to mouth," thereafter. The corresponding steady state is at the corner, in that $(w^{ss}, c^{ss}) = (0, 1)$ as $c(0) = 1$.

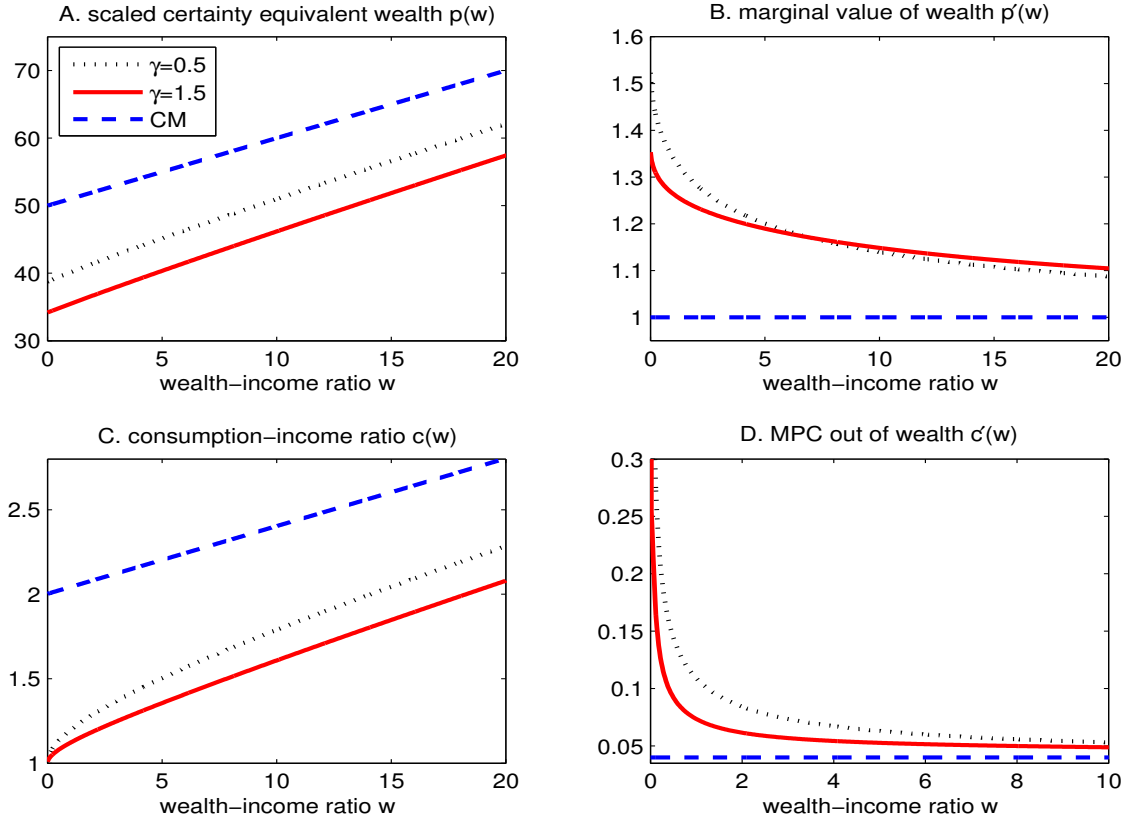It is worth emphasizing that the behaviors that we analyze in this subsection apply not

Figure 4: **Certainty equivalent wealth** $p(w)$, **marginal value of wealth** $p'(w)$, **consumption-income ratio** $c(w)$, **and the MPC** $c'(w)$: *Type B where the borrowing constraints bind, i.e., $c(0) = 1$. We plot for two levels of risk aversion: $\gamma = 0.5$ and $\gamma = 1.5$. Other parameter values are: $r = 3.5\%$, $\rho = 4\%$, $\sigma = 10\%$, $\mu = 1.5\%$, and $\psi = 1$.*

only to "hand-to-mouth" consumers (with no savings, i.e., $W = 0$) but to those (with positive wealth, i.e., $W > 0$) who are not presently constrained but may be permanently constrained in the future, i.e., $c(0) = 1$. We define an agent as *Type B* as long as her consumption when she has no wealth satisfies $c(0) = 1$. Not surprisingly, these consumers even when $W > 0$ behave very differently from those who are never constrained (i.e., *Type A*.)

Figure **??** plots $p(w)$, $p'(w)$, $c(w)$, and the MPC $c'(w)$ for $\gamma = 0.5$ and $\gamma = 1.5$ while keeping all other parameters the same as in the baseline case. For both cases, borrowing constraints bind at $W = 0$, i.e., $c(0) = 1$.

Binding borrowing constraints generate new empirically testable predictions and economic insights. First, they cause the marginal value of liquidity $p'(w)$ to *decrease* with risk aversion for low to medium values of w. As long as $w < 7.02$, $p'(w)$ for $\gamma = 0.5$ is larger

than for $\gamma = 1.5$. For example, at $w = 0$, $p'(0) = 1.53$ for $\gamma = 0.5$, which is greater than $p'(0) = 1.35$ for $\gamma = 1.5$. This is opposite to the prediction that $p'(w)$ increases with γ when the constraint does not bind, i.e., *Type A*. The intuition is as follows. When facing a binding constraint at $w = 0$, the less risk-averse agent has a higher borrowing demand against her future earnings to finance her current consumption, and therefore values liquidity more at the margin implying that $p'(w)$ decreases with γ near $w = 0$ for *Type B*, where $c(0) = 1$.

We next analytically prove this result by using the closed-form expression for $p'(0)$ and $p(0)$. Substituting $c(0) = 1$ into (??) and substituting $w = 0$ into the ODE (??) yield[14]

$$p'(0) = (m^* p(0))^{1/\psi} \,, \tag{36}$$

and

$$p(0) = \left(\frac{1}{m^*} \right)^{\frac{1}{1-\psi}} \left[\rho - (1 - \psi^{-1})\nu \right]^{\frac{\psi}{1-\psi}} \,, \tag{37}$$

where ν is a risk-aversion-adjusted and variance-adjusted income growth rate defined as

$$\nu = \mu - \frac{\gamma \sigma^2}{2} \,. \tag{38}$$

It is straightforward to show that for all admissible parameter values (i.e., regardless of values for r, ρ, and ψ,) $p(0)$ and $p'(0)$ satisfy:

$$\text{Type B:} \qquad \frac{\partial p(0)}{\partial \nu} > 0 \,, \quad \text{and} \quad \frac{\partial p'(0)}{\partial \nu} > 0 \,. \tag{39}$$

Therefore, $p(0)$ and $p'(0)$ *decrease* with γ and σ^2, and increase with μ. Because $p'(w)$ and $p(w)$ are continuous in w, $p'(w)$ and $p(w)$ must also *decrease* with γ for sufficiently low values of w. This binding borrowing-constraint channel, which is absent in *Type A*, causes $p'(w)$ to decrease with γ for low values of w in *Type B*.

The other striking and non-obvious result is that the MPC for a constrained agent near $w = 0$ is much higher than an unconstrained agent in general. For example, at $w = 0.05$, the MPC equals $c'(0.05) = 0.43$ when $\gamma = 0.5$, and $c'(0.05) = 0.23$ when $\gamma = 1.5$. These results are consistent with empirical evidence documented by Parker (1999) and Souleles (1999) who

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[14]For the case with $\psi = 1$, applying L'hopital's rule to (??), we obtain the following:

$$p'(0) = \exp \left[\frac{1}{\rho} \left(\rho - r + \mu - \frac{\gamma \sigma^2}{2} \right) \right] \quad \text{and} \quad p(0) = \frac{1}{\rho} p'(0) \quad \text{for} \quad \psi = 1 \,.$$

find that annual MPCs typically vary from 0.2 to 0.6. Why a binding borrowing constant at $W = 0$ generally causes the MPC to be high near $W = 0$? The intuition is as follows. An agent who is constrained at $W = 0$ has a strong consumption demand and wants to borrow against future. Hence when receiving a unit of wealth windfall, she optimally consumes a larger fraction of the windfall near $W = 0$ than if she were unconstrained.

What types of agents are more likely to face binding borrowing constraints when $W = 0$? They tend to have lower risk aversion, higher income growth, lower income growth volatility, and a higher subjective discount rate. For *Type B*, both frictions (incomplete markets and borrowing constraints) are important. The borrowing constraint tends to play a more important role for low/medium values of w while the standard precautionary savings motive are relatively more important for higher values of w.

Next, we use our model's non-expected utility feature to analyze the distinct effect of the EIS on consumption and wealth dispersion.

6   Elasticity of Intertemporal Substitution (EIS)

The empirical estimates for the EIS vary widely. Given the wide spectrum of the EIS value used in the literature, we choose three values: $\psi = 0.1, 1, 2$ including both a low estimate as in Hall (1988) and a high estimate as in Bansal and Yaron (2004) and other asset pricing models. We fix risk aversion at $\gamma = 3$ as in our baseline calculation.

Effects of the EIS on Consumption $c(w)$. Figure **??** plots $c(w)$ for three levels of the EIS: $\psi = 0.1$, 1, 2. Interestingly, how the EIS influences $c(w)$ depends on the level of w.

For sufficiently high levels of w, (e.g., $w \geq 19.8$), $c(w)$ *increases* with the EIS ψ. The intuition essentially follows from the CM benchmark Ramsey/Friedman's consumption rule, as in the limit, self insurance approaches perfect CM risk sharing, in that $\lim_{w \to \infty} c(w) = c^*(w) = m^*(w + h)$ and the MPC $m^* = r + \psi(\rho - r)$ increases with the EIS for a relatively impatient agent, i.e., $\rho > r$. However, for $w < 19.8$, $c(w)$ *decreases* with the EIS ψ. Here, the explicit consumption rule $c(w) = m^* p(w) p'(w)^{-\psi}$ helps us understand the mechanism. The component $p'(w)^{-\psi}$ plays a more significant role in determining consumption for low values of w as $p'(w)$ is higher for low values of w. For example, with $\psi = 0.1$, $p'(0)^{-\psi} = 0.97$ and $c(0) = 0.98$ but for $\psi = 2$, $p'(0)^{-\psi} = 0.59$ and $c(0) = 0.76$.

