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DEBT, DEFICITS AND FINITE HORIZONS: THE STOCHASTIC CASE

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I. INTRODUCTION

For the past twenty years macroeconomists have used the Real Business Cycle model (RBC) to study stochastic fluctuations in aggregate economic activity. In that model, an infinitely lived family makes decisions for all subsequent generations. The model is elegant and simple and captures many of the features of real world business cycles but it has strong properties that follow from the representative agent assumption. Among them: 1) the real interest rate in the long run is pinned down by the representative agent's rate of time preference, 2) an expansionary fiscal policy in the form of a tax financed transfer has no first order effects on aggregate economic activity and 3) in a model of multiple infinitely lived agents with time separable preferences the income distribution is degenerate.

Paul Samuelson (1958) proposed an alternative 'overlapping generations' model (OG) in which a sequence of overlapping finitely lived agents trade with each other. In Samuelson's original paper there were three generations of agents. It has since been extended to multiple generations and has been used to study optimal fiscal policy (Diamond 1965), intergenerational transfers (Kotlikoff and Summers 1981) and social security policy (Imrohoroglu, Imrohoroglu and Joines 1999). Long lived versions of

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the model have been analyzed computationally by (Rios-Rull 1996) and the dynamics of the two-period version have been analyzed by David Gale (1973).

Olivier Blanchard (1985) and Philippe Weil (1989) have studied a model populated by a measure of long-lived agents that die, and are replaced, with a fixed probability that is independent of age. This perpetual youth model combines features of the representative agent model with the OG framework. It is a particularly simple model to study since its equilibria are described by a tractable set of differential equations.

The purpose of this paper is to extend Blanchard's analysis to the stochastic discrete time case by studying a pure trade version of the perpetual youth model with aggregate endowment shocks. We introduce a technique for solving the model in the presence of a complete set of securities and we show that it behaves much like the two-period model studied by David Gale (1973). Although our focus is on solution methods, our results should be of interest to researchers interested in analytic methods for studying the impact of fiscal policy in stochastic overlapping generations models both with and without production.

II. The Model

We assume that a new cohort of individuals is born each period. Agents die with fixed probability which is independent of age. This important assumption implies that all agents discount the future in the same way and it leads to a single concept of aggregate human wealth that greatly simplifies the structure of the set of competitive equilibria.

Each household survives into the subsequent period with a fixed probability π and every period a proportion $(1-\pi)$ of households dies. At the beginning of each period, households have n children. It follows that if N_t is the number of agents alive at date t then

$$N_{t+1} = (\pi + n)N_t \tag{1}$$

is the number of agents alive at date $t+1$. Depending on whether $\pi + n$ is greater or smaller than one, the total population will increase or decrease over time. We assume that $\pi + n > 1$ and we normalize the initial population to one, $N_0 = 1$.

The combination of birth and death processes implies that at any point in time there are $(\pi + n)^t$ agents alive of whom $\pi(\pi + n)^{t-1}$ are "old", i.e. survivors from the previous period $t-1$, and $n(\pi + n)^{t-1}$ are "young", i.e. newly born in period t.

The per-period utility function of the agents is logarithmic. For a typical agent i, utility at date t is given by the expression

$$U_t^i = log(c_t^i). \tag{2}$$

We consider an exchange economy with a single consumption commodity and stochastic endowments in which uncertainty unfolds in a sequence of periods. Uncertainty each period is indexed by a finite set of states $\mathbf{S} = \{S_1, \ldots, S_n\}$. Define the set of t-period histories $\mathbf{S}^t$ recursively as follows:

$$\begin{aligned} \mathbf{S}^1 &= \mathbf{S} \\ \mathbf{S}^t &= \mathbf{S}^{t-1} \times \mathbf{S}, \ t = 2, \ldots \end{aligned} \tag{3}$$

The households in this economy trade a complete set of Arrow securities. Let $Q_t^\tau(S^\tau)$ represent the price of the security that pays one unit of the consumption commodity if and only if history $S^\tau \in \mathbf{S}^\tau$ occurs at date τ. Using this notation $Q_t^{t+1}(S')$ is the price of an Arrow security. This is a claim, sold at date t, to one unit of the consumption good for delivery at date $t+1$ if and only if state S' occurs. Let the probability that S' occurs at date $t+1$ be given by $p(S')$ and assume that this probability is independent of time.

Each period t, the agents of household i receive an endowment $w_t^i(S)$ if state $S \in \mathbf{S}$ is realized. They purchase consumption commodities $c_t^i(S)$ and they accumulate a portfolio of the n securities $a_{t+1}^i(S')$, where there is one security for each of the values of S'.

Since the household may not survive into period $t+1$, we assume, as in Blanchard (1985), that there exists an actuarially fair annuities market. The existence of this market implies that the household pays price $\pi Q_t^{t+1}(S')$ for a claim to one unit of consumption in period $t+1$ if and only if state S' occurs *and* the household is alive. We assume that this security is issued by a competitive annuity sector that earns zero profit in equilibrium. If the household dies, its claim reverts to the company that issued the annuity. We will describe the balance sheet of this sector in Section IV.

