ROBUST COMPARATIVE STATICS IN LARGE DYNAMIC ECONOMIES

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## ABSTRACT

We consider infinite horizon economies populated by a continuum of agents who are subject to idiosyncratic shocks. This framework contains models of saving and capital accumulation with incomplete markets in the spirit of works by Bewley, Aiyagari, and Huggett, and models of entry, exit and industry dynamics in the spirit of Hopenhayn's work as special cases. Robust and easy-to-apply comparative statics results are established with respect to exogenous parameters as well as various kinds of changes in the Markov processes governing the law of motion of the idiosyncratic shocks. These results complement the existing literature which uses simulations and numerical analysis to study this class of models and are illustrated using a number of examples.

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# 1 Introduction

In several settings, heterogeneous agents make dynamic choices with rewards determined by market prices or aggregate externalities. Those prices and externalities are in turn determined as the aggregates of the decisions of all agents in the market, and because there are sufficiently many agents, each ignores their impact on these aggregate variables. The equilibrium in general takes the form of a stationary distribution of decisions (or state variables such as assets), which remains invariant while each agent experiences changes in their decisions over time as a result of their type and stochastic shocks. Examples include: (1) Bewley-Aiyagari style models (e.g., Bewley (1986), Aiyagari (1994)) of capital accumulation in which each household is subject to idiosyncratic labor income shocks and make saving and consumption decisions taking future prices as given (or the related Huggett (1993) model where savings are in a zero net-supply risk-free asset). Prices are then determined as a function of the aggregate capital stock of the economy, resulting from all households' saving decisions. (2) Models of industry equilibrium in the spirit of Hopenhayn (1992), where each firm has access to a stochastically-evolving production technology, and decides how much to produce and whether to exit given market prices, which are again determined as a function of total production in the economy. (3) Models with aggregate learning-by-doing externalities in the spirit of Arrow (1962) and Romer (1986), where potentially heterogeneous firms make production decisions, taking their future productivity as given, and aggregate productivity is determined as a function of total current or past production. (4) Search models in the spirit of Diamond (1982) and Mortensen and Pissarides (1994) where current production and search effort decisions depend on future thickness of the market.[1]

Despite the common structure across these and several other models, little is known in terms of how the stationary equilibria responds to a range of shocks including changes in preference and production parameters, and changes in the distribution of (idiosyncratic) shocks influencing each agent's decisions. For example, even though the Bewley-Aiyagari model has become a workhorse in modern dynamic macroeconomics, most works rely on numerical analysis to characterize its implications.

In this paper, we provide a general framework for the study of *large dynamic economies*, nesting the above-mentioned models (or their generalizations) and show how "robust" comparative statics of stationary equilibria of these economies can be derived in a simple and tractable manner. Here "robust" comparative statics refers to results, in the spirit of those in supermodular games, that

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[1] Both models under (3) and (4) are typically set up without individual-level heterogeneity and with only limited stochastic shocks, thus stationary equilibria are often symmetric allocations. Our analysis covers significant generalizations of these papers where agents can be of different types and are subject to idiosyncratic shocks represented by arbitrary Markov processes.

hold with minimal functional form restrictions and without necessitating knowledge of specific functional forms and parameter values. Such results are particularly useful in pinpointing the economic forces at work and can naturally complement results in the existing literature on this class of economies which generally focus on simulations and numerical analysis.

Our first substantive theorem establishes monotonicity properties of fixed points of a class of mappings defined over general (non-lattice) spaces. In particular, it establishes that the set of fixed points of an upper hemi-continuous correspondence inherits the various monotonicity properties of the correspondence in question. This result is crucial for deriving comparative statics of stationary equilibria in this class of models, since strategies correspond to random variables and are thus not defined over spaces that are lattices in any natural order.

Our second set of results use this theorem to show how the stationary equilibrium of large dynamic economies respond to a range of exogenous shocks affecting a subset or all economic agents. Examples include changes in the discount factor, the borrowing limits, the parameters of the utility function (*e.g.*, the level of risk aversion), and the parameters of the production function in the Bewley-Aiyagari model, as well as changes in the fixed costs of operation and the parameters of the production function in the Hopenhayn model. In each case, we show that, under minimal and natural assumptions, changes that increase the action of individual agents for a given sequence of market aggregates translate into an increase in the greatest and least stationary equilibrium aggregates (even though as we discuss below it is generally not true that these changes will increase individual actions in equilibrium).

Our third set of results turn to an analysis of the implications of changes in the Markov processes governing the behavior of stochastic shocks. Examples here include first-order stochastic dominance changes or mean-preserving spreads of the stochastic processes affecting the labor incomes of households in Bewley-Aiyagari style models or productivity shocks in the Hopenhayn model. To the best of our knowledge, Huggett (2004) is the only other work studying comparative statics with respect to distributions or risk in this class of models, but he does so focusing on a purely partial equilibrium setup. Our results provide powerful tools for understanding how the stationary equilibrium distributions respond to changes in the law of motion of idiosyncratic shocks in general. Economically, this allows us to address important questions such as how more uncertain earnings prospects affect output-per-worker when agents are borrowing constrained.

In each case, our results are intuitive, easy to apply and robust — even though to the best of our knowledge no similar results have been derived for any of the specific models or for the general class of models under study here. A partial exception is the work by Miao (2002), which we discuss in the next section.

A noteworthy feature of our results is that in most cases, though how aggregates behave can be known robustly, very little or nothing can be said about individual behavior. Thus regularity of (market) aggregates is accompanied with irregularity of individual behavior. This highlights that our results are not a consequence of some implicit strong assumptions — in particular, large dynamic economies are *not* implicitly assumed to be monotone economies (Mirman et al (2008)) — but in the spirit of the famous correspondence principle follow because of the discipline that the market imposes on prices and aggregates.

Our paper is related to two literatures. First, we are building on and extending a variety of well-known models of large dynamic economies, including Bewley (1986), Huggett (1993), Aiyagari (1994), Jovanovic (1982), Hopenhayn (1992), Ericson and Pakes (1995). Though some of these papers contain certain specific results on how equilibria change with parameters (e.g., the effect of relaxing borrowing limits in Aiyagari (1994) and that of productivity on entry in Hopenhayn (1992)), they do not present the general approach or the robust comparative static results provided here. In particular, to the best of our knowledge, none of these papers contain comparative statics either with respect to general changes in preferences and technology or with respect to changes in distributions of shocks, in particular mean-preserving spreads.

Second, our work is related to the robust comparative statics literature including Milgrom and Roberts (1994) and Milgrom and Shannon (1994). Selten (1970) and Corchón (1994) introduced and provided comparative statics for *aggregative* games where payoffs to individual agents depends on their strategies and an aggregate of others' strategies. In Acemoglu and Jensen (2009), we provided more general comparative static results for static aggregative games, thus extending the approach of Milgrom and Roberts (1994) to aggregative games (the earlier literature on aggregative games, including Corchón (1994), exclusively relied on the implicit function theorem). In Acemoglu and Jensen (2010), we considered large static environments in which payoffs depend on aggregates (and individuals ignored their impact on aggregates). To the best of our knowledge, the current paper is the first to provide general comparative statics results for dynamic economies.

We believe that the results provided here are significant for several reasons. First, as discussed at length by Milgrom and Roberts (1994), standard comparative statics methods such as those based on the implicit function theorem often run into difficulty unless there are strong parametric restrictions, and in the presence of such restrictions, the economic role of different ingredients of the model may be blurred. The existence of multiple equilibria is also a challenge to these standard approaches. Second, most existing analyses of this class of dynamic general equilibrium models rely not on comparative statics results but on numerical analysis, *i.e.,* the model is solved numerically for two or several different sets of parameter values in order to obtain

insights about how changes in parameters or policies will impact equilibrium in general (see, for example, Sargent and Ljungqvist (2004)'s textbook analysis of Bewley-Aiyagari and the related Huggett models). The results that follow from numerical analysis may be sensitive to parameter values and the existence of multiple equilibria, and they are also silent about the role of different assumptions of the model on the results. Our approach overcomes these difficulties by providing robust comparative static results for the entire set of equilibria, in the process clarifying the role of different assumptions underpinning such results. We believe that these problems increase the utility of our results and techniques, at the very least as a complement to existing methods of analysis in these dynamic models, since they also clarify the economic role of different ingredients of the model and typically indicate how these results can be extended to other environments.

The structure of the paper is as follows: Section 2 studies some applications. Section 3 describes the basic setup and defines (stationary) equilibria in this framework, and then establishes their existence under general conditions. In Sections 4-5 we present our main comparative statics results. In Section 6 we return to the main examples from Section 2 and use our result to derive a variety of comparative statics results. Proofs are placed in Appendix I (Section 8.1). Appendix II (Section 8.2) contains a short summary of some results from stochastic dynamic programming used throughout the paper, and Appendix III (Section 8.3) discusses aggregation of risk through laws of large numbers.

## 2 Some Examples

This section describes two applications in detail, namely the Bewley-Aiyagari model of saving and capital accumulation, and Hopenhayn's model of industry equilibrium. We also discuss how our large dynamic economies framework can be applied to models from growth theory and search equilibrium.

### 2.1 The Bewley-Aiyagari Model

Let $Q_t$ denote the aggregate capital-labor ratio at date $t$. Given a standard neoclassical production sector, $Q_t$ uniquely determines the wage $w_t = w(Q_t)$ and interest rate $r_t = r(Q_t)$ at date $t$ via the usual marginal product conditions of the firm. Household $i$ chooses their assets $x_{i,t}$ and consumption $c_{i,t}$ at each date in order to maximize:

$$\mathbb{E}_0[\sum_{t=0}^{\infty} \beta^t v_i(c_{i,t})]$$

subject to the constraint:[2]

$$\tilde{\Gamma}_i(x_{i,t}, c_{i,t}, z_{i,t}, Q_t) = \{(x_{i,t+1}, c_{i,t+1}) \in [-\underline{b}_i, \bar{b}_i] \times [0, \bar{c}_i] : x_{i,t+1} \leq r(Q_t)x_{i,t} + w(Q_t)z_{i,t} - c_{i,t}\},$$

where $z_{i,t} \in Z_i \subseteq \mathbb{R}$ denotes the labor endowment of household $i$, which is assumed to follow a Markov process, which may vary across households. In addition, $\underline{b}_i$ is an individual-specific lower bound on assets capturing both natural debt limits and other borrowing constraints. This paper's framework allows for both the case where $\underline{b}_i$ is a fixed parameter, and situations with endogenous borrowing limits where typically $\underline{b}_i$ would be a function of the interest rate $r(Q)$, the wage rate $w(Q)$, or both. For notational simplicity, we shall write $\underline{b}_i$ for the borrowing limit even when it is (indirectly) a function of the capital-labor ratio. $\bar{b}_i$ is an upper bound on assets introduced for expositional simplicity (it will not bind in equilibrium); and $\bar{c}_i$ is an upper bound on consumption also introduced for expositional simplicity. The latter two ensure compactness and avoid unnecessary technical details, though it is worth noting that boundary/interiority/differentiability type assumptions play no role in our comparative statics results. Also worth noting is that the borrowing constraint $\underline{b}_i$ need not bind for a consumer even if the minimal labor endowment shock inf $Z_i$ occurs. Thus the setting nests the complete markets case as well as "mixed" cases where borrowing constraints bind on or off the equilibrium path for some but not all consumers.

More importantly, we assume throughout that there is no aggregate uncertainty, in the sense that total labor endowments in the economy is fixed, i.e.,

$$\int_{[0,1]} z_{i,t} di = 1,$$

where the mathematical meaning of this integral is discussed in Appendix III. Loosely, it can be interpreted as the "average" of the labor endowment of households in the economy.

Note that households in this economy are not assumed to be identical—they could differ with respect to their preferences, labor endowment processes, and borrowing limits. Assuming that $v_i$ is increasing, we can substitute for $c_{i,t}$ to get:

$$\mathbb{E}_0[\sum_{t=0}^{\infty} \beta^t v_i(r(Q_t)x_{i,t} + w(Q_t)z_{i,t} - x_{i,t+1})]$$

It is convenient to define:

$$u_i(x_i, y_i, z_i, Q, a_i) \equiv v_i(r(Q)x_i + w(Q)z_i - y_i) ,$$

and

$$\Gamma_i(x_i, z_i, Q) = \{y_i \in [-\underline{b}_i, \bar{b}_i] : y_i \leq r(Q)x_i + w(Q)z_i\}.$$

[2] For example, the natural debt limit in the stationary equilibrium with rate of return $r$ on the assets would be $\underline{b}_i = -\frac{z_{i,min}}{r-1} < 0$.

Finally, recalling that total labor endowment and the economy is equal to 1, the aggregate capital-labor ratio at date $t$ is defined as

$$Q_t = \int_{[0,1]} x_{i,t} di. \tag{1}$$

A stationary equilibrium will involve $Q_t = Q^*$ for all $t$, and thus will feature constant prices. Focusing on stationary equilibria, our general results establish results of the following form (in case there are more than one stationary equilibrium, the statements refer to the greatest and least aggregates). Note that in this setting any increase in $Q^*$ is also associated with an increase in output per capita:

- If agents become more patient, $Q^*$ will increase. In particular, an increase in the discount rate $\beta$ will lead to an increase in the steady-state capital-labor ratio $Q^*$ (Theorem 6).

- Any "positive shock" (to any subset of the agents not of measure zero) will lead to an increase in $Q^*$ (Theorem 5). Positive shocks are defined formally in Section 4.2, but the economic idea is simple: positive shocks are those that increase individual actions given the sequence of market aggregates, so that this result implies that any shock that encourages a subset of individuals to take greater actions given all else translates into an increase in market aggregates. Interesting economic examples of positive shocks include:

  - A "tightening" of the borrowing constraints, *i.e.*, an increase in $\underline{b}_i$.[3]
  - A decrease in marginal utilities, *i.e.*, any increase in $a_i$ when $v_i = v_i(c_i, a_i)$ and $D^2_{c_i a_i} u_i \leq 0$. For example, if $a_i$ is the rate of absolute or relative risk aversion and $D^2_{c_i a_i} u_i \leq 0$, an increase in risk aversion will be a positive shock.
  - Any technological change that increases $w(Q)$ and $r(Q)$ for any fixed capital-labor ratio $Q$. For example, if production is given by $af(Q)$ where $a$ is a scale parameter and $f$ is the intensive production function, an increase in $a$ will be a positive shock.

- Increases in "earnings risk" leads to an increase in $Q^*$ provided that $v_i$ is strictly concave and exhibits HARA (Carroll and Kimball (1996)). This class of utility functions includes, among others, Constant Relative Risk Aversion (CRRA) and Constant Absolute Risk Aversion (CARA) preferences. In particular, under these conditions, any mean-preserving spread to (any subset of) the households' stochastic processes will lead to an increase in $Q^*$ (Theorem 9).

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[3] When borrowing limits are endogenous (*i.e.*, when $\underline{b}_i$ is not a fixed parameter), a "tightening" means that $\underline{b}_i$ increases for all values of $Q$. For our results it is important that $\underline{b}_i$ is a continuous function of $Q$ (since $w(Q)$ and $r(Q)$ will be continuous this is normally not an issue), but no other conditions (*e.g.*, monotonicity or concavity) are needed.

In all of these cases, though the results are intuitive, we will also show that they cannot be derived from studying individual behavior, and in fact, while market aggregates respond robustly to these changes in the environment, very little or nothing can be said about the behavior of specific types of individuals.

The Bewley-Aiyagari model has become a workhorse framework for macroeconomic analysis, and there are many studies using it in a range of applied problems. Though much of this literature computes equilibria of this model numerically using simulations and other numerical methods (often for quantitative work or estimation), some papers, including Aiyagari's original work, study general properties of stationary equilibria. As already mentioned in the Introduction, most notable are Huggett (2004) and Miao (2002). Huggett studies the role of increased earning risk for an individual's savings decisions, so his is a partial equilibrium analysis. Our results on increased earning risk mentioned above naturally extend Huggett (2004) to a general equilibrium environment.

In addition, Aiyagari (1994) studies the impact of changing borrowing limits on the stationary equilibria, and the same arguments are used more generally in Miao (2002). The strategy of Aiyagari (1994) and Miao (2002) is as follows: First, using firms' profit maximization conditions, the wage rate is expressed in terms of the interest rate $w = w(r)$. Second, an individual consumers' savings (capital supply) can be derived as a function of the sequence of interest rates after substituting $w_t = w(r_t)$ for the wage at each date in the budget constraint. Third, focusing on an individual and keeping the interest rate stationary ($r_t = r$ all $t$), the effect of parameter changes on the capital supply can now be determined. This part involves comparing fixed points on non-lattice spaces,[4] and to achieve this both Aiyagari (1994) and Miao (2002) place strong assumptions on the problem in order to ensure that the strategy, $x^*(r)$ say, is unique *and* stable.[5] In particular, this requires cross-restrictions on preferences, technology, and the Markov

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[4]Pinning down how an individual's *stationary* strategy changes with a parameter always involves a fixed point comparative statics problem since stationary strategies are fixed points of the adjoint Markov operator in stochastic dynamic programming problems (see Stokey and Lucas (1989), p.317, or Appendix 8.2 for the more general case of Markov correspondences). The adjoint Markov operator maps a probability distribution into a probability distribution, and so its domain and range is not a lattice in any natural order (Hopenhayn and Prescott (1992)). The comparative statics strategy used by Aiyagari (1994) or Miao (2002) — discussed in the next footnote — avoids fixed point comparisons but only by assuming (restrictively) uniqueness and "stability".

[5]Uniqueness and stability are obtained by invoking a "monotone mixing" property and Theorem 12.12 in Stokey and Lucas (1989) or a variant thereof. With uniqueness and stability, the effect on stationary equilibrium of a parameter change is straight-forward to compute since for fixed $r$, the sequence of random variables $x_t^*(r)$ after a parameter change will be monotone and converge to the new stationary strategy. As mentioned in the previous footnote, this observation makes a general fixed point comparative statics result such as our theorem 3 unnecessary — but at a high cost even in the Bewley-Aiyagari framework (proving uniqueness and stability of invariant distributions is an incredibly difficult problem in general, so the approach obviously does not generalize to other frameworks). Proving uniqueness and stability requires the Markov process of idiosyncratic shocks to be monotone (this is not so in our case), borrowing limits must be exogenous and must bind for all levels of the interest rate

process governing the labor productivity shocks. An important economic consequence of this is that consumer heterogeneity (different types of consumers) cannot be addressed which excludes framework such as that of *e.g.* Guvenen (2009) from study.[6]

Finally, given unique and stable stationary capital supplies, this approach then derives equilibrium comparative statics results. The best way to understand both how this argument works and its difficulties is to consider the supply and demand for capital in the economy as a function of the interest rate. In particular, let $S(r)$ denote (mean) aggregate capital supply and $D(r)$ denote the aggregate capital demand of the firms. Under natural assumptions (e.g., constant returns to scale), $D(r)$ is downward sloping everywhere. Aiyagari (1994) and Miao (2002) draw $S(r)$ as everywhere upward sloping (Figure IIb in Aiyagari (1994) and Figure 1 in Miao (2002)).
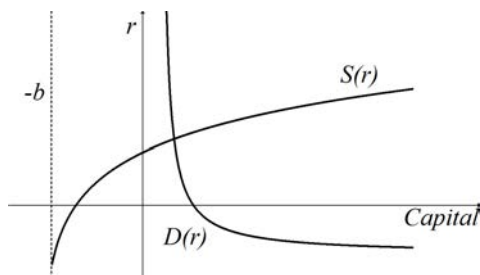


Figure 1: Demand and Supply as depicted by Aiyagari (1994) and Miao (2002).