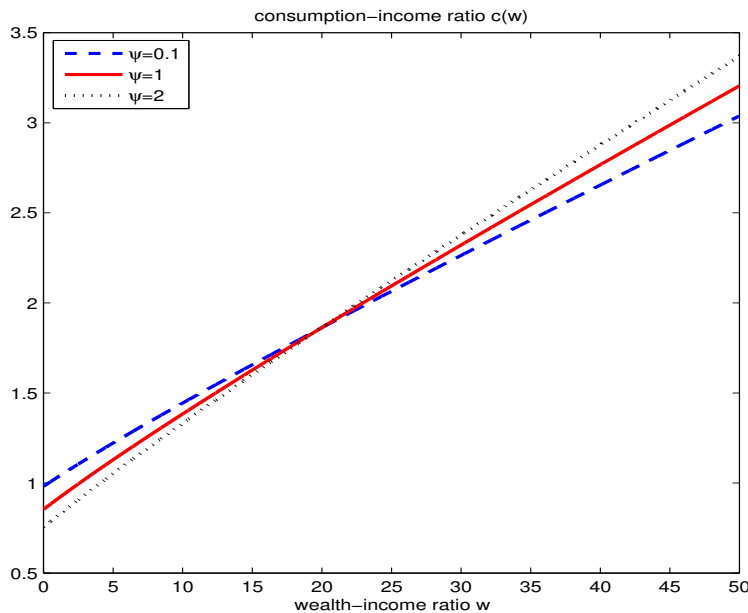Putting the opposite effects of the EIS on $c(w)$, we see that graphically $c(w)$ rotates

Figure 5: **Consumption-income ratio** $c(w)$ **where the EIS** $\psi = 0.1$, $\psi = 1$, **and** $\psi = 2$. Other parameter values are: $r = 3.5\%$, $\rho = 4\%$, $\sigma = 10\%$, $\mu = 1.5\%$, and $\gamma = 3$.

counter-clock-wise as we increase the EIS ψ. This is in sharp contrast with the monotonically decreasing relation between consumption and risk aversion. The opposing effects of the EIS on $c(w)$ have important implications on wealth dispersion to which we now turn.

Stationary Distribution of w. Table **??** reports the Gini coefficient, steady-state target w^{ss}, mean, standard deviation, and various quantiles for the stationary distribution of w for $\psi = 0.1$, 1, and 2. Risk aversion is set at $\gamma = 3$ in Panel A and $\gamma = 10$ in Panel B, respectively.

First consider the case with $\gamma = 3$ reported in Panel A. For this commonly used risk aversion, the quantitative impact of the EIS ψ on the Gini coefficient is non-monotonic but very limited, but the impact of the EIS on long-run savings and steady state w^{ss} are very large. The Gini coefficient is around 47% to 48% even as we vary the EIS over the entire economically relevant range from $\psi = 0.1$ to $\psi = 2$. Changes in the EIS drive small variations in the Gini coefficient at least partly reflects the non-monotonic effect of the EIS ψ on consumption at low and high ends of w, as we just discussed. Despite its limited effect on dispersion, increasing the EIS from $\psi = 0.1$ to $\psi = 2$ substantially increases the steady-state savings w^{ss} by 9.5 times from 0.92 to 8.72.

Table 3: STATIONARY DISTRIBUTION OF w: EFFECTS OF THE EIS ψ AND γ.

This table reports the Gini coefficient, steady-state target w^{ss}, mean, standard deviation, and various quantiles for the stationary distribution of w. In Panel A, we set $\gamma = 3$ and in Panel B, we set $\gamma = 10$ including the long-run risk case with $\psi = 2$ (the last row.)

ψ	Gini	w^{ss}	mean	std dev	1%	5%	25%	50%	75%	95%	99%
					A. $\gamma = 3$						
0.1	48.0%	**0.92**	0.96	0.59	0.30	0.39	0.59	0.80	1.12	1.97	3.12
1	48.1%	**5.96**	6.15	3.65	2.19	2.75	3.95	5.25	7.22	12.35	19.40
2	46.9%	**8.74**	8.91	4.64	3.40	4.22	5.96	7.79	10.48	17.15	25.75
					B. $\gamma = 10$						
0.1	**55.2%**	25.02	27.57	**34.05**	6.94	9.05	14.07	20.20	30.83	66.66	**134.4**
1	**50.8%**	26.46	27.73	**20.76**	8.58	10.96	16.33	22.46	32.31	60.6	**104.2**
2	**48.2%**	27.60	28.31	**15.96**	9.76	12.35	17.93	24.01	33.31	57.61	**90.7**

We find very different effects of the EIS on the steady-state w^{ss} and variation/dispersion of w for higher levels of risk aversion, e.g., $\gamma = 10$. For this case, while the impact of the EIS on steady-state savings w^{ss} is limited, the impact of the EIS on variation and dispersion of w is much greater. Panel B of Table **??** shows that as we increase the EIS from $\psi = 0.1$ to $\psi = 2$, w^{ss} does not change much (ranging from 25.0 to 27.6.) This suggests that steady-state and/or average savings for a sufficiently risk-averse agent is effectively independent of the EIS ψ. However, increasing the EIS from 0.1 to 2 discourages the savings for the super w-rich *reducing* the standard deviation of w and more generally the dispersion of w. For example, as we increase the EIS from $\psi = 0.1$ to $\psi = 2$, the top-1% w-rich quantile decreases from 134.4 to 90.7, the standard deviation of w decreases from 34.05 to 15.96, and the Gini coefficient decreases from 55.2% to 48.2%. That is, the effects for the EIS on the variation and dispersion of w are much larger for sufficiently high levels of risk aversion.

The intuition is as follows. With high risk aversion, e.g., $\gamma = 10$, the 1% w-rich quantile is super rich with w being in the range of 90 to 134 for essentially all sensible values of the EIS. With a high EIS, these super w-rich dis-saves much more as under the CM case where it is the EIS not risk aversion that influence savings hence reducing standard deviation of w and causing the distribution of w to be less dispersed.

The Case with Long-Run Risk Parameters. Explaining asset pricing facts (e.g., high equilibrium risk premium and low risk-free rate) is very challenging. One widely-used approach to match asset pricing facts is to incorporate long-run risk (LRR) into a representative-agent model with Epstein-Zin utility following Bansal and Yaron (2004). LRR models require both large risk aversion and high EIS, e.g., $\gamma = 10$ and $\psi = 2$, which is infeasible with expected utility models but is of course well captured by the Epstein-Zin recursive utility. Given the importance of LRR parameter values in the asset pricing literature, we next analyze optimal consumption and wealth dispersion with these parameter values.

The last row in Panel B of Table **??** reports the results with LRR parameters, $\gamma = 10$ and $\psi = 2$. The steady-state savings target is $w^{ss} = 27.60$, which is 3.2 times of $w^{ss} = 8.74$, the steady-state target when $\gamma = 3$ and $\psi = 2$. Not surprisingly, given high risk aversion, incentives to accumulate wealth are very strong.

In summary, we have shown that the choices of the EIS ψ and risk aversion γ have first-order quantitative effects on w^{ss} and the stationary distribution for w. Additionally, how the EIS impacts the steady-state and dispersion of w depends on the level of risk aversion and the sign for the comparative static effect of the EIS on the accumulation and dispersion of w differently significant with risk aversion, as we see from Table **??** due to the non-monotonic effect of the EIS on consumption. As the effects of risk aversion and the EIS are quite different, both qualitatively and quantitatively, our analysis in this section strongly call for separating the EIS from risk aversion in future work.

7 Comparative Statics

We discuss the comparative static effects of volatility σ, income growth μ, interest rate r, and the subjective discount rate ρ on consumption and wealth dispersion.

Volatility σ. Panel A of Figure **??** plots the marginal value of liquidity $p'(w)$ for three levels of volatility: $\sigma = 0.05, 0.1$, and 0.15. For sufficiently high values of w, the higher the volatility σ, the more valuable liquidity is to buffering against shocks implying a higher $p'(w)$. This is the standard incomplete-markets-induced precautionary savings channel. However, for low values of w, the volatility effect on $p'(w)$ depends on whether the borrowing constraint $c(0) \leq 1$ binds or not. If the borrowing constraint does not bind, i.e., *Type A*, the same precautionary savings mechanism operates. However, if the borrowing constraint binds, i.e.,
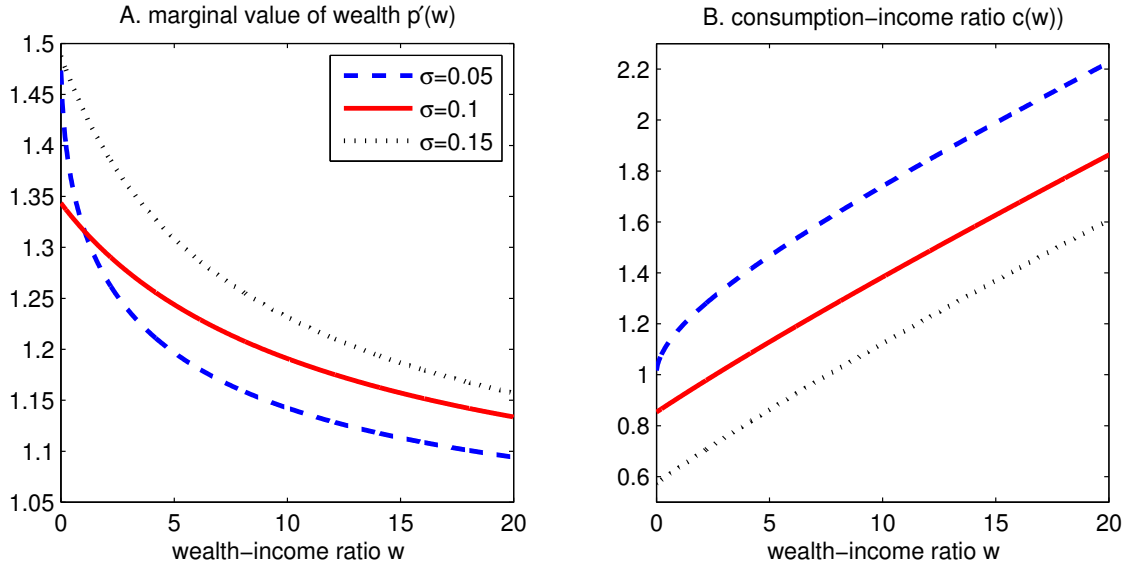
Figure 6: **The marginal (certainty equivalent) value of wealth** $p'(w)$ **and consumption-income ratio** $c(w)$ **for** $\sigma = 0.05$, $\sigma = 0.1$, **and** $\sigma = 0.15$. Other parameter values are: $r = 3.5\%$, $\rho = 4\%$, $\mu = 1.5\%$, $\psi = 1$, and $\gamma = 3$.

