MANAGING A LIQUIDITY TRAP:
MONETARY AND FISCAL POLICY

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ABSTRACT

I study monetary and fiscal policy in liquidity trap scenarios, where the zero bound on the nominal interest rate is binding. I work with a continuous-time version of the standard New Keynesian model. Without commitment, the economy suffers from deflation and depressed output. I show that, surprisingly, both are exacerbated with greater price flexibility. I examine monetary and fiscal policies that maximize utility for the agent in the model and refer to these as optimal throughout the paper. I find that the optimal interest rate is set to zero past the liquidity trap and jumps discretely up upon exit. Inflation may be positive throughout, so the absence of deflation is not evidence against a liquidity trap. Output, on the other hand, always starts below its efficient level and rises above it. I then study fiscal policy and show that, regardless of parameters that govern the value of "fiscal multipliers" during normal or liquidity trap times, at the start of a liquidity trap optimal spending is above its natural level. However, it declines over time and goes below its natural level. I propose a decomposition of spending according to "opportunistic" and "stimulus" motives. The former is defined as the level of government purchases that is optimal from a static, cost-benefit standpoint, taking into account that, due to slack resources, shadow costs may be lower during a slump; the latter measures deviations from the former. I show that stimulus spending may be zero throughout, or switch signs, depending on parameters. Finally, I consider the hybrid where monetary policy is discretionary, but fiscal policy has commitment. In this case, stimulus spending is typically positive and increasing throughout the trap.

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1   Introduction

The 2007-8 crisis in the U.S. lead to a steep recession, followed by aggressive policy responses. Monetary policy went full tilt, cutting interest rates rapidly to zero, where they have remained since the end of 2008. With conventional monetary policy seemingly exhausted, fiscal stimulus worth \$787 billion was enacted by early 2009 as part of the American Recovery and Reinvestment Act. Unconventional monetary policies were also pursued, starting with "quantitative easing", purchases of long-term bonds and other assets. In August 2011, the Federal Reserve's FOMC statement signaled the intent to keep interest rates at zero until at least mid 2013. Similar policies have been followed, at least during the peak of the crisis, by many advanced economies. Fortunately, the kind of crises that result in such extreme policy measures have been relatively few and far between. Perhaps as a consequence, the debate over whether such policies are appropriate remains largely unsettled. The purpose of this paper is to make progress on these issues.

To this end, I reexamine monetary and fiscal policy in a liquidity trap, where the zero bound on nominal interest rate binds. I work with a standard New Keynesian model that builds on Eggertsson and Woodford (2003).[1] In these models a liquidity trap is defined as a situation where negative real interest rates are needed to obtain the first-best allocation. I adopt a deterministic continuous time formulation that turns out to have several advantages. It is well suited to focus on the dynamic questions of policy, such as the optimal exit strategy, whether spending should be front- or back-loaded, etc. It also allows for a simple graphical analysis and delivers several new results. The alternative most employed in the literature is a discrete-time Poisson model, where the economy starts in a trap and exits from it with a constant exogenous probability each period. This specification is especially convenient to study the effects of suboptimal and simple Markov policies—because the

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[1] Eggertsson (2001, 2006) study government spending during a liquidity trap a New Keynesian model, with the main focus is on the case without commitment and implicit commitment to inflate afforded by rising debt. Christiano et al. (2011), Woodford (2011) and Eggertsson (2011) consider the effects of spending on output, computing "fiscal multipliers", but do not focus on optimal policy.

equilibrium calculations then reduce to finding a few numbers—but does not afford any comparable advantages for the optimal policy problem.

I examine policies that maximize welfare for the agent in the model and refer to them throughout as optimal. I consider the policy problem under commitment, under discretion and for some intermediate cases. I am interested in monetary policy, fiscal policy, as well as their interplay. What does optimal monetary policy look like? How does the commitment solution compare to the discretionary one? How does it depend on the degree of price stickiness? How can fiscal policy complement optimal monetary policy? Can fiscal policy mitigate the problem created by discretionary monetary policy? To what extent is spending governed by a concern to influence the private economy as captured by "fiscal multipliers", or by simple cost-benefit public finance considerations?

I first study monetary policy in the absence of fiscal policy. When monetary policy lacks commitment, deflation and depression ensue. Both are commonly associated with liquidity traps. Less familiar is that both outcomes are exacerbated by price flexibility. Thus, one does not need to argue for a large degree of price stickiness to worry about the problems created by a liquidity trap. In fact, quite the contrary. I show that the depression becomes unbounded as we converge to fully flexible prices. The intuition for this result is that the main problem in a liquidity trap is an elevated real interest rate. This leads to depressed output, which creates deflationary pressures. Price flexibility accelerates deflation, raising the real interest rate further and only making matters worse.

As first argued by Krugman (1998), optimal monetary policy can improve on this dire outcome by committing to future policy in a way that affects current expectations favorably. In particular, I show that, it is optimal to promote future inflation and stimulate a boom in output. I establish that optimal inflation may be positive throughout the episode, so that deflation is completely avoided. Thus, the absence of deflation, far from being at odds with a liquidity trap, actually may be evidence of an optimal response to such a situation. I show that output starts below its efficient level, but rises above it towards the end

of the trap. Indeed, the boom in output is larger than that stimulated by the inflationary promise.

There are a number of ways monetary policy can promote inflation and stimulate output. Monetary easing does not necessarily imply a low equilibrium interest rate path. Indeed, as in most monetary models, the nominal interest rate path does not uniquely determine an equilibrium. Indeed, an interest rate of zero during the trap that becomes positive immediately after the trap is consistent with positive inflation and output after the trap.[2] I show, however, that the optimal policy with commitment involves keeping the interest rate down at zero longer. The continuous time formulation helps here because it avoids time aggregation issues that may otherwise obscure the result.

Some of my results echo findings from prior work based on simulations for a Poisson specification of the natural rate of interest. Christiano et al. (2011) reports that, when the central bank follows a Taylor rule, price stickiness increases the decline in output during a liquidity trap. Eggertsson and Woodford (2003), Jung et al. (2005) and Adam and Billi (2006) find that the optimal interest rate path may keep it at zero after the natural rate of interest becomes positive. To the best of my knowledge this paper provides the first formal results explaining these findings for inflation, output and interest rates.

An implication of my result is that the interest rate should jump discretely upon exiting the zero bound—a property that can only be appreciated in continuous time. Thus, even when fundamentals vary continuously, optimal policy calls for a discontinuous interest rate path.

Turning to fiscal policy, I show that, there is a role for government spending during a liquidity trap. Spending should be front-loaded. At the start of the liquidity trap, government spending should be higher than its natural level. However, during the trap

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[2]For example, a zero interest during the trap and an interest equal to the natural rate outside the trap. This is the same path for the interest rate that results with discretionary monetary policy. However, in that case, the outcome for inflation and output is pinned down by the requirement that they reach zero upon exiting the trap. With commitment, the same path for interest rates is consistent with higher inflation and output upon exit.

spending should fall and reach a level below its natural level. Intuitively, optimal government spending is countercyclical, it leans against the wind. Private consumption starts out below its efficient level, but reaches levels above its efficient level near the end of the liquidity trap. The pattern for government spending is just the opposite.

The optimal pattern for total government spending masks two potential motives. Perhaps the most obvious, especially within the context of a New Keynesian model, is the macroeconomic, countercyclical one. Government spending affects private consumption and inflation through dynamic general equilibrium effects. In a liquidity trap this may be particularly useful, to mitigate the depression and deflation associated with these events.

However, a second, often ignored, motive is based on the idea that government spending should react to the cycle even based on static, cost-benefit calculations. In a slump, the wage, or shadow wage, of labor is low. This makes it is an opportune time to produce government goods. During the debates for the 2009 ARRA stimulus bill, variants of this argument were put forth.

Based on these notions, I propose a decomposition of spending into "stimulus" and "opportunistic" components. The latter is defined as the optimal static level of government spending, taking private consumption as given. The former is just the difference between actual spending and opportunistic spending.

I show that the optimum calls for zero stimulus at the beginning of a liquidity trap. Thus, my previous result, showing that spending starts out positive, can be attributed entirely to the opportunistic component of spending. More surprisingly, I then show that for some parameter values stimulus spending is everywhere exactly zero, so that, in these cases, opportunistic spending accounts for all of government spending policy during a liquidity trap. Of course, opportunistic spending does, incidentally, influence consumption and inflation. But the point is that these considerations need not figure into the calculation. In this sense, public finance trumps macroeconomic policy.

Another implication is that, in such cases, commitment to a path for government

spending is superfluous. A naive, fiscal authority that acts with full discretion and performs the static cost-benefit calculation chooses the optimal path for spending.

These results assume that monetary policy is optimal. Things can be quite different when monetary policy is suboptimal due to lack of commitment. To address this I study a mixed case, where monetary policy is discretionary but fiscal policy has the power to commit to a government spending path. Positive stimulus spending emerges as a way to fight deflation. Indeed, the optimal intervention is to provide positive stimulus spending that rises over time during the liquidity trap. Back-loading stimulus spending provides a bigger bang for the buck, both in terms of inflation and output. Since price setting is forward looking, spending near the end promotes inflation both near the end and earlier. In addition, any improvement in the real rate of return near the end of the liquidity trap improves the output outcome level for earlier dates. Both reasons point towards increasing stimulus spending.

If the fiscal authority can commit past the trap, then it is optimal to promise lower spending immediately after the trap, and converge towards the natural rate of spending after that. Spending features a discrete downward jump upon exiting the trap. Intuitively, after the trap, once the flexible price equilibrium is attainable, lower government spending leads to a consumption boom. This is beneficial, for the same reasons that monetary policy with commitment promotes a boom, because it raises the consumption level during the trap. Thus, the commitment to lower spending after the trap attempts to mimic the expansionary effects that the missing monetary commitments would have provided.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 studies the equilibrium without fiscal policy when monetary policy is conducted with discretion. Section 4 studies optimal monetary policy with commitment. Section 5 adds fiscal policy and studies the optimal path for government spending alongside optimal monetary policy. Section 6 considers mixed cases where monetary policy is discretionary, but fiscal policy enjoys commitment.

2 A Liquidity Trap Scenario

The model is a continuous-time version of the standard New Keynesian model. The environment features a representative agent, monopolistic competition and Calvo-style sticky prices; it abstracts from capital investment. I spare the reader another rendering of the details of this standard setting (see e.g. Woodford, 2003, or Galí, 2008) and skip directly to the well-known log-linear approximation of the equilibrium conditions which I use in the remainder of the paper.

Euler Equation and Phillips Curve.   The equilibrium conditions, log linearized around zero inflation, are

$$\dot{x}(t) = \sigma^{-1}(i(t) - r(t) - \pi(t)) \tag{1a}$$

$$\dot{\pi}(t) = \rho\pi(t) - \kappa x(t) \tag{1b}$$

$$i(t) \geq 0 \tag{1c}$$

where ρ, σ and κ are positive constants and the path $\{r(t)\}$ is exogenous and given. We also require a solution $(\pi(t), x(t))$ to remain bounded. The variable $x(t)$ represents the output gap: the log difference between actual output and the hypothetical output that would prevail at the efficient, flexible price, outcome. Inflation is denoted by $\pi(t)$ and the nominal interest rate by $i(t)$. Finally, $r(t)$ stands for the "natural rate of interest", i.e. the real interest rate that would prevail in an efficient, flexible price, outcome with $x(t) = 0$ throughout.

Equation (1a) represents the consumer's Euler equation. Output growth, equal to consumption growth, is an increasing function of the real rate of interest, $i(t) - \pi(t)$. The natural rate of interest enters this condition because output has been replaced with the output gap. Equation (1b) is the New-Keynesian, forward-looking Phillips curve. It can be restated as saying that inflation is proportional, with factor $\kappa > 0$, to the present value

of future output gaps,

$$\pi(t) = \kappa \int_0^\infty e^{-\rho s} x(t+s) ds.$$

Thus, positive output gaps stimulate inflation, while negative output gaps produce deflation. Finally, inequality (1c) is the zero-lower bound on nominal interest rates (hereafter, ZLB).