Given our assumptions, the representative family born in period h faces the following sequence of budget constraints,

$$\sum_{S' \in \mathbf{S}} \pi Q_t^{t+1}(S') a_{t+1}^i(S') = a_t^i(S) + w_t^i(S) - c_t^i(S), \qquad t = h \ldots \infty, \tag{4}$$

together with the set of no-Ponzi scheme conditions,

$$\lim_{T \to \infty} \pi^{T-h} Q_h^T(S^T) a_h^i(S^T) \geq 0, \text{ for all } S^T \in \mathbf{S}^T, \tag{5}$$

one for every possible history that might occur. These constraints imply that the household must plan to remain solvent in every possible history. The term $a^i_{t+1}(S')$ is the quantity of security S' purchased for price $\pi Q^{t+1}_t(S')$ at date t. The terms $a^i_t(S)$ and $w^i_t(S)$ on the right side of (9) are respectively the sole security that has positive value at date t and the endowment received at date t if state S is realized. $c^i_t(S)$ is the household's purchase of consumption commodities.

The human wealth of household i is defined recursively by the equation,

$$h^i_t(S) = w^i_t(S) + \sum_{S' \in \mathbf{S}} \pi Q^{t+1}_t(S') h^i_{t+1}(S'). \tag{6}$$

By iterating this expression it follows that, as long as human wealth is finite,

$$h^i_t(S) = \sum_{\tau=t}^{\infty} \left[\pi^{\tau-t} \sum_{S^\tau \in \mathbf{S}^\tau} Q^\tau_t(S^\tau) w^i_\tau(S^\tau) \right]. \tag{7}$$

The assumption that human wealth is finite requires that

$$\lim_{T\to\infty} \pi^{T-h} Q^T_h(S^T) w^i_h(S^T) < \infty, \text{ for all } S^T \in \mathbf{S}^T. \tag{8}$$

In a representative agent model with no uncertainty and a constant growth rate of endowments, a condition like this implies that the interest rate must exceed the growth rate. This condition is weaker. In the perpetual youth model the household values future assets using the factor πQ^{t+1}_t instead of Q^{t+1}_t reflecting the fact that it may not survive into the subsequent period.[1]

Equations (4) – (8) can be combined to write a single budget constraint for the household,

$$\sum_{\tau=t}^{\infty} \left[\pi^{\tau-t} \sum_{S^\tau \in \mathbf{S}^\tau} Q^\tau_t(S^\tau) c^i_\tau(S^\tau) \right] \leq h^i_t(S) + a^i_t(S). \tag{9}$$

A representative family maximizes the following intertemporal stream of discounted utilities

$$\mathbf{E}_t \left\{ \sum_{\tau=t}^{+\infty} (\pi\beta)^{\tau-t} log(c^i_\tau(S^\tau)) \right\} \tag{10}$$

subject to the budget constraint (9), where $\beta \in (0,1)$ is the discount factor. The first order condition for this maximization program is given by the following set of first order conditions, one for each of the n states $S' \in \mathbf{S}$,

$$Q^{t+1}_t(S') c^i_{t+1}(S') = \beta c^i_t(S) p(S'). \tag{11}$$

[1]This fact implies that the perpetual youth model may display inefficient equilibria in which the interest rate is less than the growth rate.

Recall that $p(S')$ is the probability that state S' occurs. As is well-known, with a logarithmic utility function, the solution to the household's problem is given by the following policy function

$$c_t^i(S) = (1 - \beta\pi)\left(a_t^i(S) + h_t^i(S)\right). \tag{12}$$

This equation instructs the household, in the optimal plan, to consume a fixed fraction of wealth each period.

III. Defining Aggregate Variables

The equilibrium of the perpetual youth model is a complicated object. The economy contains an infinite number of agents indexed by date of birth. Each of these agents takes decisions based on the realization of uncertainty at the date he was born. As with all long-lived generations models, a complete description of the equilibrium requires that one keep track of the wealth distribution across agents. This is an infinite dimensional object.

Blanchard made two assumptions that simplify the description of equilibrium. First, preferences are logarithmic.[2] Second, all agents die with the same probability that is independent of age. These assumptions imply that consumption is linear in wealth and they allow one to derive a simple set of equations in the aggregate state variables that completely characterizes their behavior. Our contribution in this paper is to extend this idea to the case of aggregate uncertainty by assuming the existence of a complete set of Arrow securities.

To describe an equilibrium, our first task is to define a set of aggregate state variables. For this purpose we will divide the population into two groups that correspond roughly to the young and the old in a standard two-period overlapping generations model.

Let $\mathcal{A}_t$ be the index set of all agents that are alive at date t. Notice that a proportion π of these agents will survive into period $t+1$. Similarly, let $\mathcal{N}_{t+1}$ denote the set of newborns at period $t+1$. For any date $t+1$ and any variable x let x_t^i be the quantity of that variable held by household i and let x_t be the aggregate quantity. Then,

$$\sum_{i\in\mathcal{A}_t} x_{t+1}^i + \sum_{i\in\mathcal{N}_{t+1}} x_{t+1}^i = \sum_{i\in\mathcal{A}_{t+1}} x_{t+1}^i = x_{t+1}. \tag{13}$$

[2]This assumption can be extended to homothetic preferences with some additional algebra. We have not pursued that complication here since it considerably increases the complexity of the mathematics needed to characterize an equilibrium.