Type B, $p'(w)$ then *decreases* with σ for low values of w. For example, as we decrease σ from 10% to 5%, $p'(0)$ increases from 1.34 to 1.47. This is because the marginal value of moving away from the borrowing constraint is higher when income volatility is lower, analogous to the impact of risk aversion on $p'(w)$ for the binding case as discussed in Subsection **??**. Next, we use closed-form expressions to confirm our intuition for *Type B*.

When the constraint binds, differentiating (**??**) with respect to σ^2 gives:

$$\frac{\partial p'(0)}{\partial \sigma^2} = -\frac{\gamma (p'(0))^\psi}{2\psi m^*} < 0 \,. \tag{40}$$

For a borrowing-constrained agent, the lower the income growth volatility σ, the more costly it is to be permanently constrained at $w = 0$, i.e., the higher the value of $p(0)$. Therefore, providing liquidity when $w = 0$ to a borrowing constrained agent with a higher value of $p(0)$ is more valuable implying a higher marginal value of liquidity $p'(0)$. This borrowing-constraint channel, which is absent in *Type A*, causes the reverse order of $p'(0)$ with respect to σ^2 for low values of w. Panel B plots $c(w)$ for the three levels of σ. Consumption decreases with σ regardless of whether the borrowing constraint binds or not. This is intuitive as precautionary savings demand is monotonically increasing in volatility σ.
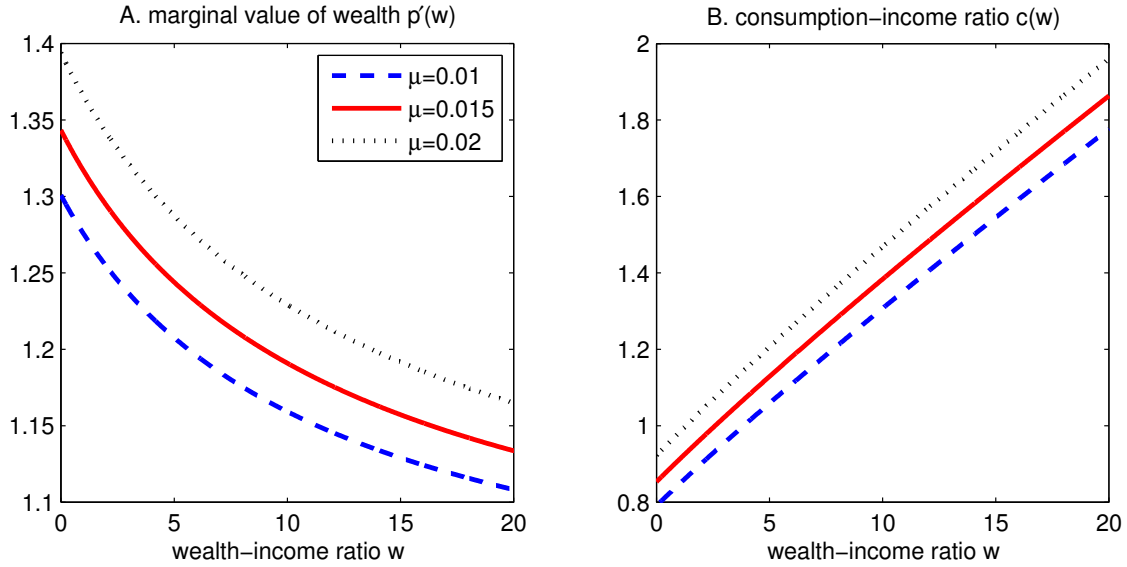
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Figure 7: **The marginal (certainty equivalent) value of wealth** $p'(w)$ **and consumption-income ratio** $c(w)$ **for** $\mu = 0.01$, $\mu = 0.015$, **and** $\mu = 0.02$. Other parameter values are: $r = 3.5\%$, $\rho = 4\%$, $\sigma = 10\%$, $\psi = 1$, and $\gamma = 3$.

Expected Growth Rate μ. Panel A of Figure **??** plots $p'(w)$ for three levels of drift: $\mu = 0.05, 0.1$, and 0.15 showing that $p'(w)$ is increasing in μ. The higher income growth μ, the more willing the agent is to borrow against her future labor incomes, and hence the higher the value of $p'(w)$. Whether the constraint binds or not does not change the comparative static effects of μ on $p'(w)$ and $c'(w)$. For *Type B*, we can show the comparative statics for low values of w in closed form. By differentiating (**??**) with respect to μ, we have:

$$\frac{\partial p'(0)}{\partial \mu} = \frac{(p'(0))^{\psi}}{\psi m^*} > 0. \tag{41}$$

Therefore, borrowing constraints have the same effect on $p'(w)$ as the incomplete-markets friction does. Panel B of Figure **??** shows that optimal consumption increases with the income growth rate μ, which is clearly intuitive.

Interest Rate r **and Subjective Discount Rate** ρ. Table **??** shows that the quantitative effects of the interest rate on savings and wealth dispersion are very large. For example, by increasing r from 3% to 4%, w^{ss} increases by almost six folds from 2.29 to 13.48, and the long run mean increases by 6.4 times from 2.32 to 14.78. The Gini coefficient for w also increases substantially from 45.1% to 54.5%. While our model is partial equilibrium with an

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Table 4: STATIONARY DISTRIBUTION OF w: EFFECTS OF INTEREST RATE r.

This table reports the Gini coefficient, steady-state target w^{ss}, mean, standard deviation, and various quantiles for the stationary distribution of w for $r = 3\%$, 3.5%, and 4%.

r	Gini	w^{ss}	mean	std dev	1%	5%	25%	50%	75%	95%	99%
3%	45.1%	2.29	2.32	1.03	0.97	1.19	1.63	2.08	2.72	4.19	5.92
3.5%	48.1%	5.96	6.15	3.65	2.19	2.75	3.95	5.25	7.22	12.35	19.40
4%	54.5%	13.48	14.78	18.61	3.84	5.01	7.73	11.00	16.59	34.91	69.08

Table 5: STATIONARY DISTRIBUTION OF w: EFFECTS OF SUBJECT DISCOUNT RATE ρ.

This table reports the Gini coefficient, steady state target w^{ss}, mean, standard deviation, and various quantiles for the stationary distribution of w for $\rho = 3.5\%$, 4%, and 4.5%.

ρ	Gini	w^{ss}	mean	std dev	1%	5%	25%	50%	75%	95%	99%
3.5%	54.5%	15.08	16.41	19.13	4.26	5.56	8.59	12.25	18.49	39.05	77.12
4%	48.1%	5.96	6.15	3.65	2.19	2.75	3.95	5.25	7.22	12.35	19.40
4.5%	44.8%	2.08	2.12	0.94	0.88	1.08	1.49	1.90	2.48	3.82	5.40

exogenous interest rate, our results indicate that in Bewley-Aiyagari equilibrium models, the equilibrium determination of interest rate will lower the Gini coefficient of w hence reducing wealth dispersion.[15] This is why the impact of risk aversion on wealth dispersion is not obvious in equilibrium models.

Table ?? demonstrates the large quantitative effects of the subjective discount rate ρ on savings and wealth dispersion. A comparison between Table ?? and Table ?? reveals that the quantitative effect of the discount rate ρ effectively negatively mirrors that of the interest rate r. For example, as ρ decreases from 4.5% to 3.5%, w^{ss} increases by 7.5 times from 2.08 to 15.08, and the long-run mean increases by 7.7 times from 2.12 to 16.41. The Gini coefficient for w also increases substantially from 44.8% to 54.5%, comparable to the increase of the Gini coefficient in response to a 1% increase of the interest rate from 3% to 4%.

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[15]Ljungqvist and Sargent (2004) offers a textbook treatment on equilibrium Bewley-Aiyagari wealth distribution models. De Nardi (2015) provides a survey on quantitative models of wealth distribution.

8 Consumption Dynamics and Empirical Puzzles

In this section, we demonstrate that our model's predictions are consistent with well known empirical regularities such as the "excess sensitivity" and "excess smoothness" properties of consumption dynamics documented in the empirical literature. These two findings are referred to as empirical puzzles as they are inconsistent with the predictions of the PIH.

8.1 Excess Sensitivity and Smoothness Puzzles

Under incomplete markets, given the labor-income process (**??**), we may write the following dynamics implied by the Friedman-Hall's PIH consumption rule (**??**):

$$dC_t = \sigma \frac{r}{r - \mu} Y_t \, d\mathcal{B}_t \,. \tag{42}$$

Consumption dynamics (**??**) generates the following two key empirical predictions:

1. Consumption is a martingale as the drift in (**??**) equals zero. However, this prediction is rejected in the data as consumption changes are predictable by anticipated changes in labor income. Flavin (1981) refers to this empirical regularity as the *excess sensitivity.*

2. In levels, the volatility of consumption changes is larger than the volatility of income changes, σY_t, in that

$$\frac{r}{r - \mu} \sigma Y_t > \sigma Y_t \,, \text{ if and only if } \mu > 0 \,. \tag{43}$$

   Note that $\mu > 0$ is the condition for labor income Y to be non-stationary. Here we have reproduced the PIH's prediction that consumption changes are *more volatile* than income changes for a non-stationary labor-income process. However, this prediction is also empirically counterfactual. Campbell and Deaton (1989) document that consumption is "excessively smooth" to unanticipated changes in labor income and refer to this inconsistency between the US data and the PIH as the *excess smoothness.*

Next, we show that our model generates predictions that are consistent with "excess sensitivity" and "excess smoothness."