Any changes in parameters, such as tighter borrowing constraints, that increase $x^*(r)$ for every level of interest $r$ will shift the $S(r)$-curve to the right, and it is then straightforward to see graphically that the aggregate capital-labor ratio will increase. It is on the basis of this argument that Aiyagari (1994) and Miao (2002) conduct their comparative statics analysis. However, an added complication—not formally tackled by these authors—arises: even with unique stationary strategies for any given $r$, $x^*(r)$, and therefore $S(r)$, need not be monotone, and we may have the situation in Figure 2 (note that this figure still assumes that individuals' stationary capital supply is uniquely determined given the interest rate). This is simply a consequence of countervailing income and substitution effects, and without additional, fairly strong, assumptions, which of these

(neither is needed in our case), extensive differentiability assumptions are required to use the envelope theorem repeatedly (which is again not necessary in our approach), and various other fairly stringent assumptions are imposed as discussed in the next footnote. Note also that stability in this context means that with $r$ held fixed, individual strategies converge to $x^*(r)$. The need to impose this condition is an artifact of specific assumptions these papers use, and is in fact not necessary as our analysis below demonstrates.

[6]Specifically, cross-restrictions are necessary because it is crucial for the Aiyagari-Miao type argument that when the minimum labor endowment shock occurs, the borrowing constraint binds *for all levels of the interest rate*. See for example Aiyagari (1993), p. 39, and also Miao (2002) whose Assumption 1.b. serves a similar purpose and is an explicit cross-restriction. As mentioned, a consequence of this is that heterogeneity of agents cannot be addressed generally. In particular, no (subset of the) agents can be borrowing unconstrained at *any* level of the interest rate. Note in this connection that Miao (2002) allows for heterogenous consumers in his general description, but Assumption 1.b. demands that consumers are identical for his general equilibrium analysis, cf. Remark 2.4.(ii).

dominates cannot be known in general (put differently, the problem is that once the wage rate has been eliminated through the relationship $w(r)$, the supply of capital is no longer increasing in the interest rate).
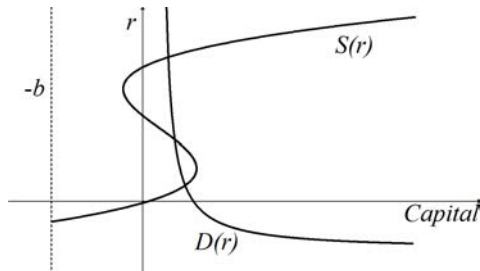


Figure 2: Possible Demand and Supply Diagram without restrictions on income and substitution effects.

This creates additional challenges for the approach that has been utilized in the literature so far. [7]

We will see that, in addition to avoiding these problems, our approach both dispenses with the strong assumptions needed for ensuring uniqueness and stability, and enables us to work with a considerably more general setup. This generality has economic content: for example, our comparative statics results still apply when borrowing constraints are endogenous or do not always bind, or when agents are heterogenous as in Guvenen (2009). This is so even though, as already noted above, there is little that can be said about individual behavior.[8] This highlights that, as we explain below, the robust comparative static results leverage the discipline that the market imposes on aggregates and prices, and the regularity of the behavior of market aggregates coexist with irregularity of individual behavior. These ideas can only be brought out by considering a more general setup as we do in this paper.

## 2.2 Hopenhayn's Model of Entry, Exit, and Firm Dynamics

Here we will study the model of Hopenhayn (1992). Hopenhayn's model of entry, exit, and firm dynamics considers a continuum of firms $\mathcal{I}$ subject to idiosyncratic productivity shocks with

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[7]In light of the proof of our Theorem 5 it is clear that what is needed to fill this gap is the results of Milgrom and Roberts (1994). If one lets $f(r) = D(r) - S(r)$ where now $S$ is parameterized by, say, the borrowing constraint, one will get the results on the greatest and least equilibria (here in terms of the interest rate). Using the interest rate as the independent variable creates some problems, however. In particular, it is now critical that for some large enough value of $r$ (with $\beta(1+r) < 1$), $S(r) > D(r)$ must hold. In the monotone case of Figure 1 this property is guaranteed by the existence of a stationary equilibrium, but in the non-monotonic case of Figure 2 it is not clear from Aiyagari (1994) or Miao (2002) how this can be established.

[8]In contrast, when a monotone relationship between the interest and savings rate is assumed as in Figure 1, individual behavior is pinned down in equilibrium from aggregate variables. In Section 4.3 we present similar "individual comparative statics" results, but for our results on aggregate variables no such individual regularity is needed.

$z_{i,t} \in Z = [0,1]$ denoting firm $i$'s shock at date $t$.

Upon entry, a firm's productivity is drawn from a fixed probability distribution $\nu$, and from then on (as long as the firm remains active), its productivity follows a monotone Markov process with transition function $\Gamma(z, A)$.[9] Let us respect attention to stationary equilibria where the sequence of (output) market prices is constant and equal to $p > 0$. Then at any point in time, the value of an active firm with productivity $z \in Z$ is determined by the value function $V$ which is the solution to the following functional equation:

$$V(p, z) = \max_{d \in \{0,1\}, x \in \mathbb{R}_+} \left\{ (px - C(x, z) - c) + d\beta \int V(p, z')\Gamma(z, dz') \right\} \tag{2}$$

Here $C$ is the cost function for producing $x$ given productivity shock $z$, and $c > 0$ a fixed cost paid each period by incumbent firms. $\beta$ is the discount rate, and $d$ a variable that captures active firms' option to exit ($d = 1$ means that the firm remains active, $d = 0$ that it exits). $C$ is continuous, strictly decreasing in $z$, and strictly convex and increasing in $x$ with $\lim_{x \to \infty} C'(x, z) = \infty$ for all $z$. This ensures that there exists a unique function $V$ that satisfies this equation. Let $d^*(z, p)$ and $x^*(z, p)$ denote the optimal exit and output strategies for a firm with productivity $z$ facing the (stationary) price $p$. It is obvious that the firm will exit if and only if $\int V(p, z')\Gamma(z, dz') \geq 0$. Since $V$ will be strictly decreasing in $z$, this determines a unique (price-dependent) exit cutoff $\bar{z}_p \in Z$ such that $d^*(z, p) = 0$ if and only if $z < \bar{z}_p$.

Any firm that is inactive at date $t$ may enter after paying an entry cost $\gamma(M) > 0$ where $M$ is the measure of firms entering at that date, and $\gamma$ is a strictly increasing function.[10] Given $p$ and the value function $V$ determined from $p$ as described above, new firms will consequently keep entering until their expected profits equals the entry cost:

$$\int V(p, z')\nu(dz') - \gamma(M) = 0, \tag{3}$$

where $\nu$ is the distribution of productivity for new entrants. Given $p$ (and from there $V$), this determines a unique measure of entrants $M_p$. Given $M_p$ and the above determined exit threshold $\bar{z}_p$, the stationary distribution of the productivities of active firms must satisfy:

$$\mu_p(A) = \int_{z_i \geq \bar{z}_p} \Gamma(z_i, A)\mu_p(dz_i) + M\nu(A) \text{ all } A \in \mathcal{B}(Z) \tag{4}$$

where $\mathcal{B}(Z)$ denotes the set of Borel subsets of $Z$.[11]

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[9]So given the shock $z_{i,t}$ at date $t$, the probability of the shock laying in the set $A \subseteq Z$ at date $t+1$ is $\Gamma(z_{i,t}, A)$. Monotonicity means that higher productivity at date $t$ makes higher productivity at date $t+1$ more likely (mathematically $\Gamma(z', \cdot)$ first-order stochastically dominates $\Gamma(z, \cdot)$ whenever $z' \geq z$).

[10]This increasing cost of entry would result, for example, because there is a scarce factor necessary for entry (e.g., land or managerial talent). Hopenhayn (1992) assumes that $\gamma(M)$ is independent of $M$. Our assumption simplifies the exposition, but it is not critical for our results.

[11] Hopenhayn (1992) refers to the measure $\mu_p$ as the *state of the industry*.

The stationary equilibrium price level $p^*$ can now be determined as

$$p^* = D[\int x^*(p^*, z_i)\mu_{p^*}(dz_i)], \tag{5}$$

where $D$ is the inverse demand function for the product of this industry, which is assumed to be continuous and strictly decreasing. This equation makes it clear that the key aggregate (market) variable in this economy, the price level $p$, is determined as an aggregate of the stochastic outputs of a large set of firms. In consequence, from an economic point of view it is intuitive that the Hopenhayn model is a special case of our framework.

From a mathematical point of view, however, there is a slight difference between (5) and equation (1) which determined the market variables in the Aiyagari-Bewley model. Specifically, the right-hand side of (5) is not an integral of stochastic variables over a set of economic agents represented by the set $\mathcal{I}$ (or some subset of $[0,1]$). Nevertheless, this differences is of no consequence. To see this, we can proceed as follows: Instead of $x(p, z_i)$, which can be seen as a random variable defined across a set of heterogeneous firms, consider the random variable $\tilde{x}_i(p)$ drawn independently for a set of $\mathcal{N}$ of firms from the distribution $\mu_p$ (on $(Z, \mathcal{B}(Z), \mu)$), where $\mathcal{N} \subseteq \mathcal{I}$ is the set of active firms. Now let the distribution of productivities across the active firms at some date $t$ be denoted by $\eta_p : \mathcal{N} \to Z$ (where this mapping potentially depends on $p$). Then the frequency distribution (image measure) is given by $\mu_p(A) = \eta_p\{i \in N : \eta_p(i) \in A\}$ where $A$ is any Borel subset of $Z$. Then

$$\int_{\mathcal{N}} \tilde{x}(\eta_p(i))di = \int_Z x^*(z, p)\mu_p(dz).$$

In words, the expected output of the "average" active firm equals the integral of $x^*(\cdot, p)$ under the measure $\mu_p$. Therefore, (5) can be equivalently written as

$$\begin{aligned} p = H((\tilde{x}_i(p))_{i \in \mathcal{I}}) &\equiv \quad D[\int_{i \in \mathcal{N}} \tilde{x}_i(p) \ di] \\ &= \quad D[\int x_i(p^*, z_i)\mu_{p^*}(dz_i)], \end{aligned}$$

which now has exactly the same mathematical form as (1), making it transparent how the Hopenhayn model is a special case of our framework.[12]

In this setting, our general results will lead to the following comparative static results for market aggregates in stationary equilibria:

- A reduction in the fixed cost of operation $c$ or an increase in the transition function $\Gamma$ will increase aggregate output and lower equilibrium price.

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[12]Hopenhayn (1992) briefly discusses the difficulties associated with integrals across random variables and the law of large numbers (Hopenhayn (1992), footnote 5 on p.1131). We discuss this issue in some detail in Appendix III, in particular we explain there why Hopenhayn's favored solution — which involves dependency across firms — will not pose any difficulties for our analysis.

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- A first-order stochastically dominant shift in the entrants' productivity distribution $\nu$ will increase aggregate output and lower the equilibrium price.

- Positive shocks to profit functions, i.e., changes in parameters that increase the desired level of production at a given price, will increase aggregate output and lower the equivalent price.

## 2.3 Additional examples

Several other models can also be cast as special cases of the framework presented here, enabling ready applications of the comparative static results developed below. We describe these models briefly in this subsection since, to economize on space and avoid repetition, we will not explicitly show how our results can be applied for these models.

1. A variety of models where a large number of firms or economic actors create an aggregate externality on others would also be a special case of our framework. A well-known example of this class is Romer's paper on endogenous growth where the aggregate capital stock of the economy determines the productivity of each firm (Romer (1986)). Though Romer's model was deterministic and did not feature any heterogeneity across firms, one could consider generalizations where such stochastic elements are important. For example, we can consider a continuum $\mathcal{I}$ of firms each with production function for a homogeneous final good given by

$$y_{i,t} = f\left(k_{i,t}, A_{i,t} Q_t\right)$$

where $f$ exhibits diminishing returns to scale, is increasing in both of its arguments, and $A_{i,t}$ is independent across producers and follows a Markov process (which can again vary across firms).[13] Each firm faces an exogenous cost of capital $R$. Romer (1986) considered an externality operating from current capital stocks, so that

$$Q_t = \int k_{i,t} di.$$

One could also consider "learning by doing" type externalities that are a function of past cumulative output, i.e.,

$$Q_t = \sum_{\tau = t - T - 1}^{t-1} \int y_{i,\tau} di,$$

for some $T < \infty$. Under these assumptions, all of the results derived below can be applied to this model.

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[13]In Romer's model $f$ exhibits constant returns to scale, which can also be allowed here, but in that case the relevant comparative statics are on the growth rate of $Q_t$.

2. Search models in the spirit of Diamond (1982), Mortensen (1982), and Mortensen and Pissarides (1994), where members of a single population match pairwise to form productive relationships, also constitute special case of this framework. In Diamond's (1982) model, for example, individuals first makes costly investments in order to produce ("collect a coconut") and then search for others who have also done so to form trading relationships. The aggregate variable, taken as given by each agent, is the fraction of agents that are searching for partners. This determines matching probabilities and thus the optimal strategies of each agent. Thus various generalizations of Diamond's model, or for that matter other search models, can also be studied using the framework presented below.

One relevant example in this context is Acemoglu and Shimer (2000), which combines elements from directed search models of Moen (1997) and Acemoglu and Shimer (1999) together with Bewley-Aiyagari style models. In this environment, each individual decides whether to apply to high wage or low wage jobs, recognizing that high wage jobs will have more applicants and thus lower offer rates (these offer rates and exact wages are determined in equilibrium is a function of applications decisions of agents). Individuals have concave preferences and do not have access to outside insurance opportunities, so use their own savings to smooth consumption. Unemployed workers with limited assets then prefer to apply to low wage jobs. Acemoglu and Shimer (2000) assumed a fixed interest rate and used numerical methods to give a glimpse of the structure of equilibrium and to argue that high unemployment benefits can increase output by encouraging more workers apply to high wage jobs. This model—and in fact a version with an endogenous interest-rate—can also be cast as a special case of our framework and thus, in addition to basic existence results, a range of comparative static results can be obtained readily.

## 3    Large Dynamic Economies

In the previous section we presented a number of special cases of the general class of *large dynamic economies*. In this section we describe this class of models in detail, and prove existence of equilibrium and stationary equilibrium. As in the model of Aiyagari (1994) and Bewley (1986), a key feature is the absence of "aggregate risk" as captured by the fact that all interaction between the agents takes place through a one-dimensional deterministic market aggregate (typically, aggregate capital or a price variable). This raises the issue of how the idiosyncratic risk is eliminated at the aggregate level, a question related to various versions of the Law of Large numbers. In this section we give a brief summary of this issue, leaving a detailed discussion until Section 8.3 in the Appendix.

## 3.1 Preferences and Technology

The setting of our analysis is infinite horizon, discrete time economies populated by a continuum of agents $\mathcal{I} = [0, 1]$.[14] Each agent $i \in [0, 1]$ is subject to (uninsurable) idiosyncratic shocks in the form of a Markov process $(z_{i,t})_{t=0}^{\infty}$ where $z_{i,t} \in Z_i \subseteq \mathbb{R}^M$. The only assumption we place on $(z_{i,t})_{t=0}^{\infty}$ is that it must have a unique invariant distribution $\mu_{z_i}$. A special case of this is when the $z_{i,t}$'s are *i.i.d.* in which case $z_{i,t}$ has the distribution $\mu_{z_i}$ for all $t$.

For given initial conditions $(x_{i,0}, z_{i,0}) \in X_i \times Z_i$, agent $i$'s action set is $X_i \subseteq \mathbb{R}^n$, and she solves:

$$\sup \mathbb{E}_0[\sum_{t=0}^{\infty} \beta^t u_i(x_{i,t}, x_{i,t+1}, z_{i,t}, Q_t, a_i)]$$
$$\text{s.t. } x_{i,t+1} \in \Gamma_i(x_{i,t}, z_{i,t}, Q_t, a_i) , \ t = 0, 1, 2, \ldots \tag{6}$$

Here (6) is a standard dynamic programming problem as treated at length in Stokey and Lucas (1989). $u_i : X_i \times X_i \times Z_i \times \mathcal{Q} \times A_i \to \mathbb{R}$ is the *instantaneous payoff function*, $\Gamma_i : X_i \times Z_i \times \mathcal{Q} \times A_i \to 2^{X_i}$ is the *constraint correspondence*, $\beta$ is the agents' common discount factor, $a_i \in A_i \subseteq \mathbb{R}^P$ is a vector of parameters with respect to which we wish to do comparative statics, and $Q_t$ is the market aggregate at time $t$ discussed below. In this setting, a *strategy* $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \ldots)$ is a sequence of random variables defined on the histories of shocks, *i.e.*, a sequence of (measurable) maps $x_{i,t} : Z_i^{t-1} \to X_i$ where $Z_i^{t-1} \equiv \prod_{\tau=0}^{t-1} Z_i$.[15] A *feasible strategy* is one that satisfies the constraints in (6), and an *optimal strategy* is a solution to (6). When a strategy is optimal, it is denoted by $\mathbf{x}_i^*$. The following standard assumptions will ensure the existence of an optimal strategy given any choice of $\mathbf{Q}$, $\mathbf{a}_i$, and $(x_{i,0}, z_{i,0})$:

**Assumption 1** $\beta \in (0, 1)$ *and for all* $i \in \mathcal{I}$: $X_i$ *is compact,* $u_i$ *is bounded and continuous, and* $\Gamma_i$ *is non-empty, compact-valued, and continuous.*

Note that optimal strategies are not necessarily unique under Assumption 1 (in particular, no convexity/quasi-concavity assumptions are imposed).

Let us next turn to the sequence of *market aggregates* $\mathbf{Q} = (Q_0, Q_1, Q_2, \ldots)$, $Q_t \in \mathcal{Q} \subseteq \mathbb{R}$. Each $Q_t$ is a deterministic real variable, and all market interaction takes place through these

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[14] Throughout, all sets are equipped with the Lebesgue measure and Borel algebra (and products of sets with the product measure and product algebra). For a set $Z$, the Borel algebra is denoted by $\mathcal{B}(Z)$ and the set of probability measures on $(Z, \mathcal{B}(Z))$ is denoted by $\mathcal{P}(Z)$.

Although we consider for simplicity only $\mathcal{I} = [0, 1]$, our results hold for any non-atomic measure space of agents. This includes a setting such as that of Al-Najjar (2004), where the set of agents is countable and the measure is finitely additive (see Section 8.3 in the Appendix for further details). In fact, our comparative statics results remain valid for economies with a finite set of agents, provided that appropriate assumptions are made to ensure the absence of aggregate risk and existence of equilibrium (here we have not imposed concavity/convexity type assumptions since the continuum plays a "convexifying" role).

