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THE ECONOMICS OF HEDGE FUNDS:  
ALPHA, FEES, LEVERAGE, AND VALUATION

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**ABSTRACT**

Hedge fund managers are compensated via management fees on the assets under management (AUM) and incentive fees indexed to the high-water mark (HWM). We study the effects of managerial skills (alpha) and compensation on dynamic leverage choices and the valuation of fees and investors' payoffs. Increasing the investment allocation to the alpha-generating strategy typically lowers the fund's risk-adjusted excess return due to frictions such as price pressure. When the manager is only paid via management fees, the manager optimally chooses time-invariant leverage to balance the size of allocation to the alpha-generating strategy against the negative impact of increasing size on the fund's alpha. When the manager is paid via both management and incentive fees, we show that (i) the high-powered incentive fees encourage excessive risk taking, while management fees have the opposite effect; (ii) conflicts of interest between the manager and investors have significant effects on dynamically changing leverage choices and the valuation of fees and investors' payoffs; (iii) the manager optimally increases leverage following strong fund performances; (iv) investors' options to liquidate the fund following sufficiently poor fund performances substantially curtail managerial risk-taking, provide strong incentives to de-leverage, and sometimes even give rise to strong precautionary motives to hoard cash (in long positions); and (v) managerial ownership concentration has incentive alignment effects.

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# 1 Introduction

Hedge funds' management compensation contracts typically feature both management fees and performance/incentive fees. The management fee is charged periodically as a fraction, e.g. 2%, of assets under management (AUM). The incentive fee, a key characteristic that differentiates hedge funds from mutual funds, is calculated as a fraction, e.g. 20%, of the fund's profits. The cost base for the profit calculation is the fund's high-water mark (HWM), which effectively keeps track of the maximum value of the invested capital and critically depends on the fund manager's dynamic investment strategies. Presumably, incentive fees are intended to reward talented managers and to align the interests of the manager and investors more closely than flat management fees do. However, incentive fees may also have unintended consequences because they tend to encourage managerial excessive risk taking and do not lead to fund value maximizing leverage choices.<sup>1</sup>

Sophisticated use of leverage is an important feature of hedge funds (Ang, Gorovyy, and van Inwegen (2010)). Hedge funds may borrow through the repo markets or from their prime brokers. Hedge funds can also use various implicit leverage, often via options and other derivatives. Skilled managers, by leveraging on strategies with positive alphas, can potentially create significant value for investors. That is, with a limited amount of capital, leverage can be rewarding for both the skilled manager and the fund's investors. However, the fund's marginal return may decrease with leverage because increasing investment exposure to the alpha generating technology potentially lowers alpha for reasons such as negative price pressure. The manager takes into account the negative impact of leverage on the portfolio's alpha when choosing dynamic investment strategies.

We develop an analytically tractable framework of leverage and valuation for hedge funds with the following features: (i) an alpha generating strategy, (ii) the management fee specified as a fraction of the AUM; (iii) the high-powered incentive fee linked to the HWM, and (iv) the fund's alpha decreasing with its leverage. The manager dynamically chooses leverage to maximize the present value (PV) of total managerial rents, the sum of the PV of both management and incentive fees.

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<sup>1</sup>Private equity funds also charge both management and incentive fees. While the compensation structure is similar in essence, institutional details such as fees and profits calculations differ significantly for hedge funds and private equity funds. Metrick and Yasuda (2010) provide an economic analysis of private equity funds.

For both conceptual and quantitative reasons, it is important to incorporate both management and incentive fees into our analysis. We will show that management and incentive fees have different, and often opposite, implications for managerial risk taking. With management fees only, the manager effectively behaves in the investors' interest because management fees make the manager a *de facto* equity investor in the fund and there is no agency issue between the manager and investors.<sup>2</sup> However, when the manager is paid via both fees, conflicts of interest between the manager and investors arise. The optionality embedded in incentive fees encourages the manager to lever more than investors would prefer. The closer the fund's AUM is to its HWM, the more leverage the manager chooses because the incentive fee is closer to being realized (i.e. the embedded call option is deeper in the money). This standard optionality argument suggests that the PV of incentive fees is increasing and *convex* in the AUM.

On the other hand, paying incentive fees lowers the AUM and hence crowds out current management fees and also potentially reduces future management fees. Thus, there is a tension between the PV of management fees and the PV of incentive fees. Importantly, the PV of management fees is increasing but *concave* in the AUM because (i) management fees effectively give the manager an "equity-like" stake in the fund, and (ii) equity investors in the fund are effectively short in a call option (embedded in incentive fees) to the manager. As a result, management fees encourage the manager to behave prudently and hence reduce leverage, *ceteris paribus*.

When the fund's performance improves (i.e. its AUM rises and gets closer to its HWM), the manager is more likely to collect the incentive fees, and thus behaves in a more risk-seeking way by increasing leverage. When the fund's performance deteriorates, incentive fees are out of the money and the manager thus puts more weight on management fees, which encourages a more prudent investment strategy. The incentive to lower leverage is stronger if the downside risk is greater. For example, as we show, the manager is even willing to go long in the risk-free asset to lower the fund's return volatility when facing serious performance-based liquidation threats from investors.

To study the impact of performance-triggered liquidation risk by investors, we extend our baseline model to allow investors to liquidate the fund when the fund loses a significant

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<sup>2</sup>See Wang and Wang (2010) for a similar result in a different setting with management fees only.

fraction (e.g. 50%) of its AUM from its HWM (i.e. effectively its maximum AUM ever achieved historically). The liquidation/withdrawal option allows investors to better manage their downside risk exposure. As we noted earlier, the fund manager is now not just long in the incentive call option but also short in investors' liquidation option. The manager rationally manages the fund's leverage in order to maximize the PV of total fees. Quantitatively, we show that liquidation options significantly reduce managerial risk-taking and sometimes cause the manager to be too conservative even from the perspective of investors. As a result, investors' liquidation options significantly curtail the manager's total rents. Moreover, management fees become much more important than incentive fees in contributing to managerial total rents when we allow investors to liquidate the fund based on performance.

In our numerical example, without the liquidation option, the total managerial rents is about 60 cents for each dollar of the AUM out of which two-thirds of total rents, i.e. 40 cents, come from incentive fees in PV. However, when investors can liquidate the fund, say after the manager loses 25% of its AUM from its HWM, the PV of total fees falls about half to 30 cents for each dollar of the AUM, out of which only one-third of total rents, i.e. 10 cents, are due to incentive fees in PV. This calculation illustrates the importance of performance-based liquidation option on managerial rents and the relative weights of incentive and management fees in contributing to managerial rents. We also find that concentrated managerial ownership helps to align the incentives between the manager and investors.

Our model provides the first elements of an analytically tractable operational framework to study the effects of managerial compensation on the fund's dynamic leverage and valuation for the manager's and investors' stakes. In addition to deriving the manager's dynamic leverage decisions, we explicitly link the PV of managerial compensation to managerial talent/alpha. Our model thus provides a valuation toolbox for investors to evaluate their investments and managerial compensation contracts given investors' beliefs about managerial skills. Once investors choose their priors about the manager's unlevered alpha and the Sharpe ratio, we can use our dynamic valuation framework to determine managerial leverage and also to quantify PVs of various fees and investors' payoffs.

**Related literature.** There are only a few theoretical papers studying the hedge fund's valuation and leverage decisions. Goetzman, Ingersoll, and Ross (2003), henceforth abbreviated GIR, provides the first quantitative inter-temporal valuation framework for management and

incentive fees in the presence of the HWM. Despite the sophisticated HWM dynamics induced by embedded optionality among other features, they derive closed-form solutions for various value functions for a fund with a constant alpha and Sharpe ratio. GIR focuses solely on valuation and thus takes AUM dynamics as given without allowing for leverage decisions. Building on GIR, we introduce investment-dependent alpha-generating technology, derive time-varying optimal endogenous leverage choice, and study the implied dynamic valuation due to time-varying incentives to manage leverage and the fund's performance.

Panageas and Westerfield (2009) study the manager's investment decisions when the manager receives incentive fees without management fees. They obtain explicit time-invariant investment strategies for a risk-neutral patient manager.<sup>3</sup> In terms of the model setup, our paper differs from theirs by incorporating management fees and also the assumption that the alpha for the levered position may not necessarily scale up with the position. We show that optimal leverage chosen by the manager critically depends on the ratio of the AUM and the HWM due to the interaction between incentive and management fees. Management and incentive fees play important but different roles on leverage and valuation. In terms of results, the value of incentive fees in our paper is convex in AUM, while it is concave in their model with no liquidation boundary.<sup>4</sup>

Wang and Wang (2010) study leverage management when the manager is compensated via management fees but not incentive fees. Compensation contracts with only management fees effectively make the manager a co-investor in the fund whose cash flows are management fees in proportion to the AUM, and thus effectively behaves in the investors' interest. Unlike their paper, our paper focuses on the interaction of management and incentive fees and other frictions such as liquidation boundaries imposed by investors. Dai and Sundaresan (2010) point out that the hedge fund is short in two important options: investors' redemption option and funding options (from prime brokers and short-term debt markets). These two short positions have significant effects on optimal leverage and risk management policies (e.g. the use of unencumbered cash). They do not model the effects of incentive fees and the HWM on leverage and valuation. Like their paper, we also model the investors' liquidation option,

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<sup>3</sup>While their portfolio strategy solution looks similar to the one in the classic portfolio choice problem for investors with constant relative risk aversion as in Merton (1971), the economic mechanism is different as they point out in their paper.

<sup>4</sup>Other differences between the two papers include payouts to investors and the allowance for a lower liquidation boundary. For more detailed comparisons with Panageas and Westerfield (2009), see Section 5.2.

but in the context of managerial compensation. Additionally, our paper also studies the importance of managerial ownership on mitigating agency conflicts.

There has been much recent and continuing interest in empirical research on hedge funds. Fung and Hsieh (1997), Ackermann, McEnally, and Ravenscraft (1999), Agarwal and Naik (2004), and Getmansky, Lo, and Makarov (2004) among others, study the nonlinear feature of hedge fund risk and return.<sup>5</sup> Lo (2008) provides a detailed treatment of hedge funds for their potential contribution to systemic risk in the economy.

## 2 The baseline model

First, we introduce the fund manager's alpha-generating investment opportunity. Second, we describe the fund's managerial compensation contracts including both management and incentive fees. Then, we discuss the dynamics for the fund's AUM, and define various value functions for the manager and investors. Finally, we state the manager's intertemporal optimization problem subject to investors' voluntary participation.

**The fund's investment technology.** The fund has a trading strategy which generates expected excess returns after risk adjustments. The fund's expected excess returns may be attributed to managerial talent and may not be traded. In reality, fund managers are sometimes secretive about their trading strategies and take measures to make replications of their strategies difficult.

In addition to investing in the risky strategy, the manager can also invest in the risk-free asset, which offers a constant rate of return  $r$ . Let the amount of investment in the risk-free asset be  $D$ . Let  $W$  denote the fund's AUM. The amount invested in the risky alpha-generating technology is then  $W - D$ . When  $D < 0$ , the fund takes on leverage, often via short-term debt. Hedge funds can obtain leverage from the fund's prime brokers, repo markets, and the use of derivatives.<sup>6</sup>

Let  $\pi$  denote the asset-capital ratio  $\pi = (W - D)/W$ . Given investment strategy  $\pi_t$ , the

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<sup>5</sup>For the presence of survivorship bias, selection bias, and back-filling bias in hedge funds databases, see Brown, Goetzmann, Ibbotson, and Ross (1992), and Brown, Goetzmann, and Ibbotson (1999), among others. For careers and survival, see Brown, Goetzmann, and Park (2001).

<sup>6</sup>Few hedge funds are able to directly issue long-term debt or secure long-term borrowing.

incremental return for the underlying alpha-generating strategy  $dR_t$  is given by

$$dR_t = \mu(\pi_t)dt + \sigma dB_t, \quad (1)$$

where  $B_t$  is a standard Brownian motion. For expositional simplicity, we assume that the volatility parameter  $\sigma$  is constant for the strategy and focus on the expected return  $\mu(\pi)$ .<sup>7</sup> Even with constant  $\sigma$ , leverage will change the volatility of the portfolio return for the standard leverage argument. The expected return  $\mu(\pi)$  is for the unlevered position and is interpreted as the expected return after the adjustment for the systematic risk. The larger the fund's position per unit of capital  $\pi$ , the lower the expected excess return on the marginal unit of investment of the unlevered position. This may be due to the price pressure, decreasing returns to scale for the fund manager's ability to scale up its alpha-generating skill/position and/or liquidity issues with the fund's increasingly large position. Motivated by these considerations, we thus assume that  $\mu(\pi)$  is decreasing in  $\pi$ , i.e.  $\mu'(\pi) < 0$ . We choose this specification to capture price pressure in an analytically tractable framework.<sup>8</sup> In addition, this homogeneity assumption for the price pressure effect ensures that the impact (i.e. price pressure) always matters and the fund will not grow out of it. Berk and Green (2004) make a similar price pressure type of assumption in their study of mutual funds industry equilibrium.

Let  $\alpha(\pi)$  denote the un-levered expected *excess* return beyond the risk-free rate

$$\alpha(\pi) = \mu(\pi) - r. \quad (2)$$

Alpha measures managerial talent in our model. We have already accounted for the systematic risk in the return generating process (1). Scarce managerial skills earn rents in equilibrium. Taking into account the effects of the investment  $\pi$  on alpha, the manager optimally chooses  $\pi$ . By increasing  $\pi$ , the firm earns alpha on a larger amount of total assets, however, each unit of asset earns a lower level of  $\alpha$ . The fund optimally trades off between these two offsetting effects in a dynamic optimizing framework as we will show.

**Managerial compensation contracts.** The typical hedge fund management compensation contract has two main components: the management fees and the incentive fees. The

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<sup>7</sup>Note that here  $dR_t$  is for the return of the alpha generating strategy, not for the return of the portfolio with leverage, which is addressed later.

<sup>8</sup>In our earlier version, we assumed costly leverage. The mathematical formulation via costly leverage is similar to the price pressure version captured by  $\mu(\pi)$  that we adopt in the paper.

management fee is specified as a constant fraction  $c$  of the AUM  $W$ :  $\{cW_t : t \geq 0\}$ . The incentive fee links compensation to the fund's performance.