Equation (13) says that any aggregate variable x can be defined as the sum over groups of people in two different ways. We can add up x^i over everyone who was alive yesterday and add it to the sum over everyone who is born today. Or we can add up x^i over everyone who is alive today. Using this notation, define A_t, W_t, C_t and H_t as follows,

$$A_t = \sum_{i \in \mathcal{A}_t} a_t^i, \quad W_t = \sum_{i \in \mathcal{A}_t} w_t^i, \quad C_t = \sum_{i \in \mathcal{A}_t} c_t^i, \quad H_t = \sum_{i \in \mathcal{A}_t} h_t^i. \tag{14}$$

We also define each of these variables in per capita terms,

$$a_t = \tfrac{A_t}{N_t}, \quad w_t = \tfrac{W_t}{N_t}, \quad c_t = \tfrac{C_t}{N_t}, \quad h_t = \tfrac{H_t}{N_t}, \tag{15}$$

where N_t evolves according to Equation (1).

IV. Government, Life Insurance and Annuities

We assume the existence of a competitive annuities sector that issues Arrow securities as liabilities. Here, we describe how this sector operates. Each period, the annuities sector issues a set of n securities. Security S' is sold to the household sector for price $\pi Q_t^{t+1}(S')$. In addition to these assets, we assume the existence of a government that issues debt B_{t+1} in the form of pure discount bonds. B_{t+1} is a claim to B_{t+1} units of the consumption good at date $t+1$ in every state of nature. The assumption of no riskless arbitrage implies that a claim of this kind will sell for price Q_t^{t+1} in period t where,

$$Q_t^{t+1} = \sum_{S' \in \mathbf{S}} Q_t^{t+1}(S'). \tag{16}$$

We will study the case where B_{t+1} may be positive (the government is in debt) or negative (the government owns claims on the private sector. Note that debt of this kind is conceptually distinct from the 'money' of Samuelson's (1958) paper since it is denominated in units of consumption. It represents an indexed bond.

We will take the initial value of debt as given: It may be positive or negative. We will study policies in which the debt is rolled over from one period to the next with neither taxation nor government purchases of commodities. We leave these additional features for future work.

In period t, the annuities sector has assets equal to B_t and liabilities of $a_t^i(S)$. It issues a set of n Arrow securities. Security S' is sold to a household for price $\pi Q_t^{t+1}(S')$. The budget constraint of the sector implies that,

$$\sum_{S' \in \mathbf{S}} \left[\pi Q_t^{t+1}(S') \sum_{i \in \mathcal{A}_t} a_{t+1}^i(S') \right] = B_{t+1}. \tag{17}$$

In period $t+1$ the company will not make a payment to the fraction $1-\pi$ of households that die. It follows from the assumption of free entry in the annuities sector that profits will be competed away by selling an Arrow security at date t for price $\pi Q_t^{t+1}(S')$ to the household. This is our assumption of actuarially fair insurance.

V. Equilibrium Relationships Between Aggregate Variables

In this section we put together the behavioral relationships and our definitions to derive a set of equations that will hold, in equilibrium, between b_t, $Q_t^{t+1}(S')$, $w_t(S)$, $a_t(S)$ and $h_t(S)$. We begin with government policy.

The assumption that government rolls over its debt in each period implies that debt each period is described by the following difference equation,

$$B_{t+1}\sum_{S'} Q_t^{t+1}(S') = B_t. \tag{18}$$

Defining per capita debt b_t as B_t/N_t, it follows from Equations (1) and (18) that

$$(\pi+n)b_{t+1}\sum_{S'} Q_t^{t+1}(S') = b_t. \tag{19}$$

Next we turn to human wealth. We will derive a recursive expression for aggregate human wealth that is similar to Equation (6), the expression for individual human wealth. We arrive at this expression by summing Equation (6) over the set $\mathcal{A}_t$ of agents alive at date t. The derivation is in Appendix A.

$$h_t(S) = w_t(S) + \pi^2 \sum_{S'} Q_{t+1}(S')h_{t+1}(S'). \tag{20}$$

Now consider the relationship between per capita consumption and wealth. It follows from summing Equation (12) over all agents alive at date t that

$$c_t(S) = (1-\beta\pi)\left(b_t + h_t(S)\right). \tag{21}$$

Using the market clearing condition, this expression implies,

$$w_t(S) = (1-\beta\pi)\left(b_t + h_t(S)\right). \tag{22}$$

Finally, we seek an equation that describes the price of an Arrow security as a function of w_t and h_t. In a representative agent model we could find an expression for this price by taking the ratio of aggregate consumption at two different dates. Something similar will work here, but we need to account for changes in the set of agents over time. Appendix A shows how to derive the following expression for $Q_t^{t+1}(S')$.

$$Q_t^{t+1}(S') = \frac{\beta w_t(S)p(S')}{(\pi+n)w_{t+1}(S') - (1-\beta\pi)nh_{t+1}(S')}. \tag{23}$$

VI. Solving the Model

In this section we write down equations in two different state variables. Either of these representations can be used to characterize the properties of equilibria. Our first variable is the ratio of human wealth to the endowment. We call this $z_t(S)$ and we define it as,

$$z_t(S) = \frac{h_t(S)}{w_t(S)}. \tag{24}$$

To derive a dynamic expression in $z_t(S)$, we first combine Equations (20) and (23), to give,

$$h_t(S) = w_t(S) + \beta\pi^2 w_t(S) \sum_{S'} \frac{h_{t+1}(S')p(S')}{(\pi+n)w_{t+1}(S') - (1-\beta\pi)nh_{t+1}(S')}. \tag{25}$$