8.2 Addressing the Puzzles

Using Ito's formula, we write the dynamics for $c_t = c(w_t)$ as:

$$
\begin{aligned}
dc_t &= c'(w_t)dw_t + \frac{1}{2}c''(w_t)d < w_t, w_t > & (44) \\
&= \left[w_t\mu_w(w_t)c'(w_t) + \frac{1}{2}c''(w_t)w_t^2\sigma^2 \right] dt - c'(w_t)w_t\sigma \, d\mathcal{B}_t & (45) \\
&= \left[\left(w_t(r - \mu + \sigma^2) + 1 - c(w_t) \right) c'(w_t) + \frac{1}{2}c''(w_t)w_t^2\sigma^2 \right] dt - c'(w_t)w_t\sigma \, d\mathcal{B}_t, & (46)
\end{aligned}
$$

where the second and third lines use the dynamics (??)-(??) for w. Again by using Ito's formula, we obtain the following process for stochastic consumption growth:

$$
\frac{dC_t}{C_t} = \frac{dc(w_t)}{c(w_t)} + \frac{dY_t}{Y_t} + \frac{dc(w_t)}{c(w_t)}\frac{dY_t}{Y_t} = g_C(w_t)dt + \sigma_C(w_t)d\mathcal{B}_t, \tag{47}
$$

where the expected consumption growth rate $g_C(w)$ (drift) is given by

$$
g_C(w) = \mu + \frac{c'(w)}{c(w)}\left[(r - \mu)w + 1 - c(w) \right] + \frac{\sigma^2 w^2}{2}\frac{c''(w)}{c(w)}, \tag{48}
$$

and the stochastic volatility of consumption growth, $\sigma_C(w)$, is given by

$$
\sigma_C(w) = \sigma\left(1 - w\frac{c'(w)}{c(w)} \right). \tag{49}
$$

Excess Sensitivity. Our model offers a natural explanation for the "excess sensitivity" puzzle via the expected consumption growth $g_C(w)$ given in (??) for consumption dynamics (??). For a fixed increment Δ, consumption changes over time interval $(t, t + \Delta)$ are predictable by w_t. To be in line with the PIH's assumption so as to compare our model's predictions with the PIH's, we set $\rho = r = 3.5\%$ for numerical analysis in this section.

Panel A of Figure ?? plots the expected consumption growth $g_C(w)$ for $\gamma = 3$. First, $g_C(w)$ decreases with w, which implies that the lower the value of w, the more predictable consumption growth. What is the intuition? Consumption changes are predictable due to financial frictions. The lower the value of w, the larger the precautionary savings demand, and hence the more predictable consumption changes are. This intuition is consistent with our model's prediction that $g_C(w)$ is decreasing.

Additionally, we may use income Y_t to predict consumption growth dC_t/C_t. The higher the value of Y, the lower the value of $w = W/Y$, and hence the higher the value of $g_C(w)$ as $g_C'(w) < 0$. Therefore, income Y_t should positively predict consumption changes, which is
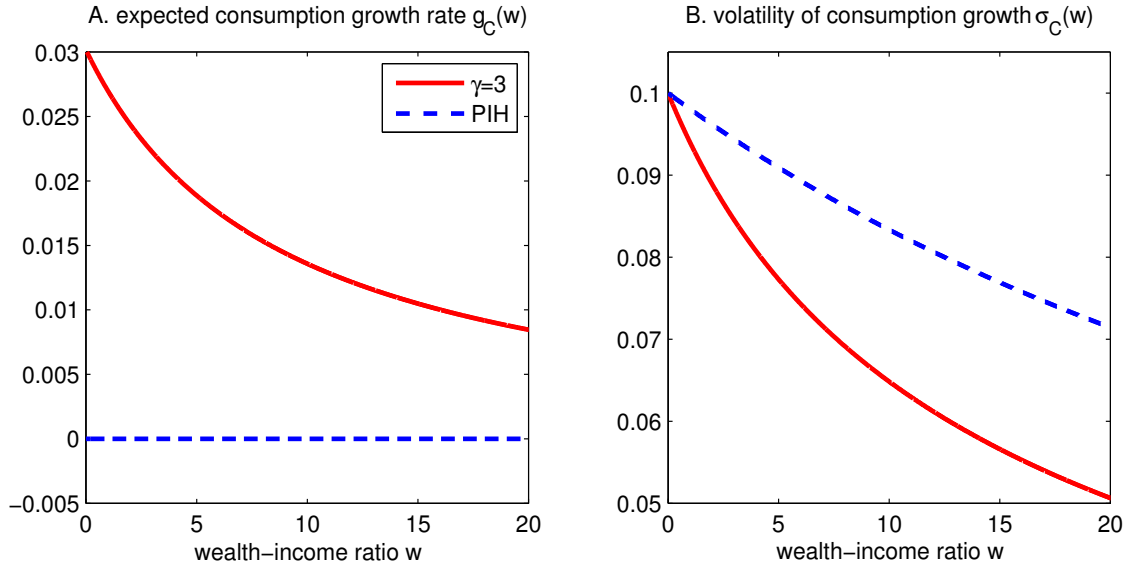
Figure 8: Understanding "excess sensitivity" and "excess smoothness" puzzles: Expected consumption growth $g_C(w)$ and consumption growth volatility $\sigma_C(w)$. Parameter values are: $r = \rho = 3.5\%$, $\sigma = 10\%$, $\mu = 1.5\%$, and $\psi = 1$.

consistent with the empirical "excess sensitivity" property of consumption. To sum up, unlike the martingale consumption property implied by the PIH, our model implies that current income positively predicts consumption growth/changes, consistent with excess sensitivity.

Panel B of Figure **??** plots $\sigma_C(w)$, the volatility of consumption growth dC/C, which decreases with w and is lower than the PIH-implied consumption growth volatility. The higher the value of w, the more effective savings buffer income shocks, and the smoother consumption growth. Additionally, precautionary savings motive weakens the impact of income shocks on consumption growth thus lowering $\sigma_C(w)$ from the PIH-implied volatility level. Next we use our model to address the "excess smoothness" puzzle.

Excess Smoothness. Using the Ito's formula comparing the diffusion coefficient in (**??**) with that in (**??**), we conclude that consumption changes are *less volatile* than income changes in levels, if and only if

the diffusion part of $dC_t = C_Y(W_t, Y_t)\sigma Y_t d\mathcal{B}_t <$ the diffusion part of $dY_t = \sigma Y_t d\mathcal{B}_t$, (50)

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where $C_Y(W_t, Y_t)$ is the MPC out of current labor income. Therefore, consumption exhibits "excess smoothness" relative to income if and only if the MPC out of labor income satisfies:

$$C_Y(W_t, Y_t) < 1. \tag{51}$$

The intuition is as follows. Equation (**??**) implies that consumption responds less to an unexpected change in income, and hence consumption is smoother than labor income in response to unexpected income shocks, if and only if $C_Y < 1$.

Quantitatively, we show that $C_Y < 1$ holds for plausible parameter values. Using the homogeneity property, we have

$$C_Y(W, Y) = \frac{\partial(c(w)Y)}{\partial Y} = c(w) - c'(w)w. \tag{52}$$

Figure **??** plots $C_Y(W, Y)$ and shows that the MPC C_Y is in the range of $(0.77, 0.84)$ for the plotted range: $0 < w < 20$. Therefore, our model's prediction is consistent with the "excess smoothness" of consumption for plausible parameter values while the PIH generates the opposite empirically counter-factual predictions.

To highlight the stark contrast between our model's and the PIH's predictions, we also plot the PIH-implied MPC out of current income, which is given by $C_Y = rh = r/(r - \mu) = 3.5\% \times 50 = 1.75$, about twice as large as the MPC C_Y in our model even for a reasonably large level of liquidity such as $w = 20$.

It is worth noting that CARA-utility-based tractable self-insurance models are also able to address these two puzzles. For example, by using different conditionally heteroskedastic income processes, Caballero (1990) and Wang (2006) show that consumption exhibits excess sensitivity and excess smoothness.[16] These CARA-utility-based models feature no borrowing constraints and consumption can be negative, while our model incorporates borrowing constraints and consumption is required to be positive. Also our model generates more plausible quantitative implications than they do as we use Epstein-Zin utility, while CARA utility misses wealth effects. Thus, CARA-utility models predict that C_Y is independent of wealth and hence wealth is not stationary due to CARA utility's lack of wealth effects, while our model features an intuitive property that C_Y increases w. CARA utility (with CARA coefficient κ) is a special case of the expected utility where the EIS equals $1/(\kappa C)$

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[16]Caballero (1990) uses ARMA processes in discrete time settings and Wang (2006) uses conditionally heteroskedastic income processes that belong to the widely-used "affine" models in finance. See Duffie (2001) for a textbook treatment on these affine models.
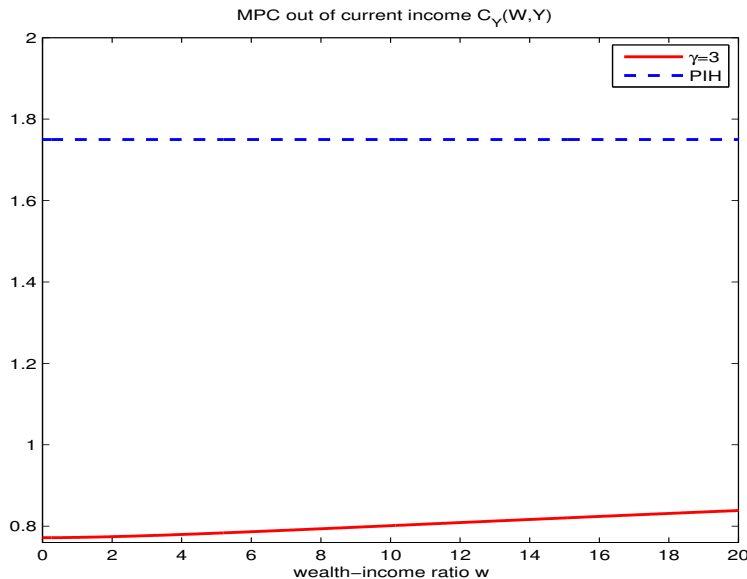
MPC out of current income $C_Y(W,Y)$

Figure 9: **Explaining excess smoothness: The MPC out of current income** $C_Y(W,Y)$**.** Parameter values are: $r = \rho = 3.5\%$, $\sigma = 10\%$, $\mu = 1.5\%$, and $\psi = 1$.

which is time-varying and hard to calibrate, while Epstein-Zin utility allows us to easily calibrate both CRRA coefficient and the EIS. For these reasons, Epstein-Zin utility (including the standard iso-elastic utility) is much preferred for quantitative and qualitative purposes especially when the use of this utility does not come at a cost of losing tractability.

9 Large Discrete (Jump) Earnings Shocks

We have so far specified the income process with diffusive permanent shocks for parsimony. Guvenen et al. (2015) document that earnings shocks display substantial deviations from log-normality and most individuals experience very small earnings shocks while a small but non-negligible number experience very large shocks. It has also been well documented that wages fall dramatically at job displacement, generating so-called "scarring" effects.[17] Wages may be low after unemployment due to fast depreciation of skills as in Ljungqvist and Sargent (1998). Specific human capital can be lost and it may be hard to replace upon re-entry as in Low, Meghir, and Pistaferri (2010). To capture large infrequent income shocks within a very short period in addition to small diffusive shocks, it is necessary to incorporate stochastic jumps into our baseline lognormal diffusion model.