To describe the incentive fees, we need to understand the fund's high-water mark (HWM) process  $\{H_t : t \geq 0\}$ . For the purpose of exposition, first consider the simplest example where the HWM  $H_t$  is the highest level that the AUM  $W$  has attained up to time  $t$ , i.e.  $H$  is the running maximum of  $W$ :  $H_t = \max_{s \leq t} W_s$ .

More generally, the HWM may also change due to indexed growth or investors' withdrawal.<sup>9</sup> Let  $g$  denote the (normal) rate at which  $H$  grows. As in GIR, investors in our model are paid continuously at a rate  $\delta W_t$  where  $\delta$  is a constant. Naturally, the fund's high-water mark is adjusted downward due to investors' withdrawal at the rate  $\delta$ .

Provided that the fund is in operation and the fund's AUM  $W$  is below its HWM ( $W < H$ ), the evolution of  $H$  is locally deterministic and is given by

$$dH_t = (g - \delta)H_t dt, \quad \text{when } W_t < H_t. \quad (3)$$

For example,  $g$  may be set to zero, the interest rate  $r$ , or other benchmark levels. The growth of  $H$  to some extent may account for the time value of money. Whenever the AUM  $W$  exceeds its HWM, the manager collects a fraction  $k$  of the fund's performance exceeding its HWM and then the HWM resets. We provide a detailed analysis for the dynamics of the HWM  $H$  on the boundary  $W = H$  in Section 4.

**The dynamic of the AUM  $W$ .** We take the initial AUM  $W_0$  as given.<sup>10</sup> In our baseline model and in GIR, investors stochastically liquidate the fund at a constant rate  $\lambda$  per unit of time. This assumption implies that the fund has a finite average duration and also keeps the model stationary and analytically tractable. Upon liquidation at exogenous stochastic time  $\tau$ , the manager receives nothing and investors collect AUM  $W$ .

While the manager runs the fund (i.e. before stochastic liquidation time  $\tau$ , i.e.  $t < \tau$ ),

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<sup>9</sup>The HWM may be reset following poor fund performance. Sometimes, it is argued that resetting the HWM helps to re-align the incentives between the manager and investors. To offset the potential increase in incentive fees collected due to the reset of the HWM, the manager may reduce management fees by lowering  $c$ .

<sup>10</sup>We leave the endogenous determination of initial  $W_0$  for future research. In reality, it is often difficult to write fully specified state contingent contracts that describe capital injection and withdrawal by investors. We assume that managers have discretion and their actions are not fully observable nor contractible. Hence, leverage will become relevant. See Titman (2010) for more discussion on this point.

the AUM  $W_t$  evolves as follows

$$\begin{aligned} dW_t &= \pi_t W_t (\mu(\pi_t)dt + \sigma dB_t) + (1 - \pi_t)rW_t dt - \delta W_t dt - cW_t dt \\ &\quad - k [dH_t - (g - \delta)H_t dt] - dJ_t, \quad t < \tau. \end{aligned} \quad (4)$$

The first and second terms in (4) describe the change of AUM  $W$  from the manager's investment strategies in its alpha-generating technology and the risk-free asset as in standard portfolio choice problem (Merton (1971)). The third term  $-\delta W dt$  gives the continuous payout rate to investors. The sum of these three terms gives the change of  $W$  in the absence of fees. The fourth term represents the management fees to the manager in flow terms (e.g.  $c = 2\%$ ). The fifth term gives the incentive/performance fees when the AUM exceeds the HWM (e.g.  $k = 20\%$ ).<sup>11</sup> The process  $J$  in the last (sixth) term is a pure jump process which describes this liquidation risk: The AUM  $W$  is set to zero with probability  $\lambda$  per unit of time when investors exogenously liquidate the fund.

**Various value functions for investors and the manager.** We next define present values for various streams of cash flows. For a given dynamic investment strategy  $\pi$ , we use  $M(W, H; \pi)$  and  $N(W, H; \pi)$  to denote the present values (PVs) of the management fees and the incentive fees, respectively. Recall that the alpha-generating technology (1)-(2) is already after the systematic risk adjustment. For example, when the market portfolio is the only source of systematic risk (i.e. CAPM holds for investments with no alpha), (1) should be interpreted as market-neutral excess return after factoring out the market risk premium. We may thus discount cash flows using the risk-free rate as follows

$$M(W, H; \pi) = \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} cW_s ds \right], \quad (5)$$

$$N(W, H; \pi) = \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} k [dH_s - (g - \delta)H_s ds] \right], \quad (6)$$

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<sup>11</sup>The manager collects the incentive fees if and only if  $dH_t > (g - \delta)H_t dt$ , which can only happen on the boundary ( $W_t = H_t$ ). However, after reaching the new HWM, the manager may not collect incentive fees if the fund does not make profits.

where the manager collects neither management nor incentive fees after stochastic liquidation time  $\tau$ .<sup>12</sup> Let  $F(W, H; \pi)$  denote the PV of total fees, which is given by

$$F(W, H; \pi) = M(W, H; \pi) + N(W, H; \pi). \quad (7)$$

Similarly, we define investors' value  $E(W, H)$  as follows

$$E(W, H; \pi) = \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} \delta W_s ds + e^{-r(\tau-t)} W_\tau \right]. \quad (8)$$

Finally, the total PV of the fund  $V(W, H)$  is given by the sum of  $F(W, H)$  and  $E(W, H)$ :

$$V(W, H; \pi) = F(W, H; \pi) + E(W, H; \pi). \quad (9)$$

**The manager's optimization problem and investors' voluntary participation.** The manager chooses the optimal dynamic investment policy  $\pi$  to maximize the PV of total fees  $F(W, H; \pi)$  as follows

$$\max_{\pi} F(W, H; \pi) = \max_{\pi} \{M(W, H; \pi) + N(W, H; \pi)\}, \quad (10)$$

subject to the investors' participation condition to which we now turn.

Anticipating that the manager behaves in own interest, investors rationally demand that the PV of their payoffs is at least higher than their time-0 investment  $W_0$  in the fund in order to break even, risk-adjusted and time-value adjusted. At time 0, by definition, the fund's HWM is set at  $H_0 = W_0$ . Given the manager's optimal dynamic investment strategy  $\pi$ , we thus require the investors' value  $E(W_0, W_0; \pi)$  at time 0 to satisfy the following condition

$$E(W_0, W_0; \pi) \geq W_0. \quad (11)$$

Intuitively, how much surplus investors collect depends on their relative bargaining power against the manager. In perfectly competitive markets where we assume that the skilled manager collects all the surplus, the above participation constraint (11) holds with equality. In moments when investors may have certain bargaining power (such as in the financial crisis

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<sup>12</sup>We may allow the manager to start a new fund after paying certain start-up costs and extend the model and analysis accordingly. In those extensions, the manager will maximize the PV of fees from the current fund and the "continuation" value from managing future funds. This important extension will significantly complicate the analysis and lengthen the paper. We leave it for future research.

periods where capital supply is potentially limited), their participation constraint may hold with slack, i.e. earning some rents by providing valuable capital to the manager.

Before analyzing the general case where the manager receives both management and incentive fees, we first study the case where the manager only receives management fees.

### 3 First-best benchmark: management fees only

With only management fees and no high-powered incentive schemes ( $k = 0$ ), there are no conflicts of interest between the manager and investors. This compensation contract effectively gives the manager an equity stake in the fund and hence managerial incentives are perfectly aligned with investors' incentives. The manager thus maximizes the fund's total surplus, which is equivalent to maximizing managerial rents or investors' interest. Leverage is valuable in the model because the manager trades off earning alpha on a larger pool of assets and lowering alpha on the marginal unit of investment due to price pressure (i.e.  $\alpha'(\pi) < 0$ ).<sup>13</sup> The following proposition summarizes the analytical results for the special case with management fees only.

**Proposition 1** *With no incentive fees ( $k = 0$ ), the PV of total fees  $F^*(W)$  is given by*

$$F^*(W) = M^*(W) = \frac{c}{c + \delta + \lambda - \alpha^*} W, \quad (12)$$

where  $\alpha^*$  is a constant given by the following equation

$$\alpha^* = \pi^* \alpha(\pi^*), \quad (13)$$

and the unique optimal time-invariant investment strategy is given by

$$\alpha(\pi^*) + \pi^* \alpha'(\pi^*) = 0. \quad (14)$$

With optimal investment strategy  $\pi^*$ , the PV of investors' payoff is given by

$$E^*(W) = \frac{\delta + \lambda}{c + \delta + \lambda - \alpha^*} W. \quad (15)$$

The PV of the total fund,  $V^*(W)$ , is given by

$$V^*(W) = M^*(W) + E^*(W) = \frac{c + \delta + \lambda}{c + \delta + \lambda - \alpha^*} W. \quad (16)$$

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<sup>13</sup>Wang and Wang (2010) use the proportional transaction cost setting in the well known portfolio choice model of Davis and Norman (1990) and obtain essentially the same result as we do when the manager is only paid via management fees. We view our results as complementary to theirs.

Note that  $\alpha^*$  can be interpreted as the level of the fund's optimally levered alpha. For convergence, we need to ensure that levered alpha  $\alpha^*$  cannot be too high, i.e.

$$\alpha^* < c + \delta + \lambda, \tag{17}$$

where  $\alpha^*$  is given by (13)-(14). With  $\alpha'(\pi) = 0$ , the optimal leverage in the  $k = 0$  benchmark is infinity, because the manager is risk neutral and there is no mechanism to discourage the manager from taking on leverage. The negative price pressure (i.e.  $\alpha'(\pi) < 0$ ) is one natural way to ensure convergence in our benchmark without incentive fees. As we will show later, when investors have options to liquidate the fund following poor fund performances, we no longer need to require  $\alpha'(\pi) < 0$  for the purpose of convergence. For the benchmark with  $k = 0$  with no frictions, we require  $\alpha'(\pi) < 0$  for convergence. Additionally, the inequality (17) ensures that investment strategy is not too good to be true so that the value of the alpha-generating technology (1) even after leverage is still finite.

Our model features managerial skills ( $\alpha > 0$  and hence  $\alpha^* > 0$ ). If the skilled manager has all the bargaining power as in perfectly competitive markets, we expect that investors break even, i.e.  $E^*(W) = W$ . Using (15), we obtain the equilibrium compensation contract which features management fee  $c = \alpha^*$ . Intuitively, the manager charges  $c$  at the managerial optimally levered skill premium  $\alpha^*$ . Consequently, using (12), we obtain that the equilibrium value of management fees to be  $F^*(W) = \alpha^*W/(\delta + \lambda)$ . Note that management fees are collected each period as a fraction of AUM, the sum of the fees can be quite significant as the formula suggests and our calculation in Section 5 shows. In markets where capital is scarce, more even distribution of bargaining power between the manager and investors may exist. In that case, we naturally expect a more balanced distribution of the surplus between investors and the manager.

**An example for the alpha-generating strategy.** Consider the following specification

$$\alpha(\pi) = \theta_0 - \theta_1\pi, \tag{18}$$

where  $\theta_0 > 0$  and  $\theta_1 \geq 0$  are constant. For the special case with  $\theta_1 = 0$ , the un-levered alpha is equal to  $\theta_0$ , a constant implying no price pressure. In general, with  $\theta_1 > 0$ , the un-levered alpha  $\alpha(\pi)$  decreases with  $\pi$ , i.e. the manager is concerned about the negative impact of  $\pi$  on the un-levered alpha. The manager optimally chooses leverage to trade off

more investments via leverage yielding excess returns against lowering alpha on the marginal unit of investment due to higher  $\pi$  (i.e. price pressure). The equilibrium optimal leverage is given by  $\pi^* = \theta_0/(2\theta_1)$ . The optimally levered alpha for the fund portfolio is given by

$$\alpha^* = \pi^* \alpha(\pi^*) = \frac{\theta_0^2}{4\theta_1}. \quad (19)$$

The manager has no conflicts of interest with investors in the benchmark without incentive fees and volatility has no influence on leverage because the manager is risk neutral. In Section 5, we will use the alpha strategy (18) to conduct our numerical analysis.

While our benchmark model is set up for the case with  $k = 0$ , it also applies to settings where the high-water mark  $H$  approaches infinity ( $H \rightarrow \infty$ ). For that case, the incentive fees are effectively completely out of the money, and hence the manager *de facto* collects only the management fees, and therefore, the solution is the same as the one given here.

We next turn to the general setting where the manager collects both management and incentive fees. The high-water mark plays a fundamental role in leverage choices and valuation, resulting in imperfect alignment of incentives between the manager and investors.

## 4 Model solution: The general case with the HWM

We solve the manager’s optimization problem using dynamic programming. First, we study the manager’s behavior and implied valuation when the AUM is below the HWM (i.e.  $W < H$ ). Second, we analyze the manager’s behavior when the evolution of the AUM leads to setting the new HWM. Third, we use the homogeneity property of our model to solve for the manager’s optimal investment strategy and various PVs for the manager and investors.

**The manager’s intertemporal decision problem when  $W < H$ .** When the fund’s AUM is below its HWM ( $W < H$ ), instantaneously the manager will not receive the incentive fees. The fund is in the “interior” region where the option component (incentive fees) is inactive over the short horizon, and the manager only collects management fees. However, being forward looking, the manager chooses dynamic investment strategy  $\pi$  to maximize  $F(W, H)$ , the PV of total fees. Using the principle of optimality, we have the following

Hamilton-Jacobi-Bellman (HJB) equation in the interior region

$$(r + \lambda)F(W, H) = \max_{\pi} cW + [\pi\alpha(\pi) + (r - \delta - c)]WF_W(W, H) + \frac{1}{2}\pi^2\sigma^2W^2F_{WW}(W, H) + (g - \delta)HF_H(W, H). \quad (20)$$

The left side of (20) elevates the discount rate from the interest rate  $r$  to  $(r + \lambda)$  to reflect the stochastic liquidation of the fund. The first term on the right side of (20) gives the management fee:  $cW$ . The second and third terms give the drift (expected change) and the volatility effects of the AUM  $W$  on  $F(W, H)$ , respectively. Finally, the last term on the right side of (20) describes the effect of the HWM  $H$  change on  $F(W, H)$ . The manager optimally chooses leverage  $\pi$  to equate the two sides of (20). Next, we analyze the properties of  $F(W, H)$  when the AUM equals the HWM and moves along the boundary  $W = H$ .