[15] Economically, the map $x_{i,t}$ is a *state-dependent contingency plan*: Given a realized history of shocks $z_i^{t-1} \in Z_i^{t-1}$ the agent will choose $x_{i,t} = x_{i,t}(z^{t-1})$ at date $t$.

aggregates. So in our setting there is *no aggregate uncertainty* (for a detailed discussion of this feature see *e.g.* Lucas (1980), Bewley (1986), and Aiyagari (1994)). To get some understanding of the definition to follow, imagine we are in the income-fluctuation setting of Aiyagari (1994) and that $Q_t$ is the capital-labor ratio at date $t$. Profit maximization of a standard neo-classical production technology entails $r_t = r(Q_t) = f'(Q_t)$ and $w_t = w(Q_t) = f(Q_t) - f'(Q_t)Q_t$ where $f$ is the intensive production technology, $r_t$ is the interest rate, and $w_t$ the real wage at date $t$. So at any date $t$, the constraint of agent $i$ depends only on $Q_t$ by way of the interest and wage rates: $\Gamma_i(x_{i,t}, z_{i,t}, Q_t, a_i) = \{y_i \in [-\underline{b}_i, \overline{b}_i] : y_{i,t} \leq r(Q_t)x_{i,t} + w(Q_t)z_{i,t}\}$. Note also that we might take $a_i = b_i$ and the comparative statics question would then be how a change in the borrowing limits affect equilibria (already mentioned in Section 2.1). Alternatively, the exogenous parameters could enter through the production technology (having $f = f(Q_t, \tilde{a})$ where then $a_i = \tilde{a}$ for all $i$).

## 3.2 Markets and Aggregates

Having now explained the economic intuition behind the market aggregates, we can turn to how they are determined. Recall that in the income-fluctuation setting just described, $Q_t$ is the capital-labor ratio at date $t$. Since $x_{i,t}$ is savings of individual $i$ at date $t$, clearing of the capital markets therefore implies that $Q_t = H((x_{i,t})_{i \in \mathcal{I}})$ where $H((x_{i,t})_{i \in \mathcal{I}})$ is the mean of the strategies:[16]

$$H((x_{i,t})_{i \in \mathcal{I}}) = \int_{[0,1]} x_{i,t} di. \tag{7}$$

For applications, (7) is by far the most important example of a so-called *aggregator* as defined next (see also Acemoglu and Jensen (2009, 2010)). For this paper's results, we allow $H$ to be a general function as long as it is continuous and increasing as explained in a moment. There is little loss of economic content in taking equation (7) as given and skip directly to the definition of an equilibrium (Definition 2).

Now to the technical details: A function $H$ that maps a vector of random variables $(\tilde{x}_i)_{i \in \mathcal{I}}$ into a real number is said to be *increasing* if it is increasing in the first-order stochastic dominance order $\succeq_{st}$, *i.e.*, if $H((\tilde{x}_i)_{i \in \mathcal{I}}) \geq H((x_i)_{i \in \mathcal{I}})$ whenever $\tilde{x}_i \succeq_{st} x_i$ for all $i \in \mathcal{I}$.[17] This property is trivially satisfied for the baseline aggregator (7). Finally, any topological statement relating to sets of random variables (probability distributions), refer to the weak $*$-topology (see *e.g.* Stokey and Lucas (1989), Hopenhayn and Prescott (1992)). The baseline aggregator above is continuous with this topology on its domain.

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[16] When $X_i$ is multi-dimensional, we would usually generalize this by taking $H((x_{i,t})_{i \in \mathcal{I}}) = M(\int_{[0,1]} x_{i,t}^1 di, \ldots, \int_{[0,1]} x_{i,t}^N di)$ where $M : \mathbb{R}^N \to \mathbb{R}$ is a continuous and coordinatewise increasing function.

[17] Let $\tilde{x}_i$ and $x_i$ be a random variables on a set $X_i$ with distributions $\tilde{\mu}_{x_i}$ and $\mu_{x_i}$. Then $\tilde{x}_i$ *first-order stochastically dominates* $x_i$, written $\tilde{x}_i \succeq_{st} x_i$ if $\int_{X_i} f(x_i)\tilde{\mu}_{x_i}(dx_i) \geq \int_{X_i} f(x_i)\mu_{x_i}(dx_i)$ for any increasing function $f : X_i \to \mathbb{R}$.

**Definition 1 (Aggregator)** *An* aggregator *is a continuous and increasing function $H$ that maps the agents' strategies at date $t$ into a real number $Q_t \in Q$ (with $Q \subseteq R$ denoting the range of $H$). The value,*

$$Q_t = H((x_{i,t})_{i \in \mathcal{I}}) \ , \tag{8}$$

*is referred to as the (market)* aggregate *at date $t$.*

**Remark 1** Note that if $H$ is an aggregator, then so is any continuous and increasing transformation of $H$. Thus (7) represents, up to a monotone transformation, the class of separable functions which is consequently a special case of this paper's aggregation concept (see for example Acemoglu and Jensen (2009) for a detailed discussion of separable aggregators).

The conditions in Definition 1 will naturally be satisfied for any reasonable aggregation procedure.[18] This does not mean that there is no ambiguity, however. In fact, even with a functional form such as (7), there is no universal agreement on how this expression should be defined. To avoid unnecessary technical discussion at this point, we have relegated a detailed discussion of this issue to Appendix III. Because we have simply *defined* an aggregator as a real-valued function, we are in effect side-stepping this issue here which has the benefit of not committing us to any specific way of integrating across random variables. In particular, our results' validity are not affected by the controversy surrounding aggregation of risk and the law of large numbers.[19]

## 3.3 Equilibrium

We are now ready to define an equilibrium in large dynamic economies:

**Definition 2 (Equilibrium)** *Let the initial conditions $(z_{i,0}, x_{i,0})_{i \in \mathcal{I}}$ as well as the exogenous variables $(a_i)_{i \in \mathcal{I}}$ be given. An equilibrium $\{\mathbf{Q}^*, (\mathbf{x}_i^*)_{i \in \mathcal{I}}\}$ is a sequence of market aggregates and a sequence of strategies for each of the agents such that:*

1. *For each agent $i \in \mathcal{I}$, $\mathbf{x}_i^* = (x_{i,1}^*, x_{i,2}^*, x_{i,3}^*, \ldots)$ solves (6) given $\mathbf{Q}^* = (Q_0^*, Q_1^*, Q_2^*, \ldots)$ and the initial conditions $(z_{i,0}, x_{i,0})$.*

2. *The market aggregate clears at each date, i.e., $Q_t^* = H((x_{i,t}^*)_{i \in \mathcal{I}})$ for all $t = 0, 1, 2, \ldots$.*

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[18]One technical detail that may be worth noting is that $Q_t$ as determined through an aggregator as above may in some cases not be a real (deterministic) variable but rather a *degenerate* random variable placing probability 1 on some number $Q_t$. Since agents maximize their expected payoffs, this distinction is of no importance.

[19]Our results are robust to all the standard specifications within this literature, including non-standard set-ups such as that of Al-Najjar (2004) (see also footnote 14).

With the baseline aggregator (7), which is simply an integral of the random variables corresponding to individual strategies, Assumption 1 is sufficient to guarantee the existence of an equilibrium due to the convexifying effect of set-valued integration (Aumann (1965)). In particular, payoff functions need not be concave, and constraint correspondences need not have convex graphs. With our general class of aggregators (not necessarily taking the simplified form of (7)), we either have to assume this convexifying feature directly, or alternatively, we must impose concavity and convex graph conditions on the agents. This is the content of the next assumption. To simplify notation we shall from now on write $u_i(x_i, y_i, z_i, Q, a_i)$ in place of $u_i(x_{i,t}, x_{i,t+1}, z_{i,t}, Q_t)$, and similarly we write $\Gamma_i(x_i, z_i, Q, a_i)$ for the constraint correspondence.

**Assumption 2** At least one of the following two conditions hold:

- For each agent, $X_i$ is convex, and given any choice of $z_i$, $Q$, and $a_i$: $u_i(x_i, y_i, z_i, Q, a_i)$ is concave in $(x_i, y_i)$ and $\Gamma_i(\cdot, z_i, Q, a_i)$ has a convex graph.

- The aggregator $H$ is convexifying, *i.e.*, for any subset $B$ of the set of joint strategies such that $H(b)$ is well-defined for all $b \in B$, the image $H(B) = \{H(b) \in \mathbb{R} : b \in B\} \subseteq \mathbb{R}$ is convex.

**Remark 2** In the previous assumption, a convexifying aggregator is defined quite generally. In most situations, the statement that $H(b)$ must be well-defined has a more specific meaning, namely that $b$ is a sequence of joint strategies that is measurable in the agents' indices or across player types (see Appendix III for further details).

We now have:

**Theorem 1 (*Existence of Equilibrium*)** *Under Assumptions 1-2, there exists an equilibrium for any choice of initial conditions $(z_{i,0}, x_{i,0})_{i \in \mathcal{I}}$ and any choice of exogenous variables $(a_i)_{i \in \mathcal{I}}$.*

As with all other results, the proof of Theorem 1 is presented in Appendix I.

## 3.4  Stationary Equilibria

Most of our results will be about stationary equilibria. The simplest way to define a stationary equilibrium in stochastic dynamic settings involves assuming that the initial conditions $(x_{i,0}, z_{i,0})$ are random variables. To simplify notation we use the symbol $\sim$ to express that two random variables have the same distribution.

**Definition 3** *(Stationary Equilibrium) Let the exogenous variables $(a_i)_{i \in \mathcal{I}}$ be given. A stationary equilibrium $\{Q^*, (x_i^*)_{i \in \mathcal{I}}\}$ is a market aggregate and a strategy for each of the agents such that:*

1. *For each agent $i \in \mathcal{I}$, $\mathbf{x}_i^* = (x_i^*, x_i^*, x_i^*, \ldots)$ solves (6) given $\mathbf{Q}^* = (Q^*, Q^*, Q^*, \ldots)$, the stationary process $z_{i,t} \sim z_i$ all $t$, and the randomly drawn initial conditions $(x_{i,0}, z_{i,0}) \sim x_i^* \times z_i$.[20]*

2. *The market aggregates clear (at all dates), i.e.,*

$$Q^* = H((x_i^*)_{i \in \mathcal{I}})$$

A market aggregate $Q^*$ of a stationary equilibrium will be referred to as an *equilibrium aggregate* and the set of equilibrium aggregates given $a = (a_i)_{i \in \mathcal{I}}$ is denoted by $\mathcal{E}(a)$. The greatest and least element in $\mathcal{E}(a)$ are referred to as the greatest and least equilibrium aggregates, respectively.

While Assumptions 1-2 imply existence of an equilibrium, they do not imply existence of a stationary equilibrium. In fact, they do not even ensure that the individual agents' decision problems will admit a stationary strategy given a stationary sequence of market aggregates. Note that in a stationary equilibrium, agent $i$ faces a stationary sequence of aggregates $(Q, Q, \ldots)$ and the stationary risk process $z_{i,t} \sim z_i$ (with distribution $\mu_{z_i}$). She then faces a stationary dynamic programming problem whose value function $v_i$ is determined by the following functional equation:[21]

$$v_i(x_i, z_i, Q, a_i) = \sup_{y_i \in \Gamma_i(x_i, z_i, Q, a_i)} [u_i(x_i, y_i, z_i, Q, a_i) + \beta \int v_i(x_i, z_i', Q, a_i) \mu_{z_i}(dz_i')] \tag{9}$$

Given $v_i$, we can then compute the *(stationary) policy correspondence*:

$$G_i(x_i, z_i, Q, a_i) = \arg \sup_{y_i \in \Gamma_i(x_i, z_i, Q, a_i)} [u_i(x_i, y_i, z_i, Q, a_i) + \beta \int v_i(x_i, z_i', Q, a_i) \mu_{z_i}(dz_i')] \tag{10}$$

The stationary strategy $\mathbf{x}_i^* = (x_i^*, x_i^*, \ldots)$ of Definition 3 is a sequence of random variables such that the distribution of $x_i^*$ is an invariant distribution for this stationary decision problem. The next assumption will ensure that such a stationary optimal strategy exists given any stationary sequence of market aggregates. As we then proceed to show, this together with assumptions 1-2, is sufficient to guarantee that a stationary equilibrium exists.

**Assumption 3** $X_i$ *is a lattice, and given any choice of $z_i$, $Q$, and $a_i$: $u_i(x_i, y_i, z_i, Q, a_i)$ is supermodular in $(x_i, y_i)$ and the graph of $\Gamma_i(\cdot, z_i, Q, a_i)$ is a sublattice of $X_i \times X_i$.*

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[20]Obviously, the probability distribution of $z_i$ is $\mu_{z_i}$ (the invariant distribution of the Markov process governing $z_{i,t}$).

[21]A solution $v_i$ exists and is unique under Assumption 1, see Stokey and Lucas (1989).

**Remark 3** Fixing and suppressing $(z_i, Q, a_i)$, $\Gamma_i$'s graph is a sublattice of $X_i \times X_i$, if for all $x_i^1, x_i^2 \in X$, $y_i^1 \in \Gamma_i(x_i^1)$ and $y_i^2 \in \Gamma_i(x_i^2)$ imply that $y_i^1 \wedge y_i^2 \in \Gamma_i(x_i^1 \wedge x_i^2)$ and $y_i^1 \vee y_i^2 \in \Gamma_i(x_i^1 \vee x_i^2)$. When $X_i \subseteq R$ (one-dimensional choice sets), this will hold if and only if the correspondence is ascending in $x_i$ (Topkis (1978)), meaning that for all $x_i^2 \geq x_i^1$ in $X_i$, $y_i^1 \in \Gamma_i(x_i^1)$ and $y_i^2 \in \Gamma_i(x_i^2)$ imply that $y_i^1 \wedge y_i^2 \in \Gamma_i(x_i^1)$ and $y_i^1 \vee y_i^2 \in \Gamma_i(x_i^2)$.

Assumption 3 implies that the policy correspondence of agent $i$, $G_i(x_i, z_i, Q, a_i)$, is ascending in $x_{i,t}$ (defined formally in the previous remark).[22] Precisely, for $x_i^2 \geq x_i^1$ and $y_i^j \in G_i(x_i^j, z_i, Q, a_i)$, $j = 1, 2$, we have $y_i^1 \wedge y_i^2 \in G_i(x_i^1, z_i, Q, a_i)$ and $y_i^1 \vee y_i^2 \in G_i(x_i^2, z_i, Q, a_i)$.[23] Economically, this means that the current decision is increasing in the last period's decision. For example in the Aiyagari model, higher past savings will increase current income and therefore lead to higher current savings. In dynamic economies, this is typically a rather weak requirement (as opposed to assuming that $G_i$ is ascending in $Q_t$ which is highly restrictive). For example, in the Bewley-Aiyagari model, the condition on $\Gamma_i$ is trivially satisfied and we have $u_i(x_i, y_i, z_i, Q, a_i) = \tilde{u}_i(r(Q)x_i + w(Q)z_i - y_i)$ where $\tilde{u}_i(c_i)$ is agent $i$'s utility from the consumption $c_i$; this implies that $u_i$ will be supermodular in $(x_i, y_i)$ if and only if the utility function $\tilde{u}_i$ is concave.[24]

**Theorem 2** *(Existence of Stationary Equilibrium) Suppose Assumptions 1-3 hold. Then there exists a stationary equilibrium and the set of equilibrium aggregates is compact. In particular, there always exist a greatest and least equilibrium aggregate.*

# 4   Main Results I: Changes in Exogenous Variables

In this section, we start with a new technical result that establishes monotonicity properties of fixed points under sufficiently general conditions to be used in the context of the analysis of comparative statics in large dynamic economies. We then use this result to derive three general comparative statics results that determine how the set of stationary equilibrium aggregates (Definition 3) changes in response to a change in various exogenous parameters. The first of these derives the effects of changes in the exogenous parameters $a = (a_i)_{i \in \mathcal{I}}$ on the greatest and least equilibrium aggregates. The second result shows how a change in the discount factor ("the level of patience") affects these equilibrium aggregates. Our third result tracks the effect of such changes

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[22] An ascending mapping is the same as a mapping that is *increasing in the strong set order*, see for example Milgrom and Shannon (1994).

[23] See Hopenhayn and Prescott (1992), Proposition 2 for a proof of this claim (Hopenhayn and Prescott consider a slightly more general situation and also ensure that $G_i$ will be ascending in $z_i$ which of course requires additional assumptions. Nonetheless, one easily sees that their proof implies that $G_i$ will be ascending in $x_i$ under assumption 3).

[24] This is true in general, but is easiest to see in the twice differentiable case: Since $D_{x_i y_i}^2 u_i = -r(Q)\tilde{u}_i''$, $D_{x_i y_i}^2 u_i \geq 0$ (supermodularity) holds if and only if $\tilde{u}_i'' \leq 0$ (concavity).

on individual strategies but in order to prove such a result much more restrictive assumptions are needed.

For our results on the aggregates, what is most striking is that we *do not* need to assume anything about how the sequence of market variables $(Q_0, Q_1, Q_2, \ldots)$ enters into the payoff functions and constraint correspondences (aside from continuity, cf. Assumptions 1-2).[25] So our assumptions do not restrict us to "monotone economies" (see *e.g.* Mirman et al (2008)). Because of this, we can *not* in general say anything about how the *individual* strategies will respond to changes in exogenous parameters. Indeed, individual strategies' response will in general be highly irregular—unless we add additional monotonicity assumptions as indeed we are forced to do for our individual comparative statics result. But at the *market level*, the irregularity of individual behavior is nonetheless restricted so as to lead to considerable regularity in the aggregate.[26]

## 4.1 Monotonicity of Fixed Points

At the heart of our substantive results is a theorem that enables us to establish monotonicity of fixed points defined over general (non-lattice) spaces. We start with this theorem. The technical problem one faces when working with large economies is that when agents' strategies are random variables (probability measures), their strategy sets will generally *not* be lattices in any natural order (Hopenhayn and Prescott (1992), p.1389).

Furthermore, for general equilibrium analysis, one cannot work with increasing selections from optimal strategies, making it necessary to study the set-valued case in general.[27] Theorem 3 enables us to derive monotonicity results on non-lattice spaces. In the special case of the Bewley-Aiyagari model this is sufficient to obtain powerful comparative static results (and also bypass the difficulties faced by other approaches as discussed in Section 2.1).[28] In general, individ-

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[25] It is useful to note that our results are valid for a finite number of agents as long as these all take the market aggregates as given. This reiterates that our results are not "aggregation" results that depend on the continuum assumption.

[26] Note here that when doing comparative statics in general equilibrium one would be inclined to *first* try to pin down the individual responses and then aggregate. What the previous discussion shows is that this is *not* a good idea, in fact it will not work simply because individual responses are not in general well-behaved. The regular comparative statics results we identify below are a feature of the market level and equilibrium forces impacting aggregate variables.

[27] In general, increasing selections may not exist in the setting of the present paper, but more importantly, even when they exist, general equilibrium analysis requires *all* invariant distributions to be taken into account (the reason is that when market variables change, a property of a specific selection, such as this being the greatest selection, may be lost). This makes it impossible to use such a result as Corollary 3 in Hopenhayn and Prescott (1992) which concerns (single-valued) increasing functions.

[28] Note that even the simplest approach in the Bewley-Aiyagari model as discussed in Section 2.1 requires knowledge of how stationary strategies (which are random variables!) change with parameters to pin down parameters' effect on the aggregate/mean capital supply. In this respect, the approach illustrated in Figure 1 is not simpler than ours - in fact, it is much more complex than our argument because it becomes necessary to establish uniqueness and stability of stationary strategies — and this is a very difficult problem to tackle, even in the basic Bewley-Aiyagari framework (cf. the Appendices in Aiyagari (1993) and Miao (2002)).

ual's stationary strategies will not be unique under our assumptions, making set-valued methods necessary. Theorem 4 replaces standard selection procedures used in the monotone comparative statics literature (where, for example, the least and greatest selections are often considered) with an argument that instead selects directly on the set of equilibrium aggregates. Mathematically, the idea of Theorem 3 is to use the fixed point results of Smithson (1971) instead of the standard Tarski fixed point theorem. The proof of Theorem 4 simply uses upper hemi-continuity and standard results on existence of a maximum. As always, proofs are placed in Appendix I.