As for the constants, ρ is the discount rate, σ^{-1} is the intertemporal elasticity of substitution and κ controls the degree of price stickiness. Lower values of κ imply greater price stickiness. As $\kappa \to \infty$ we approach the benchmark with perfectly flexible prices, where high levels of inflation or deflation are compatible with minuscule output gaps.

A number of caveats are in order. The model I use is the very basic New Keynesian setting, without any bells and whistles. Basing my analysis on this simple model is convenient because it lies at the center of many richer models, so we may learn more general lessons. It also facilitates the normative analysis, which could quickly become intractable otherwise. On the other hand, the analysis abstracts from unemployment, and omits distortionary taxes, financial constraints and other frictions which may be relevant in these situations.

Quadratic Welfare Loss. I will evaluate outcomes using the quadratic loss function

$$L \equiv \frac{1}{2} \int_0^\infty e^{-\rho t} \left(x(t)^2 + \lambda \pi(t)^2 \right) dt. \tag{2}$$

According to this loss function it is desirable to minimize deviations from zero for both inflation and the output gap. The constant λ controls the relative weight placed on the inflationary objective. The quadratic nature of the objective is convenient and can be derived as a second order approximation to welfare around zero inflation when the flexible price equilibrium is efficient.[3] Such an approximation also suggests that $\lambda = \bar{\lambda}/\kappa$ for

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[3] In order to be efficient, the equilibrium requires a constant subsidy to production to undo the monopolistic markup. An alternative quadratic objective that does not assume the flexible price equilibrium is

some constant $\bar{\lambda}$, so that $\lambda \to 0$ as $\kappa \to \infty$, as prices become more flexible, price instability becomes less harmful.

The Natural Rate of Interest. The path for the natural rate $\{r(t)\}$ plays a crucial role in the analysis. Indeed, if the natural rate were always positive, so that $r(t) \geq 0$ for all $t \geq 0$, then the flexible price outcome with zero inflation and output gap, $\pi(t) = x(t) = 0$ for all $t \geq 0$, would be feasible and obtained by letting $i(t) = r(t)$ for all $t \geq 0$. This outcome is also optimal, since it is ideal according to the loss function (2).

The situation described in the previous paragraph amounts to the case where the ZLB constraint (1c) is always slack. The focus of this paper is on situations where the ZLB constraint binds. Thus, I am interested in cases where $r(t) < 0$ for some range of time. For a few results it is useful to further assume that the the economy starts in a liquidity trap that it will eventually and permanently exit at some date $T > 0$:

$$r(t) < 0 \qquad t < T$$
$$r(t) \geq 0 \qquad t \geq T.$$

I call such a case a *liquidity trap scenario*. A simple example is the step function

$$r(t) = \begin{cases} \underline{r} & t \in [0, T) \\ \bar{r} & t \in [T, \infty) \end{cases}$$

where $\bar{r} > 0 > \underline{r}$. I use the step function case in some figures and simulations, but it is not required for any of the results in the paper.

Finally, I also make a technical assumption: that $r(s)$ is bounded and that the integral $\int_0^t r(s)ds$ be well defined and finite for any $t \geq 0$.

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efficient is $\frac{1}{2} \int_0^\infty e^{-\rho t} \left((x(t) - \bar{x})^2 + \lambda \pi(t)^2 \right) dt$ for $\bar{x} > 0$. Most of the analysis would carry through to this case.

3 Monetary Policy without Commitment

Before studying optimal policy with commitment, it is useful to consider the situation without commitment, where the central bank is benevolent but cannot credibly announce plans for the future. Instead, it acts opportunistically at each point in time, with absolute discretion. This provides a useful benchmark that illustrates some features commonly associated with liquidity traps, such as deflationary price dynamics and depressed output. I will also derive some less expected implications on the role of price stickiness. The outcome without commitment is later contrasted to the optimal solution with commitment.

3.1 Deflation and Depression

To isolate the problems created by a complete lack of commitment, I rule out explicit rules as well as reputational mechanisms that bind or affect the central bank's actions directly or indirectly. I construct the unique equilibrium as follows.[4] For $t \geq T$ the natural rate is positive, $r(t) = \bar{r} > 0$, so that, as mentioned above, the ideal outcome $(\pi(t), x(t)) = (0,0)$ is attainable. I assume that the central bank can guarantee this outcome and implements it so that $(\pi(t), x(t)) = (0,0)$ for $t \geq T$.[5] Taking this as given, at all earlier dates $t < T$ the central bank will find it optimal to set the nominal interest rate to zero. The resulting no-commitment outcome is then uniquely determined by the ODEs (1a)–(1b) with $i(t) = 0$ for $t \leq T$ and the boundary condition $(\pi(T), x(T)) = (0,0)$.[6]

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[4]In this section, I proceed informally. With continuous time, a formal study of the no-commitment case requires a dynamic game with commitment over vanishingly small intervals.

[5]Although this seems like a natural assumption, it presumes that the central bank somehow overcomes the indeterminacy of equilibria that plagues these models. Usually this can be accomplished, for example, by adherence to a Taylor rule, with appropriate coefficients. However, following such a rule requires commitment, off the equilibrium path, which is not possible here. However, note that this issue is completely separate from the zero lower bound on interest rates. Thus, the assumption that $(\pi(t), x(t)) = (0,0)$ can be guaranteed for $t \geq T$ allows us to focus on the interaction between no commitment and a liquidity trap scenario.

[6]This outcome coincides with the optimal solution with commitment if one constrains the problem by imposing $(\pi(T), x(T)) = (0,0)$. In other words, the ability to commit to outcomes within the interval $t \in [0, T)$ is irrelevant; also, the ability to commit once $t = T$ is reached is also irrelevant. What is crucial is the ability to commit ex ante at $t < T$ to outcomes for $t = T$.
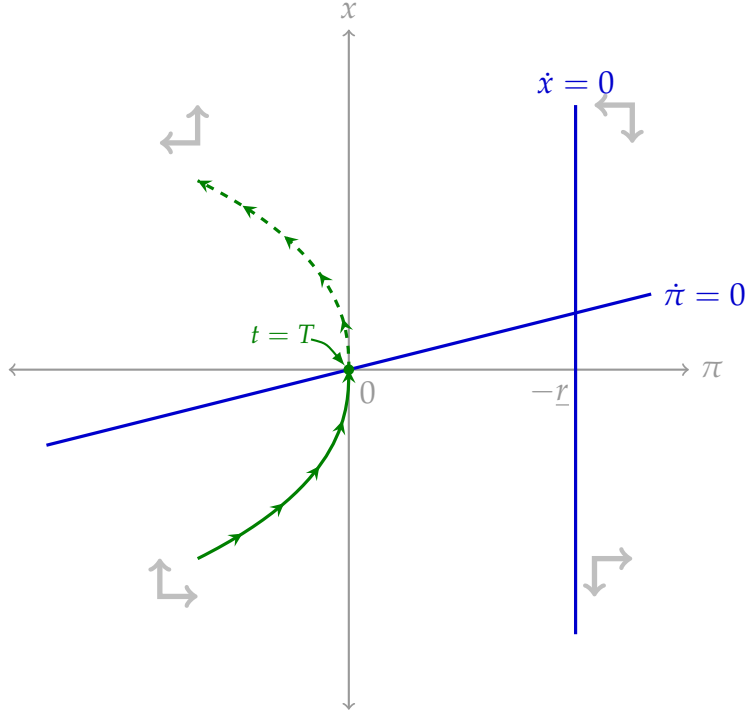
Figure 1: The equilibrium without commitment, featuring $i(t) = 0$ for $t \leq T$ and reaching $(0,0)$ at $t = T$.

This situation is depicted in Figure 1 which shows the dynamical system (1a)–(1b) with $i(t) = 0$ and depicts a path leading to $(0,0)$ precisely at $t = T$. Output and inflation are both negative for $t < T$ as they approach $(0,0)$. Note that the loci on which $(\pi(t), x(t))$ must travel towards $(0,0)$ is independent of T, but a larger T requires a starting point further away from the origin. Thus, initial inflation and output are both decreasing in T. Indeed, as $T \to \infty$ we have that $\pi(0), x(0) \to -\infty$.

Proposition 1. *Consider a liquidity trap scenario, with $r(t) < 0$ for $t < T$ and $r(t) \geq 0$ for $t \geq T$. Let $\pi^{nc}(t)$ and $x^{nc}(t)$ denote the equilibrium outcome without commitment. Then inflation and output are zero after $t = T$ and strictly negative before that:*

$$\pi^{nc}(t) = x^{nc}(t) = 0 \qquad t \geq T$$

$$\pi^{nc}(t) < 0 \quad x^{nc}(t) < 0 \qquad t < T.$$

11

Moreover, $\pi(t)$ and $x(t)$ are strictly increasing in t for $t < T$. In the limit as $T \to \infty$, if the natural rate satisfies $\int_0^T r(t; T) ds \to -\infty$, then

$$\pi^{nc}(0, T), x^{nc}(0, T) \to -\infty.$$

The equilibrium features deflation and depression. The severity of both depend, among other things, on the duration T of the liquidity trap. Both becomes unbounded as $T \to \infty$. In this sense, discretionary policy making may have very adverse welfare implications.

How can the outcome be so dire? The main distortion is that the real interest rate is set too high during the liquidity trap. This depresses consumption. Importantly, this effect accumulates over time. Even with zero inflation consumption becomes depressed by $\sigma^{-1} \int_t^T r(t) ds$. For example, with log utility $\sigma = 1$ if the natural rate is -4% and the trap lasts two years the loss in output is at least 8%. Moreover, matters are just made worse by deflation, which raises the real interest rate even more, further depressing output, leading to even more deflation, in a vicious cycle.

Note that it is the lack of commitment during the liquidity trap $t < T$ to policy actions and outcomes after the liquidity trap $t \geq T$ that is problematic. Policy commitment during the liquidity trap $t < T$ is not useful. Neither is the ability to announce a credible plan at $t = T$ for the entire future $t \geq T$. Indeed, if we add $(\pi(T), x(T)) = (0, 0)$ as a constraint, then the no commitment outcome is optimal, even when the central bank enjoys full commitment to any choice over $(\pi(t), x(t), i(t))_{t \neq T}$ satisfying (1a)–(1b) for $t < T$ and $t > T$. What is valuable is the ability to commit during the liquidity trap to policy actions and outcomes after the liquidity trap. In particular, to something other than $(\pi(T), x(T)) = (0, 0)$.

3.2 Harmful Effects from Price Flexibility

How is this bleak outcome affected by the degree of price stickiness? One might expect things to improve when prices are more flexible. After all, the main friction in New Keynesian models is price rigidities, suggesting that outcomes should improve as prices become more flexible. The next proposition, perhaps counterintuitively, shows that the reverse is actually the case.

Proposition 2. *When prices are more flexible, the outcome without commitment features lower inflation and output. That is, if $\kappa < \kappa'$ then*

$$\pi^{nc}(t, \kappa') < \pi^{nc}(t, \kappa) < 0 \quad and \quad x^{nc}(t, \kappa') < x^{nc}(t, \kappa) < 0 \quad for\ all\ t < T.$$

Indeed, for given $T > 0$ and $t < T$ in the limit as $\kappa \to \infty$

$$\pi(t, \kappa), x(t, \kappa) \to -\infty$$

and $L(\kappa) \to \infty$.

According to this result, without commitment, price stickiness is beneficial. This is punctuated by the limit as we approach perfectly flexible prices, which implies unbounded levels of deflation and depression. This upsets the common perception that severe consequences from a liquidity trap require significant levels of price stickiness. Quite the contrary, sticky prices hold back deflation and mitigate depressions.

To gain intuition for this result, note that the Phillips curve equation (1b) implies that, for a given negative output gap, a higher κ creates more deflation. More deflation, in turn, increases the real interest rate $i - \pi$. By the Euler equation (1a) this requires higher growth in the output gap $\dot{x}$; since $x = 0$ at $t = T$, this translates into a lower level of x for earlier dates $t < T$. In words, flexible prices lead to more vigorous deflation, raising the real interest rate and depressing output. Lower output reinforces the deflationary pressures,

creating a vicious cycle. The proof in the appendix echoes this intuition closely.