Dividing this expression through by $w_t(S)$ gives the expression we seek,

$$z_t(S) = +\beta\pi^2 \sum_{S'} \frac{z_{t+1}(S')p(S')}{\pi+n-(1-\beta\pi)nz_{t+1}(S')}. \tag{26}$$

We can also write the right hand side as an expectation,

$$z_t(S) = 1 + \beta\pi^2 \mathbf{E}_t \left[\frac{z_{t+1}(S')}{\pi+n-(1-\beta\pi)nz_{t+1}(S')} \right]. \tag{27}$$

Next, we turn to an equivalent expression to characterize equilibria using government debt as a state variable. Substituting equation (23) into equation (19), we obtain the following expression,

$$(\pi+n)\beta\pi w_t(S) b_{t+1} \sum_{S'} \frac{p(S')}{(\pi+n)w_{t+1}(S') - (1-\beta\pi)nh_{t+1}(S')} = b_t. \tag{28}$$

Now use Equation (22) to write human wealth as a function of debt,

$$h_{t+1}(S') = \frac{w_{t+1}(S')}{1-\beta\pi} - b_{t+1}, \tag{29}$$

and substitute this into (28) to give the equation we seek,

$$b_t = (\pi+n)\beta\pi w_t(S) b_{t+1} \mathbf{E}_t \left[\frac{1}{\pi w_{t+1}(S') + (1-\beta\pi)nb_{t+1}} \right]. \tag{30}$$

VII. Deterministic Dynamics

Although we are ultimately interested in the stochastic properties of the model, we turn first to the special case when there is no aggregate uncertainty by setting $w_t(S) =$

w for any date t and any state S. This special assumption leads to the following non-stochastic difference equation in z_t which characterizes feasible sequences of human wealth in a competitive equilibrium.

$$z_t = 1 + \frac{z_{t+1}\beta\pi^2}{\pi + n - (1-\beta\pi)nz_{t+1}}, \tag{31}$$

$$z_1 = \frac{1}{1-\beta\pi} - \frac{\overline{b}}{w_1}, \tag{32}$$

where $\overline{b}$ is initial government debt.

We can also state this as an equation that uses debt as the state variable,

$$b_t = \frac{(\pi+n)\beta\pi w b_{t+1}}{\pi w + (1-\beta\pi)nb_{t+1}}, \tag{33}$$

$$b_1 = \overline{b}. \tag{34}$$

Solving (31) at the steady state $z_t = z$ yields the following second degree polynomial

$$z^2 - z\left[\frac{\pi+n}{n} + \frac{1}{1-\beta\pi}\right] + \frac{\pi+n}{n(1-\beta\pi)} = 0, \tag{35}$$

which has two distinct steady states

$$z_1 = \frac{\pi+n}{n} \text{ and } z_2 = \frac{1}{1-\beta\pi}. \tag{36}$$

Solving Equation (33) at the steady state $b_t = b$ yields also two distinct values which are associated with the steady state values of z_1 and z_2:

$$b_1 = \frac{\pi w[\beta(\pi+n)-1]}{n(1-\beta\pi)} \text{ and } b_2 = 0. \tag{37}$$

Finally, we derive from equation (23) evaluated along a deterministic path two associated values for the price of the Arrow securities at the steady state, namely:

$$Q_1 = \frac{1}{\pi(\pi+n)} \text{ and } Q_2 = \frac{\beta}{\pi}. \tag{38}$$

We state these results as a proposition,

Proposition 1. In the deterministic version of this economy with $w_t(S) = w$ for any date t and any state S, there exist two distinct steady states for (z, b, Q) such that

$$(z_1, b_1, Q_1) = \left(\frac{\pi+n}{n}, \frac{\pi w[\beta(\pi+n)-1]}{n(1-\beta\pi)}, \frac{1}{\pi(\pi+n)}\right), \tag{39}$$

and

$$(z_2, b_2, Q_2) = \left(\frac{1}{1-\beta\pi}, 0, \frac{\beta}{\pi}\right). \tag{40}$$

Recall that the household can borrow or lend at the gross rate of interest $\frac{1}{Q\pi}$. It follows that the interest factors faced by households in these two steady state equilibria are given by,

$$\frac{1}{\pi Q_1} = \pi + n \qquad \text{and} \qquad \frac{1}{\pi Q_2} = \frac{1}{\beta}. \tag{41}$$

Since $n + \pi > 1$, by assumption, the gross interest rate faced by households in both steady state equilibria is positive and the wealth of each household is well defined. Notice however, that if $\pi < \beta$, the second of these steady state equilibria may be dynamically inefficient with $Q_2 > 1$.

David Gale (1973) studied dynamics of a two period model with a government liability that he called money. His model led to a difference equation that is very similar to the one that we have derived for the perpetual youth model. Following Gale we refer to steady state 1 as the golden rule steady state since it has the property that the interest rate paid by households is equal to the population growth rate. We refer to steady state 2 as the autarkic steady state since it has the property that each household will choose to consume its endowment in equilibrium and there is no inter-generational borrowing or lending.