Comparative statics of equilibria (whether they are represented by vectors or distributions) boil down to studying the behavior of the fixed points of some mapping $F : X \times T \to 2^X$ where $x \in X$ is the variable of interest. In most of our applications, $x$ is a probability distribution, and $t \in T$ denotes exogenous variables with respect to which comparative statics will be conducted. Mathematically, the question is how the set of fixed points

$$\Lambda(t) \equiv \{x \in X : x \in F(x,t)\}$$

varies with $t$.

When sets are not lattices, we cannot require that the mapping $F$ is ascending (increasing in the strong set order). In the setting of large dynamic economies we are in this situation because $F$ will be the adjoint Markov correspondence defined in Appendix III which maps probability measures into sets of probability measures. In Theorem 12 in Appendix III we prove that the adjoint Markov correspondence *will* satisfy a weaker type of monotonicity properties due to Smithson (1971):

**Definition 4 (Type I and Type II Monotonicity (Smithson (1971)))** *Let $X$ and $Y$ be ordered sets with order $\preceq$. A correspondence $F : X \to 2^Y$ is:*

1. *Type I monotone if for all $x_1 \preceq x_2$ and $y_1 \in F(x_1)$, there exists $y_2 \in F(x_2)$ such that $y_1 \preceq y_2$.*

2. *Type II monotone if for all $x_1 \preceq x_2$ and $y_2 \in F(x_2)$, there exists $y_1 \in F(x_1)$ such that $y_1 \preceq y_2$.*

When a correspondence $F$ is defined on a product set, $F : X \times T \to 2^Y$, where $T$ is also a partially ordered set, we say that $F$ is *type I (type II) monotone in $t$*, if $F : \{x\} \times T \to 2^Y$ is type I (type II) monotone for each $x \in X$. Type I/II monotonicity in $x$ is defined similarly by keeping $t$ fixed. If $F : X \times T \to 2^Y$ is type I (type II) monotone in $x$ as well as in $t$, we simply say that $F$ is type I (type II) monotone.

Note that for a correspondence $F$ to be type I or type II monotone, no specific order structure for the values or domain of $F$ is required.[29] As mentioned, this is critical for the study of large dynamic economies where $F$ is an adjoint Markov correspondence.

The main result, upon which all of the rest of our results built, is

**Theorem 3** *(Comparing Equilibria) Let $X$ be a compact topological space equipped with a closed order $\succeq$, $T$ a partially ordered set, and let $F : X \times \{t\} \to 2^X$ be upper hemi-continuous for each $t \in T$. Define the (possibly empty-valued) fixed point correspondence $\Lambda(t) = \{x \in X : x \in F(x, t)\}$, $\Lambda : T \to 2^X \cup \emptyset$. Then if $F$ is type I monotone, so is $\Lambda$; and if $F$ is type II monotone, so is $\Lambda$.*

The next result, discussed above, is also crucial for our results:

**Theorem 4** *Let $\Lambda(t) \subseteq X$ be the fixed point set of Theorem 3 (for given $t \in T$), assumed here to be non-empty $\Lambda(t) \neq \emptyset$ for $t \in T$, and consider a continuous and increasing function $H : X \to \mathbb{R}$. Define the greatest and least selections from $H \circ \Lambda(t)$: $\overline{h}(t) = \sup_{x \in \Lambda(t)} H(x)$ and $\underline{h}(t) = \inf_{x \in \Lambda(t)} H(x)$. Then if $\Lambda(t)$ is type I monotone, $\overline{h}(t)$ will be increasing; and if $\Lambda(t)$ is type II monotone, $\underline{h}(t)$ will be increasing.*

## 4.2 Comparative Statics of Equilibrium Aggregates

Consider the instantaneous utility function $u_i = u_i(x_i, y_i, z_i, Q, a_i)$ of an agent $i \in \mathcal{I}$. $u_i$ will exhibit *increasing differences* in $y_i$ and $a_i$ if $u_i(x_i, y_i^2, z_i, Q, a_i) - u_i(x_i, y_i^1, z_i, Q, a_i)$ is non-decreasing in $a_i$ whenever $y_i^2 \geq y_i^1$. If $X_i, A_i \subseteq \mathbb{R}$ and $u_i$ is differentiable, increasing differences in $y_i$ and $a_i$ is equivalent to having $D_{y_i a_i}^2 u_i \geq 0$. Increasing differences is of course a very well-known condition in comparative statics analysis (see for example Topkis (1998)).

Next consider the constraint correspondence $\Gamma_i(x_i, z_i, Q, a_i)$ of agent $i$. Following Hopenhayn and Prescott (1992), $\Gamma_i$ is said to have *strict complementarities* in $(x_i, a_i)$ if for any fixed choice of $(z_i, Q)$ it holds for all $x_i^2 \geq x_i^1$ and $a_i^2 \geq a_i^1$, that $y \in \Gamma(x_i^1, z_i, Q, a_i^2)$ and $\tilde{y} \in \Gamma(x_i^2, z_i, Q, a_i^1)$ implies $y \wedge \tilde{y} \in \Gamma(x_i^1, z_i, Q, a_i^1)$ and $y \vee \tilde{y} \in \Gamma(x_i^2, z_i, Q, a_i^2)$.[30] As an illustration consider the constraint correspondence of the Bewley-Aiyagari model of Section 2.1,

$$\Gamma_i(x_i, z_i, Q, a_i) = \{y_i \in [a_i, \overline{b}_i] : y_i \leq r(Q)x_i + w(Q)z_i\}.$$

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[29] This is in sharp contrast to such concepts as monotonicity with respect to the weak or strong set orders (see *e.g.* Shannon (1995)).

[30] Having strict complementarities is a weaker condition than assuming that the graph of $\Gamma_i$ is a sublattice (of $X_i \times X_i \times A_i$ for given $(z_i, Q)$). See Hopenhayn and Prescott (1992) for further details and discussion.

where we have treated the borrowing limit $-\underline{b}_i$ as the parameter (so $a_i = -\underline{b}_i$ where $\underline{b}_i$ is the agents' borrowing limit).[31] Since it is clear that when $x_i^2 \geq x_i^1$ and $a_i^2 \geq a_i^1$, $y \in [a_i^2, r(Q)x_i^1 + w(Q)z_i]$ and $\tilde{y} \in [a_i^1, r(Q)x_i^2 + w(Q)z_i]$ imply that $y \wedge \tilde{y} = \min\{y, \tilde{y}\} \in [a_i^1, r(Q)x_i^1 + w(Q)z_i]$ and $y \vee \tilde{y} = \max\{y, \tilde{y}\} \in [a_i^2, r(Q)x_i^2 + w(Q)z_i]$, this correspondence has strict complementarities in $(x_i, a_i)$. Hence a "tightening" of the borrowing limits in a Bewley-Aiyagari economy will be a positive shock according to the following definition (note that since $a_i$ does not affect the utility function in this case, the increasing differences part is trivially satisfied):

**Definition 5** *(Positive Shocks)* Consider an agent $i \in \mathcal{I}$. A (coordinatewise) increase in the exogenous parameters $a_i$ is a *positive shock* if $u_i(x_i, y_i, z_i, Q, a_i)$ exhibits increasing differences in $y_i$ and $a_i$, and $\Gamma_i(x_i, z_i, Q, a_i)$ has strict complementarities in $(x_i, a_i)$.

Definition 5 gives the "correct" notion of a positive shock: If an increase in $a_i$ is a positive shock, then the policy correspondence $G_i(x_i, z_i, Q, a_i)$ defined in (10) will be ascending in $a_i$ whenever Assumption 3 holds. In other words, it ensures that individual actions ($y_i$) increase, holding market aggregates constant, in response to a change in $a_i$. Having a determinate impact of the change in the environment, here captured by $a_i$, on individual decision problem, *i.e.*, for given aggregates and prices, is clearly a prerequisite for meaningful equilibrium comparative statics. Hence we follow Acemoglu and Jensen (2009) in presenting a positive shock as a definition rather than stating the conditions involved as an assumption.

Importantly, however, this *individual behavior* conclusion is very different from the (general) equilibrium results which are our focus. Positive shocks to the $a_i$'s of a subset of players will lead to increases in those players' strategies *for fixed market aggregates*. But in (general) equilibrium, the market variables will also change—in particular, the initial change in strategies will impact the equilibrium aggregate which will lead to further changes in everyone's strategies, further changes in equilibrium aggregates, and so on until a new equilibrium is reached. Since we have assumed *essentially nothing*—just continuity—about how the market aggregates enter into the agents' decision problems, it may at first appear that very little can be said about how aggregates will change. But as this section's main result shows, on the contrary, we can determine how market aggregates behave quite precisely:

**Theorem 5** *(Comparative Statics of the Aggregate)* *Under Assumptions 1-3, a positive shock to $a_i$ (for all players or any subset) will lead to an increase in the greatest and least stationary equilibrium aggregates.*

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[31]Note that when borrowing limits are endogenous so that $\underline{b}_i$ is a function of $Q$ (possibly by way of the interest rate $r = r(Q)$ or wage rate $w = w(Q)$), the following argument goes through without modifications as long as by a tightening we mean that $\underline{b}_i$ decreases for all $Q$.

Section 6.1 contains a number of applications of this result to the models introduced in Section 2. For example, a tightening of the borrowing limits in the Aiyagari model constitutes a positive shock and will thus increase the greatest and least equilibrium aggregates. As the proof in Section 8.1.3 shows, the fact that such a strong result can be established without any restrictions on the market variables' influence on individual decisions is due to the combined strength of Theorem 3, Theorem 4, and the equilibrium comparative statics results of Milgrom and Roberts (1994). Loosely speaking, monotonicity assumptions are not needed at the market level as long as optimal strategies are upper hemi-continuous in these variables and the aggregator is continuous. he intuition is closely related to that of famous *correspondence principle*: with sufficient regularity of the equilibrium mapping in place, a lot can be said about an economy's comparative statics properties. But whereas the correspondence principle requires one to select stable equilibria, our formulation selects the extremal equilibria (the greatest and least equilibrium aggregates).

The next comparative statics result predicts the effect of a change in agents' level of patience. This result is not a corollary of the previous result because an increase in the discount factor is not covered by our notion of a positive shock.

**Theorem 6** *(Discounting and Stationary Equilibrium) Suppose Assumptions 1-3 hold for every agent i and in addition assume that each $u_i(x_i, y_i, z_i, Q)$ is increasing in $x_i$ and that each $\Gamma_i$ is expansive in $x_i$ ($x_i \leq \tilde{x}_i \Rightarrow \Gamma_i(x_i, z_i, Q) \subseteq \Gamma_i(\tilde{x}_i, z_i, Q)$). Then an increase in the discount factor $\beta$ leads to an increase in the greatest and least stationary equilibrium aggregates.*

**Remark 4** If $u_i$ is decreasing in $x_i$ and $\Gamma_i$ is contractive in $x_i$ ($x_i \leq \tilde{x}_i \Rightarrow \Gamma_i(x_i, z_i, Q) \supseteq \Gamma_i(\tilde{x}_i, z_i, Q)$), the conclusion of the previous theorem changes to: Then an increase in the discount factor $\beta$ leads to a decrease in the greatest and least stationary equilibrium aggregates. To see this, simply substitute $\tilde{y}_i = -y_i$ and $\tilde{x}_i = -x_i$ and follow the proof using the increasing value function $v^n(\tilde{x}_i, \beta)$ in order to conclude that $y_i = -G_i(-x_i, \beta)$ will be descending in $\beta$.

## 4.3   Individual Comparative Statics

The results provided so far hold for any "positive shocks" and do not require any knowledge of how individual behavior responds to aggregates and thus to the changes in parameters Nevertheless, under stronger assumptions we can also specify what happens to individual behavior as we do in the next theorem. Crucially, note that the strategy of proof is to go from the aggregate level to the individual level rather than the other way around as in standard approaches: Once we know that the market aggregate will increase, we can simply treat this as an exogenous variable to the individuals alongside the truly exogenous parameters. The individual comparative statics

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question then becomes a standard comparative statics problem where existing, very powerful results can be made to bear (Topkis (1978), Milgrom and Shannon (1994), and Quah (2007)).

**Theorem 7** *(Individual Comparative Statics) Suppose that an increase in $Q$ is a positive shock to player $i$ (i.e., $u_i$ and $\Gamma_i$ satisfy Definition 5 with $t = Q$). Then in Theorems 5-6, we also have that $x_i$ increases in the stationary equilibrium (the increase here is in the first-order stochastic dominance sense). If instead $Q$ constitutes a "negative shock" (if Definition 5 is satisfied with $t = -Q$) and player $i$ is not affected by any change in exogenous parameters, then in Theorem 5 $x_i$ decreases in the stationary equilibrium.*

In the special case where $Q$ is a positive shock to all agents in the sense of Theorem 7, our economy would be monotone/supermodular. If instead $Q$ is a negative shock for everyone, our economy would be "submodular" (a little is known in terms of comparative statics in such economies). Notably, our main contribution in Theorem 5, requires neither (for all or even a single player).

# 5 Main Results II: Distributional Changes

In this section, we present our comparative statics results in response to changes in the distribution of the idiosyncratic shock processes. Our first result (Theorem 8) deals with first-order stochastic changes in the shock processes. Loosely speaking, first-order stochastic changes will lead to higher equilibrium aggregates if at the individual level: (i) a higher shock in a period increases the strategy in that period (Assumption 4); and (ii) given constant aggregates, a first-order stochastic increase makes the individuals increase their strategies (Assumption 5). As we explain immediately after Theorem 8, (ii) is quite stringent — for example it does not hold in the setting of the Bewley-Aiyagari model. Fortunately, (ii) plays no role for our next theorem which is this section's main result. Theorem 9 shows that, under (i) and certain concavity/convexity assumptions on the instantaneous utility function, any mean-preserving spread to the noise processes will increase the equilibrium aggregate.[32] While the result may at first look somewhat complex because the main conditions are placed on partial derivatives, it is important to keep in mind that such conditions are straight-forward to verify in concrete applications as we will illustrate in Section 6.1.

For the results in this section, the exogenous parameters $(a_i)_{i\in\mathcal{I}}$ play no role, and we suppress them to simplify notation. The following assumption, already mentioned, will be needed throughout:

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[32]In addition, this result requires relatively standard convexity and monotonicity conditions on constraint correspondences and payoff functions.

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**Assumption 4** $u_i(x_i, y_i, z_i, Q)$ *exhibits increasing differences in* $y_i$ *and* $z_i$, *and* $\Gamma_i(x_i, z_i, Q)$ *is ascending in* $z_i$.

When coupled with Assumption 3, Assumption 4 implies that the policy correspondence $G_i(x_i, z_i, Q, a_i)$ is ascending in $z_i$ (Hopenhayn and Prescott (1992)). Economically, this means that a larger value of $z_i$ will lead to an increase in actions. For example, when $z_i$ is labor productivity as in the Aiyagari model, higher labor productivity will increase income and therefore savings.

## 5.1 First-Order Stochastic Dominance Changes

We begin by looking at first-order stochastic dominance increases in the distribution of $z_{i,t}$ for all or a subset of the players. We now need an additional assumption involving once again Hopenhayn and Prescott (1992)'s notion of strict complementarities introduced at the beginning of Section 4.2. $\Gamma_i$ has strict complementarities in $(x_i, z_i)$ if (for any fixed value of $Q$), for all $x_i^2 \geq x_i^1$ and $z_i^2 \geq z_i^1$, $y \in \Gamma_i(x_i^1, z_i^2, Q)$ and $\tilde{y} \in \Gamma_i(x_i^2, z_i^1, Q)$ implies that $y \wedge \tilde{y} \in \Gamma_i(x_i^1, z_i^1, Q)$ and $y \vee \tilde{y} \in \Gamma_i(x_i^2, z_i^2, Q)$.

**Assumption 5** $\Gamma_i(x_i, z_i, Q, a_i)$ *has strict complementarities in* $(x_i, z_i)$.

Let stationary distributions of $z_i$, $\mu_{z_i}$ be ordered by first-order stochastic dominance. Then Assumptions 3-5 together ensure that the policy correspondence of player $i$, when parameterized by $\mu_{z_i}$, $G_i(x_{i,t}, z_{i,t}, \mu_{z_i})$ is ascending in $\mu_{z_i}$ (Hopenhayn and Prescott (1992)). It is intuitively clear that when this is so, a first-order stochastic dominance increase in $\mu_i$ will lead to an increase in the affected player's optimal strategy, and just as with our previous results, the main contribution of the next theorem is to show that this will translate into an increase in the aggregate *in equilibrium* (see the discussion prior to Theorem 5 on this).

**Theorem 8** *(Comparative Statics of a First-Order Stochastic Dominance Change) Under Assumptions 1-5, a first-order stochastic dominance increase in the stationary distribution of* $z_{i,t}$ *for all* $i$ *(or any subset hereof), will lead to an increase in the greatest and least stationary equilibrium aggregates.*

**Remark 5** It is straightforward to see that Theorem 7 carries over to this case to obtain individual comparative statics results once the change in the aggregate is determined.

It should be noted that Assumption 5 may be quite restrictive because a first-order stochastic dominance increase in the stationary distribution of an agent's idiosyncratic shocks may not lead

to a first-order stochastic dominance increase in her strategy. This can be seen in the context of the Bewley-Aiyagari model introduced in Section 2.1 where:

$$\Gamma_i(x_i, z_i, Q) = \{y_i \in [-\underline{b}_i, \overline{b}_i] : y_i \le r(Q)x_i + w(Q)z_i\}.$$

To illustrate this, take $r(Q) = w(Q) = 1$ *and* $x_i^1 = 1$, $x_i^2 = 2$, $z_i^1 = 1$, and $z_i^2 = 3$. Then let $y = 4 \in \Gamma_i(x_i^1, z_i^2, Q) = [-\underline{b}_i, 4]$ and $\tilde{y} = 3 \in \Gamma_i(x_i^2, z_i^1, Q) = [-\underline{b}_i, 3]$. But it is clear then that $y \wedge \tilde{y} = 3 \notin \Gamma_i(x_i^1, z_i^1, Q) = [-\underline{b}, 2]$, and so $\Gamma_i$ does not have strict complementarities in $(x_i, z_i)$. So in the Bewley-Aiyagari model any general results from first-order stochastic dominance changes in the noise environment are not possible. Nevertheless, interestingly, we will see that mean-preserving spreads lead to unambiguous changes in market aggregates without any need for strict complementarities in $(x_i, z_i)$.

## 5.2 Mean Preserving Spreads

We now investigate how mean-preserving spreads of the stationary distributions of the individual-level noise processes affect equilibrium outcomes.[33] For example, in the Bewley-Aiyagari model where $z_i$ is the labor endowment/earnings, a mean-preserving spread intuitively means that consumers face increased uncertainty about their earnings with the mean staying the same. The question we address is then whether more uncertain earning prospects will lead to higher capital- and output-per-capita in equilibrium. Note that this kind of question can be thought of as a natural extension to a general equilibrium setting of the works on precautionary saving in partial equilibrium settings (*e.g.* Huggett (2004) returned to below).