A similar result is reported in the analysis of fiscal multipliers by Christiano et al. (2011). They compute the equilibrium when monetary policy follows a Taylor rule and the natural rate of interest is a Poisson process. In this context, they show that output may be more depressed if prices are more flexible—they do not pursue a limiting result towards full flexibility.[7] My result is somewhat distinct, because it applies to a situation with optimal discretionary monetary policy, instead of a Taylor rule, and it holds for any deterministic path for the natural rate. Another difference is that in a Poisson environment an equilibrium fails to exist, when prices are too flexible. Despite these differences, the logic for the effect is the same in both cases.[8]

It is worth remarking that both the zero lower bound and the lack of commitment are not critical. The same result holds for any path of the natural rate $\{r(t)\}$ if we assume the central bank sets the nominal interest rate above the natural rate $i(t) = r(t) + \Delta$ with $\Delta > 0$ for some period of time $t \leq T$ and then switches back to the first best outcome $x(t) = \pi(t) = 0$, with $i(t) = r(t)$ for $t > T$. The zero lower bound and the lack of commitment just serve to motivate such a scenario, but it could also result from policy mistakes in interest rate setting.[9]

The conclusion that price flexibility is always harmful relies on the lack of commitment. Indeed, when the central bank can commit to an optimal policy, price flexibility may be beneficial. Interestingly, this depends on parameters. Before studying optimal policy, however, it is useful to consider the effects of commitment to simple non-optimal policies.

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[7]Basically the same Poisson calculations in Christiano et al. (2011) appear also in Woodford (2011) and Eggertsson (2011), although the effects of price flexibility are not their focus and so they do not discuss its effects.

[8]De Long and Summers (1986) make the point that, for given monetary policy rules, price flexibility may be destabilizing, even away from a liquidity trap, in the sense of increasing the variance of output.

[9]Of course a symmetric result holds for $\Delta < 0$. There is a boom in output alongside inflation. The undesirable boom and inflation are amplified when prices are more flexible, in the sense of a higher κ.

3.3 Elbow Room with a Higher Inflation Target

We now ask whether there are simple policies the central bank can commit to that avoid the depression and deflation outcomes obtained without commitment. Consider a plan that keeps inflation and output gap constant at

$$\pi(t) = -\underline{r} > 0 \qquad x(t) = -\frac{1}{\kappa}\underline{r} > 0 \qquad \text{for all } t \geq 0.$$

It follows that $i(t) = r(t) + \pi(t)$, so that $i(t) = 0$ for $t < T$ while $i(t) = \bar{r} + \bar{\pi} > \bar{r} > 0$ for $t \geq T$.

Although this policy is not optimal, it behaves well in the limit as prices become fully flexible. Indeed, in this limit as $\kappa \to \infty$ the output gap converges uniformly to zero while inflation remains constant. Thus, if we adopt the natural case where $\lambda = \bar{\lambda}/\kappa \to 0$, the loss function converges to its ideal value of zero, $L(\kappa) \to 0$. Compare this to the dire outcome without commitment in Proposition 2, where the output gap and losses converge to $-\infty$.

Just as in the case without commitment, this simple policy sets the nominal interest rate to zero during the liquidity trap, for $t < T$. Note that after the trap, for $t > T$, the nominal interest rate is actually set to a higher level than the case without commitment. Thus, the advantages of this simple policy do not hinge on lower nominal interest rates, but quite the contrary. Higher inflation here coincides with higher nominal interest rates, due to the Fischer effect. One may still describe the outcome as resulting from looser monetary policy, but the point is that the kind of monetary easing needed to avoid the deflation and depression does not require lower equilibrium nominal interest rates. As we shall see in the next section, the optimal policy with commitment does feature lower, indeed zero, nominal interest rates.

This idea is more general. For any path for the natural interest rate $\{r(t)\}$, set a con-

stant inflation rate given by

$$\pi(t) = \bar{\pi} = -\min_{t \geq 0} r(t)$$

and an output gap of $x(t) = \bar{x} = \kappa\bar{\pi}$. This plan is feasible with a non-negative nominal interest rates $i(t) \geq 0$. These simple policy capture the main idea behind calls to tolerate higher inflation targets that leave more "elbow room" for monetary policy during liquidity traps (e.g. Summers, 1991; Blanchard et al., 2010). However, given the forward looking nature of inflation in this model, what is crucial is the commitment to higher inflation after the liquidity trap. This contrasts with the conventional argument, where a higher inflation rate before the trap serves as a precautionary sacrifice for future liquidity traps.

It is perhaps surprising that commitment to a simple policy can avoid deflation and depressed output altogether. Of course, they do so at the expense of inflation and over-stimulated output. If the required inflation target $\bar{\pi}$ or output gap $\bar{x}$ are large, or if the duration of the trap T is small, these plans may be quite far from optimal, since they require a permanent sacrifice for the loss function.[10] This motivates the study of optimal monetary policy which I take up next.

4   Optimal Monetary Policy

I now turn to optimal monetary policy with commitment. The central bank's problem is to minimize the objective (2) subject to (1a)–(1c) with both initial values of the states, $\pi(0)$ and $x(0)$, free. The problem seeks the most preferable outcome, across all those compatible with an equilibrium. In what follows I focus on characterizing the optimal

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[10]The reason the output gap $\bar{x}$ is strictly positive is the New Keynesian model's non-vertical long-run Phillips curve. Some papers have explored modifications of the New Keynesian model that introduce indexation to past inflation. Some forms of full indexation imply that a constant level of inflation does not affect output nor welfare. Thus, with the right form of indexation very simple policies may be optimal or close to optimal. Of course, this is not the case in the present model without indexation.

path for inflation, output and the nominal interest rate.[11]

4.1   Optimal Interest Rates, Inflation and Output

The problem can be analyzed as an optimal control problem with state $(\pi(t), x(t))$ and control $i(t) \geq 0$. The associated Hamiltonian is

$$H \equiv \frac{1}{2}x^2 + \frac{1}{2}\lambda\pi^2 + \mu_x\sigma^{-1}(i - r - \pi) + \mu_\pi(\rho\pi - \kappa x).$$

The maximum principle implies that the co-state for x must be non-negative throughout and zero whenever the nominal interest rate is strictly positive

$$\mu_x(t) \geq 0, \tag{3a}$$

$$i(t)\mu_x(t) = 0. \tag{3b}$$

The law of motion for the co-states are

$$\dot{\mu}_x(t) = -x(t) + \kappa\mu_\pi(t) + \rho\mu_x(t), \tag{3c}$$

$$\dot{\mu}_\pi(t) = -\lambda\pi(t) + \sigma^{-1}\mu_x(t). \tag{3d}$$

Finally, because both initial states are free, we have

$$\mu_x(0) = 0, \tag{3e}$$

$$\mu_\pi(0) = 0. \tag{3f}$$

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[11]I do not dedicate much discussion to the question of implementation, in terms of a choice of (possibly time varying) policy functions that would make the optimum a unique equilibrium. It is well understood that, once the optimum is computed, a time varying interest rate rule of the form $i(t) = i^*(t) + \psi(\pi(t) - \pi^*(t)) + \psi(x(t) - x^*(t))$ ensures that this optimum is the unique local equilibrium. Eggertsson and Woodford (2003) propose a different policy, described in terms of an adjusting target for a weighted average of output and the price level, that also implements the equilibrium uniquely.

Taken together, equations (1a)–(1c) and (3a)–(3f) constitute a system for $\{\pi(t), x(t), i(t),$ $\mu_\pi(t), \mu_x(t)\}_{t\in[0,\infty)}$. Since the optimization problem is strictly convex, these conditions, together with appropriate transversality conditions, are both necessary and sufficient for an optimum. Indeed, the optimum coincides with the unique bounded solution to this system.

Suppose the zero-bound constraint is not binding over some interval $t \in [t_1, t_2]$. Then it must be the case that $\mu_x(t) = \dot{\mu}_x(t) = 0$ for $t \in [t_1, t_2]$, so that condition (3c) implies $x(t) = \kappa \mu_\pi(t)$, while condition (3d) implies $\dot{\mu}_\pi(t) = -\lambda \pi(t)$. As a result,

$$\dot{x}(t) = \kappa \dot{\mu}_\pi(t) = -\kappa \lambda \pi(t) = \sigma^{-1}(i(t) - r(t) - \pi(t)).$$

Solving for $i(t)$ gives

$$i(t) = I(\pi(t), r(t)),$$

where

$$I(\pi, r) \equiv r(t) + (1 - \kappa \sigma \lambda)\pi,$$

is a function that gives the optimal nominal rate whenever the zero-bound is not binding. This is the interest rate condition derived in the traditional analysis that assumes the ZLB never binds (see e.g. Clarida, Gali and Gertler, 1999, pg. 1683). Note that this rate equals the natural rate when inflation is zero, $I(0, r) = r$. Thus, it encompasses the well-known price stability result from basic New-Keynesian models. Away from zero inflation, the interest rate generally departs from the natural rate, unless $\sigma \lambda \kappa = 1$.

Given this result, it follows that $I^*(\pi^*(t), r(t)) \geq 0$ is a necessary condition for the zero-bound not to bind. The converse, however, is not true.

Proposition 3. *Suppose $\{\pi^*(t), x^*(t), i^*(t)\}$ is optimal. Then at any point in time t either* $i^*(t) = I(\pi^*(t), r(t))$ or $i^*(t) = 0$. Moreover

$$I(\pi^*(t), r(t)) < 0 \quad for \ t \in [t_0, t_1) \implies i^*(t) = 0 \quad for \ t \in [t_0, t_2]$$

18

with $t_2 > t_1$.

According to this result, the nominal interest rate should be held down at zero longer than what current inflation warrants. That is, the optimal path for the nominal interest rate is not the upper envelope

$$i^*(t) \neq \max\{0, I(\pi^*(t), r(t))\}.$$

Instead, the nominal interest rate should be set below this envelope for some time, at zero.

The notion that committing to future monetary easing is beneficial in a liquidity trap was first put forth by Krugman (1998). His analysis captures the benefits from future inflation only. It is based on a cash-in-advance model where prices cannot adjust within a period, but are fully flexible between periods. The first best is obtained by committing to money growth and inducing higher future inflation. Thus, inflation is easily obtained and costless in the model. Eggertsson and Woodford (2003) work with the same New Keynesian model as I do here. They report numerical simulations where a prolonged period of zero interest rates are optimal. My result provides the first formal explanation for these patterns. It also clarifies that the relevant comparison for the nominal interest rate $i^*(t)$ is the unconstrained optimum $I(\pi^*(t), r(t))$, not the natural rate $r(t)$; the two are not equivalent, unless $\kappa \sigma \lambda = 1$. The continuous time framework employed here helps capture the bang-bang nature of the solution. A discrete-time setting can obscure things due to time aggregation.

One interesting implication of my result is that the optimal exit strategy features a discrete jump in the nominal interest rate. Whenever the zero-bound stops binding the nominal interest must equal $I(\pi^*(t), r(t))$, which given Proposition 3, will generally be strictly positive. Thus, optimal policy requires a discrete upward jump, from zero, in the nominal interest rate. Even when economic fundamentals vary smoothly, so that $I(\pi^*(t), r(t))$ is continuous, the best exit strategy calls for a discontinuous hike in the

nominal interest rate.

The previous result characterizes nominal interest rates, but what can be said about the paths for inflation and output? This question is important for a number of reasons. First, output and inflation are of direct concern, since they determine welfare. In contrast, the nominal interest rate is merely an instrument to influence output and inflation. Second, as in most monetary models, the equilibrium outcome is not uniquely determined by the equilibrium path for the nominal interest rate. A central bank wishing to implement the optimum needs to know more than the path for the nominal interest rate. For example, the central bank may employ a Taylor rule centered around the target path for inflation $i(t) = i^*(t) + \psi(\pi(t) - \pi^*(t))$ with $\psi > 1$. Finally, understanding the outcome for inflation and output sheds light on the kind of policy commitment required.