The steady state value for government debt at the golden rule, b_1, can be positive or negative depending on whether $\frac{1}{\beta}$ is larger or lower than $\pi + n$. David Gale suggested the following classification. An economy where $\frac{1}{\beta} > \pi + n$ is called classical. An economy for which $\frac{1}{\beta} < \pi + n$ is called Samuelson. In a classical economy the autarkic steady state is dynamically efficient with an interest rate that exceeds the growth rate. In a Samuelson economy the autarkic steady state is dynamically inefficient with an interest rate that is less than the growth rate.[3] In a classical economy, steady state debt is negative at the golden rule. In a Samuelson economy it is positive.

The sign of government debt at the golden rule has implications for the local stability properties of the respective steady states. Consider the difference equation (33) which can be restated as a backward-looking difference equation as follows:

$$b_{t+1} = \frac{\pi w b_t}{(\pi + n)\beta\pi w - (1 - \beta\pi) n b_t} \equiv g(b_t). \tag{42}$$

Differentiating with respect to b_t, it follows that $g'(b_1) = \beta(\pi + n)$ and $g'(b_2) = 1/[\beta(\pi + n)]$. We state this result as Proposition 2:

[3]Gale chose this terminology because dynamic inefficiency is a novel feature that arises in Samuelson's overlapping generations model but is absent from classical infinite horizon models with a representative agent.

Proposition 2. In the deterministic version of our model, with $w_t(S) = w$ for any date t and any state S, the following results hold:

i) When the autarkic interest rate is less than the population growth rate, ($\frac{1}{\beta} < \pi + n$), the golden rate steady state, $b_1 > 0$, is locally unstable and the autarkic steady state, $b_2 = 0$, is locally stable. In this case autarky is both stable and dynamically inefficient.

ii) When the autarkic interest rate is greater than the population growth rate, ($\frac{1}{\beta} > \pi + n$), the golden rule steady state, $b_1 < 0$, is locally stable and the autarkic steady state, $b_2 = 0$ is locally unstable. In this case autarky is both dynamically efficient and unstable.

What does it mean for a steady state equilibrium to be stable? Recall that we have interpreted debt as an indexed bond. It follows that stability of a steady state equilibrium implies that, locally, there will exist initial values for government debt for which a policy of rolling over the debt is feasible and leads to a sequence for government debt that converges to the steady state.

If the world is Samuelson – there is a feasible policy that rolls over the debt each period. Government debt as a fraction of gdp will shrink over time as the economy grows faster than the interest rate. This policy will have bad outcomes since it causes the economy to converge to a dynamically inefficient steady state. In this world, there is a Pareto improving policy; it is a once off transfer to existing agents, financed by debt, that takes the economy to the golden rule steady state and leaves it there.

If the world is classical, a policy in which the debt is rolled over every period is infeasible. It leads to an explosive sequence of debt and, eventually, taxes will need to be raised to restore stability. In this world, there is no Pareto improving policy since equilibria are efficient. There is however, a policy that raises the welfare of all future generations at a small cost to the current generation. By taxing a small amount from existing households and lending the money back to households, the government can select an equilibrium with $\overline{b} < 0$ that converges to a golden rule steady state. In that steady state the government holds claims of b_1, a negative number, and every household receives a lump sum transfer each period, paid for by the interest on government assets.

In David Gales's (1973) analysis of a two-period model similar to ours, he interprets debt as a nominal variable. Under that interpretation of our model, the initial value, $\overline{b}$ is not pinned down since the initial price level is free. In that case, there is a connection between stability of the steady state and determinacy of equilibrium. If an equilibrium is unstable, it is locally determinate since there is a unique initial

condition associated with a path for the state variable that converges to the steady state. If the steady state is stable, it is locally indeterminate.

Local indeterminacy implies that there exists a continuum of equilibrium paths converging toward the same steady state. Since the price level is free, the initial value of debt is not given and Proposition 2 implies that there exists an equilibrium path from each admissible value for the initial debt converging either to $b_2 = 0$ or to $b_1 < 0$ depending on whether $\beta(\pi + n)$ is larger or lower than 1.