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[17]See Jacobson, LaLonde, and Sullivan (1993) for example.

9.1 Model and Solution

We model large income shocks as jumps with stochastic size occurring at a constant probability λ per unit of time (i.e. Poisson arrivals). When a jump occurs at time t, labor income changes from Y_{t-} to $Y_{t-}Z$ where Z follows a well-behaved probability density function (pdf) $q_Z(z)$ with $Z > 0$. There is no limit to the number of jump shocks and a jump does not change the likelihood of another. We write the labor-income process as:

$$dY_t = \mu Y_{t-}dt + \sigma Y_{t-}d\mathcal{B}_t - (1 - Z)Y_{t-}d\mathcal{J}_t, \ Y_0 > 0, \tag{53}$$

where $\mathcal{J}$ is a pure jump process. That is, $d\mathcal{J}_t = 1$ if the jump happens and $d\mathcal{J}_t = 0$ otherwise. For each realized jump, the expected percentage loss of income is $(1 - \mathbb{E}(Z))$. Since jumps occur with probability λ per unit of time, the expected income growth is thus lowered to $\mu - \lambda(1 - \mathbb{E}(Z))$ from μ. To ensure that human wealth H_t to be finite, we impose:

$$Condition \ 3: \ r > \mu - \lambda(1 - \mathbb{E}(Z)). \tag{54}$$

Under *Condition 3*, human wealth H, defined by (**??**), is proportional to the contemporaneous income Y_t, $H_t = h_J Y_t$, where the multiple h_J is given by

$$h_J = \frac{1}{r - [\mu - \lambda(1 - \mathbb{E}(Z))]}. \tag{55}$$

The CM benchmark consumption rule is then given by $C_t = m^* (W_t + h_J Y_t)$, where m^* is the CM MPC given in (**??**). With CM and purely idiosyncratic diffusion and jump risks, consumption C is continuous and deterministic, in that

$$C_t = e^{-\psi(\rho - r)t} C_0 = e^{-\psi(\rho - r)t} (rW_0 + h_J Y_0). \tag{56}$$

Similar to the baseline model, the agent chooses consumption C to maximize her value function $V(W, Y)$ by solving the following HJB equation:

$$0 = \max_{C>0} f(C, V) + (rW + Y - C)V_W + \mu Y V_Y + \frac{\sigma^2 Y^2}{2} V_{YY} + \lambda \mathbb{E}[V(W, ZY) - V(W, Y)]. \tag{57}$$

The expectation $\mathbb{E}(\cdot)$ is with respect to the probability density function $q_Z(z)$. Next, we summarize the main results.

Proposition 3 *The optimal consumption-income ratio $c(w)$ is given by (??), the same as in the baseline model. The scaled certainty equivalent wealth $p(w)$ solves the following ODE:*

$$0 = \left(\frac{m^*(p'(w))^{1-\psi} - \psi\rho}{\psi - 1} + \mu - \frac{\gamma\sigma^2}{2} \right) p(w) + p'(w) + (r - \mu + \gamma\sigma^2)wp'(w)$$

$$+ \frac{\sigma^2 w^2}{2} \left(p''(w) - \gamma\frac{(p'(w))^2}{p(w)} \right) + \frac{\lambda}{1-\gamma}\mathbb{E}\left[\left(\frac{Zp(w/Z)}{p(w)} \right)^{1-\gamma} - 1 \right] p(w). \quad (58)$$

The above ODE for $p(w)$ is solved subject to the following conditions:

$$\lim_{w\to\infty} p(w) = p^*(w) = w + h_J = w + \frac{1}{r - [\mu - \lambda(1 - \mathbb{E}(Z))]}. \quad (59)$$

Additionally, we require $0 < c(0) \leq 1$, the same condition as (??).

Jumps have two effects. First, when $\mathbb{E}(Z) \neq 1$, jumps change the expected value of future labor incomes and hence reduce h_J if $\mathbb{E}(Z) < 1$. Second, jumps also generate additional precautionary savings demand since jump risk is not spanned and/or borrowing constrained. The last term in (??) captures both mean and higher-order moments effects.[18]

9.2 An Example

The solution presented above applies to any well-behaved distribution for Z. For the numerical example, we consider the case where jumps always lead to losses, i.e. $Z < 1$. We assume that Z follows a power distribution over $[0, 1]$ with parameter $\beta > 0$. Thus, the density function is $q_Z(z) = \beta z^{\beta-1}$ where $0 \leq z \leq 1$. A large value of β implies a small expected income loss of $\mathbb{E}(1 - Z) = 1/(\beta+1)$ in percentages. For $\beta = 1$, Z follows a uniform distribution. For any $\beta > 0$, $-\ln Z$ is exponentially distributed with mean $\mathbb{E}(-\ln Z) = 1/\beta$.

Figure ?? demonstrates the effects of the jump's mean arrival rate λ on $p(w)$, $p'(w)$, $c(w)$, and the MPC $c'(w)$. We set $\beta = 4$ implying an average loss $\mathbb{E}(1 - Z)$ is 20% when a jump occurs. An annual rate of $\lambda = 0.05$ implies a jump every twenty years on average. Therefore, the unconditional expected annual loss is then $\lambda\mathbb{E}(1 - Z) = 0.05 \times 20\% = 1\%$ per year.

Given these parameter values, jumps lower the certainty equivalent wealth at $w = 0$ by 30% from $p(0) = 28.7$ under pure diffusion to $p(0) = 20.07$ and reduce consumption at $w = 0$ by 44% from $c(0) = 0.85$ under pure diffusion to $c(0) = 0.48$.

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[18]Similarly, the continuity of the ODE (??) for $p(w)$ implies an additional condition at $w = 0$:

$$0 = \left[\frac{m^*(p'(0))^{1-\psi} - \psi\rho}{\psi - 1} + \mu - \frac{\gamma\sigma^2}{2} + \frac{\lambda}{1-\gamma}\mathbb{E}\left(Z^{1-\gamma} - 1 \right) \right] p(0) + p'(0).$$
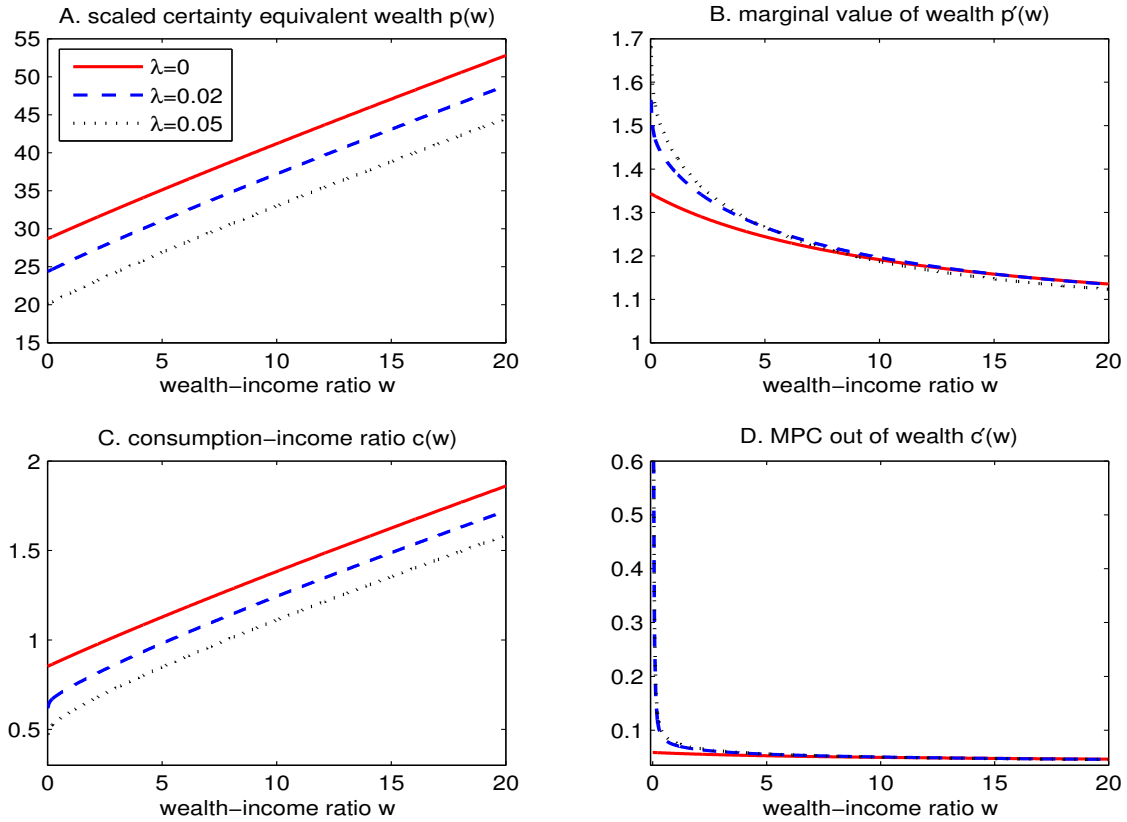
Figure 10: **Large income shocks: The effects of the mean arrival rate** λ. The expected income loss in percentage upon jumps is $\mathbb{E}(1 - Z) = 1/(1 + \beta) = 20\%$. Other parameter values are: $r = 3.5\%$, $\rho = 4\%$, $\sigma = 10\%$, $\mu = 1.5\%$, $\gamma = 3$, and $\psi = 1$.

Additionally, jumps substantially increase the MPC $c'(w)$ even for the unconstrained case. For example, with jumps at the rate of $\lambda = 0.05$, the MPC at $w = 0.04$ increases from $c'(0.04) = 0.06$ under pure diffusion to $c'(0.04) = 0.62$. Parker (1999) and Souleles (1999) report empirical estimates of MPCs in the range of 0.2 to 0.6. Even though these jumps only happen with 5% probability per year and the average loss for each jump event is 20%, jumps are very costly in terms of consumption smoothing as it is very hard to self insure against these jump shocks in the absence of complete markets. Unlike diffusion shocks which are continuous locally, jump shocks cause discrete random changes in both liquidity w and consumption C due to missing markets to manage these jump risks. Therefore, consumption profiles are particularly steep at low values of w, implying that with jumps the MPCs can be particularly large near the origin.

In summary, large downward income shocks even when occurring with low probability

can be very costly in terms of consumption smoothing causing the consumption profile $c(w)$ to be very steep especially for low values of w and hence a high MPC in that region.