The next proposition provides results for inflation and output. Inflation must be positive at some point in time. Indeed, in some cases, inflation is always positive, despite the liquidity trap. Output, on the other hand, must switch signs. Thus, a future boom in output is created, but the initial recession is never completely avoided.

Proposition 4. *Suppose the first-best outcome is not attainable and that $\{\pi^*(t), x^*(t), i^*(t)\}$ is optimal. Then inflation must be strictly positive at some point in time. Output is initially negative, but becomes strictly positive at some point. If $\kappa\sigma\lambda = 1$ then inflation is initially zero and is nonnegative throughout, $\pi^*(0) = 0$ and $\pi^*(t) \geq 0$ for all $t \geq 0$.*

There are two things optimal monetary policy accomplishes. First and most obvious, it promotes inflation. This helps mitigate the deflationary spiral during the liquidity trap. Lower deflation, or even inflation, lowers the real rate of interest, which is the true root of the problem in a liquidity trap. Second, due to the non-neutrality of money, it stimulates future output, after the trap. This percolates back in time, increasing output during the trap. Anticipating a boom, consumers lower their saving and increase current consumption, mitigating the negative output gap.

In this model the two goals are related, since inflation requires a boom in output. Thus,

pursuing the first goal already leads, incidentally, to the second, and vice versa. Importantly, the nominal interest rate path implied by Proposition 3 stimulates a larger boom than that required by the inflation promise. To see this, suppose that along the optimal plan $I(\pi^*(t), r(t)) \geq 0$ for $t \geq t_1$, and $I(\pi^*(t), r(t)) < 0$ otherwise. The optimal plan then calls for $i^*(t) = 0$ over some interval $t \in [t_1, t_2]$. However, consider an alternative plan that has the same inflation at t_1, so that $\pi(t_1) = \pi^*(t_1)$, but, in contradiction with Proposition 3, features $i(t) = I(\pi(t), r(t))$ for all $t \geq t_1$.[12] Suppose also that, for both plans, the long-run output gap is zero: $\lim_{t \to \infty} x(t) = \lim_{t \to \infty} x^*(t) = 0$. It then follows that $x(t_1) < x^*(t_1)$. In this sense, holding down the interest rate to zero stimulates a boom that is greater than the one implied by the inflation promise.

Figure 2 plots the equilibrium paths for a numerical example. The parameters are set to $T = 2$, $\sigma = 1$, $\kappa = .5$ and $\lambda = 1/\kappa$. These choices are made for illustrative purposes and to ensure that $\kappa\sigma\lambda = 1$. They do not represent a calibration. The choices are tilted towards a flexible price situation. Relative to the New Keynesian literature, the degree of price stickiness is low (high κ) and the planner is quite tolerant of inflation (low λ). It is also common to set a lower value for σ, on the grounds that investment, which may be quite sensitive to the interest rate, has been omitted from the analysis.

The black line represents the equilibrium with discretion; the blue line, the optimum with commitment. With discretion output is initial depressed by about 11%, at the optimum this is reduced to just under 4%. The optimum features a boom which peaks at about 3% at $t = T$. The discretionary case features significant deflation. In contrast, because $\kappa\sigma\lambda = 1$ optimal inflation starts at zero and is always positive. Both paths end at origin, which represents the ideal first-best outcome. However, although the optimum reaches it later at $\hat{T} = 2.7$, it circles around it, managing to stay closer to it on average. This improves welfare.

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[12]Note that, depending on the value of $\kappa\sigma\lambda$, the interest rate may even be greater than the natural rate $r(t)$. The fact that this policy is consistent with positive inflation and output after the trap even though it may have higher interest rates than the discretionary solution underscores, once again, that monetary easing does not necessarily manifest itself in lower equilibrium interest rates.
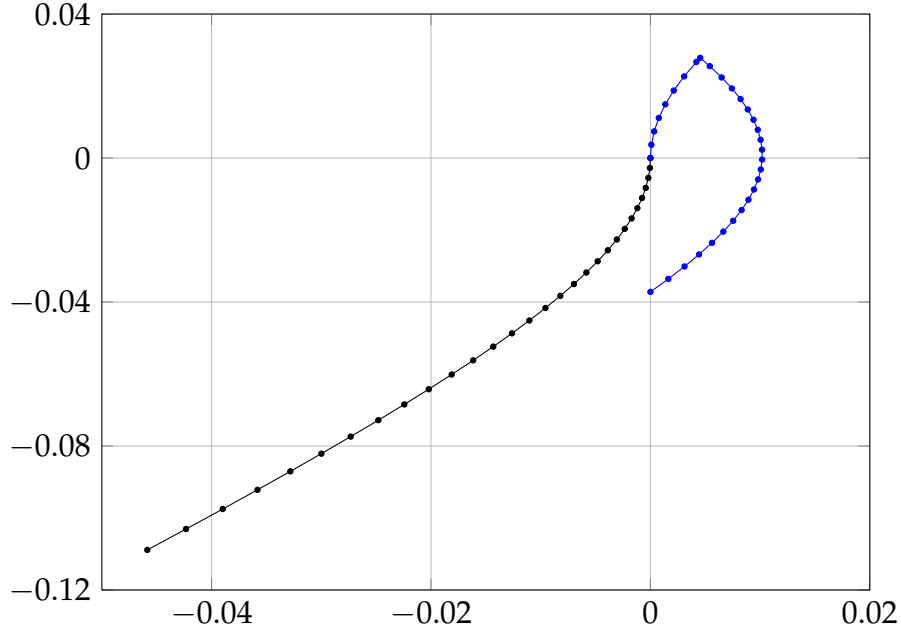
Figure 2: A numerical example showing the full discretion case (black) and optimal commitment case (blue).

One implication of Proposition 4 is that, whenever the first best is unattainable, optimal monetary policy requires commitment. Output is initially negative $x^*(0) \leq 0$, but must turn strictly positive $x^*(t') > 0$ at some future date for $t' > 0$. This implies that, if the planner can reoptimize and make a new credible plan at time t', then this new plan would involve initially negative output $x^*(t') \leq 0$. Hence, it cannot coincide with the original plan which called for positive output.

Note that the kind of commitment needed in this model involves more than a promise for future inflation, at time T, as in Krugman (1998). Indeed, my discussion here emphasizes commitment to an output boom. More generally, the planning problem features both π and x as state variables, so commitment to deliver promises for both inflation and output are generally required.

Proposition 4 highlights the case with $\kappa\sigma\lambda = 1$, where inflation starts and ends at zero and is positive throughout. This case occurs when the costate $\mu_\pi(t)$ on the Phillips curve is zero for all $t \geq 0$. This case turns out to be an interesting benchmark. Numerical results

show the following pattern, which I state as a conjecture.[13]

Conjecture. *Suppose $\{\pi^*(t), x^*(t), i^*(t)\}$ is optimal and not equal to the first best. If $\kappa\sigma\lambda < 1$ then $\pi^*(t) > 0$ for all t. If $\kappa\sigma\lambda > 1$ then $\pi^*(0) < 0$.*

Liquidity traps are commonly associated with deflation, but these results suggest that the optimum completely avoids deflation in some cases. This is more likely to be the case if prices are less flexible (low κ), if the intertemporal elasticity of substitution is high (low σ), or if the central bank is not too concerned about inflation (low λ). Note that if we set $\lambda = \bar{\lambda}/\kappa$, then $\kappa\sigma\lambda = \lambda\sigma$, so the degree of price flexibility κ drops out of the condition determining the sign of initial inflation. In the other case, when $\kappa\sigma\lambda < 1$, the optimum does feature deflation initially, but transitions through a period of positive inflation as shown by Proposition 4. Numerical simulations return to deflation and a negative output gap.

It is worth noting that prolonged zero nominal interest rates are not needed to promote positive inflation and stimulate output after the trap. Indeed, there are equilibria with both features and a nominal interest rate path given by $i(t) = \max\{0, I(\pi(t), r(t))\}$. In the liquidity trap scenario, the same is true for the interest rate path considered under pure discretion, $i(t) = 0$ for $t < T$ and $i(t) = r(t)$ for $t \geq T$. Without commitment, a unique equilibrium was obtained by adding the condition that the first best outcome $\pi(t) = x(t) = 0$ was implemented for $t \geq T$. However, positive inflation and output, $\pi(T), x(T) \geq 0$ are also compatible with this very same interest rate path. This is possible because equilibrium outcomes are not uniquely determined by equilibrium nominal interest rates. Policy may still be described as one of monetary easing, even if this is not necessarily reflected in equilibrium nominal interest rates.[14]

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[13]I verified this conjecture numerically for a very wide set of the parameter values in the step-wise liquidity trap scenario. My procedure solves the optimum in near closed form as a solution to an ODE with boundary conditions. Thus, it is very fast, essentially instantaneous for a single parametrization. This makes checking the conjecture automatically over a large set of parameters feasible. To do so, for each parameter I set up a dense and wide grid of values. Using loops, I then had the conjecture checked over the Cartesian product of these grids.

[14]To be specific, suppose policy is determined endogenously according to a simple Taylor rule, with a

4.2 A Simple Case: Fully Rigid Prices

To gain intuition it helps to consider the extreme case with fully rigid prices, where $\kappa = 0$ and $\pi(t) = 0$ for all $t \geq 0$.[15] Consider the liquidity trap scenario, where $r(t) < 0$ for $t < T$ and $r(t) > 0$ for $t > T$, and suppose we keep the nominal interest rate at zero until some time $\hat{T} \geq T$, and implement $x(t) = \pi(t) = 0$ after $\hat{T}$. Output is then

$$x(t; \hat{T}) \equiv \sigma^{-1} \int_t^{\hat{T}} r(s) ds.$$

Note that if $\hat{T} = T$ then $x(t, T) < 0$ for $t < T$, a special case of Proposition 1. More generally, output rises up to time T, and then falls and reaches zero at time $\hat{T}$. Higher $\hat{T}$ increases the path for output $x(\cdot, \hat{T})$ in a parallel fashion, so that, as long as $\hat{T}$ is greater than T, but not too large, output starts out strictly negative and then turns strictly positive for a while. Larger values of $\hat{T}$ shrink the initially negative output gaps, but lead to larger positive gaps later.

It follows that, starting from $\hat{T} = T$ an increase in $\hat{T}$ improves welfare, since the loses from creating positive output gaps are second order, while the gains from reducing the pre-existing negative output gaps are first order. More formally, the optimum minimizes the objective $V(\hat{T}) \equiv \frac{1}{2} \int_0^\infty e^{-\rho t} x(t; \hat{T})^2 dt$, implying

$$V'(\hat{T}^*) = r(\hat{T}^*) \sigma^{-1} \int_0^{\hat{T}^*} e^{-\rho t} x(t; \hat{T}^*) dt = 0.$$

It follows that $T < \hat{T}^* < \bar{T}$ where $x(0, \bar{T}) = 0$. Monetary easing goes beyond the liquidity trap, but stops short of preventing a recession. Indeed, the optimality condition implies

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time varying intercept, $i(t) = \bar{i}(t) + \phi_\pi \pi(t)$ with $\phi_\pi > 1$. In the unique bounded equilibrium, a temporarily low value for $\bar{i}(t)$ typically leads to higher inflation $\pi(t)$, but not necessarily a lower equilibrium interest rate $i(t)$. The outcome for the nominal interest rate $i(t)$ depends on various parameters. Either way, the situation with temporarily low $\bar{i}(t)$ may be described as one of "monetary easing".

[15]The same conditions we will obtain for $\kappa = 0$ here can be obtained if we consider the limit of the general optimality conditions derived above as $\kappa \to 0$. However, it is more revealing to derive the optimality condition from a separate perturbation argument.

that the present value of output is zero $\int e^{-\rho t} x(t) dt = 0$, so that the recession and the subsequent boom average out.