**The manager's intertemporal decision problem when  $W = H$ .** Our reasoning for the boundary behavior ( $W = H$ ) essentially follows GIR and Panageas and Westerfield (2009). A positive return shock increases the AUM from  $W = H$  to  $H + \Delta H$ . The PV of total fees for the manager is then given by  $F(H + \Delta H, H)$  before the HWM adjusts. Immediately after the positive shock, the HWM adjusts to  $H + \Delta H$ . The manager then collects the incentive fees in flow terms  $k\Delta H$ , and consequently the AUM is lowered from  $H + \Delta H$  to  $H + \Delta H - k\Delta H$ . The PV of total managerial fees is then equal to  $F(H + \Delta H - k\Delta H, H + \Delta H)$ . Using the continuity of  $F(\cdot, \cdot)$  before and after the adjustment of the HWM, we have

$$F(H + \Delta H, H) = k\Delta H + F(H + \Delta H - k\Delta H, H + \Delta H). \quad (21)$$

By taking the limit as  $\Delta H$  approaches zero and using Taylor's expansion rule, we obtain

$$kF_W(H, H) = k + F_H(H, H). \quad (22)$$

The above is the value-matching condition for the manager on the boundary  $W = H$ . By using essentially the same logic, we obtain the following boundary conditions for the PV of management fees  $M(W, H)$  and the PV of incentive fees  $N(W, H)$  at the boundary  $W = H$ :  $kM_W(H, H) = M_H(H, H)$  and  $kN_W(H, H) = k + N_H(H, H)$ .

Finally, we turn to the left boundary condition. When the fund runs out of assets ( $W = 0$ ), there is no more AUM in the future and hence no fees:  $F(0, H) = 0$ . In Section 6,

we extend our model to allow investors to have more control rights. For example, investors may prevent the AUM from falling below a certain fraction of the HWM by being able to contractually liquidate the fund. We show that this lower liquidation boundary has significant effects on the manager’s dynamic investment strategies and valuation.

As we will show, our model captures dynamic leverage decisions in a tractable framework. The tractability of our model solution critically rests on the homogeneity property. That is, if we simultaneously double the AUM  $W$  and the HWM  $H$ , the PV of total fees  $F(W, H)$  will accordingly double. The effective state variable is thus the ratio between the AUM  $W$  and the HWM  $H$ :  $w = W/H$ . We will use the lower case to denote the corresponding variable in the upper case scaled by the contemporaneous HWM  $H$ . For example,  $f(w) = F(W, H)/H$ . The following theorem summarizes the main results on  $\pi(w)$  and the PV of total fees  $F(W, H)$ .

**Theorem 1** *The scaled PV of total fees  $f(w)$  solves the following ODE*

$$(r + \lambda - g + \delta)f(w) = cw + [\pi(w)\alpha(\pi(w)) + r - g - c]wf'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2f''(w), \quad (23)$$

where the optimal investment strategy  $\pi(w)$  solves the following equation

$$\pi(w) = \frac{\alpha(\pi(w))}{\sigma^2\psi(w) - \alpha'(\pi(w))}, \quad (24)$$

and  $\psi(w)$  is given by

$$\psi(w) = -\frac{wf''(w)}{f'(w)}. \quad (25)$$

The ODE (23) for  $f(w)$  is solved subject to the following boundary conditions

$$f(0) = 0, \quad (26)$$

$$f(1) = (k + 1)f'(1) - k. \quad (27)$$

Equation (24) gives the manager’s optimal dynamic investment strategy. While the manager is risk neutral, incentive issues and other frictions generate curvatures for the managerial value function  $f(w)$ . The numerator in (24) gives the fund’s optimally chosen levered alpha  $\alpha(\pi(w))$ . The denominator has two components. First, the curvature of the manager’s value function  $f(w)$ , i.e. the manager’s effective risk attitude  $\psi(w) = -wf''(w)/f'(w)$ , matters for the portfolio choice  $\pi(w)$  due to an “effective” risk attitude for the manager,  $\psi(w)$ , which

works in an interactive way with the variance of the return process,  $\sigma^2$ . Second, with market pressure (i.e.  $\alpha'(\pi) < 0$ ), the manager chooses investment  $\pi$  by anticipating its diminishing effect on total expected excess return  $\alpha(\pi)$ . Provided that the sum of the two terms is positive, i.e.  $\sigma^2\psi(w) - \alpha'(w) > 0$ , the optimal portfolio rule  $\pi(w)$  has an interior solution.

The formula (25) for  $\psi(w)$  measures the curvature of the manager's value function  $f(w)$ , and we may intuitively refer to  $\psi(w)$  as the manager's effective risk attitude. When  $f(w)$  is convex (due to the optionality induced by the HWM), the risk-neutral manager behaves effectively in a risk-loving manner. When  $f(w)$  is concave, the risk-neutral manager behaves effectively in a risk-averse manner for precautionary reasons to mitigate investors' liquidation likelihood. Whether the manager behaves in a risk-averse or risk-seeking manner depends on various constraints and managerial incentives (See Sections 5 and 6 for details).

Finally, the ODE (23) is a Black-Merton-Scholes type of valuation equation adapted for the manager's dynamic optimal leverage and management/incentive fees. Equation (26) states that  $w = 0$  is an absorbing boundary with  $f(0) = 0$ . Equation (27) gives the condition at the upper boundary  $w = 1$  when the manager is close to collecting incentive fees.

We now turn to the implied dynamic valuation for various components of the fund. We summarize the main results in the following proposition.

**Proposition 2** *Given investment strategy  $\pi(w)$ , the dynamics of  $w = W/H$  is given by*

$$dw_t = \mu_w(w_t)dt + \sigma\pi(w_t)w_tdB_t - dJ_t, \quad (28)$$

where  $J$  is a pure jump process which sets  $w = 0$  upon the stochastic arrival of the jump with intensity  $\lambda$ , and the drift function  $\mu_w(w)$  is given by

$$\mu_w(w) = [\pi(w)\alpha(\pi(w)) + r - g - c]w. \quad (29)$$

The value functions  $M(W, H)$ ,  $N(W, H)$ , and  $E(W, H)$  are all homogeneous with degree one in AUM  $W$  and HWM  $H$ , i.e.  $M(W, H) = m(w)H$ ,  $N(W, H) = n(w)H$ ,  $E(W, H) = e(w)H$ , where  $m(w)$ ,  $n(w)$ ,  $e(w)$  solve the following ODEs respectively

$$(r + \lambda - g + \delta)m(w) = cw + \mu_w(w)m'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2m''(w), \quad (30)$$

$$(r + \lambda - g + \delta)n(w) = \mu_w(w)n'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2n''(w), \quad (31)$$

$$(r + \lambda - g + \delta)e(w) = (\delta + \lambda)w + \mu_w(w)e'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2e''(w), \quad (32)$$

with the following boundary conditions

$$m(0) = n(0) = e(0) = 0, \quad (33)$$

$$m(1) = (k+1)m'(1), \quad (34)$$

$$n(1) = (k+1)n'(1) - k, \quad (35)$$

$$e(1) = (k+1)e'(1). \quad (36)$$

The total fund's value is  $V(W, H) = v(w)H$ , where  $v(w) = m(w) + n(w) + e(w) = f(w) + e(w)$ .

**Investors' voluntary participation.** In order for investors to voluntarily participate, we need to ensure that investors do not lose money in present value by investing with the manager, i.e.  $E(W_0, W_0) \geq W_0$  as given in (11). It is equivalent to require  $e(1) \geq 1$  due to the homogeneity property of  $E(W, H)$ . In our baseline model with perfectly competitive capital markets, the manager collects all the rents from their managerial skills, and therefore, we have the equilibrium outcome for the investors' value

$$e(1) = 1. \quad (37)$$

In general, depending on the relative bargaining power between the manager and investors, investors may collect some surpluses especially when capital supply is limited as in financial crisis periods. We now analyze the results for the general case with the HWM.

## 5 Results: Alpha, fees, leverage, and valuation

We first choose the parameter values and then analyze the model's results.

**Parameter choices.** All rates are annualized and continuously compounded, whenever applicable. For the baseline calculation, we set the interest rate  $r = 5\%$ , the payout rate to investors  $\delta = 5\%$ , the annual liquidation probability  $\lambda = 5\%$ , and the target growth rate of the HWM  $H$   $g = 5\%$  so that the net growth rate of the HWM  $H$  in the interior region is zero in the absence of stochastic liquidation, i.e.  $g - \delta = 0$ . That is, the HWM  $H$  is the running maximum of the AUM  $W$ :  $H_t = \max\{W_s : s \in [0, t]\}$ . We specify the alpha strategy:  $\alpha(\pi) = \theta_0 - \theta_1\pi$  as in (18) and choose  $\theta_0 = 2.5\%$ ,  $\theta_1 = 0.25\%$ , and  $\sigma = 9\%$ .

## 5.1 The HWM $H$ , leverage $\pi$ , and managerial rents $f(w)$

**The benchmark with no incentive fees ( $k = 0$ ).** With the baseline parameter values, the optimal leverage is time-invariant and is equal to  $\pi^* = 5$ . That is, for each dollar of the AUM, the fund borrows four dollars and invests total five dollars in the risky positive-alpha generating technology delivering an optimally levered alpha for the fund:  $\alpha^* = 6.25\%$ . To ensure that investors break even under perfectly competitive capital markets ( $e(1) = 1$ ), we need to set the equilibrium management fee to the value of the portfolio's levered alpha:  $c = \alpha^* = 6.25\%$  using (15) so that the manager collects all the surplus. The managerial rents are significant in this example: 62.5% of the AUM, i.e.  $F^*(W) = 0.625W$ . Investors always value his investments at AUM, i.e.  $E^*(W) = W$ . The dashed line in Figure 1 plots the time-invariant optimal leverage  $\pi^* = \theta_0/(2\theta_1) = 5$ .

**The effect of incentive fees ( $k > 0$ ).** We choose the 2-20 management contract:  $c = 2\%$  and  $k = 20\%$ . As we will show later, with other parameter values, they imply  $e(1) = 1$  so that investors break even in PV and the manager collects all the surplus for skills.

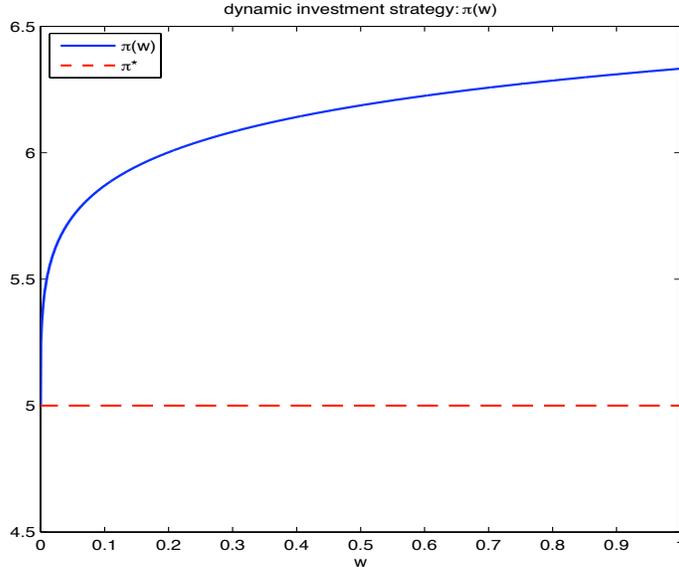


Figure 1: **Dynamic investment strategy  $\pi(w)$ .** The solid line corresponds to the 2-20 compensation contract:  $c = 2\%$  and  $k = 20\%$ ; and the dashed line corresponds to the benchmark contract with no incentive fees:  $c = 6.25\%$  and  $k = 0$ . Both contracts satisfy  $e(1) = 1$ . The other parameter values are:  $r = 5\%$ ,  $\delta = 5\%$ ,  $\lambda = 5\%$ ,  $g = 5\%$ ,  $\theta_0 = 2.5\%$ ,  $\theta_1 = 0.25\%$ , and  $\sigma = 9\%$ .

The solid line in Figure 1 plots  $\pi(w)$  as a function of  $w$  for a typical 2-20 compensation contract ( $c = 2\%$  and  $k = 20\%$ ). Reducing management fees  $c$  from 6.25% to 2% to accompany the introduction of incentive fees from  $k = 0$  to  $k = 20\%$  ensures that investors break even at the moment of participation, i.e.  $e(1) = 1$ .

Compared with the  $k = 0$  benchmark, the manager invests more in the positive-alpha technology due to incentive fees:  $\pi(w) > \pi^* = 5$  for  $w > 0$ . Intuitively, the manager now maximizes the sum of an “equity” like stake (management fees) and an “option” like stake (incentive fees). The manager’s overall position is thus more levered than investors’ pure equity stake. The call optionality embedded in incentive fees ( $k > 0$ ) encourages *excessive* managerial risk taking, the degree of which increases with  $w$  because the moneyness of the incentive fees increases with  $w$ . In the limit as  $w \rightarrow 1$ , we obtain the maximal leverage  $\pi(1) = 6.33$ , which is about 27% higher than the no-incentive-fee benchmark leverage  $\pi^* = 5$ . At the other end as  $w \rightarrow 0$ , excessive risk taking disappears and optimal leverage  $\pi(0)$  converges to  $\pi^* = 5$  because the call option is completely out of money.

The left panel of Figure 2 plots the PV of total fees  $f(w)$  under the 2-20 contract and compares with  $f^*(w)$  for the  $k = 0$  benchmark. Greater risk taking induced by incentive fees lowers total managerial rents:  $f(w) < f^*(w)$ . Note that this result follows from the following two results: (i) the no-incentive fee ( $k = 0$ ) benchmark yields the highest total value of the fund  $v(1)$  and (ii) the management compensation contracts adjust so that  $e(1) = 1$  under both settings and hence the manager collects all the surpluses:  $f(1) = v(1) - 1$ . In our example, under the 2-20 contract,  $f(1) = 0.589$ , which is about 5.76% smaller than the first-best benchmark level  $f^*(1) = 0.625$ .