For the result to follow we need additional structure on the individuals' decision problems. Recall that a correspondence $\Gamma : X \to 2^X$ has a *convex graph* if for all $x, \tilde{x} \in X$ and $y \in \Gamma(x)$ and $\tilde{y} \in \Gamma(\tilde{x})$: $\lambda y + (1 - \lambda)\tilde{y} \in \Gamma(\lambda x + (1 - \lambda)\tilde{x})$ for all $\lambda \in [0, 1]$.

**Assumption 6**

1. $X_i \subseteq \mathbb{R}$ for all $i$.[34]

2. $\Gamma_i(\cdot, z_i, Q) : X_i \to 2^{X_i}$ and $\Gamma_i(x_i, \cdot, Q) : Z_i \to 2^{X_i}$ have convex graphs and $u_i(x_i, y_i, z_i, Q)$ is concave in $(x_i, y_i)$, strictly concave in $y_i$, and is increasing in $x_i$.

Assumption 6 is of course entirely standard (see for example Stokey and Lucas (1989)), and it is easily satisfied in all of this paper's applications. The following definition is less familiar.

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[33] $\mu_{z_i}$ is a *mean-preserving spread* of $\mu_{z_i}$ if and only if $\mu_{z_i} \succeq_{cx} \mu'_{z_i}$ where $\succeq_{cx}$ is the convex order ($\mu_{z_i} \succeq_{cx} \mu'_{z_i}$ if and only if $\int f(\tau)\mu(\tau) \ge \int f(\tau)\mu'(\tau)$ for all convex functions $f$).

[34] This part of the assumption is imposed for notational convenience and can be relaxed.

**Definition 6** *Let $k \geq 0$. A function $f : X \to \mathbb{R}_+$ is $k$-convex [$k$-concave] if:*

- *When $k \neq 1$, the function $\frac{1}{1-k}[f(x)]^{1-k}$ is convex [concave].*

- *When $k = 1$, the function $\log f(x)$ is convex [concave] (i.e. $f$ is log-convex [log-concave]).*

A detailed treatment of the concepts of $k$-convexity and $k$-concavity can be found in Jensen (2012b). The essence of the concepts is that $k$-convexity is a strengthening of (conventional) convexity, while $k$-concavity is a weakening of concavity. So in terms of the conditions on the derivatives in this section's main result which follows next, the requirement is loosely that some derivatives must be "a little more than convex" while others must be "a little less than concave". In light of the literature on precautionary savings (again see *e.g.* Huggett (2004) and references therein), there is of course nothing at all surprising in the fact that our results place conditions on the curvature of the partial derivatives (third derivatives). The economic intuition should be straight-forward to grasp in light of our previous results: Under the theorem's conditions, mean-preserving spreads will amount to "positive shocks" in the sense that, given equilibrium aggregates, they will make the affected individuals increase their strategies (in the income-allocation setting this is precisely the precautionary savings motive). Then we again use the results on the monotonicity of fixed points from Section 4.1 to ensure that equilibrium aggregates change in the same direction. As usual, this happens *despite* the fact that at the individual level there is no regularity (for example, it may easily happen that some of the agents that are subjected to increased risk end up *lowering* their strategies in equilibrium).

**Theorem 9** *(**Comparative Statics Effect of Mean-Preserving Spreads**) Suppose that Assumptions 1-4, and 6 hold for all agents, and in addition assume that each $u_i$ is differentiable and satisfies the following upper boundary condition $\lim_{y_i^n \uparrow \sup \Gamma_i(x_i, z_i, Q)} D_{y_i} u_i(x_i, y_i^n, z_i, Q) = -\infty$ (which ensures that $\sup \Gamma_i(x_i, z_i, Q)$ will never be optimal given $(x_i, z_i, Q)$). Then a mean-preserving spread to the invariant distribution $\mu_{z_i}$ of any subset of agents $\mathcal{I}' \subseteq \mathcal{I}$ will lead to an increase in the greatest and least stationary equilibrium aggregates if for each $i \in \mathcal{I}$, there exists a $k_i \geq 0$ such that $-D_{y_i} u_i(x_i, y_i, z_i, Q)$ is $k_i$-concave in $(x_i, y_i)$ as well as in $(y_i, z_i)$; and $D_{x_i} u_i(x_i, y_i, z_i, Q)$ is $k_i$-convex in $(x_i, y_i)$ as well as in $(y_i, z_i)$.*

Theorem 9 provides a fairly easy-to-apply result showing how changes in the individual-level noise affect market aggregates (for an example that explicitly verifies the various conditions see Section 6.1). Mathematically, mean-preserving spreads increase individual level actions whenever the policy correspondence defined in (10) is *convex* in $x_i$ (note that the policy correspondence

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will be single-valued/a function under Assumption 6, so this statement is unambiguous). The assumptions imposed in Theorem 9 ensure such convexity of policy functions.[35]

# 6  Applications

In this section we apply the comparative statics results already announced in Section 2. In both cases, we emphasize how the assumptions of the approach developed so far can be easily verified and in consequence, the theorems above lead to general comparative static results.

## 6.1  Comparative Statics in the Bewley-Aiyagari Model

To exploit this paper's comparative statics results, we must verify Assumptions 1-3. Assumption 1 this is trivially satisfied under the general conditions (continuity, compactness) described in Section 2.1. Assumption 2 holds because the aggregator in the Aiyagari-model - which is our baseline aggregator - is convexifying.

Assumption 3 requires that $u_i$ is supermodular in $(x_i, y_i)$ and that the graph of $\Gamma_i(\cdot, z_i, Q)$ is a sublattice of $X_i \times X_i$.[36] Beginning with supermodularity, this will hold if and only if the period utility function $v_i$ is concave. This equivalence is true in general, but it is particularly easy to see when $v_i$ is twice continuously differentiable since then $D^2 v_i \leq 0$ (concavity) $\Leftrightarrow D^2_{x_i y_i} u_i \geq 0$ (supermodularity). Next turning to the sublattice property, as noted in Remark 3, $\Gamma_i(\cdot, z_i, Q)$ will be a sublattice of $X_i \times X_i$ if and only if $\Gamma_i(x_i, z_i, Q)$ is ascending in $x_i$ (this is true in general when $X_i$ is one-dimensional). Recall from Section 2.1 that

$$\Gamma_i(x_i, z_i, Q) = \{y_i \in [-\underline{b}_i, \bar{b}_i] : y_i \leq r(Q) x_i + w(Q) z_i\}.$$

This correspondence is ascending in $x_i$ if (for any fixed choice of $(z_i, Q)$) whenever $x_i^2 \geq x_i^1$, $y_i^1 \in \Gamma_i(x_i^1, z_i, Q)$, and $y_i^2 \in \Gamma_i(x_i^2, z_i, Q)$, we have $\max\{y_i^1, y_i^2\} \in \Gamma_i(x_i^1, z_i, Q)$ and $\min\{y_i^1, y_i^2\} \in \Gamma_i(x_i^2, z_i, Q)$. It is straight-forward to see that this will indeed be the case, intuitively because $\Gamma_i$ is "increasing in $x_i$".

We also note that $u_i$ is increasing in $x_i$ and that $\Gamma_i$ is expansive in $x_i$ (these additional properties are used in Theorem 6, where an expansive correspondence is also defined). Finally we recall from the discussion immediately prior to Definition 5, that a tightening of the borrowing limits (a decrease in the $b_i$'s) will be positive shocks.

Using our first set of comparative statics theorems in Section 4 (Theorems 5-6), we can then straightforwardly conclude:

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[35]See Jensen (2012) for a detailed treatment of this issue. See also Carroll and Kimball (1996) and Huggett (2004) for the special case of income-allocation problems.

[36]In addition, the choice set $X_i \subseteq \mathbb{R}$, must be a lattice. But this is is trivially satisfied whenever the choice set is one-dimensional.

**Proposition 1** *Consider the generalized Bewley-Aiyagari model described in Section 2.1. The following then follow:*

- *An increase in the discount rate $\beta$ will lead to an increase in the greatest and least capital-labor ratios in equilibrium, as well as an increase in the associated greatest and least equilibrium output per capita.*

- *Any tightening of the borrowing limits (a decrease in $b_i$ for all or a subset of households) is a positive shock and consequently leads to an increase in the greatest and least capital-labor ratios in equilibrium, as well as an increase in the associated greatest and least equilibrium output per capita. This statement remains valid when borrowing limits are endogenous ($\underline{b}_i$ is a function of $Q$) where a tightening means that $\underline{b}_i$ decreases for any fixed value of $Q$.*

- *Let $a_i$ parameterize the instantaneous utility function $v_i = v_i(c_i, a_i)$ where $c_i$ denotes consumption at a point in time, and consider the effect of a decrease in marginal utility, i.e., assume that $D^2_{c_i a_i} v_i \leq 0$. Then an increase in $a_i$ (for any subset of the agents not of measure zero) will lead to an increase in the greatest and least capital-labor ratios in equilibrium, as well as an increase in the associated greatest and least equilibrium output per capita.*

An immediate consequence of Proposition 1 is that the conclusion of Aiyagari (1994) and Miao (2002) that tightened borrowing limits increases output per capita is valid under much more general conditions than in these works.[37] For example, the conclusion remains valid under endogenous borrowing constraints as well as with heterogenous consumers. For a slightly deeper consequence, consider two economies where one is more "credit rationed" than the other in the sense that the borrowing constraints bind for a larger fraction of the agents (the relevant case is when the borrowing constraints bind at the smallest labor endowments $z_{i,min}$). Then by Proposition 1 the more credit rationed economy will have the higher capital-labor and out-per-labor ratios in equilibrium. In particular, a complete market economy (where borrowing limits never bind) will have a lower output per capita than an economy with "partial" credit rationing (where some borrowing limits bind) which in turn will have a lower output per capita than a completely credit rationed economy of the type studied by Aiyagari (1994). Economically, all of the previous conclusions follow from the fact that increased credit rationing forces agents to increase their precautionary savings levels when they face the prospect of being borrowing constrained at the "disaster-outcome" $z_{i,min}$.

We can also use the results in Proposition 1 to briefly discuss why in general very little can be said about individual behavior even though we can obtain quite strong results on aggregates.

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[37]For a detailed discussion of these works see section 2.1.

Consider, for example, an increase in $\beta$. At given $Q$, this will increase the savings (asset holdings) of all individuals and thus correspond to a positive shock in terms of our terminology. This will naturally tend to increase the aggregate capital-labor ratio. As the aggregate capital-labor ratio increases, the wage rate increases and the interest rate declines. But this might discourage savings by at least some of individuals. In fact, even a small increase in $Q$ may have a significant impact on the savings of some individuals depending on income and substitution effects. Thus at the end a subset of individuals may end up reducing their savings and a subset may end up raising savings (where for any specific agent, the outcome depends on the current level of assets and her underlying preferences). In fact, it is in general very difficult to say which individuals will reduce and which will increase their savings, because this will depend on the exact changes in the wage and interest rates. However, even though some individuals might reduce their savings and the extent of this is quite irregular, we know that in the aggregate savings and thus $Q$ must go up.

A second case that illustrates the previous point even more sharply is that of a population of consumers all of whose payoff functions exhibit decreasing differences in $Q$ and $y_i$. When this holds, any consumer will lower his savings when $Q$ increases. Now imagine that a subset (not of measure zero) of the consumers have their borrowing constraints tightened. Proposition 1 applies and tells us that the equilibrium aggregate will increase. But clearly by what was just said, any consumer whose borrowing constraint remains the same must then *lower* his savings (and some of the consumers who do experience tightened borrowing constraints may lower their equilibrium savings as well). On aggregate all such falls in savings must be counter-acted by agents who save more however, since otherwise the capital-labor ratio could not increase.

For our second set of comparative statics results in Section 5, we first need to verify Assumption 4. This requires that $u_i(x_i, y_i, z_i, Q)$ must exhibit increasing differences/be supermodular in $y_i$ and $z_i$, and that $\Gamma_i(x_i, z_i, Q)$ is ascending in $z_i$. But this follows from the exact same argument as that used above to verify that $u_i$ is supermodular in $x_i$ and $y_i$ and that $\Gamma_i$ ascending in $x_i$ (this is simply because $x_i$ and $z_i$ enter in an entirely "symmetric" way in $u_i$ and $\Gamma_i$).

As discussed at the end of Section 5.1, $\Gamma_i$ in the current model does not satisfy strict complementarities in $(x_i, z_i)$. Hence we cannot say anything about first-order stochastic dominance changes in the invariant distributions of the households' stochastic processes.

Nevertheless, the effects of mean preserving spreads (in particular, a mean-preserving spread to $\mu_{z_i}$ for any subset of the agents) can be determined using the result of Section 5.2. Beginning with Assumption 6, it is straightforward to verify that $\Gamma_i$ has a convex graph as required. The concavity parts of Assumption 6 will all hold if we take $v_i$ to be strictly concave (note that this corresponds to assuming that households are *risk averse*). Next let us turn to the required $k$-

concavity and $k$-convexity conditions of Theorem 9. Specifically, there must for each household $i$ exist an $k_i \geq 0$ such that $-D_{y_i} u_i(x_i, y_i, z_i, Q)$ is $k_i$-concave in $(x_i, y_i)$ as well as $(y_i, z_i)$ and $D_{x_i} u_i(x_i, y_i, z_i, Q)$ is $k_i$-convex in $(x_i, y_i)$ as well as in $(z_i, y_i)$. Now, because the argument of $v_i$ is linear in $y_i$, $x_i$, and $z_i$, it is straightforward to verify that all of these conditions will be satisfied simultaneously if and only if $Dv_i(c_i)$ is $k_i$-concave as well as $k_i$-convex. In other words, $\frac{1}{1-k_i}[Dv_i(c_i)]^{1-k_i}$ must be linear in $c_i$. Clearly, strict concavity in addition requires that $k_i > 0$. Differentiating twice, setting it equal to zero, and rearranging this yields the condition:

$$\frac{D^3 v_i(c_i) D v_i(c_i)}{(D^2 v_i(c_i))^2} = k_i > 0 \qquad (11)$$

This condition on a utility function is well known, and a function that satisfies it is said to belong to the HARA class (Carroll and Kimball (1996)). Most commonly used utility functions are in fact in the HARA class, including those that exhibit either constant absolute risk aversion (CARA) or constant relative risk aversion (CRRA). Note that, conveniently, such functions will also satisfy the boundary condition of Theorem 9. So picking $v_i$ in the HARA class is sufficient for all of the conditions of Theorem 9 to hold, and so we get:

**Proposition 2** *Consider the generalized Bewley-Aiyagari model of Section 2.1, and assume that $v_i$ belongs to the HARA class for all $i$. Then a mean-preserving spread to (any subset of) the households' noise environments will lead to an increase in the greatest and least equilibrium capital-labor ratios and an increase in the associated greatest and least equilibrium per capita outputs.*

Proposition 2 shows that an observation made by Aiyagari (1994) (p. 671) with reference to an example is in fact true under very general conditions: an economy with idiosyncratic shocks will lead to higher savings and output per capita than a parallel economy without any uncertainty.[38] Proposition 2 is also closely related to Huggett (2004), who shows that an *individual* agent's accumulation of wealth will increase if she is subjected to higher earnings risk (in particular, this result is valid for preferences that are a subset of the HARA class, cf. Huggett (2004), p.776). Proposition 2 can be seen as generalizing Huggett's individual-level result to the market/general equilibrium level. Note in this context that a crucial common component is that when utility belongs to the HARA class, the savings function will be convex, a result proved by Carroll and Kimball (1996) in the setting without borrowing constraints, and extended to the setting with borrowing constraint in Huggett (2004) and Jensen (2012).

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[38]To see this, simply note that the movement from a deterministic model to one with uncertainty amounts to subjecting all agents' labor endowments to mean preserving spreads.

## 6.2 Comparative Statics in the Hopenhayn Model

As explained in Section 2.2, Hopenhayn's model of entry, exit, and firm dynamics can be cast as a large dynamic economy with the following aggregator $H$:

$$H((\tilde{x}_i(p))_{i\in\mathcal{I}}) = D(\int_{i\in\mathcal{N}} \tilde{x}_i(p) \ di),$$

Here $\tilde{x}_i(p)$ is the strategy of a firm given the stationary price level $p$. The only difference from the Bewley-Aiyagari model is that $\tilde{x}_i(p)$ is now a random variable $x^*(\cdot, p)$ defined on the probability space $(Z, \mathcal{B}(Z), \mu_p)$, where $\mu_p$ (the frequency distribution of the active firms' productivities) in general will depend not only on $p$ but on any exogenous parameters of the model. Therefore shocks will affect $\tilde{x}_i(p)$ through two channels: directly through $x^*$, and indirectly through the change in the distribution $\mu_p$.

It is straightforward to verify that Assumptions 1-2 hold. Assumption 3 is also satisfied since for a given productivity level $z$, a firm will choose output to maximize $px - C(x, z, a) - c$ (here $a$ is an exogenous parameter affecting costs), and thus the payoff function only depends on $x$ and thus trivially satisfies the supermodularity assumption. Since there is no constraint other than $x \geq 0$ on this problem, the assumption that the graph of the constraint correspondence is a sublattice of $X_i \times X_i$ is also immediately satisfied.[39] From this observation, it also follows that an increase in $a$ will be a positive shock if and only if $D^2_{xa}C(x, z, a) \leq 0$. In other words, a positive shock is one that lowers the marginal cost (given $p$ and $z$). Let us also impose the natural restriction that $D_aC(x, z, a) \leq 0$ which implies that $V(z, p, a)$ is increasing in $a$.

Next, note that, as outlined in Section 2.2, $\mu_p$ is determined from the exit cutoff $\bar{z}_p$ and the measure of entrants $M$ as a solution to equation (4). The right-hand side of (4) is type I and type II monotone in $\mu_p$ as well as in $-\bar{z}_p$ and $M$.[40] Therefore Theorem 3 implies that an increase $M$ or a decrease in $\bar{z}_p$ will lead to a (first-order stochastic dominance) increase in the distribution $\mu_p$.[41]

It follows that the aggregate in this case, $\int_Z x^*(z, p)\mu_p(dz)$, will increase not only with positive shocks as defined above but also with other changes in parameters that lowers $\bar{z}_p$ or raises $M$.[42]

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[39]These observations also show that an interesting generalization of Hopenhayn's model with learning by doing at the firm level—where current productivity depends on past production—is also a special case of our framework and will yield essentially the same comparative static results provided that the interaction between current output and past output satisfies supermodularity.

[40]In this statement $\mu_p$ is ordered by first-order stochastic dominance. The right-hand side of (4), $F(\mu(\cdot), \bar{z}_p, M) = \int_{z_i \geq \bar{z}_p} \Gamma(z_i, \cdot)\mu(dz_i) + M\nu(\cdot)$, is single-valued, so type I and type II monotonicity coincide with monotonicity in the usual sense. Note that $\int_{z_i \geq \bar{z}_p} \Gamma(z_i, \cdot)\mu(dz_i)$ is simply the adjoint of $\Gamma$ imputed at $\bar{z}_p$. From this follows immediately that $F$ will be monotone in $\mu_p$ since $\Gamma$ is monotone (and it also easily follows that a decrease in $\bar{z}_p$ will lead to a first-order stochastic increase in $F$). That $F$ is monotone in $M$ (as well as in $\nu$ ordered by first-order stochastic dominance) is straightforward to verify.