Thus, with fully rigid prices the promise to hold nominal interest rates at zero longer is optimal because it creates an output boom that peaks at time T and helps mitigate the earlier recession. Inflation is zero regardless of monetary policy in this extreme case, so it isolates only one of the two motives for monetary easing discussed in the previous subsection. In this sense, it is the polar opposite of Krugman's example, which isolated the inflationary motive only. Next I turn to a graphical analysis of intermediate cases, where both motives are present.

4.3   Stitching a Solution Together: A Graphical Representation

To see the solution graphically, consider the particular liquidity trap scenario with the step function path for the natural rate of interest: $r(t) = \underline{r} < 0$ for $t < T$ but $r(t) = \bar{r} \geq 0$ for $t \geq T$. It is useful to break up the solution into three separate phases, from back to front. I first consider the solution after some time $\hat{T} > T$ when the ZLB constraint is no longer binding (Phase III). I then consider the solution between time T and $\hat{T}$ with the ZLB constraint (Phase II). Finally, I consider the solution during the trap $t \in [0, T]$ (Phase I).

After the Storm: Slack ZLB Constraint (Phase III).   Consider the problem where the ZLB constraint is ignored, or no longer binding. If this were true for all time $t \geq 0$ then the solution would be the first best $\pi(t) = x(t) = 0$. However, here I am concerned with a situation where the ZLB constraint is slack only after some date $\hat{T} > T > 0$, at which point the state $(\pi(\hat{T}), x(\hat{T}))$ is given and no longer free, so the first best is generally not feasible.

The planning problem now ignores the ZLB constraint but takes the initial state (π_0, x_0) as given. Because the ZLB constraint is absent, the constraint representing the Euler equa-

tion is not binding. Thus, it is appropriate to ignore this constraint and drop the output gap $x(t)$ as a state variable, treating it as a control variable instead. The only remaining state is inflation $\pi(t)$.[16] Also note that the path of the natural interest rate $\{r(t)\}$ is irrelevant when the ZLB constraint is ignored.

I seek a solution for output x as a function of inflation π. Using the optimality conditions with $\mu_x(t) = 0$ one can show that $i(t) = I^*(\pi(t), r(t))$ as discussed earlier, with output satisfying

$$x(t) = \phi \pi(t)$$

and costate $\mu_\pi(t) = \frac{\phi}{\kappa} \pi(t)$, where $\phi \equiv \frac{\rho + \sqrt{\rho^2 + 4\lambda\kappa^2}}{2\kappa}$ so that $\phi > \rho/\kappa$. The last inequality implies that the ray $x = \phi\pi$ is steeper than that for $\dot{\pi} = 0$. Thus, starting with any initial value of π the solution converges over time along the loci $x = \phi\pi$ to the origin $(\pi(t), x(t)) \to (0,0)$. These dynamics are illustrated in Figure 3.

Just out of the Trap (Phase II). Consider next the problem for $t \geq T$ incorporating the ZLB constraint for any arbitrary starting point $(\pi(T), x(T))$. The problem is stationary since $r(t) = \bar{r} > 0$ for $t \geq T$.

If the initial state lies on the loci $x = \phi\pi$, then the solution coincides with the one above. This is essentially also the case when the initial state satisfies $x < \phi\pi$, since one can engineer an upward jump in x to reach the loci $x = \phi\pi$.[17] After this jump, one proceeds with the solution that ignores the ZLB constraint. In contrast, the optimum features an initial state that satisfies $x > \phi\pi$. Intuitively, the optimum attempts to reach the red line as quickly as possible, by setting the nominal interest rate to zero until $x = \phi\pi$.

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[16]One can pick any absolutely continuous path for $x(t)$ and solve for the required nominal interest rate as a residual: $i(t) = \sigma\dot{x}(t) + \pi(t) + r(t)$. Discontinuous paths for $x(t)$ can be approximated arbitrarily well by continuous ones. Intuitively, it is as if discontinuous paths for $\{x(t)\}$ are possible, since upward or downward jumps in $x(t)$ can be engineered by setting the interest rate to ∞ or $-\infty$ for an infinitesimal moment in time. Formally, the supremum for the problem that ignores the ZLB constraint, but carries both $\pi(t)$ and $x(t)$ as states, is independent of the current value of $x(t)$. Since the current value of $x(t)$ does not meaningfully constrain the planning problem, it can be ignored as a state variable.

[17]For example, set $i(t) = \Delta/\varepsilon > 0$ for a short period of time $[0, \varepsilon)$ and choose Δ so that $x(\varepsilon) = \phi\pi(\varepsilon)$. As $\varepsilon \downarrow 0$ this approximates an upward jump up to the $x = \phi\pi$ loci at $t = 0$.
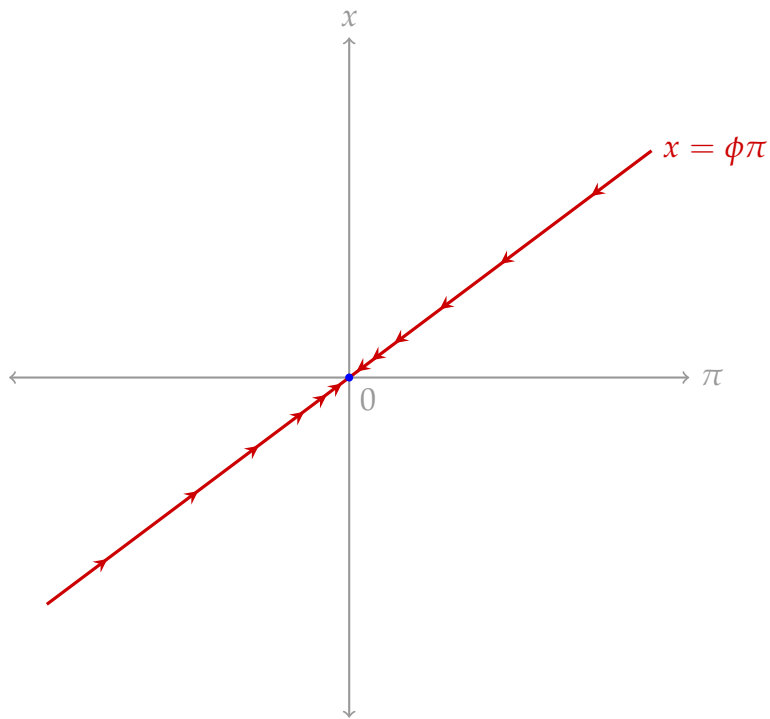
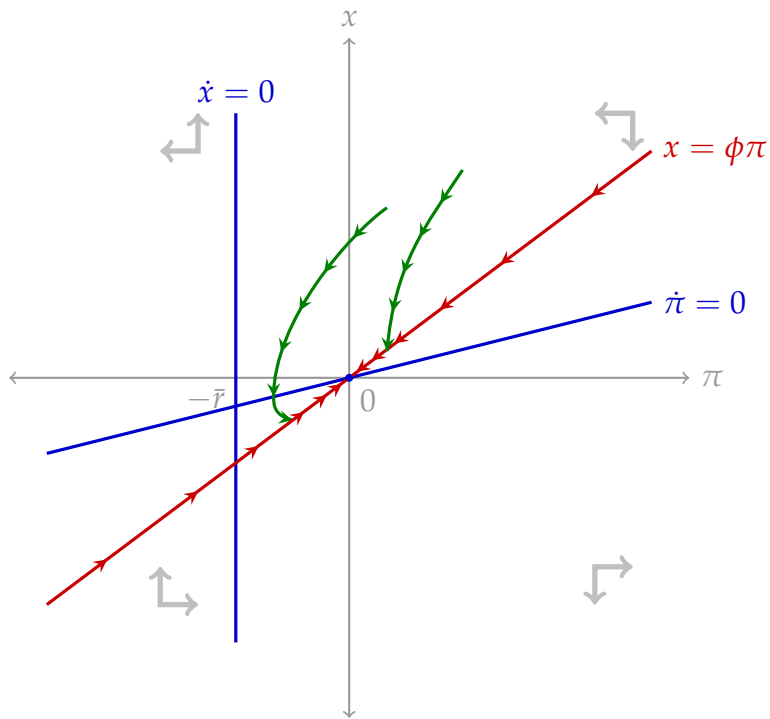Figure 3: The solution without the ZLB constraint.

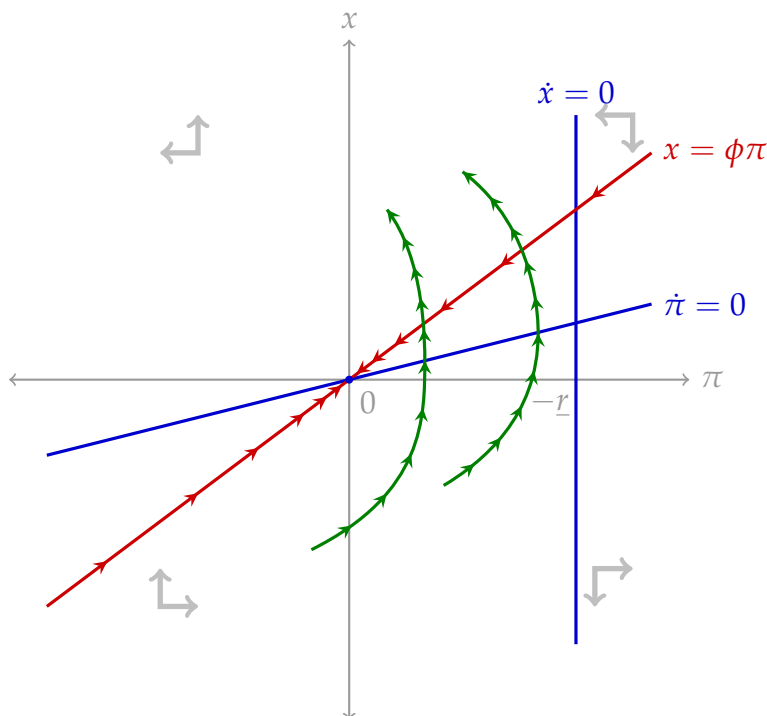Figure 4: The solution for $t > T$ with the ZLB constraint.

Figure 5: The solution for $t \leq T$ and $r(t) = -\bar{r} < 0$ with the ZLB constraint binding.

These dynamics are illustrated in Figure 3 using the phase diagram implied by the system (1a)–(1b) with $i(t) = 0$. The steady state with $\dot{x} = \dot{\pi} = 0$ involves deflation and a negative output gap: $\pi = -\bar{r} < 0$ and $x = -\frac{\rho}{\kappa}\bar{r} < 0$. As a result, for inflation rates near zero the output gap falls over time. As before, the red line denotes the loci $x = \phi\pi$, for the solution to the problem ignoring the ZLB constraint. For two initial values satisfying $x > \phi\pi$, the figure shows the trajectories in green implied by the system (1a)–(1b) with $i(t) = 0$. Along these paths $x(t)$ and $\pi(t)$ fall over time, eventually reaching the loci $x = \phi\pi$. After this point, the state follows the solution ignoring the ZLB constraint, staying on the $x = \phi\pi$ line and converges towards the origin.

During the Liquidity Trap (Phase I) During the liquidity trap $t \leq T$ the ZLB constraint binds and $i(t) = 0$. The dynamics are illustrated in Figure 5 using the phase diagram implied by equations (1a) and (1b) setting $i(t) = 0$. For reference, the red line denoting the optimum ignoring the ZLB constraint is also show.

28

Unlike the previous case, the steady state $\dot{x} = \dot{\pi} = 0$ for this system now has positive inflation and a positive output gap: $\pi = -\underline{r} > 0$ and $x = -\frac{\rho}{\kappa}\underline{r} > 0$. In contrast to the previous phase diagram, also featuring $i(t) = 0$, for inflation rates near zero the output gap rises over time. Two trajectories are shown in green. Both trajectories start at $t = 0$ below the red line are above it at $t = T$. In one case the inflation rate is initially negative, while in the other it is positive. In both cases the output gap is initially negative and becomes positive some time before $t = T$.

Figure 6 puts the three phases together to display two possible optimal paths for all $t \geq 0$. The two trajectories illustrated in the figure are quite representative and illustrate the possibilities described in Proposition 4.