10 Concluding Remarks

In this section, we first summarize the main quantitative results of our model and then discuss our plan for future work on wealth distribution by building on the insights and lessons that we have learned from the model developed here. For the quantitative implications, the following results are potentially useful in guiding our future work on wealth distribution.

Summary on Quantitative Results. First, we find that increasing risk aversion (while holding the EIS fixed) substantially increases the steady state (scaled) savings and generates a more dispersed distribution for w. Second, the impact of the EIS on dispersions of scaled wealth and consumption is ambiguous as the EIS has opposing effects on consumption at low and high ends of w. For high levels of risk aversion, increasing the EIS tends to discourage the super rich to save causing the dispersion of w to decrease. This is not desirable given that the empirical facts on wealth dispersion call for the rich to save a lot. For standard values of risk aversion, e.g., around three, the impact of the EIS on dispersion of w can be limited. Third, the impact of the subjective discount rate on wealth dispersion is large, which is line with findings in Krusell and Smith (1998), and the interest rate effect on wealth dispersion essentially mirrors the negative impact of the discount rate ρ, which make intuitive sense, as the wedge $\rho - r$ has a first-order effect on buffer-stock savings.

Fourth, uninsurable labor-income continuous diffusion shocks (under incomplete markets) alone generate annual MPCs that are typically around 4-7% (even when liquidity w is low), which are substantially lower than the empirical range of 0.2 to 0.6 as in Parker (1999) and Souleles (1999). This suggests that pure precautionary savings demand does not generate sufficient consumption demand for the wealth-poor. We can generate MPCs in the empirically plausible range by either choosing parameters to make the borrowing constraint eventually binding (when the agent runs out of her wealth, i.e., $W = 0$,) or incorporating empirically sensible jump features into her labor-income process. Guvenen et al. (2015) report that most individuals experience small earnings shocks and a small but non-negligible number experience very large shocks, which are consistent with our generalized labor-income process with both diffusive shocks and discrete jumps that occur with low probability but

induce sufficiently large movements conditioning on the arrival of a jump.

Next we discuss our future work. Inevitably, we may be a bit more subjective and perhaps speculative than the main body of the paper, where we stay closely to our model.

Some Thoughts on Future Work. While our model's implications on the dispersion of w cannot be directly interpreted as those for the dispersion of wealth, insights that we have gained from the self insurance model are useful for constructing models to understand wealth distribution. A key challenge for equilibrium models is that the wealthy do not save enough and hence the wealth concentration among the rich is too low compared to the data. Based on the 1989 survey of consumer finances (SCF), the top 1% wealth-rich own 29% of the total wealth in the U.S., but Aiyagari (1994) only generates 3.2% wealth holdings by the top 1% wealthy. How do we provide strong incentives for the wealthy to save?

We find that permanent earnings shocks have much more significant effects on savings than transitory shocks do because permanent shocks are much harder to self insure against than the transitory shocks causing the agent to hold much larger stocks of savings. As a first step, permanent earnings shocks in our model increase savings demand in equilibrium. This is different from the standard approach as in Aiyagari (1994) where earnings shocks are transitory hence self insurance is effective causing wealth to be less dispersed than income in equilibrium. Additionally, jumps capture some key features of the earnings processes as shown in Guvenen et al. (2015) and may also be helpful in further increasing savings motives also for the rich as these large discrete shocks are much harder to self insure against.

How do we reconcile a non-stationary earnings process at the micro level with a stationary cross-sectional distribution of wealth? We need an overlapping-generations model with finitely-lived agents, as in Huggett (1993) and De Nardi (2004), for example. While we have focused on the infinite-horizon setting, we can tractably extend our model to life-cycle settings. Although standard finite-horizon OLG models and infinite-horizon models (with stationary income shocks) may not be able to incentivize the rich to save a lot hence failing to match the wealth concentration by the rich, our model may be different as we have permanent earnings shocks with jumps. These risks will be large even for the wealthy over their finite lives and hence their savings motives can remain very strong generating a more concentrated wealth holdings by the wealthy. Additionally, in a life-cycle setting, we can incorporate bequest motives and inter-generational links (e.g., transmission of ability) as in De Nardi (2004) to further strengthen incentives for the rich to save inducing a more

persistent intergenerational wealth dynamics and concentrating wealth among the rich.

In Aiyagari (1994) and the follow-up literature on wealth distribution, the interest rate r will be lower than the subjective discount rate ρ to clear the market, and hence the rich in equilibrium dis-saves as $\rho > r$. How do we encourage the wealthy to continue saving much? This is feasible in OLG models with *ex ante* heterogeneity for preferences and earnings. Here is the intuition. Provided that the wealthy have a low wedge $\rho - r$ (e.g., being close to zero) in equilibrium, their savings demand will remain sufficiently strong causing them to continue saving at a high rate. Then, how do we make the equilibrium wedge $\rho - r$ low for the wealthy? One possibility is to let a small fraction of agents with low ρ and/or high risk aversion γ to face high earnings risks (e.g., large downward earnings jumps) so that in equilibrium this group has strong savings demand but exerts little impact on the equilibrium interest rate which is primarily determined by other agents (large population but less wealthy per capita, however contributing to the majority of aggregate savings and hence exerting first-order effects on the determination of the equilibrium interest rate.) In summary, by incorporating some forms of *ex ante* heterogeneity (preferences and earnings) across agents, we may be able to generate more concentrated wealth holdings by the wealthy who have strong incentives to continue saving.

Finally, incorporating entrepreneurship into our baseline model in an equilibrium setting is potentially a fruitful direction of research that allows us to tie models closer to data. Indeed, the importance of entrepreneurship for wealth distribution has been emphasized and studied in Quadrini (1999), Cagetti and De Nardi (2006), and Buera and Shin (2011), among others. While our model takes the earnings process as exogenously given, we can generalize our model by allowing career choices (e.g., whether being an entrepreneur or an employee.) Intuitively speaking, as entrepreneurs are often exposed to greater business risks than employees and also face capital-intensive investments, they tend to have higher marginal valuation of wealth for both consumption smoothing and investment purposes. We thus expect entrepreneurs have strong savings motives and therefore that incorporating entrepreneurship can help generating substantial wealth concentration among the rich, as shown earlier. Our tractable model may allow us to provide new insights from an entrepreneur's perspective.

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Appendices

We provide various technical details for the main results in appendices.

A   Complete Markets (CM)

First, we compete markets by introducing a traded financial asset that is perfectly correlated with labor income. As labor income risks are idiosyncratic and can be fully diversified away at no premium, we write the dynamics of price process for this new asset as:

$$dS_t = S_t(rdt + \sigma_S d\mathcal{B}_t),$$ (A.1)

where σ_S is the volatility parameter and $\mathcal{B}$ is the same Brownian motion driving the labor-income process (??). Let η_t denote the fraction of the agent's wealth allocated to this risky asset. Then, wealth W accumulates as follows:

$$dW_t = (rW_t + Y_t - C_t)dt + \sigma_S \eta_t W_t d\mathcal{B}_t.$$ (A.2)

Using the standard principle of optimality, we may write the HJB equation as follows:

$$0 = \max_{C,\eta}\ f(C,V) + (rW + Y - C)V_W(W,Y) + \mu Y V_Y(W,Y) + \frac{\eta^2 \sigma_S^2 W^2}{2} V_{WW}(W,Y)$$

$$+ \eta \sigma_S \sigma W Y V_{WY}(W,Y) + \frac{\sigma^2 Y^2}{2} V_{YY}(W,Y).$$ (A.3)

Unlike the incomplete-markets setting, we have V_{WW} due to the stochastic returns for the newly introduced risky asset and also the cross-partial term V_{WY} due to dynamic hedging. Obviously, W and Y are perfectly (negatively) correlated.

By using the homogeneity property of the value function, which holds for all the cases in our paper, we conjecture that the value function is given by

$$V(W,Y) = \frac{(bP(W,Y))^{1-\gamma}}{1-\gamma},$$ (A.4)

where b is given in (??) and $p(w) = P(W,Y)/Y$. Therefore, we have

$$V_W = b^{1-\gamma}(p(w)Y)^{-\gamma}p'(w),$$ (A.5)

$$V_Y = b^{1-\gamma}(p(w)Y)^{-\gamma}(p(w) - wp'(w)),$$ (A.6)

$$V_{WW} = b^{1-\gamma}(p(w)Y)^{-1-\gamma}\left(p(w)p''(w) - \gamma(p'(w))^2\right),$$ (A.7)

$$V_{WY} = b^{1-\gamma}(p(w)Y)^{-1-\gamma}\left(-wp(w)p''(w) - \gamma p'(w)(p(w) - wp'(w))\right),$$ (A.8)

$$V_{YY} = b^{1-\gamma}(p(w)Y)^{-1-\gamma}\left(w^2 p(w)p''(w) - \gamma(p(w) - wp'(w))^2\right).$$ (A.9)

The FOCs for consumption C and hedging demand η imply

$$f_C(C,V) = V_W(W,Y), \tag{A.10}$$

$$\eta = -\frac{\sigma}{\sigma_S}\frac{YV_{WY}(W,Y)}{WV_{WW}(W,Y)}. \tag{A.11}$$

Substituting the value function (**??**), (**??**), (**??**) and (**??**), into the FOCs given in (**??**) and (**??**) we obtain the following decision rules:

$$c(w) = m^*p(w)(p'(w))^{-\psi}, \tag{A.12}$$

$$\eta(w) = \frac{\sigma}{\sigma_S}\left(1 - \frac{\gamma p(w)p'(w)}{w(\gamma(p'(w))^2 - p(w)p''(w))}\right), \tag{A.13}$$

where m^* is defined by

$$m^* \equiv b^{1-\psi}\rho^\psi. \tag{A.14}$$

Substituting the consumption rule (**??**), the hedging demand (**??**) and the value function (**??**) into the HJB equation (**??**), and simplifying, we obtain:

$$0 = \left(\frac{m^*(p'(w))^{1-\psi} - \psi\rho}{\psi - 1} + \mu - \frac{\gamma\sigma^2}{2}\right)p(w) + [(r - \mu)w + 1]\,p'(w) + \frac{\gamma^2\sigma^2 p(w)}{2\left(\gamma - \frac{p(w)p''(w)}{(p'(w))^2}\right)}. \tag{A.15}$$

By using the perfect risk-sharing and consumption smoothing insights for the CM case, we substitute the conjecture $p(w) = p^*(w) = w + h$ into the ODE (**??**) and obtain:

$$0 = \left(\frac{m^* - \psi\rho}{\psi - 1} + \mu - \frac{\gamma\sigma^2}{2}\right)(w + h) + [(r - \mu)w + 1] + \frac{\gamma\sigma^2}{2}(w + h) \tag{A.16}$$

$$= \left(\frac{m^* - \psi\rho}{\psi - 1} + \mu\right)(w + h) + [(r - \mu)w + 1] \tag{A.17}$$

$$= \left(\frac{m^* - \psi\rho}{\psi - 1} + r\right)w + \left(\frac{m^* - \psi\rho}{\psi - 1} + \mu\right)\frac{1}{r - \mu} + 1. \tag{A.18}$$

As (**??**) must hold for all w, we obtain $m^* = r + \psi(\rho - r)$ as given by (**??**). And then using (**??**), we obtain the explicit formula (**??**) for the coefficient b.