[41]When $V(z, p, a)$ is increasing in $a$—which our assumption that $D_aC(x, z, a) \leq 0$ guarantees— an increase in $a$ will lead to an increase in $M$ (which can be directly seen from equation (3)), and thus to an increase in $\mu_p$.

[42]The fact that this aggregate, $\int_Z x^*(z, p)\mu_p(dz)$, increases when $\mu_p(z)$ undergoes a type I and/or type II increase

Combining the previous observation and applying Theorem 5, we obtain:

**Proposition 3**     *1. A decrease in the fixed cost of operation c or a (first-order) increase in the transition function $\Gamma$ increases aggregate output and lowers the equilibrium price.*

    *2. A first-order stochastic increase in the entrants' productivity distribution $\nu$ increases aggregate output and lowers the equilibrium price.*

    *3. A positive shock to the firms' profit functions, i.e., an increase in a with $D_a C \leq 0$ and $D^2_{xa} C \leq 0$, increases aggregate output and lowers the equilibrium price.*

It is also useful to note that the effects on individual firms are uncertain, and may easily go in the opposite direction. Take a decline in the fixed costs of operation $c$ to illustrate this for the first part of the proposition. Such a decline leaves the profit-maximizing choice of output for incumbents, $x(p, z)$, unchanged for any given price and level of productivity. The conclusion in part 1 of Proposition 3 instead follows the effect of this cost reduction on the equilibrium distribution $\mu_p$—"state of the industry". This is because as $c$ declines, the value of a firm with any given productivity $V(p, z)$ increases and the exit cutoff $\bar{z}_p$ also decreases, making it less likely that any active firm will exit in any period. The increase in $V(p, z)$ (for all $z$) also leads to greater entry, which together with the decline in $\bar{z}_p$ leads to an increase in $\mu_p$, raising aggregate output. But as aggregate output increases, the equilibrium price will fall which leads to counteracting effects on $V(p, z)$ as well as $\bar{z}_p$ (a decrease and an increase, respectively). The combined consequence for any firm with a given productivity level $z$ is uncertain—for many types of firms the indirect effects may dominate, reducing their output, and some types of firms might choose to exit. Nevertheless, aggregate output necessarily increases and the equilibrium price necessarily declines. Similarly in part 2, the result is again driven by the impact of the shift in $\nu$ on $\mu_p$; the resulting decline in $p$ is a counteracting effect, reducing firm-level output at given productivity level $z$. In part 3, a positive shock directly raises $x(p, z, a)$ for all $p, z$ and also raises the value function $V$, increasing $\mu_p$, and thus also increasing aggregate output and lowering the equilibrium price. Because the resulting decrease in $p$ counteracts this effect, the overall impact on a firm of a given productivity level $z$ is again uncertain. This discussion therefore illustrates that the types of results contained in Proposition 3 would not have been possible by studying comparative statics at the *individual* firm level—indeed, similar with some of the results discussed in Proposition 1, there will generally be no regularity at the individual level.

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is a consequence of Theorem 4.

# 7 Conclusion

There are relatively few known comparative static results on the structure of equilibria in dynamic economies. Many existing analytic results, such as those in endogenous growth models (overviewed in Acemoglu (2009)), are obtained using closed-form characterizations and rely heavily on functional forms. Many other works study the structure of such models using numerical analysis. In this paper, we developed a general and fairly easy-to-apply framework for robust comparative statics about the structure of stationary equilibria in such dynamic economies. Our results are "robust" in the sense defined by Milgrom and Roberts (1994) in that they do not rely on parametric assumptions but on qualitative economic properties, such as utility functions exhibiting increasing differences in choice variables and certain parameters. Nevertheless, and importantly from the viewpoint of placing the contribution within the broader literature, none of our main results exploit standard supermodularity or monotonicity results—and in fact, our key technical result, which underlies all of our results, is introduced to enable us to work with spaces that are not lattices.

Some of the well-known models that are special cases of our framework are models of saving and capital accumulation with incomplete markets along the lines of work by Bewley, Aiyagari, and Huggett, and models of industry equilibrium along the lines of work by Hopenhayn. In all cases, our results enable us to establish—to the best of our knowledge—much stronger and more general results than those available in the literature. They also lead to a new set of comparative static results in response to first-order and second-order stochastic dominance shifts in distributions representing uncertainty in these models. All of the major comparative static results provided in the paper are truly about the structure of equilibrium—not about individual behavior. This is highlighted by the fact that in most cases, while robust and general results can be obtained about how market outcomes behave, little can be said about individual behavior, which is in fact often quite irregular.

We believe that our framework and methods are useful both because they clarify the underlying economic forces, for example in demonstrating that robust comparative statics applies to aggregate market variables, and because they can be applied readily in a range of problems.

# 8    Appendices

## 8.1    Appendix I: Proofs of Results from the Text

In this Appendix, we present the proofs of the main results from the text. Some of these proofs rely on technical results presented in Appendixes II and III.

### 8.1.1    Proofs from Section 3

**Proof of Theorem 1.**    Only a brief sketch will be provided. For agent $i$, let $\mathcal{X}_i$ denote the set of strategies (these are infinite sequences of random variables as described in the main text), and let $\gamma_i(\mathbf{Q}) \subseteq \mathcal{X}_i$ denote that set of optimal strategies for agent $i$ given the sequence of aggregates $\mathbf{Q} \in \prod_{t=0}^{\infty} \mathcal{Q}$. $\prod_{t=0}^{\infty} \mathcal{Q}$ with the supremum norm $\|\mathbf{Q}\| = \sup_t |Q_t|$, is a compact and convex topological space. $\mathcal{X}_i$ is equipped with the topology of pointwise convergence where each coordinate converges if and only if the random variable converges in the weak $*$-topology. Under Assumption 1, $\gamma_i : \prod_{t=0}^{\infty} \mathcal{Q} \to 2^{\mathcal{X}_i}$ will be non-empty valued and upper hemi-continuous. Let $\mathcal{H}(\mathbf{Q}) = \{H((\mathbf{x}_i)_{i \in \mathcal{I}}) : \mathbf{x}_i \in \gamma_i(\mathbf{Q}) \text{ for } i \in \mathcal{I}\}$. Since $H$ is continuous and convexifying, $\mathcal{H}$ will be upper hemi-continuous and convex valued. A fixed point $\mathbf{Q}^* \in \mathcal{H}(\mathbf{Q}^*)$ exists by the

Kakutani-Glicksberg-Fan Theorem. It is easy to see that such a $\mathbf{Q}^*$ corresponds to an equilibrium with $\mathbf{x}_i^* \in \gamma_i(\mathbf{Q}^*)$ for each agent $i$.  ∎

**Proof of Theorem 2.**    Rather than proving this theorem directly, we refer to the proof of Theorem 5 from which existence of a stationary equilibrium follows quite easily. Indeed, in that proof it is shown that $Q$ is an equilibrium aggregate given $a$ if and only if $Q \in \hat{H}(Q, a)$ where $\hat{H}$ is an upper hemi-continuous and convex valued correspondence that maps a compact and convex subset of the reals into itself. Existence therefore follows from Kakutani's fixed point theorem. The set of equilibrium aggregates will be compact as a direct consequence of the boundedness of the set of feasible equilibrium aggregates (a consequence of continuity of $H$ and assumption 1) and the upper hemi-continuity of $\hat{H}$. Consequently, a least and a greatest equilibrium aggregate will always exist.  ∎

### 8.1.2    Proofs from Subsection 4.1

The proof of Theorem 3 relies on a generalization of the following result from Smithson (1971).

**Theorem 10 (Smithson (1971))** *Let $X$ be a chain-complete partially ordered set, and $F :$ $X \to 2^X$ a type I monotone correspondence. Assume as follows: For any chain $C$ in $X$, and any monotone selection from the restriction of $F$ to $C$, $f : C \to X$ (if one exists!); there exists*

$y_0 \in F(\sup C)$ *such that* $f(x) \leq y_0$ *for all* $x \in C$. *Then, if there exists a point* $e \in X$ *and a point* $y \in F(e)$

such that $e \preceq y$, F has a fixed point.

The generalization, which to the best of our knowledge is new, is presented and proved next.

**Theorem 11** *In Theorem 10, the conclusion may be strengthened to: F has a fixed point* $x^*$ *with* $x^* \succeq e$.

**Proof.** Let $F : X \to 2^X$ and $e \in X$ be as described, and set $\hat{X} \equiv \{x \in X : x \succeq e\}$. Note that since $X$ is chain-complete, so is $\hat{X}$. Then define a correspondence on $\hat{X}$ by $\hat{F}(x) \equiv F(x) \cap \{z \in X : z \succeq e\}$. We begin by showing that $\hat{F}$ has non-empty values. So pick any $x \in \hat{X}$. By type I monotonicity, there exists $y' \in F(x)$ with $y' \succeq y \succeq e$ where $y$ is the element in $F(e)$ with $e \preceq y$ guaranteed to exist by assumption. But then $y' \in \hat{F}(x)$. Next, $\hat{F} : \hat{X} \to 2^{\hat{X}}$ is type I monotone, for if $x_1 \preceq x_2$ and $y_1 \in \hat{F}(x_1)$ ($\subseteq F(x_1)$), there will exist $y_2 \in F(x_2)$ such that $y_1 \preceq y_2$; and since $e \preceq y_1 \preceq y_2$, $y_2 \in \hat{F}(x_2)$ also. That $\hat{F}$ satisfies the condition on the supremum of chains in Theorem 10 is trivial to show and we omit the proof. Now all we have to do is apply Smithson (1971)'s Theorem in order to conclude that $\hat{F}$ has a fixed point $x^* \in \hat{X}$. But it is clear that any fixed point for $\hat{F}$ is also a fixed point for $F$, and since by construction $x^* \succeq e$, this completes the proof. ∎

Both of the previous results have parallel statements for type II monotone correspondences. In particular (see Smithson (1971), Remark p. 306), the conclusion of Theorem 10 (existence of a fixed point) remains valid for type II monotone correspondences if the hypothesis are altered as follows: (i) $X$ is assumed to be lower chain-complete rather than chain-complete (a partially ordered set is lower chain complete if each non-empty chain has an infimum). (ii) The condition on monotone selections on chains is altered to: For any chain $C$ in $X$, and any monotone selection from the restriction of $F$ to $C$, $f : C \to X$ (if any); there exists $y_0 \in F(\inf C)$ such that $f(x) \geq y_0$ for all $x \in C$. (iii) Instead of elements $e \in X$ and $y \in F(e)$ with $e \preceq y$; there must exist $e \in X$ and $y \in F(e)$ with $e \succeq y$. Theorem 11 also holds as before with the only modification that the conclusion is now the existence of a fixed point $x^*$ with $x^* \preceq e$.

**Proof of Theorem 3.** We prove only the type I monotone case (the type II monotone case is similar). Compactness of $X$ together with the fact that the order $\succeq$ is assumed to be closed, ensures the chain-completeness as well as lower chain-completeness of $(X, \succeq)$.[43] The condition in

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[43] A partially ordered set where all chains have an infimum as well as a supremum is usually simply said to be *complete* (*e.g.*, Ward (1954), p.148). In the present setting where $X$ is topological and the order $\succeq$ is closed, the

Theorem 10 on the supremum (and infimum in the type II case) of chains is satisfied because $F$ is upper hemi-continuous. Indeed, let $C$ be a chain with supremum $\sup C \in X$, and let $f : C \to X$ be a monotone selection from $F : C \to 2^X$. There will exist an increasing sequence $(c_n)_{n=1}^{\infty}$, $c_{n+1} \succeq c_n$, from $C$ with $\lim_{n \to \infty} c_n = \sup C$. It follows then from upper hemi-continuity of $F$ that $f(\sup C) = \lim_{n \to \infty} f(c_n) \in F(\sup C)$. In addition, since $(f(c_n))_n$ is increasing with supremum $f(\sup C) = \lim_{n \to \infty} f(c_n)$; $f(\sup C) \succeq f(x)$ for all $x \in C$. This proves the claim.

Pick $t_1 \le t_2$ and a fixed point $x_1 \in \Lambda(t_1)$. We must show that there will exist an $x_2 \in \Lambda(t_2)$ with $x_1 \preceq x_2$. To this end we apply Theorem 11 to the correspondence $F(\cdot, t_2)$. The only thing we need to verify is that there exists $e \in X$ and $y \in F(e, t_2)$ with $y \succeq e$. But taking $e = x_1$, it is clear that $e \in F(e, t_1)$, and since $F$ is type I monotone in $t$, there will for our $t_2 \ge t_1$ exist $y \in F(e, t_2)$ with $y \succeq e$. This is exactly what we needed. We conclude that $F(\cdot, t_2)$ has a fixed point "above" $e = x_1$, i.e., there exists $x_2 \in \Lambda(t_2)$ with $x_2 \succeq x_1$. $\blacksquare$

**Proof of Theorem 4.** We prove only that $\overline{h}(t)$ is increasing (the other case is similar). $\overline{h}(t)$ is well-defined because $H$ is continuous and $\Lambda(t)$ is compact (the fixed point set of an upper hemi-continuous correspondence on a compact set is always compact). Pick $t_1 \le t_2$, and let $x_1 \in \Lambda(t_1)$ be an element such that $\overline{h}(t) = H(x_1)$. Since $\Lambda(t)$ is type I monotone, there will exist $x_2 \in \Lambda(t_2)$ such that $x_2 \succeq x_1$. Since $H$ is monotone, $\overline{h}(t_2) = \sup_{x \in \Lambda(t_2)} H(x) \ge H(x_2) \ge H(x_1) = \overline{h}(t_1)$. $\blacksquare$

### 8.1.3 Proofs from Subsection 4.2

We first provide a brief roadmap for the proof of Theorem 5. The proof has three steps: In the first step Theorem 3 is used to show that for any *fixed* equilibrium aggregate $Q$, the set of stationary distributions for each individual will be type I and type II increasing in the exogenous variables $a$. In step two, a map $\hat{H}$ that for each $Q$ and $a$ gives a set of aggregates is constructed. The fixed points of this map are precisely the set of equilibrium aggregates given $a$. Crucially, the least and greatest selections from $\hat{H}$ will be increasing in $a$ by Theorem 4. Using this, the third and final step uses an argument from Acemoglu and Jensen (2009) and Milgrom and Roberts (1994) to show that the equilibrium aggregates must also be increasing in $a$.

**Proof of Theorem 5.** As explained above, the proof has three steps.

*Step 1:* Fix $Q \in \mathcal{Q}$. Under Assumption 3, the policy correspondence of each player $G_i : X_i \times Z_i \times \{Q\} \times A_i \to 2^{X_i}$ will have a least and a greatest selection and both of these will be increasing in $x_i$. For given $Q$ and $a_i$, let $T_{Q,a_i}^* : \mathcal{P}(X_i) \to 2^{\mathcal{P}(X_i)}$ denote the adjoint Markov

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claim that compactness implies completeness follows from Theorem 3 in Ward (1954) because any closed chain will be compact (any closed subset of a compact set is compact).

correspondence induced by $G_i$. By Theorem 12, $T^*_{Q,a_i}$ will be type I and type II monotone when $\mathcal{P}(X_i)$ is equipped with the first-order stochastic dominance order $\succeq_{st}$. Since $(\mathcal{P}(X_i), \succeq_{st})$ has an infimum (namely the degenerate distribution placing probability 1 on $\inf X_i$), this implies that the invariant distribution correspondence $F_i : \mathcal{Q} \times A_i \to 2^{\mathcal{P}(X_i)}$, given by $F_i(Q, a_i) = \{\mu \in \mathcal{P}(X_i) : \mu \in T^*_{Q,a_i}\mu\}$ is non-empty valued and upper hemi-continuous (Theorem 13). Now the results from Section 4.1 come into play. Since, again by Theorem 12, $T^*_{Q,a_i}$ is also type I and type II monotone in $a_i$, we can use Theorem 3 to conclude that the invariant distribution correspondence $F_i$ will be type I and type II monotone in $a_i$. This is true for every $i \in \mathcal{I}$ hence the joint correspondence: $F = (F_i)_{i \in \mathcal{I}} : \mathcal{Q} \times (\prod_{i \in \mathcal{I}} A_i) \to 2^{\prod_{i \in \mathcal{I}} \mathcal{P}(X_i)}$ is type I and type II monotone in $a = (a_i)_{i \in \mathcal{I}}$.

*Step 2:* Next consider:

$$\hat{H}(Q, a) = \{H(x) \in \mathbb{R} : x \in F(Q, a) \text{ for all } i\}$$

It is clear from the definition of a stationary equilibrium, that $Q^*$ is a stationary equilibrium aggregate given $a \in A$ if and only if $Q^* \in \hat{H}(Q^*, a)$. Under assumption 2, either (i) $G_i$ will be convex valued for all $i$ and therefore $F$ will be convex valued, or (ii) $H$ will be convexifying. In either case, $\hat{H}$ will have convex values. Since $H$ is continuous and each $F_i(Q, a_i)$ is upper hemi-continuous, $\hat{H}$ will in addition be upper hemi-continuous (in particular, it has a least and a greatest selection). Since $F$ is type I and type II monotone, and $H$ is increasing, we can next use Theorem 4 to conclude that $\hat{H}$'s least and greatest selections will be increasing.

*Step 3:* Let $Q_{min} \equiv H((\delta_{\inf X_i})_{i \in \mathcal{I}})$ and $Q_{min} \equiv H((\delta_{\sup X_i})_{i \in \mathcal{I}})$ where $\delta_{x_i}$ denotes the degenerate measure on $X_i$ with its mass at $x_i$. It is then clear that $Q \geq Q_{min}$ for all $Q \in \hat{H}(Q_{min})$ and $Q \leq Q_{max}$ for all $Q \in \hat{H}(Q_{max})$. It follows that for every $a \in A$, $\hat{H}(\cdot, a) : [Q_{min}, Q_{max}] \to 2^{[Q_{min}, Q_{max}]}$.