The right panel of Figure 2 plots the manager’s risk attitude  $\psi(w) = -wf''(w)/f'(w)$ , which is defined in (25). In the benchmark with  $k = 0$ , we have  $\psi^*(w) = 0$  for all  $w$ , i.e. the manager is risk neutral. When  $k > 0$ , the risk-neutral manager takes on excessive risk. Using the optimal leverage formula (24), we see that  $\pi(w) > \pi^*$  because the manager is effectively risk seeking ( $f(w)$  is convex and  $\psi(w) < 0$ ). Moreover, as  $w$  increases, the call option embedded in incentive fees becomes deeper in the money, the manager becomes more risk seeking ( $\psi(w) < 0$  and  $|\psi(w)|$  increases with  $w$ ), and therefore, leverage  $\pi(w)$  increases with  $w$  as we see in Figure 1.

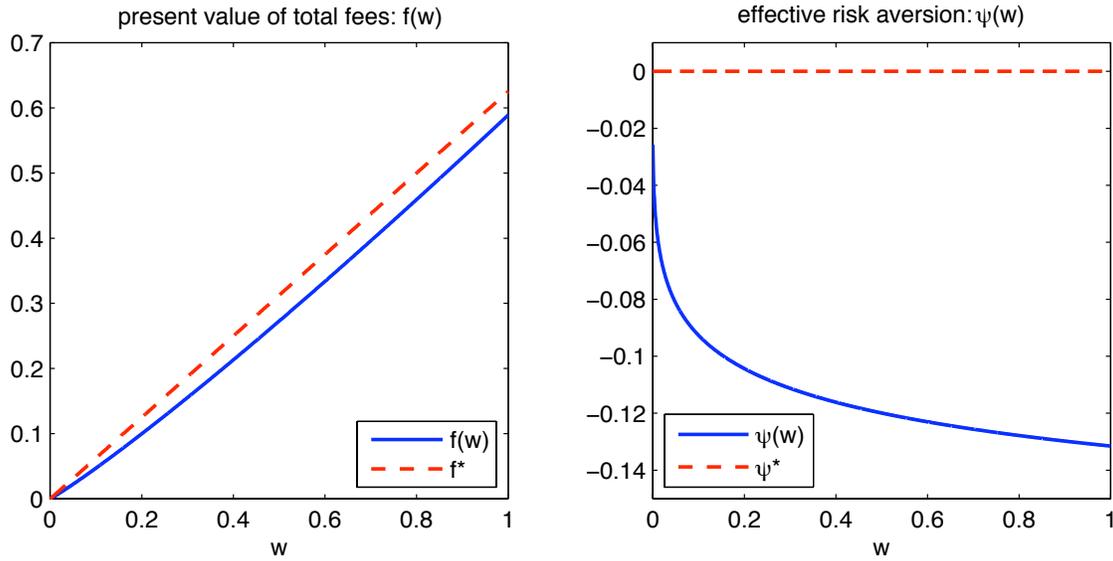


Figure 2: **The PV of total fees  $f(w)$  and the “effective” risk attitude:  $\psi(w) = -wf''(w)/f'(w)$ .** The manager’s value  $f(w)$  is convex, indicating that there is excessive managerial risk seeking ( $\psi(w) < 0$ ) with incentive fees ( $k > 0$ ) than without ( $k = 0$ ).

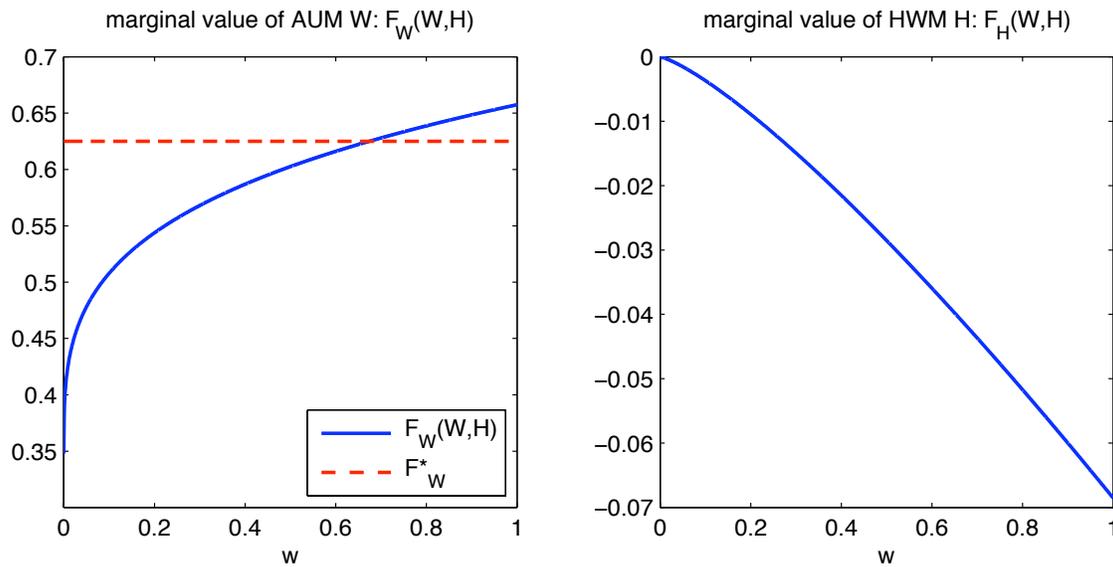


Figure 3: **The sensitivities of the PV of total fees  $F(W, H)$  with respect to the AUM  $W$  and the HWM  $H$ :  $F_W(W, H)$  and  $F_H(W, H)$ .**

**The marginal values of the AUM  $W$  and of the HWM  $H$ :  $F_W(W, H)$  and  $F_H(W, H)$ .**

Using the homogeneity property of the PV of total fees  $F(W, H)$ , we have the following results for the marginal values of the AUM  $W$  and of the HWM  $H$  on managerial rents  $F(W, H)$

$$F_W(W, H) = f'(w), \quad (38)$$

$$F_H(W, H) = f(w) - wf'(w). \quad (39)$$

The solid line in the left panel of Figure 3 plots  $F_W(W, H) = f'(w)$  for the case with incentive fees ( $k > 0$ ). For comparison purposes, the dashed horizontal line in the left panel of Figure 3 depicts the corresponding constant marginal value of the AUM for the  $k = 0$  benchmark at  $F_W^*(W) = f_w^*(w) = 0.625$  for all  $w$ , which implies that the managerial rents are equal to 62.5% of the AUM, a significant amount. Unlike the  $k = 0$  benchmark, the marginal value of the AUM  $F_W(W, H) = f'(w)$  is increasing in  $w$ , which implies that  $F(W, H)$  is convex in the AUM  $W$  (i.e.  $f''(w) > 0$ ) due to the call option feature embedded in incentive fees.

The marginal value of the AUM  $F_W(W, H)$  varies significantly with  $w$ . If  $w$  is sufficiently high, the marginal value of the AUM  $F_W(W, H)$  is high because of the high powered incentives from the incentive fee structure. For example,  $F_W(W, H)$  reaches \$0.658 at  $w = 1$ :  $f'(1) = 0.658$ , which is above the horizontal line for the  $k = 0$  benchmark. For sufficiently low  $w$ , the marginal value of AUM  $F_W$  can be quite low. For example, when  $w = 0$  (say due to  $H \rightarrow \infty$ ), the incentive fee is completely out of the money and  $F_W(0, H) = f'(0) = c/(\delta + \lambda + c - \alpha^*) = 0.02/(0.05 + 0.05 + 0.02 - 0.0625) = 0.348$ , significantly below the horizontal line (note that  $c = 2\%$  in the 2-20 contract compared with  $c = 6.25\%$  in the  $k = 0$  benchmark).

The right panel of Figure 3 plots  $F_H(W, H)$ , the marginal effect of the HWM  $H$  on  $F(W, H)$ . The HWM  $H$  is effectively the strike price of  $F(W, H)$ , however, it is stochastic and endogenous. Increasing the HWM  $H$  reduces the optionality and hence lowers the PV of total fees  $F(W, H)$  giving rise to the intuitive result:  $F_H(W, H) < 0$ . See the negatively valued  $F_H(W, H)$  in the right panel of Figure 3. Because  $F_{HH}(W, H) = w^2 f''(w)/H > 0$  and  $f''(w) > 0$  with  $k > 0$ ,  $F(W, H)$  is thus convex in the HWM  $H$ , which is similar to the result that option value is convex in the strike price.

## 5.2 Valuing various components for the manager and investors

**PVs of incentive fees  $N(W, H)$  and management fees  $M(W, H)$ .** Figure 4 plots the scaled PVs of incentive fees  $n(w)$  and management fees  $m(w)$ . Not surprisingly, both  $n(w)$

and  $m(w)$  increase with  $w$ . For this numerical example, the PV of incentive fees at  $w = 1$  is  $n(1) = 0.39$ , which is approximately twice as much as the PV of management fees:  $m(1) = 0.20$ . Incentive fees matter much more than management fees in this example. The total surplus  $F(W, W)$  is about 59% of the AUM  $W$ , ie.  $f(1) = 0.59$ , which is significant. When we allow investors to contractually liquidate the fund conditional on managerial performances, we show that both absolute and relative quantitative significance of incentive and management fees fundamentally change.

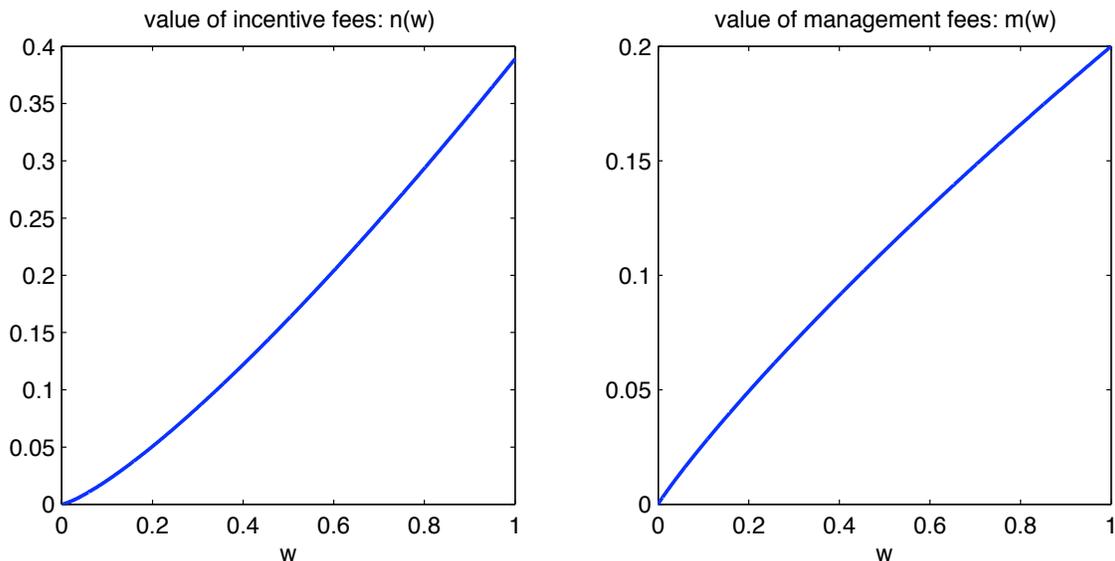


Figure 4: **Decomposing managerial rents  $f(w)$  into the PV of incentive fees  $n(w)$  and the PV of management fees  $m(w)$ :  $f(w) = n(w) + m(w)$ .**

The upper and lower left panels of Figure 5 plot  $N_W(W, H) = n'(w)$  and  $M_W(W, H) = m'(w)$ , respectively. First,  $n'(w)$  is increasing in  $w$ . That is, the value of incentive fees  $N(W, H)$  is convex in the AUM  $W$  due to the embedded optionality of incentive fees.

Second,  $m'(w)$  is decreasing in  $w$ . That is, the value of management fees  $M(W, H)$  is concave in the AUM  $W$ . Incentive fees crowd out management fees, and the payment of incentive fees lowers the AUM. The substitution of incentive fees for management fees effectively makes the value of management fees  $M(W, H)$  concave in  $W$ .<sup>14</sup>

<sup>14</sup>Of course, there is an overall risk-taking effect induced by the incentive fees, i.e. the value of total fees  $F(W, H)$  is convex in  $W$ . The crowding-out effect is stronger than the overall risk-taking effect which makes  $F(W, H)$  and  $N(W, H)$  convex in  $W$  but  $M(W, H)$  concave in  $W$ .

We now turn to the effects of the HWM  $H$  on  $N(W, H)$  and  $M(W, H)$ . Intuitively, the higher the HWM  $H$ , the less likely the manager collects the incentive fees, which suggests that the PV of incentive fees  $N(W, H)$  decreases with the HWM  $H$ :  $N_H(W, H) < 0$ . Indeed, the upper right panel in Figure 5 plots  $N_H(W, H)$ , which is negative for all  $w$ .