Substituting $p(w) = p^*(w) = w + h$ into (**??**) gives the CM consumption rule (**??**). And substituting $p(w) = p^*(w) = w + h$ into (**??**) gives the following hedge portfolio:

$$\eta(w) = \frac{\sigma}{\sigma_S}\left(1 - \frac{p(w)}{w}\right) = -\frac{\sigma h}{\sigma_S w}. \tag{A.19}$$

Note that the hedge portfolio position has to be short in the risky asset to hedge against the idiosyncratic labor-income risk. The total position in the hedge portfolio is then $\eta(w)W =$

$-\sigma h Y/\sigma_S$, which is negative and proportional to contemporaneous income Y. Using Ito's formula, we obtain the following dynamics for wealth W_t^*:

$$
\begin{aligned}
dW_t^* &= \left[rW_t^* + Y_t - C_t^* \right] dt + \sigma_S \eta_t^* W_t^* d\mathcal{B}_t \,, \\
&= \left[-\psi(\rho - r)W_t^* + (1 - m^* h)\, Y_t \right] dt - \sigma h Y_t d\mathcal{B}_t \,. \qquad \text{(A.20)}
\end{aligned}
$$

Note that W_t^* is negatively correlated with labor income Y, so that the total wealth $P^*(W_t, Y_t)$ is deterministic as the risk-averse agent engages in perfect risk sharing. Therefore,

$$
\begin{aligned}
dP^*(W_t, Y_t) = d(W_t^* + hY_t) &= \left[-\psi(\rho - r)W_t^* - \psi(\rho - r)hY_t \right] dt \,, \\
&= -\psi(\rho - r)P^*(W_t, Y_t)\, dt \,. \qquad \text{(A.21)}
\end{aligned}
$$

Special case: $Y = 0$. Note that $Y = 0$ is an absorbing state, as both the drift and volatility of Y for a GBM process are zero at $Y = 0$. Therefore, the terms in (**??**) involving $V_Y(W, 0)$, $V_{YY}(W, 0)$ and $V_{WY}(W, 0)$ are zero. And we write the HJB equation as:

$$
0 = \max_{C, \eta} \ f(C, V) + (rW - C)V_W(W, 0) + \frac{\eta^2 \sigma_S^2 W^2}{2} V_{WW}(W, 0)\,. \qquad \text{(A.22)}
$$

The FOC for consumption C is $f_C(C, V) = V_W(W, 0)$ and there is no hedging demand, $\eta = 0$. It is then straightforward to show that $V(\cdot, 0)$ have the following closed-form solution:

$$
V(W, 0) = \frac{(bW)^{1-\gamma}}{1 - \gamma} \,, \qquad \text{(A.23)}
$$

where b is given by (**??**).

B   Incomplete Markets

We provide a sketch of proofs for main results involving incomplete markets.

Proof of Proposition ??.   Using the homogeneity property $P(W, Y) = p(w)Y$ and substituting the conjectured value function given by (**??**, (**??**), (**??**), and (**??**) into the HJB equation (**??**), we obtain:

$$
\begin{aligned}
0 &= \left(\frac{\psi\rho \left(\frac{c(w)}{bp(w)} \right)^{1-\psi^{-1}} - \psi\rho}{\psi - 1} + \mu - \frac{\gamma\sigma^2}{2} \right) p(w) + (1 - c(w))p'(w) + (r - \mu + \gamma\sigma^2)wp'(w) \\
&\quad + \frac{\sigma^2 w^2}{2} \left(p''(w) - \gamma \frac{(p'(w))^2}{p(w)} \right) . \qquad \text{(B.1)}
\end{aligned}
$$

Substituting the value function (**??**) into the FOC condition for C given in (**??**), we obtain the optimal consumption-income ratio that is given by (**??**). Substituting (**??**) into (**??**), using $m^* = b^{1-\psi}\rho^\psi$, and simplifying, we obtain (**??**), the ODE for $p(w)$.

Next, we turn to the boundary conditions. As $w \to \infty$, it is straightforward to verify $\lim_{w\to\infty} p(w) = p^*(w) = w + h$. Substituting $w = 0$ into the ODE (**??**), we have

$$0 = \left(\frac{m^*(p'(0))^{1-\psi} - \psi\rho}{\psi - 1} + \mu - \frac{\gamma\sigma^2}{2} \right) p(0) + p'(0) \,.$$

Finally, borrowing constraints imply $0 < c(0) \leq 1$.

Conditions for Stationarity of w. How do we ensure that w is stationary in the region $[0, \infty)$? First, the drift in (**??**) should turn negative for sufficiently high w, so that w eventually reverts back to the mean in expectation. As $w \to \infty$, self insurance achieves the CM outcome, $c_t = c^*(w_t)$. Therefore,

$$
\begin{aligned}
\lim_{w\to\infty} \mu_w(w) &= \lim_{w\to\infty} \frac{1}{w} \left(1 + \left(r - \mu + \sigma^2\right) w - m^*(w + h) \right) , \\
&= r - \mu + \sigma^2 - m^* + \lim_{w\to\infty} \left(\frac{1 - m^*h}{w} \right) , \\
&= -\left(\mu - \sigma^2 + \psi(\rho - r)\right) ,
\end{aligned}
\tag{B.2}
$$

where we use $\lim_{w\to\infty} c(w) = c^*(w) = m^*(w+h)$ and the CM MPC result $m^* = r + \psi(\rho - r)$. Note that the borrowing constraint implies that the drift is nonnegative at $w = 0$, $\mu_w(0) \geq 0$. Therefore, there must exist a value of $w \geq 0$ such that $\mu_w(w) = 0$. We denote this value w^{ss}.

As $w_t \to \infty$, its process converges to a GBM with drift given by (**??**) and volatility coefficient $-\sigma$. The following condition is sufficient for the mean of w to exist as $t \to \infty$:

$$\text{Condition 4}: \ \mu - \sigma^2 + \psi(\rho - r) > 0 \,. \tag{B.3}$$

The following ensures that the second moment for the stationary distribution of w exists:

$$\text{Condition 5}: \ \alpha - \sigma^2 + \psi(\rho - r) > 0 \,. \tag{B.4}$$

As $\alpha = \mu - \sigma^2/2 < \mu$, *Condition 5* is stronger than *Condition 4*.

Condition 4 and *Condition 5* suggest that in order to generate a stationary distribution of w, we want one of or a combination of the following three: (i) a sufficiently high growth rate μ lowering $w = W/Y$, (ii) a sufficiently low volatility σ decreasing the probability for low

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Y and hence high w (the denominator effect via Jensen's correction), and (iii) a sufficiently large MPC in excess of the interest rate, $\psi(\rho - r)$ reducing savings (the numerator effect.)

It is worth noting that risk aversion γ appears in neither *Condition 4* nor *Condition 5*. Why? Because stationarity of w only requires that w mean reverts in the limit as $w \to \infty$ and risk aversion has no effect on consumption when $w \to \infty$ as self insurance achieves the CM consumption rule. Note that the standard impatience condition $\rho > r$ in the buffer-stock savings literature is neither necessary nor sufficient to ensure that w is stationary.

Solution: Type B. When $c(0) = 1$, the value function $V(0, Y)$ satisfies the following:

$$0 = f(Y, V) + \mu Y V_Y(0, Y) + \frac{\sigma^2 Y^2}{2} V_{YY}(0, Y).$$ (B.5)

Substituting $V(0, Y) = \frac{(bp(0)Y)^{1-\gamma}}{1-\gamma}$ into (**??**) and simplifying, we obtain (**??**) and (**??**).

Proof of Proposition ??. We extend the methodology for the baseline model to account for jumps. We substitute the value function $V(W, Y)$ given by (**??**) into the HJB equation (**??**) and using (**??**)-(**??**), we obtain the ODE (**??**) for $p(w)$ and (**??**) for $c(w)$ as in the baseline case. Similarly, as $w \to \infty$, $p(w)$ takes the following form

$$\lim_{w \to \infty} p(w) = w + h_J.$$ (B.6)

Substituting the above into ODE (**??**), and simplifying the expression, we obtain (**??**).

C Transitory and Permanent Shocks

Empirical labor-income specifications often feature both permanent and transitory shocks. Meghir and Pistaferri (2011) provide a comprehensive survey. We next generalize the income process to have both permanent and transitory components. We show that transitory income shocks also have an important effect on consumption, especially for the wealth-poor.

We continue to use Y given in (**??**) to denote the permanent component of income. Let x denote the transitory component of income. The total income (in levels), denoted by X, is given by the product of Y and x, $X_t = x_t Y_t$. Empirical researchers often express the income process in logs, $\ln X_t = \ln Y_t + \ln x_t$. In our model, the logarithmic permanent component $\ln Y$ given by (**??**) follows an arithmetic Brownian motion.

Let $\{s_t : t \geq 0\}$ denote the transitory income state. For simplicity, we suppose that s_t is in one of the two states, G and B, which we refer to the good and bad state respectively. The transitory income value x equals x_G in state G and equals x_B in state B, with $x_B < x_G$. Over a small time period $(t, t + \Delta t)$, if the current state is G, the transitory state switches from x_G to x_B with probability $\phi_G \Delta t$, and stays unchanged with the remaining probability $1 - \phi_G \Delta t$. Similarly, the transition probability from B to G over a small time period Δt is $\phi_B \Delta t$. Technically, we model the transitory income state via a two-state Markov chain.[19]

We first summarize the CM setting where the modified PIH consumption rule is optimal. To construct a CM setting, we need to introduce Arrow securities to span the jump risks.