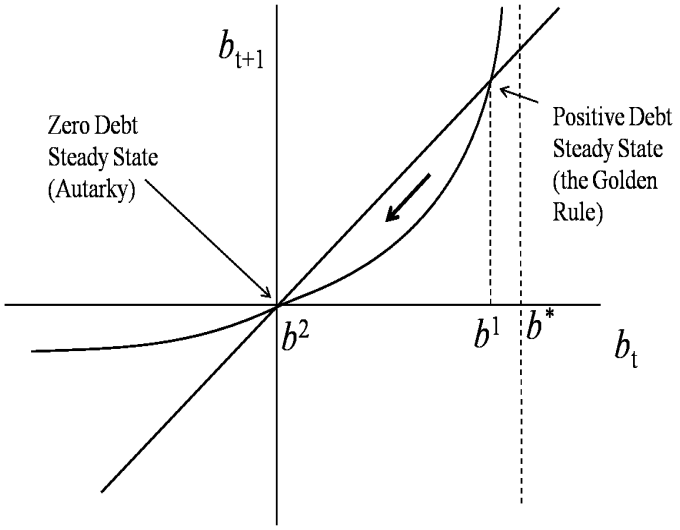

FIGURE 1. Equilibrium in the Samuelson Case

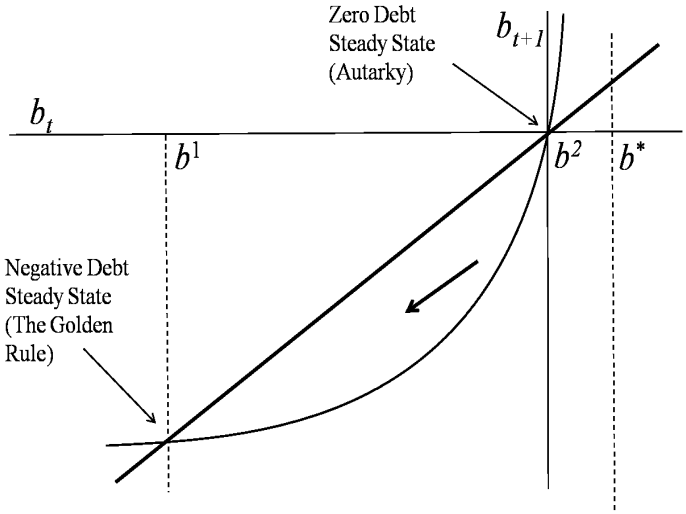

FIGURE 2. Equilibrium in the Classical Case

In a Samuelson economy in which b_t is interpreted as a nominal variable, there is a one parameter family of perfect-foresight equilibria converging asymptotically to the

zero debt steady state. At this steady state households can borrow and lend at an interest rate of $1/\beta$ (see Figure 1). The only perfect foresight equilibrium converging to the positive debt golden rule steady state, is the one that begins with a price level that exactly supports that steady state as an equilibrium.

In a classical economy in which b_t is interpreted as a nominal variable, there exists an unstable (determinate) no-debt steady state, and a stable (indeterminate) negative debt steady state. Any initial interest rate, different from $1/\beta$, generates a perfect foresight equilibrium converging to the negative-debt steady state equilibrium (see Figure 2).

VIII. Stochastic Dynamics: the Case of Indexed Debt

Equation (30) must hold each period in any rational-expectation equilibrium. To parameterize uncertainty we assume that w_t is *i.i.d.*, and drawn each period from a distribution $F(w)$ with bounded support $[1, A]$. We assume further that $F(w)$ is common knowledge. We normalize the lower bound of the support to unity since it simplifies some algebra in the proofs.

To analyze equilibria of the stochastic model, we will consider two cases. Suppose first that debt is denominated in units of the consumption commodity and that the economy begins with per-capita debt of $\overline{b}$ which may be positive or negative. In this case, since debt is chosen at date t, b_{t+1} is in the date t information set. An equilibrium sequence of values of per-capita debt must satisfy the equation,

$$b_t = (\pi + n)\beta\pi w_t(S)\Psi(b_{t+1}), \tag{43}$$

$$b_1 = \overline{b}, \tag{44}$$

where the function $\Psi(b_{t+1})$ is defined by integrating the right hand side of Equation (45) over future uncertainty.

$$\Psi(b_{t+1}) = \int \left[\frac{b_{t+1}}{\pi w' + (1-\beta\pi)nb_{t+1}}\right] dF(w'). \tag{45}$$

By a straightforward application of the inverse function theorem we obtain the following lemma which allows us to invert Equation (43) and to write b_{t+1} as a function of the endowment w_t and the current level of debt b_t.

Lemma 1. There exists an increasing function $f(x) : [-\infty, b^*] \to [-\infty, +\infty]$ with $b^* = (\pi + n)\beta\pi/(1-\beta\pi)n$ and the following properties

(i) $f(0) = 0$,

(ii) $\lim_{x\to b^*} f(x) = +\infty$,

(iii) $\lim_{x\to-\infty} f(x) = -\infty$.

Any bounded sequence of debt that satisfies the equation

$$b_{t+1} = f\left(\frac{b_t}{w_t}\right), \tag{46}$$

for $b_1 = \overline{b}$ is an equilibrium.