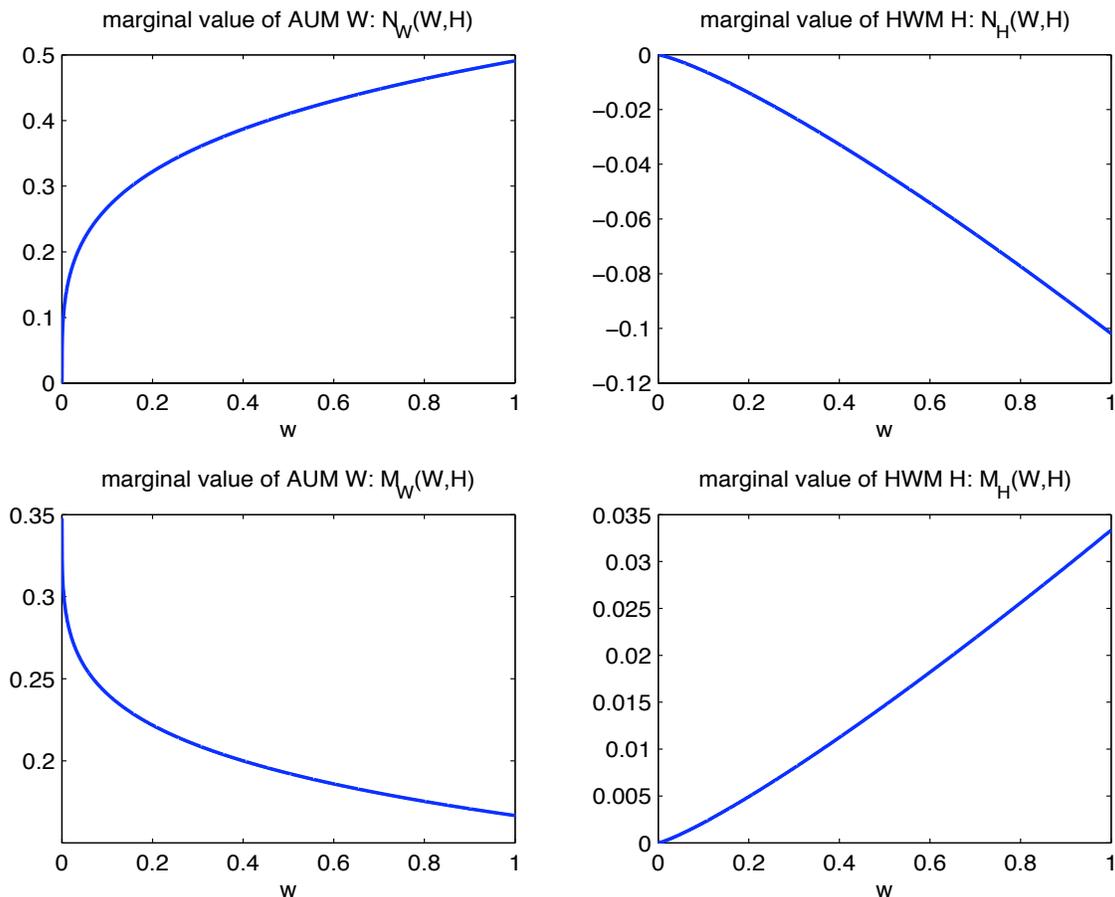


Figure 5: **Sensitivities of the PV of management fees  $M(W, H)$  and the PV of incentive fees  $N(W, H)$  with respect to the AUM  $W$  and the HWM  $H$ :  $N_W(W, H)$ ,  $N_H(W, H)$ ,  $M_W(W, H)$ , and  $M_H(W, H)$ .**

The higher the HWM  $H$ , the less likely the manager gets paid via the incentive fees, which implies a higher AUM for a given dynamic investment strategy and hence a larger PV of management fees  $M(W, H)$ , i.e.  $M_H(W, H) > 0$ . The lower right panel in Figure 5 plots  $M_H(W, H)$ , which is indeed positive for all  $w$ .

It is worth pointing out the differences between our model and Panageas and Westerfield

(2009). In their model, the manager is only compensated with incentive fees and is more patient than investors.<sup>15</sup> The main differences are as follows. First, our model predicts that the PV of incentive fees  $N(W, H)$  is convex in the AUM  $W$  while  $N(W, H)$  is *concave* in the AUM  $W$  in Panageas and Westerfield (2009). Second, our model predicts that the PV of incentive fees  $N(W, H)$  decreases with the HWM  $H$ . Third, our model predicts that *ceteris paribus*  $N(W, H)$  increases with the incentive fee parameter  $k$ . These predictions are opposite of those given by Panageas and Westerfield (2009). In their model, the manager prefers keeping money inside the fund due to the fund's alpha-generating technology and managerial patience. Increasing the HWM keeps the manager's future fees inside the fund longer, which allows the manager to generate more expected excess returns by building up the AUM. As a result, the PV of incentive fees  $N(W, H)$  *increases* with the HWM  $H$ . The mechanism in our model fundamentally differs from theirs.

**Investors' value  $E(W, H)$  and the total value of the fund  $V(W, H)$ .** The upper row of Figure 6 plots scaled investors' value  $e(w)$ ,  $E_W(W, H)$  and  $E_H(W, H)$ . The scaled investors' value  $e(w)$  is increasing and concave in  $w$  because investors are short in the incentive fee induced call option, whose value is convex in the AUM  $W$ . The mid-panel shows that  $E_W(W, H)$  is decreasing and hence implies a concave  $E(W, H)$  in  $W$ . The right panel shows that the higher the HWM  $H$ , the less valuable the incentive fees and as a result, the higher the investors' value  $E_H(W, H) > 0$ . The lower row of Figure 6 plots the scaled total fund value  $v(w)$ ,  $V_W(W, H)$ , and  $V_H(W, H)$ . While investors' value  $e(w)$  is concave and the manager's value  $f(w)$  is convex, the sum of the two,  $v(w) = f(w) + e(w)$ , is concave. The net effect of paying the manager incentive fees encourages excessive risk taking which in turn lowers the total fund value. Therefore, the total fund is short of a risk-seeking option *ex post* and hence its *ex ante* value is concave in the AUM  $W$ . Finally, the total fund value  $V(W, H)$  increases with the HWM  $H$  because the gain to investors from a higher  $H$  (an increase in  $E(W, H)$ ) outweighs the corresponding loss to the manager (decreases in  $N(W, H)$  and  $F(W, H)$ ).

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<sup>15</sup>There is no payout from the fund to investors other than exogenous liquidation of the fund which stochastically arrives at a constant rate.

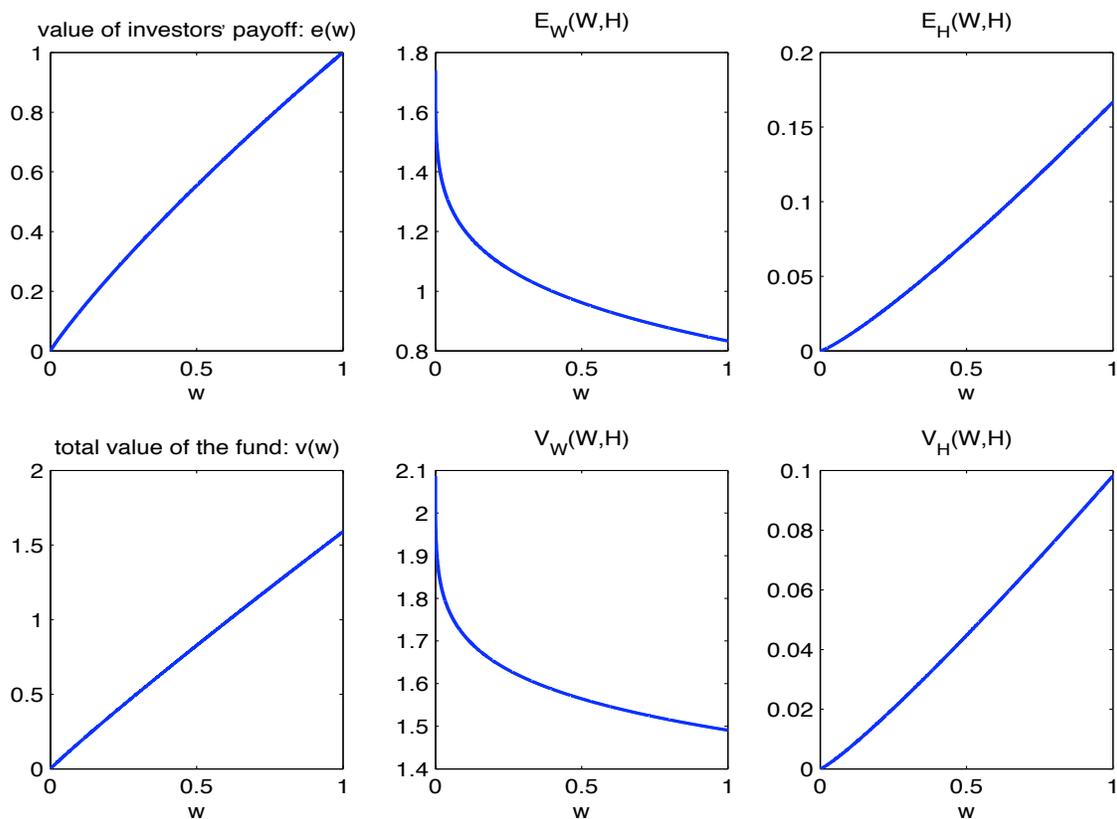


Figure 6: The scaled investors' value  $e(w)$ , the scaled total fund value  $v(w)$ , and sensitivities of investors' value  $E(W,H)$  and the total fund value  $V(W,H)$  with respect to the AUM  $W$  and the HWM  $H$ .

## 6 Extensions: Liquidation and managerial ownership

In order to focus on economic intuition, we have so far intentionally kept our baseline model parsimonious. However, there are two important features that we often observe in the real world hedge fund contracts that are missing in the baseline model. One key simplification in the baseline model is that the fund has no performance-triggered withdrawal. In our baseline model, the manager still manages the fund even after losing 99% of the AUM from the peak performance. In reality, when the fund performance deteriorates sufficiently (for example, if the fund loses 50% or more of its AUM from its running maximum, i.e. the HWM), investors are likely to withdraw from and sometimes even liquidate the fund.

We will show that investors' option to liquidate or withdraw from the fund significantly

changes managerial leverage and also has important quantitative implications on the valuation of fees. We model liquidation by allowing investors to liquidate the fund if the manager performs sufficiently poorly. Specifically, we assume that the fund is liquidated whenever the AUM  $W$  falls by a fixed fraction  $(1 - b)$  from the fund's HWM  $H$ , i.e. the fund's running maximum/best record in the simple example. GIR makes the same assumption on the liquidation boundary but the AUM evolution in their paper is exogenously given.

The manager is now potentially averse to performance-linked withdrawal/liquidation because the manager loses both management and incentive fees if the fund is liquidated.<sup>16</sup> The manager rationally manages the fund's leverage to influence the AUM evolution and hence the liquidation likelihood. As we will show, the manager will significantly delever the fund when it is sufficiently close to the liquidation boundary in order to preserve the fund as a going-concern and collect future management and incentive fees.

One way to mitigate managerial agency is to require managers to have their own skins in the game. In equilibrium, rational investors take into account managerial ownership in the fund when making investment decisions. We next extend our baseline model of Section 2 to incorporate both performance-based liquidation and managerial ownership.

## 6.1 A generalized setup and solution

Let  $\phi$  denote the managerial ownership in the fund. For simplicity, we assume that  $\phi$  is constant over time. Let  $Q(W, H)$  denote the manager's total value, which includes both the value of total fees  $F(W, H)$  and  $\phi E(W, H)$ , the manager's share as an investor

$$Q(W, H) = F(W, H) + \phi E(W, H). \quad (40)$$

The manager dynamically chooses the investment policy  $\pi(w)$  to maximize (40). The following theorem gives the optimal  $\pi(w)$  and the scaled manager's total value  $q(w) = f(w) + \phi e(w)$ .

**Theorem 2** *The manager's scaled total value  $q(w)$  solves the following ODE*

$$(r + \lambda - g + \delta)q(w) = [c + \phi(\delta + \lambda)]w + \mu_w(w)q'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2q''(w), \quad (41)$$

where  $\mu_w(w)$  is given by (29). The ODE (41) is solved subject to the boundary conditions

$$q(b) = \phi b, \quad (42)$$

$$q(1) = (k + 1)q'(1) - k. \quad (43)$$

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<sup>16</sup>See Dai and Sundaresan (2010) on this point in a model without the high-water mark and incentive fees.

With the alpha-generating technology (18), the optimal investment strategy  $\pi(w)$  is given by

$$\pi(w) = \frac{\pi^*}{1 + \sigma^2 \psi_q(w)/(2\theta_1)}, \quad (44)$$

where  $\pi^* = \theta_0/(2\theta_1)$  is the first-best leverage ratio and  $\psi_q(w)$  is the manager's effective risk aversion defined by

$$\psi_q(w) = -\frac{wq''(w)}{q'(w)}. \quad (45)$$

There are two key differences between Theorem 2 and Theorem 1: concentrated managerial ownership  $\phi$  in the fund and the lower liquidation boundary  $b$ . The cash flow to the manager (prior to liquidation) in the region  $b \leq w < 1$  is given by  $(c + \phi\delta)w$ . With probability  $\lambda$  per unit of time, the fund is exogenously liquidated. Therefore, the expected cash flow per unit of time (including exogenous liquidation risk) is  $(c + \phi(\delta + \lambda))w$ . When the fund's performance is sufficiently poor, reaching the lower liquidation boundary  $b$ , the manager loses all fees but collects *pro rata* share of the liquidating AUM:  $q(b) = \phi b$ .

Requiring the risk-neutral manager to have skin in the game via equity ownership improves the alignment between managerial and investors' interests. As in the baseline model, the manager's optimal investment strategy  $\pi(w)$  depends on the manager's effective risk attitude. When calculating the manager's effective risk aversion  $\psi_q(w)$ , we need to use the manager's total value  $q(w)$ , which includes both managerial *pro rata* share of the fund  $\phi e(w)$  and the PV of total fees  $f(w)$ . The next proposition summarizes the effects of a liquidation boundary and managerial ownership on dynamic valuation.