Human wealth and the PIH. As before, we define human wealth H_t under state s_t as the present value of future labor incomes, discounted at the risk-free rate,

$$H_t = \mathbb{E}_t \left(\int_t^\infty e^{-r(u-t)} x_u Y_u du \right) . \tag{C.1}$$

Note that transitory income $\{x_u : u \geq 0\}$ follows a stochastic process. Transitory income shocks affect human wealth in an economically relevant and interesting way. We denote by h_t the agent's human wealth scaled by the permanent component Y, i.e. $h_t = H_t/Y_t$.

We show that h_B and h_G have explicit forms given by

$$h_G = \frac{x_G}{r - \mu} \left(1 + \frac{\phi_G}{r - \mu + \phi_G + \phi_B} \frac{x_B - x_G}{x_G} \right), \tag{C.2}$$

$$h_B = \frac{x_B}{r - \mu} \left(1 + \frac{\phi_B}{r - \mu + \phi_G + \phi_B} \frac{x_G - x_B}{x_B} \right). \tag{C.3}$$

As the formula for h_G is symmetric to that for h_B, we only discuss h_G. First consider the special case where state G is absorbing, i.e. the probability of leaving state G is zero, $\phi_G = 0$. Transitory shocks become permanent and $h_G = x_G/(r-\mu)$. More generally, transitory shocks ($\phi_G > 0$) induce mean reversion between G and B, which we see from the second term in (??) for h_G. The higher the mean arrival rate ϕ_G from state G to B, the lower the value of h_G. Also, the larger the gap $(x_G - x_B)$, the lower the value of h_G.

With CM, consumption is given by the PIH rule, $c_s^*(w) = m^*(w + h_s)$, which implies that scaled consumption is proportional to total wealth $x + h_s$. As expected, the MPC is the same as (??) for the baseline case. Note that (unscaled) consumption C is continuous under the

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[19]Markov chain specifications of the income process are often used in macro consumption-savings literature. Our model can be generalized to allow for multiple discrete states for the transitory income component.

CM setting when transitory regime switching risks are purely idiosyncratic. We next solve for the general incomplete-markets case with both permanent and transitory income shocks.

Incomplete-markets solution. Let $V(W, Y; s)$ denote the value function with wealth W, the permanent component of income Y, and the transitory income state s. In the interior region with positive wealth, i.e. $W_t > 0$, we have the following HJB equation,

$$0 = \max_{C > 0} \; f(C, V) + (rW + x_s Y - C)V_W(W, Y; s) + \mu Y V_Y(W, Y; s) + \frac{\sigma^2 Y^2}{2} V_{YY}(W, Y; s)$$
$$+ \phi_s \left[V(W, Y; s') - V(W, Y; s) \right], \tag{C.4}$$

where s' denotes the other discrete state. Note that the total income flow is $X = x_s Y$. The last term in (??) gives the conditional expected change of $V(W, Y; s)$ due to transitory income shocks. Consumption satisfies the FOC, $f_C(C, V) = V_W(W, Y; s)$. Using the homogeneity property, we write the certainty equivalent wealth $P(W, Y; s) = p(w; s)Y$. The following proposition summarizes the main results.

Proposition 4 *The optimal consumption-income ratio $c_s(w)$ is given by*

$$c_s(w) = m^* p_s(w)(p_s'(w))^{-\psi}, \; s = G, \, B, \tag{C.5}$$

where m^* is given in (??) and $p_s(w)$ solves the following system of ODEs:

$$0 = \left(\frac{m^*(p_s'(w))^{1-\psi} - \psi\rho}{\psi - 1} + \mu - \frac{\gamma\sigma^2}{2} \right) p_s(w) + x_s p_s'(w) + (r - \mu + \gamma\sigma^2) w p_s'(w)$$
$$+ \frac{\sigma^2 w^2}{2} \left(p_s''(w) - \gamma \frac{(p_s'(w))^2}{p_s(w)} \right) + \phi_s \left(p_{s'}(w) - p_s(w) \right), \; s, s' = G, B. \tag{C.6}$$

The above system of ODEs is solved with the following boundary conditions:

$$\lim_{w \to \infty} p_s(w) = w + h_s, \; s = G, \, B, \tag{C.7}$$

where h_s is given by (??) and (??) for state G and B, respectively.

    Additionally, consumption at the origin cannot exceed total income which implies

$$0 < c_s(0) \le x_s, \; s = G, \, B. \tag{C.8}$$

We now have two inter-linked ODEs that jointly characterize $p_G(w)$ and $p_B(w)$. Liquidity constraints in the two states are now different. In state G, $c_G(0)$ can possibly exceed one as
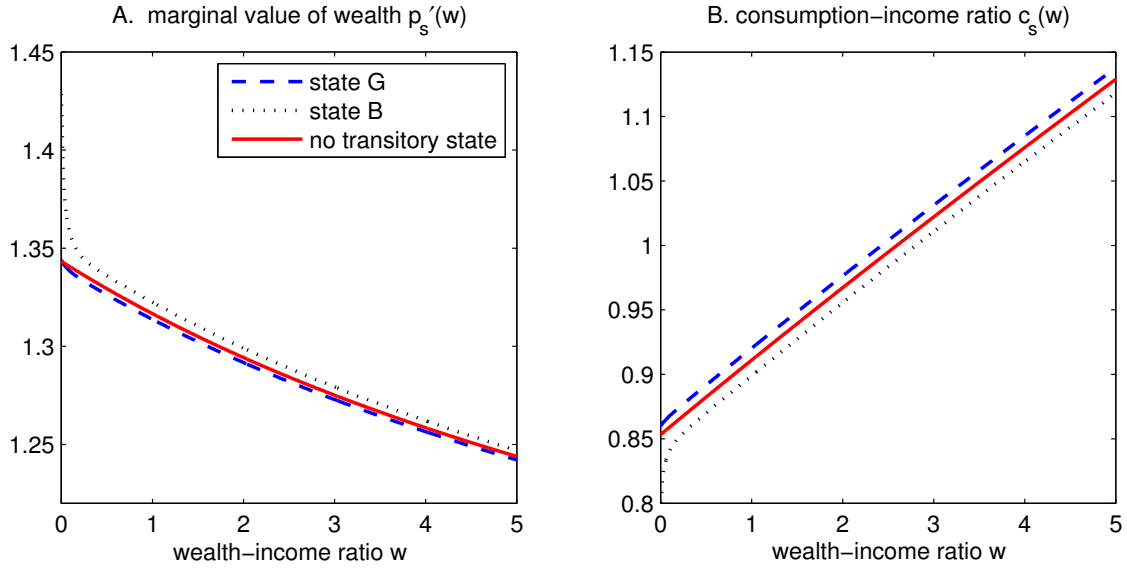
Figure 11: **Permanent and transitory income shocks.** Parameter values are: $r = 3.5\%$, $\rho = 4\%$, $\sigma = 10\%$, $\mu = 1.5\%$, $\gamma = 3$, $\psi = 1$, $\phi_G = 0.5$, $\phi_B = 0.5$, $x_G = 1.2$, and $x_B = 0.8$.

the transitory income shock $x_G > 1$. In State B, $c_B(0) < x_B < 1$. Therefore, the borrowing constraint is tighter in state B than in state G.[20]

Figure **??** demonstrates the effects of transitory income shocks on marginal value of liquidity $p'_s(w)$ and consumption $c_s(w)$ in both G and B states. We choose $x_G = 1.2$ in state G and $x_B = 0.8$ in state B. The mean transition rates from state B to G and from G to B are set at $\phi_B = \phi_G = 0.5$ which implies that the expected durations for both state B and G are 2 year. Using the formulas for h_B and h_G, we obtain $h_B = 49.80$ and $h_G = 50.20$. Because $\phi_G = \phi_B$, the probability mass for the stationary distribution is $\pi_G = \pi_B = 1/2$.

Intuitively, one can view our exercise in this section as a dynamic "mean-preserving" spread of transitory income shocks around the baseline case where $x = 1$. The precautionary savings demand induced by this mean-preserving spread of transitory shocks generates large curvatures for consumption rules $c_s(w)$ in both state G and B especially for low values of w.

The agent becomes constrained at $w = 0$ in state B, in that $c_B(0) = x_B = 0.8$. In-

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[20]Similarly, following the continuity of the ODE (**??**) for $p_s(w)$ at $w = 0$ there is an additional condition at $w = 0$:

$$0 = \left[\frac{m^*(p'_s(0))^{1-\psi} - \psi\rho}{\psi - 1} + \mu - \frac{\gamma\sigma^2}{2} - \phi_s \right] p_s(0) + x_s p'_s(0) + \phi_s p_{s'}(0), \quad s, s' = G, B \,.$$

terestingly, when the transitory income switches out of state B and transitions to state G, consumption jumps from $c_B(0) = x_B = 0.8$ to $c_G(0) = 0.86$. Once in state G, the agent saves $x_G - c_G(0) = 1.2 - 0.86 = 0.34$ per unit of permanent component Y, which is much larger than $0.15 \times Y$, savings in the benchmark case with permanent earnings shocks only. Panel A shows that $p'(w)$ is high for low values of w in state B. For example, $p'(0) = 1.43$. Intuitively speaking, in state B, the agent anticipates her future earnings to be higher and wants to smooth her consumption by borrowing against her future (higher) earnings, which may cause the borrowing constraint to bind. As a result, liquidity not only buffers against uninsurable shocks but also mitigates the impact of a binding borrowing constraint, both of which contribute to a higher $p'(w)$ in state B near $w = 0$. Panel B confirms the intuition that consumption's curvature is larger especially near $w = 0$ in state B than in state G. In summary, transitory income shocks are critically important in understanding consumption and savings for the poor and can generate large consumption responses in state B especially for the w-poor (low w). Finally, we provide a sketch of the proof for Proposition **??**.

Proof of Proposition ??. We conjecture that the value function is given by

$$V(W, Y; s) = \frac{(bP(W, Y; s))^{1-\gamma}}{1 - \gamma}, \tag{C.9}$$

where b is given in (**??**) and $P(W, Y; s)$ is the certainty equivalent wealth. Using (**??**) and the consumption FOC, we jointly solve $p_s(w)$ and the consumption $c_s(w)$ via (**??**) and the interrelated ODEs (**??**) for G and B. As $w \to \infty$, $p_s(w) \to w + h_s$, where h_s is the corresponding human wealth under state s. Substituting $p_s(w) = w + h_s$ into (**??**), we obtain (**??**) and (**??**) for h_G and h_B, respectively. At $w = 0$, we have (**??**). Using the borrowing constraint at $w = 0$, i.e. $C(0, Y; s) \leq X = x_s Y$, we have (**??**).