As these figures suggest one can prove that the nominal interest rate should be kept at zero past T. The following proposition follows from Proposition 3 and elements of the dynamics captured by the phase diagrams.

Proposition 5. *Consider the liquidity trap scenario with $r(t) = \underline{r} < 0$ for $t < T$ and $r(t) = \bar{r} > 0$ for $t \geq T$. Suppose the path $\{\pi^*(t), x^*(t), i^*(t)\}$ is optimal. Then there exists a $\hat{T} > T$ such that*

$$i(t) = 0 \qquad \forall t \in [0, \hat{T}].$$

There are two ways of summarizing the optimal plan. In the first, the central bank commits to a zero nominal interest rate during the liquidity trap, for $t \in [0, T]$. It also makes a commitment to an inflation rate and output gap target $(\pi^*(T), x^*(T))$ after the trap. However, note that here

$$x^*(T) > \phi\pi^*(T)$$

so that the promised boom in output is higher than that implied by the inflation promise. Commitment to a target at time T is needed not just in terms of inflation, but also in terms of the output gap.

Another way of characterizing policy is as follows. The central bank commits to set-
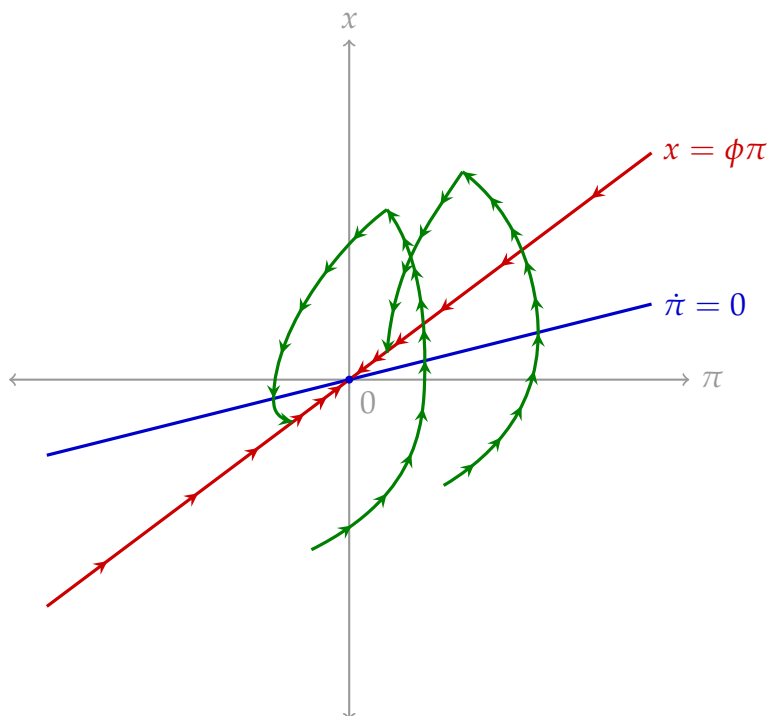
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Figure 6: Two possible paths of the solution for $t \geq 0$.

ting a zero interest rate at zero for longer than the liquidity trap, so that $i(t) = 0$ for $t \in [0, \hat{T}]$ with $\hat{T} > T$. It also commits to implementing an inflation rate $\pi(\hat{T})$ upon exit of the ZLB, at time $\hat{T}$. In this case, no further commitment regarding $x(\hat{T})$ is required, since $x(\hat{T}) = \phi\pi(\hat{T})$ is ex-post optimal given the promised $\pi(\hat{T})$. Note that the level of inflation promised in this case may be positive or negative, depending on the sign of $1 - \kappa\sigma\lambda$. A commitment to positive inflation once interest rates become positive is not necessarily a feature of all optimum.

5   Government Spending: Opportunistic and Stimulus

I now introduce government spending as an additional instrument. I first consider the full optimum over both fiscal and monetary policy. I then turn to a more restricted case, where monetary policy is conducted with complete discretion and is, thus, suboptimal. Fiscal policy, on the other hand, is chosen with commitment. This captures the notion that,

30

for both technical and political reasons, announcements of future government spending may be more credible than those for monetary policy. Finally, I briefly discuss the case where both fiscal and monetary policy are conducted with full discretion.

The planning problem is now

$$\min_{c,\pi,i,g} \frac{1}{2} \int_0^\infty e^{-\rho t} \left((c(t) + (1 - \Gamma)g(t))^2 + \lambda \pi(t)^2 + \eta g(t)^2 \right) dt$$

subject to

$$\dot{c}(t) = \sigma^{-1}(i(t) - r(t) - \pi(t))$$

$$\dot{\pi}(t) = \rho \pi(t) - \kappa \left(c(t) + (1 - \Gamma)g(t) \right)$$

$$i(t) \geq 0$$

$$x(0), \pi(0) \text{ free.}$$

Here the constants satisfy $\eta > 0$ and $\Gamma \in (0, 1)$; the variable $c(t) = (C(t) - C^*(t))/C^*(t) \approx \log(C(t)) - \log(C^*(t))$ represents the private consumption gap, while $g(t) = (G(t) - G^*(t))/C^*(t)$ represents the government consumption gap, normalized by private consumption.

The coefficient $\Gamma \in (0, 1)$ represents the first best, or flexible-price equilibrium, government spending multiplier, i.e. for each unit increase in spending, output increases by Γ units, consumption is reduced by $1 - \Gamma$ units. The loss function captures this, because given spending g, the ideal consumption level is $c = -(1 - \Gamma)g$. The Phillips curve shows that $c = -(1 - \Gamma)g$ also corresponds to a situation with zero inflation, replicating the flexible-price equilibrium.

The potential usefulness of the additional spending instrument g can be easily seen noting that spending can zero out the first two quadratic terms in the loss function, ensuring $c(t) + (1 - \Gamma)g(t) = \pi(t) = 0$ for all $t \geq 0$. This requires a particular path for

spending satisfying

$$\dot{g}(t) = \frac{\sigma^{-1}}{1 - \Gamma}(r(t) - i(t)).$$

For simplicity, suppose we set $i(t) = 0$ for $t < T$ and $i(t) = r(t)$ for $t \geq T$. Then spending is declining for $t < T$ and given by

$$g(t) = \frac{\sigma^{-1}}{1 - \Gamma} \int_0^t r(s)ds + g(0).$$

After this, spending is flat $g(t) = g(T)$ for $t \geq T$. To minimize the quadratic loss from spending, the optimal initial value $g(0)$ is set to ensures that $g(t)$ takes on both signs: $g(0)$ is positive and $g(T)$ is negative. The same is true for consumption, since $c(t) = -(1 - \Gamma)g(t)$.

Although this plan is not optimal, it is suggestive that optimal spending may take on both positive and negative values during a liquidity trap. We prove this result in the next subsection.

5.1   The Optimal Pattern for Spending

It will be useful to transform the planning problem by a change variables. In fact, I will use two transformations. Each has its own advantages.

For the first transformation, define the output gap $x(t) \equiv c(t) + (1 - \Gamma)g(t)$. The planning problem becomes

$$\min_{x,\pi,i,g} \frac{1}{2} \int_0^\infty e^{-\rho t} \left(x(t)^2 + \lambda \pi(t)^2 + \eta g(t)^2 \right) dt$$

subject to

$$\dot{x}(t) = (1 - \Gamma)\dot{g}(t) + \sigma^{-1}(i(t) - r(t) - \pi(t))$$

$$\dot{\pi}(t) = \rho\pi(t) - \kappa x(t)$$

$$i(t) \geq 0$$

$$x(0), \pi(0) \text{ free.}$$

This is an optimal control problem with $i(t)$ and $g(t)$ as controls and x and π as states. According to the objective, the ideal level of government spending, given the state variables $x(t)$ and $\pi(t)$, is always zero, $g(t)$. However, because spending also appears in the constraints, it may help relax them. In particular, spending enters the constraint associated with the consumer's Euler equation. Indeed, the change in spending $\dot{g}(t)$ plays a role that is analogous to the nominal interest rate $i(t)$. Unlike the latter, the former is not restricted to being nonnegative.

Since government spending relaxes the Euler equation, it should be zero whenever the zero-bound constraint is not binding, which is the case whenever $i(t) > 0$. Conversely, if the zero-bound constraint binds and $i(t) = 0$ then government spending is not generally zero. As the next proposition shows, spending is initially positive, then becomes negative, and finally returns to zero.

Proposition 6. *Suppose the zero lower bound binds over the interval (t_0, t_1) and is slack in a neighborhood outside it. Then $g(t_0) > 0$, $g(t_1) = 0$ with $g(t) < 0$ for $t < t_1$ in a neighborhood of t_1.*

This result confirms the notion that government spending should be front loaded. It may seem surprising, however, that optimal spending takes on both positive and negative values. The intuition is as follows. Initially, higher spending helps compensate for the negative consumption gap at the start of a liquidity trap. However, recall that optimal
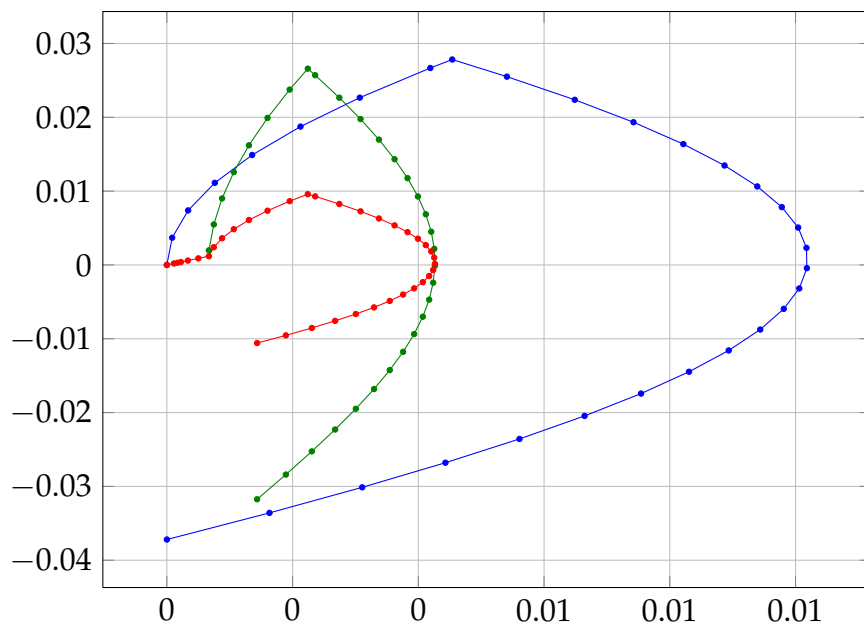
Figure 7: A numerical example. The optimum without spending (blue) vs. the optimum with spending for output (red) and consumption (orange).

monetary policy eventually engineers a consumption boom. If government spending leans against the wind, we should expect lower spending. The next subsection refines this intuition by decomposing spending into an opportunistic and a stimulus component.

Figure 7 provides a numerical example, following the same parametrization used for the example in Section 4, with the additional parameters $\Gamma = 0.5$ and $\eta = .5$. The figure shows both consumption and output. As we see from the figure consumption is not as affected as output is in this case.

5.2   Opportunistic vs. Stimulus Spending

Even a shortsighted government that ignores dynamic general equilibrium effects on the private sector, finds reasons to increase government spending during a slump. When the economy is depressed, the wage, or shadow wage, is lowered. This provides a cheap opportunity for government consumption.

Based on this notion, I define an opportunistic component of spending, the level that

is optimal from a simple static, cost-benefit calculation. I then define the stimulus compo-
nent of spending as the difference between actual spending and the opportunistic com-
ponent. More precisely, given private consumption c, define opportunistic spending by
minimizing the loss function,

$$g^*(c) \equiv \arg\max_g \left\{ (c + (1 - \Gamma)g)^2 + \eta g^2 \right\},$$

Define stimulus spending as the difference between actual and opportunistic spending,

$$\hat{g}(t) \equiv g(t) - g^*(c(t)).$$

Note that
$$g^*(c) = -\frac{1 - \Gamma}{\eta} \psi c,$$

$$c + (1 - \Gamma)g^*(c) = \psi c,$$

with the constant $\psi \equiv \eta / \left(\eta + (1 - \Gamma)^2 \right) \in (0, 1)$. Thus, opportunistic spending leans
against the wind, $\psi < 1$, but does not close the gap, $\psi > 0$.