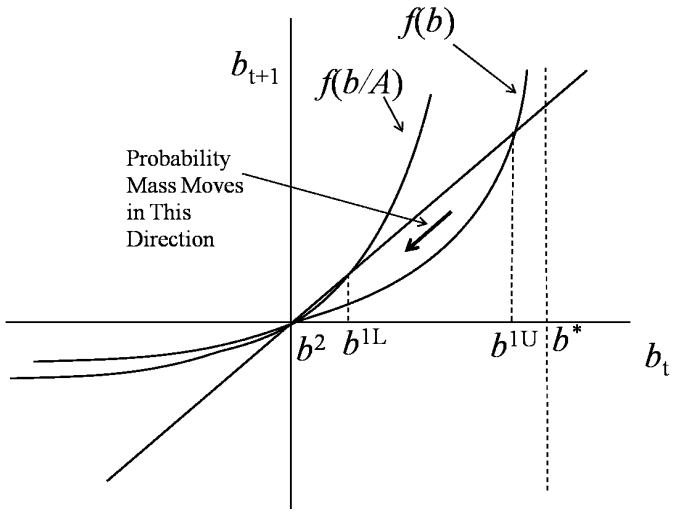

FIGURE 3. Stochastic Equilibrium in the Samuelson Case

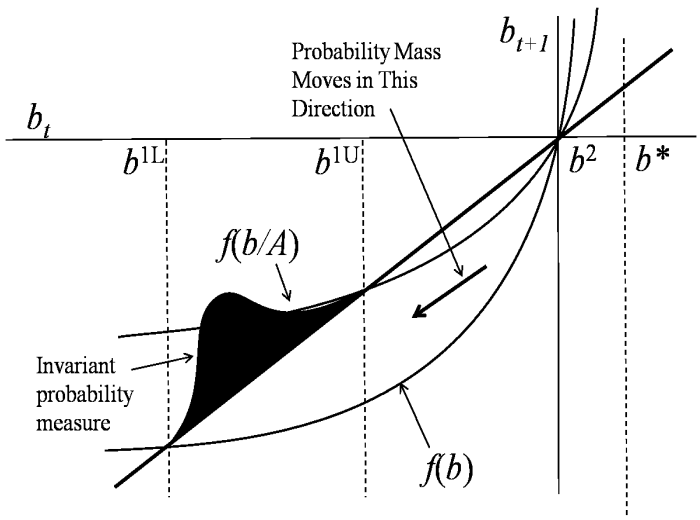

FIGURE 4. Stochastic Equilibrium in the Classical Case

Using this lemma we state the characteristics of an equilibrium for this economy in the following proposition.[4]

[4]We use the same type of argument as in Farmer and Woodford (1997). This proposition is derived from Theorem 4.6, Definition 2.1, Theorem 3.3 and Theorem 2.9 in Futia (1982).

Proposition 3. *(i)* If the economy is Samuelson and the initial debt level satisfies the restriction $0 < \overline{b} < b^*$, the policy of rolling over the debt can be supported as an equilibrium in which debt follows Equation (46) and

$$\lim_{t\to\infty} Prob(b_t > 0) \to 0. \tag{47}$$

If $\overline{b} \in [-\infty, 0] \cup [b^*, +\infty]$, the policy of rolling over debt cannot be supported as an equilibrium.

(ii) If the economy is Classical, there exist numbers b^{1L} and b^{1U} where $-\infty < b^{1U} < 0$, and an initial debt $\tilde{b}$ such that if $b^{1L} < \tilde{b} < 0$ the policy of rolling over the debt can be supported as an equilibrium in which debt follows Equation (46). Further, there exists an invariant measure $\phi(x)$ such that

$$\lim_{t\to\infty} Prob(a < b_t < b) \to \int_a^b \phi(x)dx. \tag{48}$$

If $\overline{b} > 0$, the policy of rolling over debt cannot be supported as an equilibrium.

Proposition 3 is illustrated in Figures 3 and 4. The proofs are standard and are omitted.

IX. Stochastic Dynamics: the Case of Money

There is a second case of interest in which b_t is denominated in units of account. This is what Samuelson (1958) called money. David Gale has analyzed the dynamics of the non-stochastic two-period model and Farmer and Woodford (1997) have studied a stochastic version of it. Our perpetual youth model behaves very much like the stochastic two-period case studied by Farmer and Woodford .

When b_t is denominated in nominal units, Equation (30) must still hold each period in a rational-expectation equilibrium. But now there is no initial condition and b_{t+1} is no longer in the date t information set. It is a random variable that depends on the realization of the period $t+1$ price level. Consider the following change of variables

$$x_t = \frac{b_t}{w_t}. \tag{49}$$

Using this new variable we can write Equation(30) as follows,

$$x_t = \lambda_1 \mathbf{E}_t \left[\frac{x_{t+1}}{\pi + \lambda_2 x_{t+1}} \right], \tag{50}$$

where $\lambda_1 \equiv (\pi + n)\beta\pi$ and $\lambda_2 \equiv (1 - \beta\pi)n$. The fact that b_{t+1} is not in the date t information set means that we can treat x_{t+1} as a random variable, determined at

date $t+1$. As in Proposition 3, we derive the characteristics of an equilibrium in terms of debt to endowment ratio.[5]

Proposition 4. *(i)* If the economy is Samuelson then the number $l^* = \frac{\lambda_1 - \pi}{\lambda_2}$ is positive. In this case, if the initial debt to endowment ratio satisfies the restriction $0 < \overline{x} < l^*$, the policy of rolling over the debt can be supported as an equilibrium in which debt follows Equation (50) and

$$\lim_{t\to\infty} x_t = 0. \tag{51}$$

If $\overline{x} \in [-\infty, 0] \cup [l^*, +\infty]$, the policy of rolling over debt cannot be supported as an equilibrium.

(ii) If the economy is Classical, then the number $l^* = \frac{\lambda_1 - \pi}{\lambda_2}$ is negative. In this case, if the initial debt to endowment ratio satisfies the restriction $-\infty < \overline{x} < 0$, the policy of rolling over the debt can be supported as an equilibrium in which debt follows Equation (50) and

$$\lim_{t\to\infty} x_t = \frac{\lambda_1 - \pi}{\lambda_2}. \tag{52}$$

If $\overline{x} > 0$, the policy of rolling over debt cannot be supported as an equilibrium.

Although x_t is non-stochastic in the equilibria described in this proposition, debt itself is a random variable that fluctuates in proportion to the endowment. Since the convergent steady states are indeterminate in this proposition, the techniques discussed in Farmer and Woodford (1997) allow the construction of sunspot equilibria in the neighborhood of the steady state. For the monetary economy, we can also support equilibria at the determinate steady states.