**Proposition 3** *The scaled value functions  $m(w)$ ,  $n(w)$ , and  $e(w)$  solve (30), (31), and (32), subject to the following lower boundary conditions*

$$m(b) = n(b) = 0, \quad e(b) = b. \quad (46)$$

*The upper boundary conditions at  $w = 1$  remain the same as in Proposition 2, i.e. conditions (34), (35), and (36), respectively. The total fund's value is  $V(W, H) = v(w)H$ , where  $v(w) = m(w) + n(w) + e(w) = f(w) + e(w)$ . The manager's scale value is  $q(w) = f(w) + \phi e(w)$ .*

## 6.2 Performance-triggered liquidation

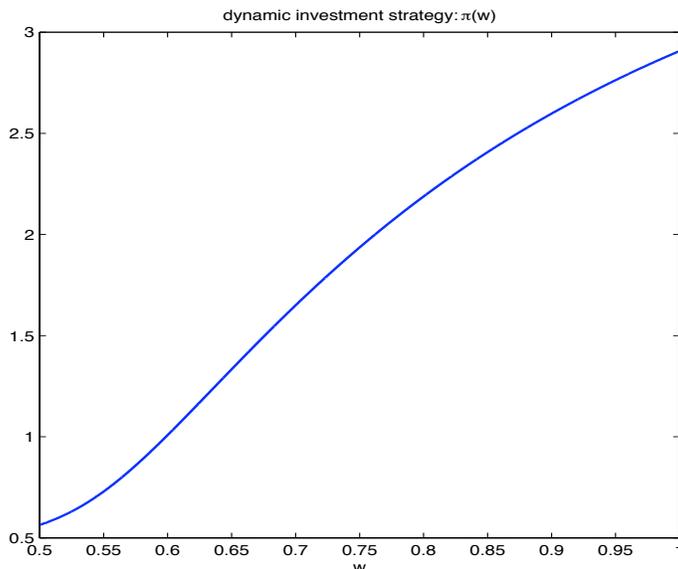


Figure 7: **Dynamic investment strategy  $\pi(w)$ : The case with liquidation boundary  $b = 0.5$  and no managerial ownership ( $\phi = 0$ ).** The managerial compensation contract  $(c, k)$  is set at (2.48%, 20%) to ensure that investors break even:  $e(1) = 1$ . Other parameter values are:  $r = 5\%$ ,  $\delta = 5\%$ ,  $\lambda = 5\%$ ,  $g = 5\%$ ,  $\theta_0 = 2.5\%$ ,  $\theta_1 = 0.25\%$ , and  $\sigma = 9\%$ .

**Dynamic investment strategy  $\pi(w)$  under a liquidation boundary  $b > 0$ .** Figure 7 plots the manager's optimal leverage  $\pi(w)$  when the liquidation boundary is set at  $b = 0.5$  and no managerial ownership ( $\phi = 0$ ). A new equilibrium compensation contract which ensures  $e(1) = 1$ , i.e. investors break even, is given by  $c = 2.48\%$  and  $k = 20\%$ . All other parameter values remain unchanged.

Recall that in our baseline model (with  $c = 2\%$  and  $k = 20\%$ ) and no liquidation boundary ( $b = 0$ ), leverage is between 5 and 6.33 (See Figure 1). With the liquidation boundary ( $b = 0.5$ ), the overall leverage is much lower due to the manager's aversion to liquidation. For sufficiently high  $w$  (i.e.  $w \geq 0.6$ ), the manager takes on leverage ( $\pi > 1$ ). However, the leverage level is much lower with liquidation boundary  $b = 0.5$  than without ( $b = 0$ ). For example, at  $w = 1$ , leverage is  $\pi(1) = 2.91$  when  $b = 0.5$  compared to  $\pi(1) = 6.33$  when  $b = 0$ .

When  $w$  is sufficiently close to the liquidation boundary  $b = 0.5$  (i.e.  $0.5 < w < 0.6$ ), fighting for survival is the manager's primary concern. The manager rationally allocates

some of the fund’s AUM to the risk-free asset ( $\pi(w) < 1$ ) by hoarding cash in order to lower the volatility of the AUM return even though the risky investment strategy has a positive alpha. That is, unlike the case without liquidation ( $b = 0$ ), the optimal investment strategy  $\pi(w)$  can sometimes be less than unity. The manager is not only long an incentive-fees-induced call option but also short a liquidation option. This liquidation option is closer to being in the money following sufficiently poor performance, i.e.  $w$  drifts leftward toward the liquidation boundary  $b = 0.5$ . The short position in the liquidation option induces the manager to behave in an effectively risk averse manner and hence de-leveraging is often optimal. Casual observations seem consistent with this prediction. Fund managers tend to reduce leverage in bad times following poor performances.

Liquidation is costly because the manager loses future rents from the alpha generating technology.<sup>17</sup> By reducing exposure to the risky positive alpha technology, the manager gives up some current and near-term fees in exchange for increasing their long term survival probability. Dai and Sundaresan (2010) also show that hedge funds may use conservative levels of leverage if they properly recognize the short option positions due to contractual arrangements with investors and prime brokers.

**Managerial rents  $f(w)$  and risk attitude  $\psi(w)$ .** The left panel of Figure 8 plots  $f(w)$  when the liquidation boundary is set at  $b = 0.5$ . Compared with the case of  $b = 0$ , the PV of total fees  $f(w)$  is substantially lower. Liquidation shortens the duration of the fund’s life and also reduces the manager’s incentive to take on risk, both of which lower  $f(w)$ . With  $b = 0.5$ , the manager makes a rent of 29.6 cents for each dollar in AUM  $f(1) = 0.296$ , which is substantially lower than a rent of 58.9 cents per dollar of AUM  $f(1) = 0.589$  when  $b = 0$ . Note that this comparison has already accounted for the adjustment of managerial compensation from  $c = 2\%$  for  $b = 0$  to  $c = 2.48\%$  for  $b = 0.5$  while keeping  $k = 20\%$  so that investors break even under perfectly competitive capital markets,  $e(1) = 1$ .

The right panel of Figure 8 plots the manager’s effective risk aversion  $\psi(w)$  for the case with  $b = 0.5$ . The lower liquidation boundary turns the risk-neutral manager from behaving in a risk-seeking way ( $\psi(w) < 0$  for  $b = 0$ ) to behaving in an effectively risk-averse way ( $\psi(w) > 0$  when  $b = 0.5$ ). The manager’s “effective” risk aversion at the liquidation

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<sup>17</sup>We do not consider the manager’s option to start a new fund. While costly, it is potentially an important consideration for the manager. We leave the modeling for future research.

boundary  $b = 0.5$  is  $\psi(0.5) = 4.91$ , which is quantitatively significant. Even when incentive fees are close to being in the money (when  $w = 1$ ), the manager continues to behave in an effectively risk-averse manner with an effective risk aversion:  $\psi(1) = 0.45 > 0$ .

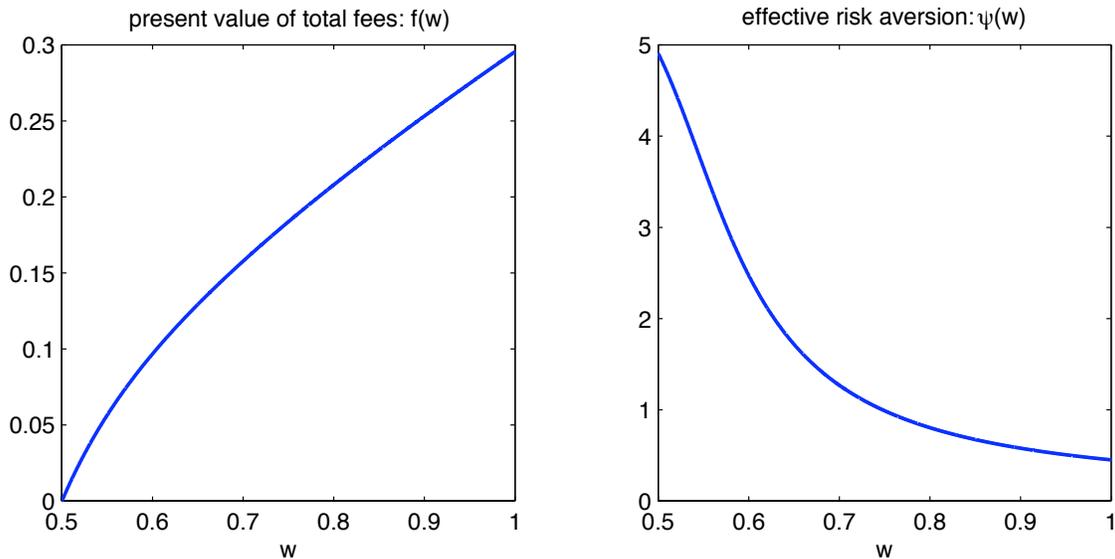


Figure 8: **Managerial rents  $f(w)$  and the manager’s effective risk aversion,  $\psi(w) = -wf''(w)/f'(w)$ : The case with liquidation boundary  $b = 0.5$  and no managerial ownership ( $\phi = 0$ ).** Increasing the liquidation boundary from  $b = 0$  to  $b = 0.5$  substantially lowers the PV of total fees  $f(w)$  and makes the manager effectively risk averse ( $\psi(w) > 0$ ).

Our results challenge the conventional wisdom that high-powered incentive fees often encourage excessive risk taking. In isolation, the high-powered incentives encourage excessive risk taking as we have shown in the baseline model with  $b = 0$ . However, other frictions, such as the liquidation option held by investors, fundamentally change the managerial incentive to take on leverage. Indeed, the manager sometimes behaves too conservatively from the investors’ perspective. Compared with the first-best leverage  $\pi^* = 5$ , the fund is underlevered for all  $w$  when  $b = 0.5$ . This provides one explanation why sometimes we see that investors require a lower bound for the fund’s leverage. Managers may sometimes choose excessively low leverage primarily for survival and fee collection. Next, we characterize various value functions and provide some quantitative results from this numerical example.

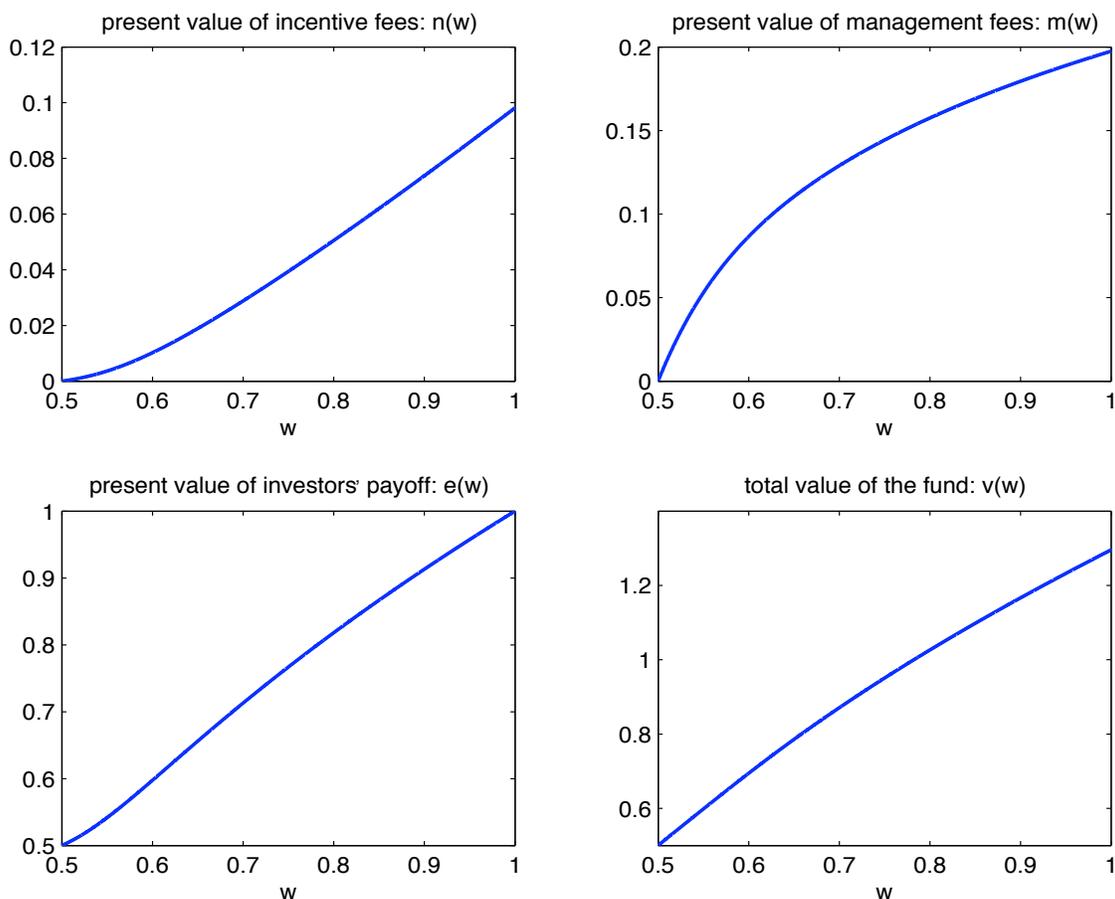


Figure 9: **The value of incentive fees  $n(w)$ , the value of management fees  $m(w)$ , investors' value  $e(w)$ , and the total fund value  $v(w)$ : The case with lower liquidation boundary  $b = 0.5$ .**

**Various value functions:  $n(w)$ ,  $m(w)$ ,  $e(w)$ , and  $v(w)$ .** Figure 9 plots the PV of incentive fees  $n(w)$ , the PV of management fees  $m(w)$ , investors' value  $e(w)$ , and the total fund value  $v(w)$  for the case with  $b = 0.5$  and no managerial ownership ( $\phi = 0$ ). Intuitively, we find the PV of management fees  $m(w)$  is increasing and concave, and the PV of incentive fees  $n(w)$  is increasing and convex. These results are consistent with those in Section 5 for the baseline model with  $b = 0$ .

Quantitatively, the liquidation option significantly changes the economics of managerial fees and valuation. For the baseline model with  $b = 0$ , in our numerical example, the PV of incentive fees  $n(1) = 0.39$ , which is approximately twice as much as the PV of management fees  $m(1) = 0.2$ . Increasing  $b = 0$  to  $b = 0.5$ , the value of incentive fees drops to  $n(1) = 0.1$

which is about half of the value of management fees  $m(1) = 0.2$ . Therefore, increasing the liquidation boundary from  $b = 0$  to  $b = 0.5$  not only lowers the PV of total fees from  $f(1) = 0.39 + 0.2 = 0.59$  to  $f(1) = 0.1 + 0.2 = 0.3$  by about half, it also changes the relative weights of incentive fees to management fees from 2:1 to 1:2. With liquidation boundary  $b = 0.5$ , the manager acts much more conservatively with leverage and management fees become the larger portion of the total fees.