Using these transformations, I rewrite the planning problem as

$$\min_{\hat{x}, \pi, i, \hat{g}} \frac{1}{2} \int_0^\infty e^{-\rho t} \left(c(t)^2 + \hat{\lambda}\pi(t)^2 + \hat{\eta}\hat{g}(t)^2 \right) dt$$

subject to

$$\dot{c}(t) = \sigma^{-1}(i(t) - r(t) - \pi(t))$$

$$\dot{\pi}(t) = \rho\pi(t) - \kappa \left(\psi c(t) + (1 - \Gamma)\hat{g}(t) \right)$$

$$i(t) \geq 0,$$

$$c(0), \pi(0) \text{ free.}$$

35

where $\hat{\lambda} = \lambda/\psi$ and $\hat{\eta} = \eta/\psi^2$. According to the loss function, the ideal level of stimulus spending is zero. However, stimulus may help relax the Phillips curve constraint.

This problem is almost identical to the problem without spending. The only new optimality condition is

$$\hat{g}(t) = \frac{\kappa(1-\Gamma)}{\hat{\eta}}\mu_\pi(t). \tag{4}$$

This gives a first result. Unlike total spending, stimulus spending is initially zero.

Proposition 7. *Stimulus spending is always initially zero $\hat{g}(0) = 0$.*

Thus, total spending at the start of a liquidity trap is entirely opportunistic.[18]

To say more, note that the costate for the Phillips curve, unlike the costate for the Euler equation, is not restricted to being nonnegative and the path it takes actually depends on parameters. Indeed, my main result for stimulus spending exploits this fact, providing a benchmark where stimulus spending is always zero.

Proposition 8. *Suppose $\kappa\sigma\lambda = 1$. Then at an optimum $\hat{g}(t) = 0$ for all $t \geq 0$.*

Thus, under the conditions of the proposition, spending is entirely determined by its opportunistic considerations. It is as if spending were chosen by a purely static cost-benefit calculation, with no regards for its dynamic general equilibrium impact on the economy. By implication, in this case government spending could be determined by a naive agency, lacking commitment, that performs a static cost-benefit calculation, ignoring the dynamic effects this has on the private sector.

Figure 8 displays the optimal paths for total, opportunistic and stimulus spending for our numerical example (with the same parameters as those behind Figure 7). Spending starts at 2% of output above it's efficient level. It then falls at a steady state reaching

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[18] Another implication of equation (4) is that stimulus spending, unlike total spending, may be nonzero even when the zero lower bound constraint is not currently binding and will never bind in the future. This occurs whenever inflation is nonzero. Indeed, since total spending must be zero, stimulus spending must be canceling out opportunistic spending. This makes sense. If we have promised positive inflation, for example, then we require a positive gap. Opportunistic spending would call for lower spending, but doing so would frustrate stimulating the promised inflation.
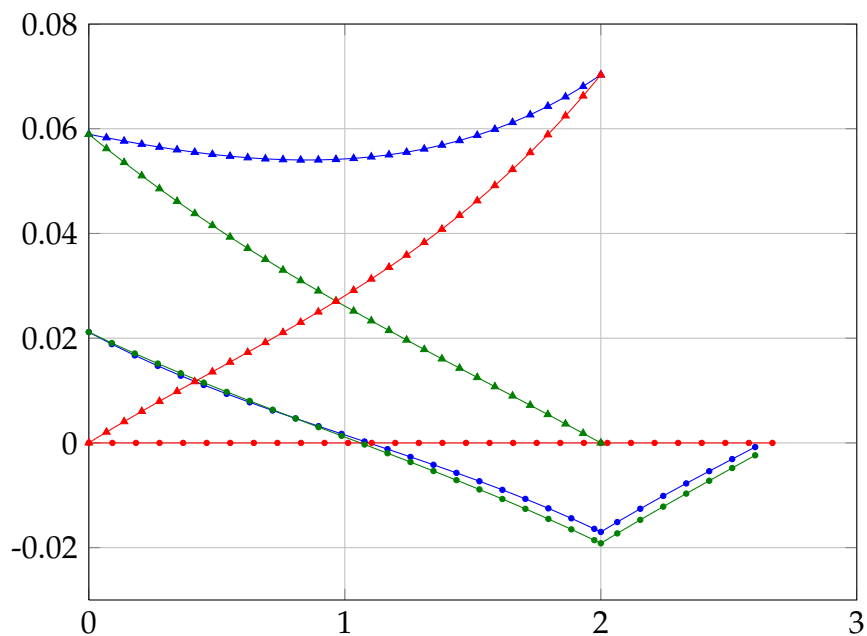
Figure 8: Total spending (blue), opportunistic spending (green) and stimulus spending (red) for a numerical example. Both the case with monetary commitment (circles) and discretion (triangles) are shown.

almost 2% below its efficient level of output. In this example, spending is virtually all opportunistic. Stimulus spending is virtually zero.

Away from this benchmark, numerical simulations show that stimulus starts at zero, it has a sinusoidal shape, switching signs once. When $\kappa\sigma\lambda > 1$ it first becomes positive, then turns negative, eventually asymptoting to zero from below; when $\kappa\sigma\lambda < 1$ the reverse pattern obtains: first negative, then turns positive, eventually asymptoting to zero from above. In most cases, stimulus spending is a small component of total spending.

The results highlight that positive stimulus spending is just not a robust feature of the optimum for this model. Opportunistic spending does affect private consumption, by affecting the path for inflation. In particular, by leaning against the wind, it promotes price stability, mitigates both deflations and inflations. However, the effects are incidental, in that they would be obtained by a policy maker choosing spending that ignores these effects.

6 Spending with Discretionary Monetary Policy

I now relax the assumption of full commitment and consider a mixed case, where monetary policy is discretionary, as in Section 3, while government spending is carried out with commitment during the trap.

More specifically, consider the liquidity trap scenario, where $r(t) < 0$ for $t < T$ and $r(t) \geq 0$ for $t \geq T$. Once the liquidity trap is over, monetary policy will implement the flexible-price equilibrium, so that $c(t) + (1 - \Gamma)g(t) = 0$ and $\pi(t) = 0$ for all $t \geq T$. During the trap, the nominal interest rate is set to zero, $i(t) = 0$ for $t < T$. In contrast, the government spending can be credibly announced, at least for some time. I initially assume that spending after T is chosen with discretion, implying that $g(t) = 0$ for $t \geq T$. I then consider the case with commitment on the entire path for government spending, for all $t \in [0, \infty)$.

Government spending may be a powerful tool in this scenario. Absent spending, deflation and depression prevail. But, as I argued above, spending can avoid both, achieving a zero output gap and inflation rate, $c(t) + (1 - \Gamma)g(t) = \pi(t) = 0$ for all $t \geq 0$. It does so by filling in the gap left by consumption. Of course, this simple plan is suboptimal. Next, I study optimal spending commitments.

6.1 Commitment to spending during the liquidity trap

The planning problem is essentially the same as before[19] with the additional constraint that

$$\pi(T) = c(T) = 0.$$

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[19]Except that we may impose $i(t) = 0$ for $t < T$ since this is chosen by the monetary authority. However, the optimum will also feature this interest rate path.

Once again the optimality condition gives

$$\hat{g}(t) = \frac{\kappa(1-\Gamma)}{\hat{\eta}}\mu_\pi(t)$$

with the law of motion for the co-states as before. Thus, just as before, stimulus spending is initially zero.

It is difficult to formally characterize the rest of the solution. I make progress by considering small stimulus spending interventions, starting from spending. Specifically, consider appending the constraint

$$\int_0^\infty e^{-\rho t}g(t)^2dt \leq G,$$

to the planning problem. Here G is a parameter. Setting $G = 0$ implies the no commitment outcome, without spending or stimulus, which involves deflation and depression. For $G > 0$ large enough the constraint no longer binds. The idea is to characterize the optimum for small enough $G > 0$. This allows us to use the above formula for spending, with the costates evaluated at the original no spending and no discretion equilibrium.

Proposition 9. $\hat{g}(0) = 0$. *For small enough $G > 0$, $\hat{g}(t) \geq 0$ and is strictly increasing in t. Moreover, total spending is positive $g(t) > 0$ for all $t \in [0, T)$.*

Simulations support that this pattern generally carries over for the case where G is chosen freely. Figure 8 confirms this for the numerical example from the previous sections. In this example, total spending is quite large and relatively flat. As opportunistic spending falls, stimulus spending rises and compensates.

6.2 Commitment to spending after the liquidity trap

I now relax the assumption that fiscal policy cannot commit past T. Thus, I now consider a situation where spending is chosen for the entire future $\{g(t)\}_{t=0}^\infty$, allowing for $g(t) \neq 0$

for $t \geq T$.

This problem can be simplified by looking at the subproblem from $t \geq T$. Clearly for any positive consumption $c(T) > 0$ the optimum calls for $i(t) = 0$ and $g(t) = \tilde{g}(t)$ for $t \in [T, T + \Delta]$ and $g(t)$ for $t > T + \Delta$, where

$$\tilde{g}(t) = -(1 - \Gamma)c(T) + \frac{\sigma^{-1}}{1 - \Gamma} \int_T^t r(s)ds$$

and Δ is defined so that $\tilde{g}(T + \Delta) = 0$. Such a plan implies a tail cost

$$\Psi(c(T)) \equiv \eta \int_0^\Delta e^{-\rho s} \eta \tilde{g}(T + s)^2 ds.$$

The planning problem can be rewritten as

$$\min_{\hat{x}, \pi, i, \hat{g}} \frac{1}{2} \int_0^T e^{-\rho t} \left(c(t)^2 + \hat{\lambda} \pi(t)^2 + \hat{\eta} \hat{g}(t)^2 \right) dt + e^{-\rho T} \Psi(c(T))$$

subject to

$$\dot{c}(t) = \sigma^{-1}(i(t) - r(t) - \pi(t))$$

$$\dot{\pi}(t) = \rho \pi(t) - \kappa \left(\psi c(t) + (1 - \Gamma)\hat{g}(t) \right)$$

$$i(t) \geq 0,$$

$$\pi(T) = 0.$$

Under this formulation $c(T)$ is a free variable, but the planner incurs a cost $\Psi(c(T))$. The new optimality condition (replacing $c(T) = 0$) is the transversality condition

$$\mu_c(T) = \Psi'(c(T)).$$

A similar result applies in this case. For a small intervention, stimulus spending is pos-

itive and increasing. Now, however, spending after the trap is negative, to promote a boom in consumption $c(T)$, which helps raise the level of consumption at earlier dates, during the trap.

7 Conclusion

This paper has revisited monetary policy during a liquidity trap. The continuous time setup up offers some distinct advantages in terms of the analysis and results that are obtained. Some of my results support the findings from prior work based on simulations. Pptimal monetary policy in the model is engineered to promote inflation and an output boom. It does so, in part, by commiting to holding the nominal interest rate at zero for an extended period of time.

To the best of my knowledge, my results on government spending have no clear parallel in the literature. In particular, the decomposition between opportunistic and stimulus spending is novel and leads to unexpected results.

When both fiscal and monetary policy are coordinated, I find that optimal government spending starts at a positive level, but declines and become negative. However, I show that most of these dynamics are explained by a cost-benefit motive for spending, which, by definition, ignores the effects this spending has on private consumption and inflation. At the model's optimum, stimulus spending is always initially zero. Moreover, depending on parameters. stimulus may be identically zero throughout or deviate from zero changing signs. However, simulations show stimulus spending playing a modest role.

This situation can be very different when monetary policy is suboptimal due to the lack of commitment. In this case, the model's optimal policy calls for positive and increasing stimulus spending during the trap and lower spending after the trap.