Proposition 5. *(i)* If the economy is Samuelson, and if the initial value of debt is non-negative, there exists a stochastic golden rule equilibrium in which

$$x_t = \frac{\lambda_1 - \pi}{\lambda_2}. \tag{53}$$

(ii) If the economy is Classical, there exists a stochastic autarkic equilibrium in which

$$x_t = 0. \tag{54}$$

In the Samuelson case, the price level adjusts each period to keep the ratio of debt to the endowment constant. In the classical case, existing debt is repudiated by an instantaneous hyperinflation that wipes it out. In this equilibrium money has no value.

[5]Define the function $G(x) = \pi x/(\lambda_1 - \lambda_2 x)$. We easily get $G'(0) = \pi/\lambda_1$ and $G'((\lambda_1 - \pi)/\lambda_2) = \lambda_1/\pi$.

X. Conclusion

Our main contribution in this paper was to introduce a technique for solving a discrete time version of the perpetual youth model by assuming the existence of complete financial markets. We believe that our model is an excellent vehicle for studying the effects of fiscal policy. We showed that, when preferences are logarithmic, the model behaves a lot like a pure-trade version of the two-period overlapping generations model and, like that model, government debt has important effects on the real interest rate and on intertemporal allocations. The solution technique introduced here should prove useful in empirical analysis of macroeconomic models that drop the representative agents assumption, a step that is critical to an assessment of debt and deficits.

Appendix A

Deriving the Human Wealth Equation. Using Equation (1) and Definitions (14) and (15),

$$\begin{aligned}
\sum_{i\in\mathcal{A}_t} h_t^i(S) &= \sum_{i\in\mathcal{A}_t} w_t^i(S) + \sum_{S'\in\mathbf{S}} Q_t^{t+1}(S')\pi \sum_{i\in\mathcal{A}_t} h_{t+1}^i(S') \\
\Leftrightarrow \quad N_t h_t(S) &= W_t(S) + \sum_{S'} Q_t^{t+1}(S')\pi[N_{t+1} - nN_t]h_{t+1}(S') \\
\Leftrightarrow \quad N_t h_t(S) &= N_t w_t(S) + \pi^2 \sum_{S'} Q_{t+1}(S')N_t h_{t+1}(S') \\
\Leftrightarrow \quad h_t(S) &= w_t(S) + \pi^2 \sum_{S'} Q_{t+1}(S')h_{t+1}(S').
\end{aligned} \tag{55}$$

Line 2 follows from line 1 using the definitions of aggregate wealth and aggregate human wealth and by recognizing that the human wealth of the old at date $t+1$ is equal to total human wealth $N_{t+1}h_{t+1}$ minus the human wealth of the new generation which has nN_t members. Line 3 uses equation (1) to replace $(N_{t+1} - nN_t)$ by πN_t and the final line follows from canceling N_t.

Deriving the Pricing Kernel. We begin with Equation (11), the agent's Euler equation. Aggregating this equation over the set $\mathcal{A}_t$ of agents that are alive at date t, we obtain the expression

$$Q_t^{t+1}(S') \sum_{i\in\mathcal{A}_t} c_{t+1}^i(S') = \beta p(S') \sum_{i\in\mathcal{A}_t} c_t^i. \tag{56}$$

We can obtain an expression for the right hand side of this equation in terms of the aggregate endowment since from market clearing we know that

$$\sum_{i\in\mathcal{A}_t} c_t^i = \sum_{i\in\mathcal{A}_t} w_t^i = W_t. \tag{57}$$

To find a simple equation that describes equilibrium, we need an expression for $\sum_{i\in\mathcal{A}_t} c_{t+1}^i(S')$ in terms of the aggregate variables H_t, W_t and B_t. We will use two facts. First, the consumption of all agents is the same linear function of their wealth. Second, every aggregate variable can be split into a sum over new born agents and existing ones.

Using the aggregation definition, Equation (13), and market clearing, it follows that

$$W_{t+1}(S') = \sum_{i\in\mathcal{A}_{t+1}} c_{t+1}^i(S') = \sum_{i\in\mathcal{A}_t} c_{t+1}^i(S') + \sum_{i\in\mathcal{N}_{t+1}} c_{t+1}^i(S'). \tag{58}$$

Recall that newborns do not own Arrow securities. Their wealth is in the form of human wealth. Using the policy function (12) evaluated at date $t+1$ we can find an expression for newborn consumption in terms of human wealth. Since the number of newborns at date $t+1$ is nN_t, we can write this expression as follows,

$$\sum_{i\in\mathcal{N}_{t+1}} c_{t+1}^i(S') = (1-\beta\pi)nN_t h_{t+1}(S'). \tag{59}$$

Combining Equations (57), (56), (58) and (59) leads to the expression

$$Q_t^{t+1}(S') = \frac{\beta p(S')W_t(S)}{[W_{t+1}(S') - (1-\beta\pi)nN_t h_{t+1}(S')]}. \tag{60}$$

Dividing top and bottom by N_t gives an expression for $Q_t^{t+1}(S')$ in terms of per capita variables w_t and $h_t(S)$.

$$Q_t^{t+1}(S') = \frac{\beta w_t(S)p(S')}{(\pi+n)w_{t+1}(S') - (1-\beta\pi)nh_{t+1}(S')} \tag{61}$$

which is Equation (23) in the body of the paper. This equation gives the equilibrium value of the price of an Arrow security.

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