That the least and greatest solutions to the fixed-point problem $Q^* \in \hat{H}(Q^*, a)$ are increasing in $a$ now follows from the argument used in the proof of Lemma 2 in Acemoglu and Jensen (2009). There is was shown that any correspondence $\hat{H}(\cdot, a) : [Q_{min}, Q_{max}] \to 2^{[Q_{min}, Q_{max}]}$ that is upper hemi-continuous, convex valued, and has least and greatest selections that are increasing in $a$, will satisfy the conditions of Corollary 2 in Milgrom and Roberts (1994). Milgrom and Roberts' result in turn says that the least and greatest fixed points $Q \in \hat{H}(Q, a)$ will be increasing in $a$. This completes the proof of the theorem since the least and greatest fixed points of $\hat{H}$ are the greatest and least equilibrium aggregates. ∎

**Proof of Theorem 6.** The value function of agent $i$ will, given a stationary sequence for the aggregate $Q_t = Q$ all $t$, and the stationary distribution for $z_{i,t}$, $z_{i,t} \sim \mu_{z_i} \in \mathcal{P}(Z_i)$ all $t$, equal the

pointwise limit of the sequence $(v_i^n)_{n=0}^{\infty}$ determined by:

$$v_i^{n+1}(x_i, z_i, \beta) = \sup_{y_i \in \Gamma_i(x_i, z_i)} [u_i(x_i, y_i, z_i) + \beta \int v_i^n(y_i, z_i', \beta) \mu_{z_i}(dz_i')] \tag{12}$$

Here $v^0$ may be picked arbitrarily and we have suppressed $Q$'s entry to simplify notation. Pick $v^0(x_i, z_i, \beta)$ that is increasing and supermodular in $x_i$ and exhibits increasing differences in $x_i$ and $\beta$. Since integration preserves supermodularity and increasing differences, $\int v_i^0(y_i, z_i', \beta) \mu_{z_i}(dz_i')$ will be supermodular in $y_i$ and exhibit increasing differences in $y_i$ and $\beta$. It immediately follows from Topkis' Theorem on preservation of supermodularity under maximization (Topkis (1998), Theorem 2.7.6), that $v^1$ will be supermodular in $x_i$. By recursion then, $v^2, v^3, \ldots$ are all supermodular in $x_i$ and so is consequently the pointwise limit $v^*$ (Topkis (1998), Lemma 2.6.1). It is trivial to show that when $u_i$ is increasing in $x_i$ and $\Gamma_i$ is expansive in $x_i$ in the sense of the theorem's assumption, $v^{n+1}$ will be increasing in $x_i$, hence the pointwise limit $v^*$ will also be increasing in $x_i$. Since $\int v_i^0(y_i, z_i', \beta) \mu_{z_i}(dz_i')$ exhibits increasing differences in $y_i$ and $\beta$ and is increasing in $y_i$, $\beta \int v_i^0(y_i, z_i', \beta) \mu_{z_i}(dz_i')$ will exhibit increasing differences in $y_i$ and $\beta$.[44] It follows from Hopenhayn and Prescott (1992), Lemma 1, that $v_i^1$ will exhibit increasing differences in $x_i$ and $\beta$, and again this property recursively carries over to the pointwise limit $v^*$. By Topkis's Monotonicity Theorem, we conclude that the policy correspondence $G_i(x_i, z_i, \beta)$ will be increasing in $\beta$. The conclusion of the Theorem now follows by running through the proof of Theorem 5 with an increase in $\beta$ playing the role of a positive shock rather than an increase in $a$. $\blacksquare$

**Proof of Theorem 7.** The conclusions are trivial consequences of the comparative statics results of Topkis (1978) and the first part of the proof of Theorem 5. This is because $Q$ can now be treated as an exogenous variable (alongside $a$) so that we in effect are dealing with just the question of how an individual's set of stationary strategies varies with $Q$ and $a$. The details are left to the reader. $\blacksquare$

### 8.1.4 Proofs from Section 5.1

**Proof of Theorem 8.** This proof is nearly identical to the proof of theorem 5. As mentioned after Assumption 5, $G_i(x_{i,t}, z_{i,t}, \mu_{z_i})$ will be ascending in $\mu_{z_i}$ under the Theorem's assumptions when stationary distributions are ordered by first-order stochastic dominance (Hopenhayn and Prescott (1992)). Therefore first-order stochastic increases in $\mu_{z_i}$ for (a subset of) agents will correspond to "positive shocks" in exactly the same way as increases in exogenous parameters in

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[44]Let $f(y, \beta)$ exhibit increasing differences and be increasing in $y$. Then $\beta f(\tilde{y}, \beta) - \beta f(y, \beta)$ is clearly increasing in $\beta$ for $\tilde{y} \geq y$, showing that $\beta f(y, \beta)$ exhibits increasing differences.

the proof of Theorem 5. Once we realize this, Theorem 8 follows from the exact same argument that was used to prove Theorem 5. ∎

### 8.1.5 Proofs from Section 5

In this section we prove Theorem 9. The basic idea is to show that a mean-preserving spread to the distributions of the agents' environment constitutes a "positive shock" in the sense of this paper: it leads to an increase in individuals' stationary strategies for any fixed equilibrium aggregate $Q$. Once again Theorem 3 plays a critical role because the spaces we work with have no lattice structure. Once it has been established that mean-preserving spreads are positive shocks, the proof follows the proof of Theorem 5 line by line.

To prove that mean-preserving spreads constitute positive shocks we need to introduce a bit of additional notation as well as an intermediate lemma. We begin by noting that under Assumption 6, the policy correspondence of (10) will be single-valued, *i.e.*, $G_i(x_i, z_i, Q) = \{g_i(x_i, z_i, Q)\}$ where $g_i$ is the (unique) policy function. For a given stationary market aggregate $Q \in Q$, an agent's optimal strategy is therefore described by the following stochastic difference equation:

$$x_{i,t+1} = g_i(x_{i,t}, z_{i,t}, Q, \mu_{z_i}) \tag{13}$$

Note that here we have made $g_i$'s dependence on the distribution of $z_{i,t}$ explicit. We already know that $g_i$ will be increasing in $x_i$ and $z_i$ (Assumptions 3-4). By Theorem 8 of Jensen (2012), $g_i$ will in addition be convex in $x_i$ as well as in $z_i$ under the conditions of the theorem. We now turn to proving that $g_i$ will be $\succeq_{cx}$-increasing in $\mu_{z_i}$ (precisely, this means that $g_i(x_{i,t}, z_{i,t}, Q, \tilde{\mu}_{z_i}) \geq g_i(x_{i,t}, z_{i,t}, Q, \mu_{z_i})$ whenever $\tilde{\mu}_{z_i} \succeq_{cx} \mu_{z_i}$). From Jensen (2012) (corollary in the proof of Theorem 8 applied with $k = 0$), $D_{x_i} v_i(x_i, z_i, Q)$ will (in the sense of agreeing with a function with these properties almost everywhere) be convex in $z_i$ because $D_{x_i} u_i(x_i, y_i, z_i, Q)$ is non-decreasing in $y_i$ and convex in $(z_i, y_i)$ (the latter is true because $k_i$-convexity is stronger than convexity). This is precisely one of the conditions of the following lemma (the other is supermodularity, already used). The lemma is stated in some generality because it is of independent interest (note that $Q$ is suppressed in the lemma's statement).

**Lemma 1** *Assume that $u_i(x_i, y_i, z_i)$ is supermodular in $(x_i, y_i)$ and denote the value function by $v_i(x_i, z_i, \mu_{z_i})$ where $\mu_{z_i}$ is the stationary distribution of $z_i$. Let $x_i$ be ordered by the usual Euclidean order and $\mu_{z_i}$ be ordered by $\succeq_{cx}$. Then the value function exhibits increasing differences in $x_i$ and $\mu_{z_i}$ if for all $\tilde{x}_i \geq x_i$ the following function is convex in $z_i$ (for all fixed $\mu_{z_i}$):*

$$v_i(\tilde{x}_i, z_i, \mu_{z_i}) - v_i(x_i, z_i, \mu_{z_i})$$

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*When the value function $v_i(x_i, z_i, \mu_{z_i})$ exhibits increasing differences in $x_i$ and $\mu_{z_i}$ it in turn follows that $\int v_i(y_i, z'_i, \mu_{z_i})\mu_{z_i}(dz'_i)$ exhibits increasing differences in $y_i$ and $\mu_{z_i}$ and so if $v_i$ is supermodular in $y_i$, the policy function $g_i(x_i, z_i, \mu_i)$ will be increasing in $\mu_i$.*

**Proof.** Let $v_i^n$ denote the $n$'th iterate of the value function and consider the $n + 1$'th iterate $v_i^{n+1}(x, z, \mu_{z_i}) = \sup_{y \in \Gamma_i(x,z)}\{u_i(x, y, z) + \beta \int v_i^n(y, z', \mu_{z_i})\mu_{z_i}(dz')\}$. Assume by induction that $v_i^n$ exhibits increasing differences in $(y, \mu_{z_i})$ and that the hypothesis of the theorem holds for $v_i^n$. When $\tilde{y} \geq y$ and $\mu_{z_i} \succeq_{cx} \mu'_{z_i}$ we then have $\int v_i^n(\tilde{y}, z', \mu_{z_i}) - v_i^n(y, z', \mu_{z_i})\mu_{z_i}(dz') \geq \int v_i^n(\tilde{y}, z', \mu_{z_i}) - v_i^n(y, z', \mu_{z_i})\mu'_z(dz') \geq \int v_i^n(\tilde{y}, z', \mu'_{z_i}) - v_i^n(y, z', \mu'_{z_i})\mu'_z(dz')$. Here the first inequality follows from the definition of the convex order, and the second inequality follows from increasing differences of $v_i^n$ in $(y, \mu_{z_i})$. Note that this evaluation implies the second conclusion of the lemma once the first has been established. Since $u_i(x, y, z) + \beta \int v_i^n(y, z', \mu_{z_i})\mu_{z_i}(dz')$ is supermodular in $(x, y)$ by assumption and trivially exhibits increasing differences in $(x, \mu_{z_i})$ it follows from the preservation of increasing differences under maximization that $v^{n+1}(x, z, \mu_{z_i})$ exhibits increasing differences in $(x, \mu_{z_i})$. The first conclusion of the lemma now follows from a standard argument (increasing differences is a property that is pointwise closed and the value function is the pointwise limit of the sequence $v^n$, $n = 0, 1, 2, \ldots$).  ∎

**Proof of Theorem 9.** We begin with some notation. For a set $Z$, let $\mathcal{P}(Z)$ denote the set of probability distributions on $Z$ with the Borel algebra. A distribution $\lambda \in \mathcal{P}(Z)$ is larger than another probability distribution $\tilde{\lambda} \in \mathcal{P}(Z)$ in the monotone convex order (written $\lambda \succeq_{cxi} \tilde{\lambda}$) if $\int_Z f(\tau)\lambda(d\tau) \geq \int_Z f(\tau)\tilde{\lambda}(d\tau)$ for all convex and increasing functions $f : Z \to \mathbb{R}$ for which the integrals exist (see Huggett (2004) and Shaked and Shanthikumar (2007), Chapter 4.A). The stochastic difference equation (13) gives rise to a transition function $P_{Q,\mu_{z_i}}$ in the usual way (here $x_i \in X_i$ and $A_i$ is a Borel subset of $X_i$):

$$P_{Q,\mu_{z_i}}(x_i, A) \equiv \mu_{z_i}(\{z_i \in Z_i : g_i(x_i, z_i, Q, \mu_{z_i}) \in A\}) \tag{14}$$

This in turn determines the adjoint Markov operator:

$$T^*_{Q,\mu_{z_i}}\mu_{x_i} = \int P_{Q,\mu_{z_i}}(x_i, \cdot)\mu_{x_i}(dx_i) \tag{15}$$

$\mu^*_{x_i}$ is an invariant distribution for (13) if and only if it is a fixed point for $T^*_{Q,\mu_{z_i}}$, *i.e.*, $\mu^*_{x_i} = T^*_{Q,\mu_{z_i}}\mu^*_{x_i}$. We are first going to use that $g_i$ is convex and increasing in $x_i$ to show that $T^*_{Q,\mu_{z_i}}$ will be a $\succeq_{cxi}$-monotone operator, *i.e.*, we are going to show that $\tilde{\mu}_{x_i} \succeq_{cxi} \mu_{x_i} \Rightarrow T^*_{Q,\mu_{z_i}}\tilde{\mu}_{x_i} \succeq_{cxi} T^*_{Q,\mu_{z_i}}\mu_{x_i}$. The statement that $T^*_{Q,\mu_{z_i}}\tilde{\mu}_{x_i} \succeq_{cxi} T^*_{Q,\mu_{z_i}}\mu_{x_i}$ by definition means that for all convex and increasing

functions $f : X_i \to \mathbb{R}$:

$$\int f(\tau)\, T^*_{Q,\mu_{z_i}}\tilde{\mu}_{x_i}(d\tau) \geq \int f(\tau)\, T^*_{Q,\mu_{z_i}}\mu_{x_i}(d\tau)$$

But since this is equivalent to,

$$\int_{Z_i} [\int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i}))\tilde{\mu}_{x_i}(dx_i)]\mu_{z_i}(dz_i) \geq \int_{Z_i} [\int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i}))\mu_{x_i}(dx_i)]\mu_{z_i}(dz_i) \ ,$$

we immediately see that this inequality will hold whenever $\tilde{\mu}_{x_i} \succeq_{cxi} \mu_{x_i}$ (the composition of two convex and increasing functions is convex and increasing). This proves that $T^*_{Q,\mu_{z_i}}$ is a $\succeq_{cxi}$-monotone operator.

Our next objective is to prove that $\tilde{\mu}_{z_i} \succeq_{cx} \mu_{z_i} \Rightarrow T^*_{Q,\tilde{\mu}_{z_i}}\mu_{x_i} \succeq_{cxi} T^*_{Q,\mu_{z_i}}\mu_{x_i}$ for all $\mu_{x_i} \in \mathcal{P}(X_i)$. As above, we can rewrite the statement that $T^*_{Q,\tilde{\mu}_{z_i}}\mu_{x_i} \succeq_{cxi} T^*_{Q,\mu_{z_i}}\mu_{x_i}$:

$$\int_{Z_i} [\int_{X_i} f(g_i(x_i, z_i, Q, \tilde{\mu}_{z_i}))\mu_{x_i}(dx_i)]\tilde{\mu}_{z_i}(dz_i) \geq \int_{Z_i} [\int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i}))\mu_{x_i}(dx_i)]\mu_{z_i}(dz_i) \quad (16)$$

Since $f$ is increasing and $g_i$ is $\succeq_{cx}$-increasing in $\mu_{z_i}$, it is obvious that for all $z_i \in Z_i$:

$$\int_{X_i} f(g_i(x_i, z_i, Q, \tilde{\mu}_{z_i}))\mu_{x_i}(dx_i) \geq \int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i}))\mu_{x_i}(dx_i)$$

Hence:

$$\int_{Z_i} [\int_{X_i} f(g_i(x_i, z_i, Q, \tilde{\mu}_{z_i}))\mu_{x_i}(dx_i)]\tilde{\mu}_{z_i}(dz_i) \geq \int_{Z_i} [\int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i}))\mu_{x_i}(dx_i)]\tilde{\mu}_{z_i}(dz_i) \quad (17)$$

But we also have:[45]

$$\int_{Z_i} [\int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i}))\mu_{x_i}(dx_i)]\tilde{\mu}_{z_i}(dz_i) \geq \int_{Z_i} [\int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i}))\mu_{x_i}(dx_i)]\mu_{z_i}(dz_i) \quad (18)$$

Combining (17) and (18) we get (16) under the condition that $\tilde{\mu}_{z_i} \succeq_{cx} \mu_{z_i}$. This is what we wanted to prove.

We are now ready to use Theorem 3 to conclude that $F_i(Q, \mu_{z_i}) \equiv \{\mu_{x_i} \in \mathcal{P}(X_i) : \mu_{x_i} = T^*_{Q,\mu_{z_i}}\mu_{x_i}\}$ will be type I and type II monotone in $\mu_{z_i}$ when $\mathcal{P}(Z_i)$ is equipped with the order $\succeq_{cx}$ and $\mathcal{P}(X_i)$ is equipped with $\succeq_{cxi}$.[46] Note that in the language of Theorem 3, $F$ equals $\{T^*_{Q,\mu_{z_i}}\}$ and $t$ corresponds to $\mu_{z_i}$.

The rest of the proof proceeds exactly as the proof of Theorem 5 with $(\mu_{z_i})_{i \in \mathcal{I}}$ replacing the exogenous variables $(a_i)_{i \in \mathcal{I}}$ in that proof. To be a bit more specific, we let $F(Q, \mu_z) = (F_i(Q, \mu_{z_i})_{i \in \mathcal{I}}$ where $\mu_z = (\mu_{z_i})_{i \in \mathcal{I}}$ and consider:

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[45]To verify (18), reverse the order of integration and use the convexity of $f(g_i(x_i, \cdot, Q, \tilde{\mu}_{z_i}))$ and the definition of $\succeq_{cx}$.

[46]$\succeq_{cxi}$ is a closed order on $\mathcal{P}(X_i)$.

$$\hat{H}(Q, a) = \{H(x) \in \mathbb{R} : x \in F(Q, \mu_z) \text{ for all } i\}$$

The rest of the proof then follows the proof of Theorem 5 line-by-line except that, as mentioned, $\mu_z$ replaces $a$. We are thus able to conclude that a mean-preserving spread to (any subset of) the agents leads to an increase in the greatest and least equilibrium aggregates. ■

## 8.2 Appendix II: Dynamic Programming with Transition Correspondences

Consider a standard recursive stochastic programming problem with functional equation:

$$v(x, z) = \sup_{y \in \Gamma(x,z)} [u(y, x, z) + \beta \int v(y, z')\mu_z(dz')] \tag{19}$$

As is well known, (19) has a unique solution $v^* : X \times Z \to \mathbb{R}$ (and this will be a continuous function) when $u : X^2 \times Z \to \mathbb{R}$ and $\Gamma : X \times Z \to 2^X$ are continuous, $X$ and $Z$ are compact, and $\beta \in (0, 1)$ (Stokey and Lucas (1989)). From $v^*$, the *policy correspondence* $G : X \times Z \to 2^X$ is then defined by,

$$G(x, z) = \arg \sup_{y \in \Gamma(x,z)} u(y, x, z) + \beta \int v^*(y, z')\mu_z(dz') \tag{20}$$

Clearly, $G$ will be upper hemi-continuous under the above assumptions. A *policy function* is a measurable selection from $G$, *i.e.*, a measurable function $g : X \times Z \to X$ such that $g(x, z) \in G(x, z)$ in $X \times Z$. Throughout it is understood that $X \times Z$ is equipped with the product $\sigma$-algebra, $\mathcal{B}(X) \otimes \mathcal{B}(Z)$. Recall that a correspondence such as $G$ is *(upper) measurable* if the inverse image of every open set is measurable, that is if $G^{-1}(O) \equiv \{(x, z) \in X \times Z : G(x, z) \cap O \neq \emptyset\} \in \mathcal{B}(X) \otimes \mathcal{B}(Z)$, whenever $O \subseteq X$ is open. An upper hemi-continuous correspondence is measurable (Aubin and Frankowska (1990), Proposition 8.2.1.).[47] Since a measurable correspondence has a measurable selection (Aubin and Frankowska (1990), Theorem 8.1.3.), any upper hemi-continuous policy correspondence admits a policy function $g$. Let $\mathcal{G}$ denote the *set* of measurable selections from $G$, which was just shown to be non-empty.

Given a policy function $g \in \mathcal{G}$, an $x \in X$, and a measurable set $A \in \mathcal{B}(X)$ let:

$$P_g(x, A) \equiv \mu_z(\{z \in Z : g(x, z) \in A\}) \quad \left(= \int_Z \chi_A(g(x, z))\mu_z(dz)\right) \tag{21}$$

For fixed $x \in X$, $P_g(x, \cdot)$ is a measure and for fixed $A \in \mathcal{B}(X)$, $P_g(\cdot, A)$ is measurable (the last statement is a consequence of Fubini's Theorem). So $P_g$ is a transition function.

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[47]Specifically, this is true when $X \times Z$ is a metric space with the Borel algebra and a complete $\sigma$-finite measure (see Aubin and Frankowska (1990) for details and a proof).

The family of policy correspondences $\mathcal{G}$ then gives rise to the *transition correspondence*:

$$P(x, \cdot) = \{P_g(x, \cdot) : g \in \mathcal{G}\}$$

Intuitively, given a state $x_t$ at date $t$, there is a set of possible probability measures $P(x, \cdot)$ each of which may describe the probability of being in a set $A \in \mathcal{B}(X)$ at date $t + 1$.