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A Proof of Proposition 1

Recall that $\pi(t) = x(t) = 0$ for $t \geq T$. In integral form the equilibrium conditions for $t < T$ are

$$x(t) = \int_t^T \sigma^{-1}(r(s) + \pi(s))ds,$$

$$\pi(t) = \kappa \int_t^T e^{-\rho(s-t)}x(s)ds.$$

Substituting inflation $\pi(t)$ we can write this as a single condition for the output path $\{x(t)\}$

$$x(t) = \sigma^{-1} \int_t^T \left(r(s) + \kappa \int_s^T e^{-\rho(z-s)}x(z)dz \right) ds.$$

Define the operator associated with the right hand side of this expression:

$$T[x](t) = \sigma^{-1} \int_t^T \left(r(s) + \kappa \int_s^T e^{-\rho(z-s)}x(z)dz \right) ds = a(t) + \kappa \int_t^T m(z-t)x(z)dz$$

43

with $a(t) \equiv \sigma^{-1} \int_t^T r(z)dz$ and $m(s) \equiv (\sigma\rho)^{-1}(1 - e^{-\rho s})$, note that m is nonnegative, strictly increasing, with $m(0) = 0$ and $\lim_{s\to\infty} m(s) = M \equiv (\sigma\rho)^{-1} > 0$.

The operator T maps the space of continuous functions on $(-\infty, T]$ onto itself. An equilibrium x^* is a fixed point $T[x^*] = x^*$. Since an equilibrium represents a solution to an initial value problem for a linear ordinary differential equation, there is a unique fixed point x^*. The T operator is linear and monotone (since $m \geq 0$), so that if $x^a \geq x^b$ then $T[x^a] \geq T[x^b]$.

Fix an interval $[\hat{t}, T]$. Although T is not a contraction, starting from any continuous function x_0 that is bounded on $[\hat{t}, T]$ and defining $x_n \equiv T^n[x_0]$ we obtain a sequence that converges uniformly on $[\hat{t}, T]$ to the unique fixed point $x_n \to x^*$. To prove this claim, note that since $|x_0(t)| \leq B$ and $|r(t)| \leq R$ then

$$|x_1(t) - x_0(t)| \leq |a(t)| + \kappa \int_t^T m(z - t) |x_0(t)| \leq \left(\frac{R}{\kappa M} + B \right) \kappa M |T - t|.$$

In turn

$$|x_2(t) - x_1(t)| = |T[x_1](t) - T[x_0](t)| \leq \kappa \int_t^T m(z - t) |x_1(z) - x_0(z)| \, dz$$

$$\leq \kappa M \int_t^T |x_1(z) - x_0(z)| \, dz \leq \left(\frac{R}{\kappa M} + B \right) (\kappa M)^2 \frac{|T - t|^2}{2}.$$

By induction it follows that

$$|x_n(t) - x_{n-1}(t)| \leq \left(\frac{R}{\kappa M} + B \right) (\kappa M)^n \frac{|T - t|^n}{n!}.$$

It follows that

$$\sum_{n=1}^{\infty} |x_n - x_{n-1}(t)| \leq \left(\frac{R}{\kappa M} + B \right) e^{\kappa M |T - t|}.$$

44

As a consequence for any $m \geq n$

$$|x_n - x_m(t)| \leq \sum_{j=n+1}^{m} |x_j - x_{j-1}(t)| \leq \sum_{j=n+1}^{\infty} |x_j - x_{j-1}(t)|$$

Since the right hand converges to zero as $n \to \infty$, for any $\varepsilon > 0$ and $t' < T$ there exists an N such that for all $n, m \geq N$ we have

$$|x_n(t) - x_m(t)| \leq \varepsilon$$

for all $t \geq t'$. Thus, $\{x_n\}$ is a Cauchy sequence on a complete metric space, implying that $x_n \to x^*$ where x^* is a bounded function. Since the operator T is continuous it follows that x^* is a fixed point $x^* = T[x^*]$.

Note that starting from the zero function $x_0(t) = 0$ for all t we obtain $x_1(t) = a(t) < 0$ for $t < T$. Since the operator T is monotone, the sequence $\{x_n\}$ is decreasing $x_0 \geq x_1 \geq \cdots \geq x_n \geq \cdots$ Thus, $x^*(t) < x_1(t) < 0$ for all $t < T$. This implies $\pi^*(t) < 0$ for $t < T$.

Next I establish that both inflation and output $x^*(t)$ and $\pi^*(t)$ are monotone. Note that $\frac{\partial}{\partial t} T[x](t) = a'(t) - \int_t^T m'(z-t)x(z)dz$. Since $a'(t) = -\sigma^{-1}r(t) > 0$ and $x^*(t) < 0$ for $t < T$ we have that $T[x^*](t) = x^*(t)$ is strictly increasing in t for $t < T$. Likewise, differentiating we find $\dot{\pi}(t) = \rho\kappa \int_0^\infty e^{-\rho z}(x(t+s) - x(t))ds$, which implies $\dot{\pi}^*(t) > 0$ for $t < T$ given that $x^*(t+s) \geq x^*(t)$ with strict inequality for $s > 0$ and $t + s \leq T$.

Finally note that

$$x^*(t; T) = \sigma^{-1} \int_t^T (r(s; T) - \pi^*(s; T))ds \leq \sigma^{-1} \int_t^T r(s; T)ds$$

and since $\int_0^T r(s; T)ds \to -\infty$ as $T \to \infty$ it follows that $\int_t^T r(s; T)ds \to -\infty$, implying

45

$x^*(t;T) \to -\infty$ for $t < T$. Using that $x^*(t,T) \le 0$, this implies that

$$\pi^*(t,T) = \kappa \int_t^\infty e^{-\rho(s-t)} x^*(s;T)ds \le \kappa \int_t^{t+1} e^{-\rho(s-t)} x^*(s;T)ds$$

$$\le \kappa x^*(t+1;T) \int_t^{t+1} e^{-\rho(s-t)}ds$$

As $T \to \infty$ we have that $x^*(t+1;T) \to -\infty$, so it follows that $\pi^*(t;T) \to -\infty$ for any t.

B   Proof of Proposition 2

Consider two values $\kappa_0 < \kappa_1$ with associated equilibria x_0^* and x_1^*. Let $T[\cdot;\kappa]$ be the operator defined in the proof of Proposition 1 associated with κ. Define the sequence $x_n = T^n[x_0^*;\kappa_1]$. Since $x_0^*(t) < 0$ for $t < T$, it follows that $x_1(t) = T[x_0^*;\kappa_1] < T[x_0^*;\kappa_0] = x_0(t)$ for $t < T$. Since the operator $T[\cdot;\kappa_0]$ is monotone, this implies that $\{x_n\}$ is a declining sequence. Since the sequence converges to x_1^*. This proves that $x_0^*(t) > x_1(t) > \cdots > x_n(t) > \cdots > x_1^*(t)$ for $t < T$. This implies that $\pi_1^*(t) < \pi_0^*(t)$ for $t < T$.

To prove the limit result as $\kappa_1 \to \infty$ it suffices to show that $T[x_0^*;\kappa_1](t) \to -\infty$ for all $t < T$. This follows from

$$T[x_0^*;\kappa_1](t) = a(t) + \kappa \int_t^T m(z-t)x_0^*(z)dz \le \kappa_1 \int_t^T m(z-t)a(z)dz$$

and the result follows from the fact that $\int_t^T m(z-t)a(z)dz < 0$ for all $t < T$. The same implication for $\pi^*(t;\kappa)$ then follows.

The following estimate for the loss function

$$L \ge \int_0^\infty e^{-\rho t} x(t)^2 dt \ge \int_0^{T/2} e^{-\rho t} x(t;\kappa)^2 dt \ge x(T/2;\kappa)^2 \int_0^{T/2} e^{-\rho t}dt$$

where I have used that x is nonpositive and increasing. The result then follows since $x(T/2;\kappa) \to -\infty$.

C   Proof of Proposition 3

Suppose $i(t) > 0$ for $t \in (t', t' + \varepsilon)$ with $\varepsilon > 0$. Then it must be that $\mu(t) = \dot{\mu}(t) = 0$ for $t \in [t', t' + \varepsilon)$. However, whenever $\dot{\mu}_x(t) \leq 0$ we have

$$\dot{\mu}_x(t) = -\dot{x}(t) + \kappa \dot{\mu}_\pi(t) + \rho \dot{\mu}_x(t)$$
$$\leq -\sigma^{-1}(i(t) - r(t) - \pi(t)) - \kappa\lambda\pi(t)$$
$$= \sigma^{-1}\left(I^*(\pi(t), t) - i(t)\right)$$

where I have used that $\mu_x(t) \geq 0$.

It follows that if $I^*(\pi(t), t) < 0$ for $t \in (t' - \varepsilon, t')$ then $\dot{\mu}_x(t) < 0$. Since $\dot{\mu}_x(t') = 0$ this implies $\dot{\mu}_x(t) > 0$ for $t \in (t' - \varepsilon, t')$. Given that $\mu_x(t') = 0$ this then implies that $\mu_x(t) < 0$ for $t \in (t' - \varepsilon, t')$, a contradiction with the optimality conditions.

D   Proof of Proposition 4

The initial conditions require $\mu_x(0) = \mu_\pi(0) = 0$. The non-negativity requirement for μ_x then requires $\dot{\mu}_x(0) = -x(0) \geq 0$, implying $x(0) \leq 0$.

To establish that inflation must be positive at some point, consider the perturbation $x(t, \varepsilon) = x(t) + \varepsilon$ and

$$\pi(t, \varepsilon) = \kappa \int_0^\infty e^{-\rho s} x(t + s, \varepsilon) ds = \kappa \int_0^\infty e^{-\rho s} x(t + s) ds + \kappa \int_0^\infty e^{-\rho s} \varepsilon ds = \pi(t) + \frac{\kappa}{\rho}\varepsilon.$$

Note that the perturbation is feasible for all $\varepsilon > 0$ since higher inflation relaxes the ZLB constraint $\sigma\dot{x}(t) \geq -r(t) - \pi(t)$. In terms of the loss function

$$L(\varepsilon) = \frac{1}{2} \int_0^\infty e^{-\rho t} \left(x(t, \varepsilon)^2 + \lambda\pi(t, \varepsilon)^2\right) dt$$

47

we have

$$L'(0) = \int_0^\infty e^{-\rho t}\left(x(t) + \lambda\frac{\kappa}{\rho}\pi(t)\right)dt = \frac{1}{\kappa}\pi(0) + \lambda\frac{\kappa}{\rho}\int_0^\infty e^{-\rho t}\pi(t)dt \leq 0.$$

Hence, negative inflation $\pi(t) \leq 0$ for all $t \geq 0$ with strict inequality over some range, implies $L'(0) < 0$, a contradiction with optimality.

Now suppose $\kappa\sigma\lambda = 1$. Suppose $(\pi(t), x(t))$ satisfy the ODE system (1a)–(1b) with $i(t) = 0$ for $t \in [0, \hat{T}]$ with $\hat{T} > T$. Now set $\mu_\pi(t) = 0$ and define $\mu_x(t) = \sigma\lambda\pi(t)$ so that

$$\dot{\mu}_\pi = 0 = -\lambda\pi + \sigma^{-1}\mu_x$$

$$\dot{\mu}_x(t) = \sigma\lambda(\rho\pi(t) - \kappa x(t)) = -x(t) + \rho\sigma\lambda\pi(t)$$

are both satisfied. It follows that $(\pi(t), x(t))_{t\in[0,\hat{T}]}$ together with $\pi(t) = x(t) = 0$ for $t > \hat{T}$ is optimal if and only if $\pi(t) \geq 0$ and $\pi(0) = \pi(\hat{T}) = 0$ and $x(\hat{T}) = 0$.

To establish that $x(t)$ must be positive for some $t \geq 0$, proceed by contradiction. Suppose $x(t) \leq 0$ for all $t \geq 0$. This implies that $\pi(t) \leq 0$ for all $t \geq 0$, a contradiction.