Unlike in the baseline model with  $b = 0$ , the investors' value  $e(w)$  is no longer globally concave. It is convex in  $w$  for sufficiently low  $w$  ( $0.5 \leq w \leq 0.617$ ) and concave in  $w$  for sufficiently high  $w$  ( $0.617 \leq w \leq 1$ ). The intuition is as follows. For any given  $w$ , the investors are simultaneously short in the incentive call option and long the liquidation option, in addition to the unlevered "equity" claim in the fund's after-fees cash flow process. For lower values of  $w$ , the long position of the liquidation option is quantitatively more important than the short position of the incentive option. The net effect is that  $e(w)$  is convex in  $w$  for low values of  $w$ . For higher values of  $w$ , the opposite holds and hence the net effect is that  $e(w)$  is concave in  $w$  at the right end of  $w$ . The high non-linearity of  $e(w)$  reflects various option features embedded in the hedge fund management compensation contracts. The lower left panel of Figure 9 depicts  $e(w)$  which is convex in  $w$  for  $0.5 \leq w \leq 0.617$  and concave in  $w$  for  $0.617 \leq w \leq 1$ . Finally, the total fund's value  $v(w)$  is increasing and concave. The concavity follows from the net effects of total fees and investors' value.

To summarize, the liquidation boundary makes the manager's investment strategy much more conservative because preserving the fund as a going-concern is valuable for the manager. Management fees carry more weight when the liquidation option becomes more valuable for investors. Liquidation boundary can significantly lower the value of total fees by truncating the fund's horizon and also substantially lowering the managerial incentive to take on leverage. We next analyze the effects of managerial ownership.

### 6.3 Managerial ownership and liquidation threshold

Table 1 reports the investment strategy  $\pi(1)$ , the PV of management fees  $m(1)$ , the PV of incentive fees  $n(1)$ , the PV of total fees  $f(1)$ , the investor's payoff  $e(1)$ , and the total value of the fund  $v(1)$  under various levels of managerial ownership  $\phi$  and liquidation boundaries

Table 1: **The effects of managerial ownership and the liquidation boundary on fees, leverage, and valuation at  $w = 1$ .**

This table reports the investment strategy  $\pi(1)$ , the PV of the management fees  $m(1)$ , the PV of incentive fees  $n(1)$ , the PV of total fees  $f(1)$ , the investor's payoff  $e(1)$  and the total value of the fund  $v(1)$  under various levels of managerial ownership  $\phi$  and liquidation boundaries  $b$ . The compensation contract is fixed at  $c = 2.61\%$  and  $k = 20\%$ . Other parameter values are:  $r = 5\%$ ,  $\delta = 5\%$ ,  $\lambda = 5\%$ ,  $g = 5\%$ ,  $\theta_0 = 2.5\%$ ,  $\theta_1 = 0.25\%$ , and  $\sigma = 9\%$ .

	$\pi(1)$	$m(1)$	$n(1)$	$f(1)$	$e(1)$	$v(1)$
<u>Managerial ownership <math>\phi</math></u>	Panel A. Liquidation boundary $b = 0.5$					
0	2.8381	0.2055	0.0947	0.3002	0.9906	1.2908
0.05	2.7646	0.2046	0.0955	0.3001	0.9958	1.2958
<b>0.10</b>	<b>2.7069</b>	<b>0.2035</b>	<b>0.0963</b>	<b>0.2998</b>	<b>1.0000</b>	<b>1.2998</b>
0.25	2.5921	0.2002	0.0981	0.2983	1.0090	1.3072
0.50	2.4971	0.1953	0.1002	0.2956	1.0165	1.3121
<u>Liquidation boundary <math>b</math></u>	Panel B. Managerial ownership $\phi = 0.1$					
0	5.7732	0.2642	0.3452	0.6095	1.0116	1.6210
0.25	4.7106	0.2407	0.2262	0.4669	1.0504	1.5173
<b>0.50</b>	<b>2.7069</b>	<b>0.2035</b>	<b>0.0963</b>	<b>0.2998</b>	<b>1.0000</b>	<b>1.2998</b>
0.75	0.3044	0.1742	0.0083	0.1826	0.9071	1.0897

$b$ . The managerial compensation contract is fixed at  $c = 2.61\%$  and  $k = 20\%$  for all cases.<sup>18</sup> Other parameters are the same as in Section 5.

Panel A of Table 1 reports the results at  $w = 1$  (e.g. when the fund is started) for various levels of managerial ownership  $\phi$  with  $b = 0.5$ . Recall that when  $\phi = 10\%$ , investors break even (i.e.  $e(1) = 1$ ) under the given compensation contract  $c = 2.61\%$  and  $k = 20\%$ . The third row highlighted in bold summarizes the results for this base case: the optimal leverage  $\pi(1) = 2.7$ , the PV of total fees is about 30% of the AUM, i.e.  $f(1) = 0.3$ , out of which management fees and incentive fees account for two-thirds and one-third, respectively:  $m(1) = 0.2$  and  $n(1) = 0.1$ .

As we increase managerial ownership  $\phi$  from 0 to 50%, leverage  $\pi(w)$  falls from 2.8 to

<sup>18</sup>We use the condition that capital markets are perfectly competitive and investors break even (i.e.  $e(1) = 1$ ) for the case with liquidation boundary  $b = 0.5$  and managerial ownership  $\phi = 0.1$  to pin down the management fee at  $c = 2.61\%$ . We set  $k = 20\%$  for this calculation.

2.5.<sup>19</sup> The more concentrated the managerial ownership  $\phi$ , the less the fund will be levered. The value of incentive fees  $n(1)$  thus also falls with ownership  $\phi$ , but the value of management fees  $m(1)$ , the value of investors' payoff  $e(1)$ , and the total value of the fund  $v(1)$  all increase.

Next, we turn to the impact of liquidation boundary  $b$ . Panel B of Table 1 reports the results for various levels of  $b$  with  $\phi = 0.1$ . As we increase the liquidation boundary  $b$  from 0 to 0.5, the manager significantly decreases leverage  $\pi(1)$  from 5.77 to 2.71. The PV of total fees  $f(1)$ , the PV of incentive fees  $n(1)$ , and the PV of management fees  $m(1)$  all decrease with the liquidation boundary  $b$  because liquidation shortens the fund's horizon and discourages leverage. Comparing the case of  $b = 0$  with that of  $b = 0.5$ , the PV of total fees  $f(1)$  drops by about half from 0.61 to 0.30, while the PV of incentive fees  $n(1)$  drops from 0.35 to 0.1 by more than 70% and the PV of management fees  $m(1)$  decreases from 0.26 to 0.20 by 23%. The relative weights of management and incentive fees also significantly change with the liquidation boundary  $b$ . For example, while incentive fees account for 57% of  $f(1)$  with  $b = 0$ , incentive fees only account for less than 33% of  $f(1)$  with  $b = 0.50$ .

With sufficiently high liquidation risk (e.g.  $b = 0.75$ ), the manager optimally places a significant fraction of the AUM, about 70%, in cash to earn the risk-free rate:  $\pi(1) = 0.3$ . The manager's strong desire for the fund's survival dominates the dynamic investment decision. As a result, the total fund's value  $v(1)$  is significantly lower, only about 1.09, which implies that the fund is about 9 percent more valuable than its AUM  $W$ . For the manager, the PV of total fees  $f(1) = 0.18$ , out of which the vast majority is from management fees:  $m(1) = 0.17$  leaving only the value of incentive fees being  $n(1) = 0.01$ .

The preceding analyses fix the managerial compensation contract at  $c = 2.61\%$  and  $k = 20\%$ , keep all other parameter values unchanged, and focus on the comparative effects of managerial ownership  $\phi$  and liquidation boundary  $b$  on leverage and valuation. However, it is important to note that the surplus created from more concentrated managerial ownership and/or better incentive alignments is not going to be captured by investors but rather by the manager under perfectly competitive capital markets,  $e(1) = 1$ . In this case, the compensation contract will adjust to reflect the equilibrium outcome where the manager collects all the surplus.<sup>20</sup>

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<sup>19</sup>Note that in this exercise, as we vary managerial ownership  $\phi$ , we no longer impose the restriction that  $e(1) = 1$ . We are focusing on the impact of managerial ownership  $\phi$  and later liquidation boundary  $b$  on leverage and valuation for a given fixed compensation contract.

<sup>20</sup>Due to space constraints, we leave out the details of the analysis when we adjust the equilibrium com-

## 6.4 A special case: Constant un-levered alpha ( $\theta_1 = 0$ )

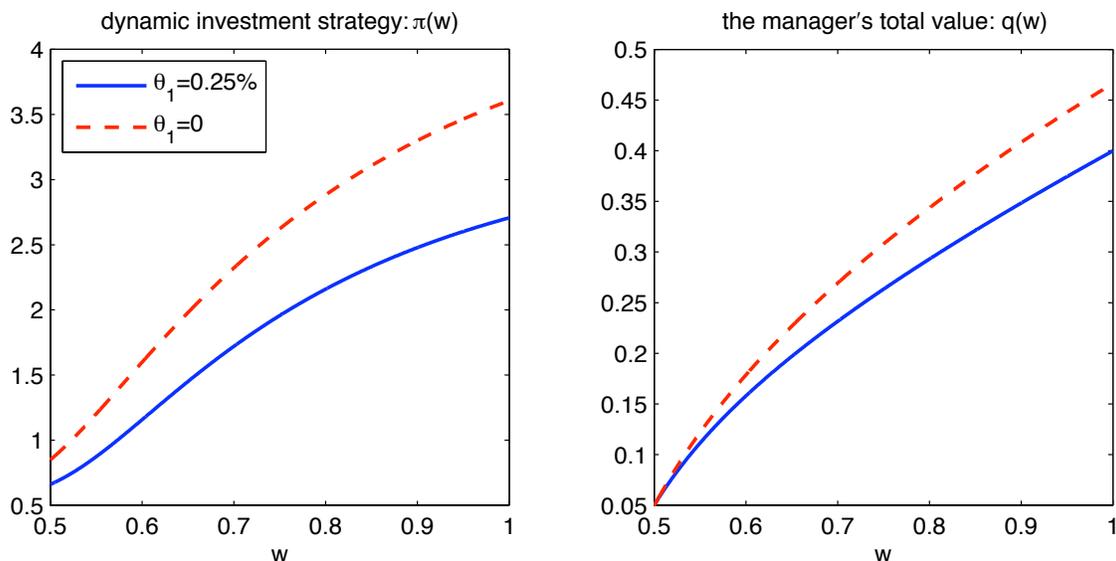


Figure 10: **Dynamic leverage  $\pi(w)$  and the the manager's total value  $q(w) = f(w) + \phi e(w)$  with and without price pressure:  $\theta_1 = 0.25\%, 0$ .**

Throughout the paper, we have mainly considered the cases where  $\pi$  lowers the expected excess returns in the un-levered alpha strategy ( $\alpha'(\pi) < 0$  and  $\theta_1 > 0$  for the alpha-generating strategy (18)). In Section 3, we have noted that for the benchmark with management fees only,  $\alpha(\pi) < 0$  is necessary to ensure convergence because the manager is risk neutral and there are no other constraints on managerial decision making.

However, for the general case where investors can liquidate the fund ( $b > 0$ ), the convergence condition also holds even when  $\alpha'(\pi) = 0$ , say under a constant alpha generating strategy:  $\alpha(\pi) = \theta_0$ . Figure 10 plots dynamic leverage  $\pi(w)$  and the manager's total value  $q(w) = f(w) + \phi e(w)$ , including both the PV of fees and *pro rata* share of fund equity, for two levels of  $\theta_1$ : 0.25% and 0. The former is the case analyzed earlier and the latter corresponds to the case with a constant unlevered alpha  $\theta_0 = 2.5\%$ . Without negative price pressure ( $\theta_1 = 0$ ), leverage  $\pi(w)$  is higher and the manager's value  $q(w)$  is also higher.

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pensation contract. The details are available upon request.

## 7 The value of dynamically adjusting leverage

Leverage in our model creates value because the manager has skills and there is an optimal scale of investments. We can decompose the PV of managerial rents  $f(w)$  and the total fund value  $v(w)$  into an unlevered component and the net increase in value due to the use of leverage. First, we summarize the results in the benchmark case where the manager takes no leverage, i.e. a pure valuation model as in GIR.

### 7.1 A pure valuation model ( $\pi(w) = 1$ ): GIR

GIR incorporates the effects of the HWM on the valuation of management fees and incentive fees. But their model does not allow for managerial leverage decisions. Their model can thus be viewed as a special case of ours with  $\pi(w)$  fixed at unity for all  $w$ . The following theorem summarizes the main results in GIR.

**Theorem 3** *Fixing  $\pi(w) = 1$  at all times and for all  $w$ , we have the following closed-form solutions for various value functions*

$$n_{gir}(w) = \frac{k(w^\eta - b^{\eta-\zeta}w^\zeta)}{\eta(k+1) - 1 - b^{\eta-\zeta}(\zeta(1+k) - 1)}, \quad (47)$$

$$f_{gir}(w) = \frac{c}{c + \delta + \lambda - \alpha(1)}w + \frac{(\delta + \lambda - \alpha(1))k + (\zeta(1+k) - 1)cb^{1-\zeta}}{(c + \delta + \lambda - \alpha(1))(\eta(k+1) - 1 - b^{\eta-\zeta}(\zeta(1+k) - 1))}w^\eta - \frac{b^{\eta-\zeta}(\delta + \lambda - \alpha(1))k + (\eta(1+k) - 1)cb^{1-\zeta}}{(c + \delta + \lambda - \alpha(1))(\eta(k+1) - 1 - b^{\eta-\zeta}(\zeta(1+k) - 1))}w^\zeta, \quad (48)$$

$$e_{gir}(w) = \frac{\delta + \lambda}{c + \delta + \lambda - \alpha(1)}w - \frac{(\delta + \lambda)k + (\zeta(1+k) - 1)(c - \alpha(1))b^{1-\zeta}}{(c + \delta + \lambda - \alpha(1))(\eta(k+1) - 1 - b^{\eta-\zeta}(\zeta(1+k) - 1))}w^\eta + \frac{b^{\eta-\zeta}(\delta + \lambda)k + (\eta(1+k) - 1)(c - \alpha(1))b^{1-\zeta}}{(c + \delta + \lambda - \alpha(1))(\eta(k+1) - 1 - b^{\eta-\zeta}(\zeta(1+k) - 1))}w^\zeta. \quad (49)$$

where  $\alpha(1) = \theta_0 - \theta_1$ ,  $\eta$  and  $\zeta$  are given by

$$\eta = \frac{1}{2} - \frac{\alpha(1) + r - g - c}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\alpha(1) + r - g - c}{\sigma^2}\right)^2 + \frac{2(r + \lambda - g + \delta)}{\sigma^2}} > 1, \quad (50)$$

and

$$\zeta = \frac{1}{2} - \frac{\alpha(1) + r - g - c}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\alpha(1) + r - g - c}{\sigma^2}\right)^2 + \frac{2(r + \lambda - g + \delta)}{\sigma^2}} < 0. \quad (51)$$

In addition, we have  $m_{gir}(w) = f_{gir}(w) - n_{gir}(w)$  and  $v_{gir}(w) = e_{gir}(w) + f_{gir}(w)$ .