**Lemma 2 (The Transition Correspondence is Upper Hemi-Continuous)** *Consider a sequence $(x_n)_{n=0}^{\infty}$ in $X$ that converges to a limit point $x \in X$. Let $P_{g_n}(x_n, \cdot) \in P(x_n, \cdot)$ be an associated sequence of transition functions from the transition correspondence $P$. Then for any weakly convergent subsequence $P_{g_{n_m}}(x_{n_m}, \cdot)$ there exists a $P_g(x, \cdot) \in P(x, \cdot)$ such that $P_{g_{n_m}}(x_{n_m}, \cdot) \to_w P_g(x, \cdot)$.*

**Proof.** We loose no generality by assuming that the original sequence actually converges, $P_{g_n}(x_n, \cdot) \to_w \mu$, where $\mu$ is a probability measure on $(X, \mathcal{B}(X))$. Precisely, this means that for all $f \in \mathcal{C}(X)$ (the set of continuous real-valued functions on $X$):

$$\lim_{n \to \infty} \int f(z) P_{g_n}(x_n, dz) = \int f(z) \mu_z(dz)$$

We must show that that this equality holds with $\mu_z(\cdot) = P_g(x, \cdot)$ for some $g \in \mathcal{G}$. Fix $z \in Z$ and consider the sequence $g_n(x_n, z)$, $n = 0, 1, 2, \dots$. By the upper hemi-continuity of $G$, $\lim_{n \to \infty} g_n(x_n, z) \in G(x, z)$ (passing, if necessary to a subsequence which we index here again by $n$ to simplify notation). Then let $g(x, z) = \lim_{n \to \infty} g_n(x_n, z) \in G(x, z)$ for all $z$. Since each $g_n(x_n, \cdot)$ is measurable (in $z$), so is $g(x, z)$ (it is the pointwise limit of the sequence of functions $(g_1(x_1, \cdot), g_2(x_2, \cdot), \dots)$). Since $f$ is continuous, $f \circ g_n(x_n, \cdot)$ is measurable for all $n$, and so we have:

$$\lim_{n \to \infty} \int f(z) P_{g_n}(x_n, dz) = \lim_{n \to \infty} \int f \circ g_n(x_n, z) \mu_z(dz)$$

Since $f \circ g_n(x_n, z) \to f \circ g(x, z)$ for all $z$ (pointwise), it follows by Lebesgue's Dominated Convergence Theorem that:

$$\lim_{n \to \infty} \int f \circ g_n(x_n, z) \mu_z(dz) = \int f \circ g(x, z) \mu_z(dz)$$

Combining the above expressions we conclude that $\lim_{n \to \infty} \int f(z) P_{g_n}(x_n, dz) = \int f \circ g(x, z) \mu_z(dz) = \int f(z) P_g(x, dz)$ which is what we wanted to show. ∎

**Remark 6** Since an upper hemi-continuous correspondence is measurable, we get what Blume (1982) calls a multi-valued stochastic kernel $K : X \to 2^{\mathcal{P}(X)}$ by taking $P(x, \cdot) = K(x)$ for all $x \in X$.

Given $g \in \mathcal{G}$, define the adjoint Markov operator in the usual way from the transition function $P_g$:

$$T_g^* \lambda = \int P_g(x, \cdot) \lambda(dx) \qquad (22)$$

Next define the *adjoint Markov correspondence*:

$$T^* \lambda = \{T_g^*\}_{g \in \mathcal{G}} \qquad (23)$$

To clarify, $T^*$ maps a probability measure $\lambda$ into a set of probability measures, namely the set $\{T_g^* \lambda : g \in \mathcal{G}\}$. A probability measure $\lambda^*$ is *invariant* if:

$$\lambda^* \in T^* \lambda^*$$

Of course this is the same as saying that there exists $g \in \mathcal{G}$ such that $\lambda^* = T_g^* \lambda^*$.

**Lemma 3** *(The Adjoint Markov Correspondence is Upper Hemi-Continuous) Let $\lambda_n \to_w \lambda$ and consider a sequence $(\mu_n)$ with $\mu_n \in T^* \lambda_n$. Then for any convergent subsequence $\mu^{n_m} \to_w \mu$, it holds that $\mu \in T^* \lambda$.*

**Proof.** Although easy to prove directly, we shall not because it is a direct consequence of Proposition 2.3. in Blume (1982) (see Remark 6). ∎

One way to prove existence of an invariant distribution with transition correspondences is based on convexity, upper hemi-continuity, and the Kakutani-Glicksberg-Fan Theorem (Blume (1982)). Alternatively, one can look at suitable increasing selections and prove existence along the lines of Hopenhayn and Prescott (1992) using the Knaster-Tarski Theorem. However, for this paper's developments, we need a set-valued existence result that integrates with the results of Section 4.1. Mathematically, this can be accomplished by using the set-valued fixed point theorem of Smithson (1971), and this is what we shall do below. The order is the first-order stochastic dominance order.

We begin by proving a new result saying that if the policy correspondence $G(x, z)$ has an increasing and measurable greatest (respectively, least) selection in $x$ (for fixed $z$), then the adjoint Markov correspondence will be type I (respectively, type II) monotone in the sense of Definition 4.

**Theorem 12** *Assume that the policy correspondence $G : X \times \{z\} \to 2^X$ has an increasing greatest [least] selection for each fixed $z \in Z$. Then the adjoint Markov correspondence $T^*$ is type I [type*

*II] monotone. If G depends on an exogenous variable $a \in A$ so that $G : X \times \{z\} \times A \to 2^X$ and the greatest [least] selection from G is increasing in a, then $T_a^*$ will in addition be type I [type II] monotone in a.*

**Proof.** We prove the greatest/type I case only (the second case is similar). Consider probability measures $\mu_2 \succeq \mu_1$. We wish to show that for any $\lambda_1 \in T^*\mu_1$, there exists $\lambda_2 \in T^*\mu_2$ such that $\lambda_2 \succeq \lambda_1$. $\lambda_1 \in T^*\mu_1$ if and only if there exists a measurable selection $g_1 \in \mathcal{G}$ such that:

$$\lambda_1(\cdot) = \int_X P_{g_1}(x, \cdot) \mu_1(dx)$$

where,

$$P_{g_1}(x, A) = \int_Z \chi_A(g_1(x, z)) \mu_z(dz) \text{ , for } A \in \mathcal{B}(X)$$

Similarly for $\lambda_2 \in T^*\mu_2$ where we denote the (not yet determined) measurable selection by $g_2 \in \mathcal{G}$. Given these measurable selections, we have $\lambda_2 \succeq \lambda_1$ if and only if for every increasing function $f$:

$$\int_X f(x)\lambda_2(dx) \geq \int_X f(x)\lambda_1(dx) \Leftrightarrow$$

$$\int_X \int_Z f \circ g_2(x, z) \mu_z(dz) \mu_2(dx) \geq \int_X \int_Z f \circ g_1(x, z) \mu_z(dz) \mu_1(dx) \tag{24}$$

But taking $g_2$ to be the greatest selection from $G$ (which is measurable), it is clear that,

$$\int_X \int_Z f \circ g_2(x, z) \mu_z(dz) \mu_1(dx) \geq \int_X \int_Z f \circ g_1(x, z) \mu_z(dz) \mu_1(dx) \tag{25}$$

In addition, since $g_2$ is increasing in $x$, the function $x \mapsto \int_Z f \circ g_2(x, z) \mu_z(dz)$ is increasing in $x$. Since $\mu_2 \succeq \mu_1$ it follows that,

$$\int_X \int_Z f \circ g_2(x, z) \mu_z(dz) \mu_2(dx) \geq \int_X \int_Z f \circ g_2(x, z) \mu_z(dz) \mu_1(dx) \tag{26}$$

Now simply combine (25) and (26) to get (24). Thus we have proved that if $G$ has an increasing maximal selection, $T^*$ will be type I monotone.

The statements concerning the variable $a \in A$ are proved by essentially the same argument and is omitted. ∎

We now get the following existence result. Note that unless $T^*$ is also convex valued (which is not assumed here), the set of invariant distributions will generally not be convex.

**Theorem 13** *(Existence in the Type I/II Monotone Case) Assume that the adjoint Markov correspondence is either type I (or type II) order preserving. In addition assume that the state space (strategy set) has an infimum. Then $T^*$ has a fixed point (there exists an invariant measure). In addition, the fixed point correspondence will be upper hemi-continuous if $T^*$ is upper hemi-continuous in $(\mu, \theta)$ where $\theta$ is a parameter.*

**Proof.** By Proposition 1 in Hopenhayn and Prescott (1992), $(\mathcal{P}(X), \succeq)$ is chain complete (meaning that any chain $C$ in $\mathcal{P}(X)$ has a supremum in $\mathcal{P}(X)$). In order to apply Theorem 1.1. of Smithson (1971) we need therefore only verify his "Condition III" and establish the existence of some $\mu \in \mathcal{P}(X)$ such that there exists a $\lambda \in T^*\mu$ with $\mu \preceq \lambda$. The first of these ("Condition III") follows directly from upper hemi-continuity of $T^*$ (proof omitted). For the second, we do as Hopenhayn and Prescott (1992), proof of Corollary 2, and pick a measure $\delta_a$ from $\mathcal{P}(X)$ that places probability one on the infimum $\{a\} \equiv \inf X \in X$. Then $\lambda \succeq \delta_a$ for all $\lambda \in \mathcal{P}(X)$. It is then clear that if we take $\mu = \delta_a$ we have $\lambda \succeq \mu$ for (in fact, every) $\lambda \in T^*\mu$. The upper hemi-continuity claim is trivial under the stated assumptions. ∎

## 8.3 Appendix III: Aggregation of Risk and Laws of Large Numbers

This appendix is devoted to possible mathematical interpretations of the baseline aggregator (7) of Section 3:

$$H((x_{i,t})_{i \in \mathcal{I}}) = \int_{[0,1]} x_{i,t} di \tag{27}$$

Since the integrands on the right-hand-side of (27) are random variables, we must *define* what it means to integrate across them. And the fact is that there simply is not a uniformly accepted way to define this. In addition, we must ensure that some law of large numbers supports the assertion that the function's values are real numbers. There is a large and growing theoretical literature on how this can be done. The following are some of the most popular approaches to eliminating risk at the aggregate level.

- **(The Sampling Approach)** If one defines the integral $\int_{[0,1]} x_{i,t} di$ as the limiting average over an infinite (randomly drawn) subset of agents (Bewley (1986)), a law of large numbers will immediately apply and $H$ will take values in $\mathbb{R}$.

- **(Stochastic Integrals)** This approach is originally due to Uhlig (1996). Integrals of random functions with respect to deterministic measures is a special case of integrals of random

functions with respect to random measures, also known as stochastic integrals.[48] Viewing $\int_{[0,1]} x_i di$ as a stochastic integral, we have:

$$\int_{[0,1]} x_i di \equiv \lim_{n \to \infty} \sum_{i=1}^{n} x_{t_i}(t_i - t_{i-1}) \tag{28}$$

where the convergence is usually taken to be in $L^2$-norm, and as $n \to \infty$, the lengths of the subdivision $0 = t_1 < t_2 < \ldots < t_n = 1$ tends to zero. Given this interpretation, $\int_{[0,1]} x_{i,t} di$ will itself be a random variable, but when the $x_i$'s satisfy assumptions of some appropriate law of large numbers, the distribution will be degenerate. We may then identify it with a real number $H((x_{i,t})_{i \in \mathcal{I}})$ equal to the degenerate distribution's point of unit-mass as explained above. See the discussion below for further details on the stochastic integral approach.

- **(Pathwise Integration and Dependency Settings)** Another interpretation of (7) is that of Judd (1985) and Feldman and Gilles (1985) who suggest integrating over the set of sample paths (or rather, the measurable ones). As Judd (1985) and Feldman and Gilles (1985) explain, this approach runs into technical difficulties, however, making it inappropriate for the present purposes. Instead, Feldman and Gilles (1985) suggest looking at shocks across agents that are not independent of each other. This potentially solves the problems associated with pathwise integration in the i.i.d. case. Since independence of shock plays no other role for the results in our paper than that of ensuring that a law of large numbers applies, such "dependency" assumptions pose no problems as long as they lead to a well-defined aggregator.

- **(Discrete Set of Players)** In some contexts it may be unappealing to look at a continuum of agents, but one still wishes formalize the notion that each player is infinitely small relative to the market so that aggregate risk disappears by a law of large numbers type of argument. A way to model this is to look at a countable set of agents $\mathcal{I} \subseteq [0,1]$ (think of an infinitely fine "grid" such as the set of rational numbers) and equip this set with a non-atomic measure. Such a measure cannot be countably additive (or else the measure of the entire set of players would be 0). The setting thus becomes non-standard, but the advantage is that pathwise integration over sample paths becomes well-defined and the integral over a sample path will equal the sample average almost surely (the difficulties mention in the previous case thus disappear). See Al-Najjar (2004) for more on this idea. In terms of (27), the expression $\int_{[0,1]} x_{i,t} di$ must now be interpreted as the integral over sample paths. When a law of large

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[48]One can think of the former case as a stochastic integral where the random measures being integrated with respect to has a degenerate distribution. See for example Gourieroux (1997), page 71-72, who develops stochastic integrals from precisely this perspective (beginning with the deterministic measure case considered here.

numbers applies we once again get a well-defined aggregator. As may be verified, this paper's results nowhere make explicit use of our favored assumption of a continuum of agents - so if the reader prefers the approach of Al-Najjar (2004) (or any other non-standard approach for that matter), this is easily accommodated by taking $\mathcal{I}$ to be an uncountable but discrete set throughout.

The stochastic integral approach of Uhlig (1996) is further detailed in Acemoglu and Jensen (2010). Here we wish to expand upon this approach. To repeat, the idea is to take $\int_{[0,1]} x_{i,t} di$ to be equal to the random variable that is given by the limit in $L^2$-norm of the sequence of "Riemann sums", $\sum_{i=1}^{n} x_{\tau_i,t}(\tau_i - \tau_{i-1})$, $n = 1, 2, 3, \ldots$, for a narrowing sequence of subdivisions $0 = \tau_1 < \tau_2 < \ldots < \tau_n = 1$, $n = 1, 2, 3, \ldots$. Whenever the random variables considered are bounded (which they will be in our setting, cf. Assumption 1), convergence in $L^2$-norm is equivalent to convergence in probability.[49] Another thing worth mentioning is that sums of the type $\sum_{i=1}^{n} x_{\tau_i,t}(\tau_i - \tau_{i-1})$ may seem less general than the standard Riemann sums considered by Uhlig (1996) (precisely, the standard definition of the Riemann integral uses tagged partitions of the type $\sum_{i=1}^{n} x_{\rho_i,t}(\tau_i - \tau_{i-1})$ where $\rho_i \in [\tau_{i-1}, \tau_i]$ for all $i$). However, it is well know that using the "left-hand" Riemann sum is no less general than general tagged partitions for the simple reason that any tagged partition can be subdivided into a new finer partition whose subdivisions' left end-points are precisely the original tags. The following lemma is useful for determining when the limit of the Riemann sums is well-defined, and evaluating the integral.

**Lemma 4** *Consider the integral in (27) defined as the $L^2$-norm limit of the Riemann sums as described above. Then the limit is well-defined (i.e., it exists and is independent on the subdivisions) if the following condition is met:*

$$\int_{[0,1]} \mathbb{E}[(x_{i,t})^2] di < +\infty \tag{29}$$

*Furthermore, under this condition (or any other condition that implies that the limit is well-defined), the integral can be calculated as:*

$$\int_{[0,1]} x_{i,t} di = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} X_{i,t} \tag{30}$$

*where $(X_{1,t}, X_{2,t}, X_{3,t}, \ldots)$ is the sequence of random variables defined by recursively halving the interval $[0,1]$, i.e., $X_{1,t} \equiv x_{1,t}$, $X_{2,t} \equiv x_{\frac{1}{2},t}$, $X_{3,t} \equiv x_{\frac{1}{4},t}$, $X_{4,t} \equiv x_{\frac{3}{4},t}$, $X_{5,t} \equiv x_{\frac{1}{8},t}$, $\ldots$.*

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[49]Since convergence almost surely implies convergence in probability, $L^2$-norm convergence is consequently weaker than convergence almost surely in the present setting. This will be used repeatedly below.

**Proof.** The first claim of the lemma is found in Gourieroux (1997), p.71. As for (30), we begin by observing that when the subdivisions do not matter, we may focus attention on a convenient sequence of subdivisions such as the even subdivisions, $0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n-1}{n} < 1$, $n = 1, 2, 3, \ldots$. With this subdivision, (27) becomes:

$$\int_{[0,1]} x_{i,t} di = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} x_{\frac{i}{n},t} \tag{31}$$

In the summation we get for a given $n$ a sum over the set of random variables $\{x_{\frac{1}{n},t}, x_{\frac{2}{n},t}, \ldots, x_{\frac{n-1}{n},t}, x_{1,t}\}$. If we look at the subsequence $n = 1, 2, 4, 8, \ldots$ (which we clearly may do without loss of generality), we get an expanding sequence of random variables: $\{x_{1,t}\} \subseteq \{x_{\frac{1}{2},t}, x_{1,t}\} \subseteq \{x_{\frac{1}{4},t}, x_{\frac{1}{2},t}, x_{\frac{3}{4},t}, x_{1,t}\} \subseteq \{x_{\frac{1}{8},t}, x_{\frac{1}{4},t}, x_{\frac{3}{8},t}, x_{\frac{1}{2},t}, x_{\frac{5}{8},t}, x_{\frac{3}{4},t}, x_{\frac{7}{8},t}, x_{1,t}\} \subseteq \ldots$. In terms of (31) (slightly modified to the subsequence $n = 1, 2, 4, \ldots$), this *exactly* brings us to the sequence of random variables $X_{1,t}, X_{2,t}, X_{3,t}, \ldots$. ■

The upshot of the previous lemma is that (30) allows us to appeal to a standard version of the law of large numbers such as that of Chebyshev (1867) (see Acemoglu and Jensen (2010)) in order to conclude that:

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} X_i$$

will be a degenerate random variable with unit mass at:

$$\lim_{n \to \infty} \mathbb{E}[\int \sum_{i=1}^{n} \frac{1}{n} X_i]$$

The only remaining problem then is to ensure that this limit exists.

**Lemma 5** *If the function $i \mapsto \mathbb{E}[X_i]$ is Riemann-integrable, $\mu_n$ converges to the limit $\int \mathbb{E}[X_i] di$ as $n \to \infty$ where the integral is the Riemann integral.*[50]

**Proof.** Since $\mu_n = \mathbb{E}[A_n] = \sum_{i=1}^{n} \frac{1}{n} \mathbb{E}[X_i]$ is a Riemann sum, the existence of a limit follows directly from the definition of the Riemann integral. ■

If there is an at most countable number of types, the previous lemma applies. This is because the function $i \mapsto \mathbb{E}[X_i]$ will in this case be continuous almost everywhere (in fact, it will be piecewise constant). Since a bounded and continuous almost everywhere function is Riemann integrable, the conclusion follows.

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[50]When the Riemann integral exists, as it does here by assumption, the Riemann and Lebesgue integrals coincide. So it would be equally true to write that $\mu_n \to \int \mathbb{E}[X_i] di$ where the integral is the Lebesgue integral.

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