Table 2: **The value of leverage for various levels of managerial ownership  $\phi$  and liquidation boundaries  $b$  at  $w = 1$ .**

This table reports the PV of total fees  $f_{gir}(1)$  and the total fund value  $v_{gir}(1)$  in the GIR benchmark and then quantifies the NPV of leverage by calculating  $\Delta f(1) = f(1) - f_{gir}(1)$  and  $\Delta v(1) = v(1) - v_{gir}(1)$  for various levels of managerial ownership  $\phi$  and liquidation boundaries  $b$ . The managerial compensation contract is fixed at  $c = 2.61\%$  and  $k = 20\%$  for all cases. Other parameter values are:  $r = 5\%$ ,  $\delta = 5\%$ ,  $\lambda = 5\%$ ,  $g = 5\%$ ,  $\theta_0 = 2.5\%$ ,  $\theta_1 = 0.25\%$ , and  $\sigma = 9\%$ .

Panel A. Liquidation boundary $b = 0.5$					
Managerial ownership	GIR Benchmark		NPV of Leverage		
$\phi$	$f_{gir}(1)$	$v_{gir}(1)$	$\Delta f(1)$	$\Delta v(1)$	
0	0.1459	0.8580	0.1543	0.4328	
0.05	0.1459	0.8580	0.1542	0.4378	
<b>0.10</b>	<b>0.1459</b>	<b>0.8580</b>	<b>0.1539</b>	<b>0.4418</b>	
0.25	0.1459	0.8580	0.1524	0.4492	
0.50	0.1459	0.8580	0.1497	0.4541	

Panel B. Managerial ownership $\phi = 0.1$					
Liquidation boundary	GIR Benchmark		NPV of Leverage		
$b$	$f_{gir}(1)$	$v_{gir}(1)$	$\Delta f(1)$	$\Delta v(1)$	
0	0.1792	0.8230	0.4303	0.7981	
0.25	0.1735	0.8289	0.2934	0.6884	
<b>0.50</b>	<b>0.1459</b>	<b>0.8580</b>	<b>0.1539</b>	<b>0.4418</b>	
0.75	0.0850	0.9220	0.0976	0.1677	

In our model, the manager creates value in two ways: the alpha-generating technology and the use of leverage. Note that M&M does not hold in our model. Using the GIR as the benchmark, we can quantify the value purely due to leverage.

## 7.2 The value of leverage

Let  $\Delta f(w) = f(w) - f_{gir}(w)$  denote the difference between the PV of total fees  $f(w)$  in our model and  $f_{gir}(w)$ , the PV of total fees in GIR. Similarly, let  $\Delta v(w) = v(w) - v_{gir}(w)$  denote the difference between the PV of total fund's value  $v(w)$  in our model and  $v_{gir}(w)$ , the PV

of total fund's value in GIR. Table 2 is based on the same set of calculations as Table 1.

Panel A of Table 2 calculates the net present value (NPV) of leverage for various levels of ownership  $\phi$  with the liquidation boundary fixed at  $b = 0.5$ . First, by construction,  $f_{gir}(1)$  and  $v_{gir}(1)$  in the GIR benchmark are independent of managerial ownership  $\phi$ ,  $f_{gir}(1) = 0.15$  and  $v_{gir}(1) = 0.86$  for all  $\phi$ . Second, incentive alignments between the manager and investors improve with managerial ownership  $\phi$ . As a result, the net gain due to leverage for the total fund value,  $\Delta v(1)$ , increases with ownership  $\phi$ . However, the net gain due to leverage for the value of total fees,  $\Delta f(1)$ , decreases.

Panel B of Table 2 calculates the NPV of leverage for various levels of liquidation boundary  $b$  with managerial ownership fixed at  $\phi = 0.1$ . For the GIR benchmark,  $f_{gir}(1)$  decreases with liquidation boundary  $b$ . The manager becomes much more prudent in choosing leverage due to concerns of losing future fees, and as a result, the NPV of leverage for the manager  $\Delta f(1)$  drops significantly from 0.43 to 0.098 when the liquidation boundary  $b$  increases from 0 to 0.75. Similarly, the added value from leverage for the total fund,  $\Delta v(1)$ , also decreases significantly, from 0.80 to 0.17 as we increase  $b$  from 0 to 0.75.

## 8 Conclusions

Hedge fund managers are paid via both management fees and high-powered incentive fees. Management fees are often specified as a fraction of the assets under management (AUM), and incentive fees are indexed to the stochastic and endogenously evolving high-water mark (HWM), which is effectively the running maximum of the AUM. While this compensation structure potentially rewards managerial talent, it also invites agency conflicts between the manager and investors. Incentive fees provide the manager a sequence of call options on the AUM and hence encourage the manager to take on excessive risk. However, management fees provide an unlevered equity-like financial claim in the fund and thus mitigate the managerial risk seeking induced by incentive fees. The manager dynamically manages leverage to maximize the present value of total fees, and investors rationally anticipate dynamically changing managerial incentives and accordingly price their investments in the fund.

The two key state variables in our model are the AUM and the HWM. In our homogenous framework, the ratio between the AUM and the HWM,  $w$ , determines the fund's dynamic leverage decisions and various value functions including those for management and incentive

fees. As the fund's performance improves, i.e. as  $w$  increases, the manager increases leverage as incentive fees become deeper in the money. When the performance deteriorates, i.e. as  $w$  decreases, the manager reduces leverage as management fees become more important, *ceteris paribus*. This performance-dependent dynamic leverage strategy has important implications on valuing management fees and incentive fees. While both management fees and incentive fees contribute significantly to managerial compensation in general, they carry different weights in contributing to total managerial rents under different circumstances.

In our model, leverage contributes significantly to the present value of managerial rents and the total fund value because leverage allows the manager to capitalize on skills using a larger pool of capital in a non-M&M world. Granting investors options to liquidate the fund substantially curtails the manager's risk-taking incentives, effectively making the manager behave in a risk-averse manner. Having the manager co-invest in the fund as a limited partner also helps to align incentives and reduce agency conflicts.

In reality, managerial skills may be unknown and time-varying. Extending the model to allow investors to learn about time-varying unknown managerial skills is left for future research. Additionally, the recent crisis reveals that market liquidity and funding liquidity are first-order issues (Brunnermeier and Pedersen (2009)), and that funding costs can increase significantly in crisis periods. Anticipating these liquidity problems, the manager rationally chooses the fund's leverage and prudently manages risk in a state contingent way (Dai and Sundaresan (2010)). In future work, we plan to integrate market and funding illiquidity.

In order to make dynamic leverage and valuation results practical and operational, we have taken real-world hedge fund management compensation contracts as given and focus on the space of compensation contracts which feature management and incentive fees. Within the set of contracts featuring management and incentive fees, we are able to provide a direct link from managerial skills to managerial compensation contracts. For example, our model predicts that a more talented manager charges more. James Simons' Renaissance charges a management fee at 5% of the AUM and an incentive fee at 36%, while the industry standard charges management and incentive fees at 2% and 20%, respectively. An interesting direction for future research is to developing an industry equilibrium model of hedge funds with heterogenous managerial skills. We expect that in equilibrium, managers with better skills run larger funds and potentially charge higher fees.

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# Appendices

## A Technical details

**Proposition 1.** Without incentive fees, the manager chooses optimal investment  $\pi^*$  to maximize his value  $F^*(W)$  by solving the following HJB equation:

$$(r + \lambda)F^*(W) = \max_{\pi} cW + (\pi\alpha(\pi) + r - \delta - c)WF_W^*(W) + \frac{1}{2}\pi^2\sigma^2W^2F_{WW}^*(W). \quad (\text{A.1})$$

With no incentive fees, we can conjecture that  $F^*(W) = zW$ , where  $z$  is a constant to be determined. The FOC for leverage is given by

$$\alpha(\pi^*) + \pi^*\alpha'(\pi^*) = 0, \quad (\text{A.2})$$

Substituting the optimal investment leverage into the HJB equation (A.1), we have

$$F^*(W) = M^*(W) = \frac{c}{c + \delta + \lambda - \alpha^*}W, \quad (\text{A.3})$$

where  $\alpha^*$  is given in (13). We then obtain explicit linear value functions for  $E^*(W)$  and  $V^*(W)$  given in Proposition 1.

**Theorem 1.** Using the FOC for  $\pi$  in the HJB equation (20), we have

$$\pi = \frac{\alpha(\pi)}{-\sigma^2WF_{WW}(W, H)F_W^{-1}(W, H) - \alpha'(\pi)}. \quad (\text{A.4})$$

We have also verified the second-order condition. We conjecture that the value function  $F(W, H)$  takes the following homogeneous form in  $W$  and  $H$

$$F(W, H) = f(w)H. \quad (\text{A.5})$$

The above conjecture implies the following results:  $F_W(W, H) = f'(w)$ ,  $F_{WW}(W, H) = f''(w)/H$ , and  $F_H(W, H) = f(w) - wf'(w)$ . Substituting (A.4) into the HJB equation (20) and boundary conditions (22), we obtain (23)-(27) in Theorem 1.

**Proposition 2.** Applying the Ito's formula to (3) and (4), we obtain the following dynamics for  $w$

$$dw_t = d\frac{W_t}{H_t} = \frac{dW_t}{H_t} - \frac{W_t dH_t}{H_t^2} = \mu_w(w)dt + \pi(w_t)w_t\sigma dB_t - dJ_t, \quad (\text{A.6})$$

where  $\mu_w(w)$  is given in (29). Applying the standard differential equation pricing methodology to the present values  $M(W, H)$  given in (5),  $N(W, H)$  given in (6), and  $E(W, H)$  given in (8), we obtain (30), (31), and (32) for  $m(w)$ ,  $n(w)$ , and  $e(w)$ , respectively.

Now consider the boundary behavior at  $W_t = H_t$ . We use the same argument as the one for  $F(W, H)$ . Consider the scenario where the asset value increases by  $\Delta H$  over a small time interval  $\Delta t$ , the high-water mark is then re-set to  $H + \Delta H$ . Because value functions are continuous, we thus have

$$M(H + \Delta H, H) = M(H + \Delta H - k\Delta H, H + \Delta H), \quad (\text{A.7})$$

$$N(H + \Delta H, H) = k\Delta H + N(H + \Delta H - k\Delta H, H + \Delta H), \quad (\text{A.8})$$

$$E(H + \Delta H, H) = E(H + \Delta H - k\Delta H, H + \Delta H). \quad (\text{A.9})$$

Therefore, by taking limits, we have

$$kM_W = M_H, \quad kN_W = k + N_H, \quad kE_W = E_H. \quad (\text{A.10})$$

Similarly, using the property that value functions are homogeneous with degree one in  $W$  and  $H$ , we may write  $M(W, H) = m(w)H$ ,  $N(W, H) = n(w)H$ , and  $E(W, H) = e(w)H$ . Finally,  $W = 0$  is an absorbing state and therefore  $M(0, H) = N(0, H) = E(0, H) = 0$ . Substituting these results into valuation equations, we obtain results reported in Proposition 2.

**Theorem 3.** GIR is a special case of our model where the investment strategy fixed at  $\pi(w) = 1$  for all  $w$ . Using our earlier results, we calculate  $f(w)$  as follows

$$(r + \lambda - g + \delta)f(w) = cw + (\alpha(1) + r - g - c)wf'(w) + \frac{\sigma^2 w^2 f''(w)}{2}, \quad (\text{A.11})$$

where  $\alpha(1) = \theta_0 - \theta_1$ . The linear ODE (A.11) has the following closed-form solution

$$f(w) = xw + y_1 w^\eta + y_2 w^\zeta, \quad (\text{A.12})$$

where  $x$ ,  $y_1$ , and  $y_2$  are constant coefficients which we will determine next, and  $\eta$  and  $\zeta$  are given by (50) and (51), respectively. Note that  $\eta$  and  $\zeta$  are the positive and negative roots of the following quadratic equation, respectively

$$(r + \lambda - g + \delta) = (\alpha(1) + r - g - c)z + \frac{\sigma^2}{2}z(z - 1). \quad (\text{A.13})$$

To ensure convergence, we require  $c + \delta + \lambda - \alpha(1) > 0$ , i.e. equivalently  $\eta > 1$ . Using the boundary conditions  $f(b) = 0$  and  $f(1) = (k + 1)f'(1) - k$ , we obtain

$$x = \frac{c}{c + \delta + \lambda - \alpha(1)}, \quad (\text{A.14})$$

$$y_1 = \frac{(\delta + \lambda - \alpha(1))k + (\zeta(1 + k) - 1)cb^{1-\zeta}}{(c + \delta + \lambda - \alpha(1))(\eta(k + 1) - 1 - b^{\eta-\zeta}(\zeta(1 + k) - 1))}, \quad (\text{A.15})$$

$$y_2 = -\frac{b^{\eta-\zeta}(\delta + \lambda - \alpha(1))k + (\eta(1 + k) - 1)cb^{1-\zeta}}{(c + \delta + \lambda - \alpha(1))(\eta(k + 1) - 1 - b^{\eta-\zeta}(\zeta(1 + k) - 1))}. \quad (\text{A.16})$$

Closed-form expressions for other value functions can be similarly derived.