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ABSTRACT

This paper introduces several variations of the Eaton and Kortum (1999) model of technological change and characterizes their long run implications. Both exogenous and endogenous growth examples are studied.

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1. Introduction

This note describes some models that may prove useful in thinking about technological change. We think of the technology of an economy as described by a probability distribution of available costs—in the sense of labor or other resource requirements—for producing different goods. Kortum (1997) calls such a distribution a *technology frontier*. An individual producer is characterized by his current cost level—a random variable drawn from the frontier distribution—and is also subject to a stochastic flow of new ideas—new cost levels. When he receives a cost idea that is better than the one he is now producing with he adopts it and this new cost becomes his state. If he receives a higher cost idea, or no idea at all, his cost state remains unchanged.

The evolution of a technological frontier in this sense can be decribed by an ordinary differential equation, the exact form of which depends on the way the flow of incoming ideas is modeled. We consider two possible formulations. The first is a version of the basic differential equation for the technology frontier derived in Eaton and Kortum *(*1999), under the assumption that the arrival of new ideas is stochastic, described by a Poisson process. In the second version, the arrival of new ideas in deterministic. In both cases, the *quality* of a new cost idea is modeled as a random variable.

Poisson Idea Arrivals

Consider an economy in which each producer at t has an inherited cost level $\widetilde{x}$ (the inverse of TFP, say) drawn from a distribution on $\mathbf{R}_+$ characterized by

$$G(x,t) = \Pr\{\widetilde{x} \geq x\}.$$

(If this distribution has a density, it is $-G_x(x,t)$.) Each producer receives new ideas from his own economy, characterized by the entire distribution $G(x,t)$, at a Poisson arrival rate α and also receives ideas $\widetilde{z}$ from an external source, characterized by

$$H(x,t) = \Pr\{\widetilde{z} \geq x\},$$

at the arrival rate β.

For fixed x, we motivate an ordinary differential equation for $G(x,t)$ with

$$G(x,t+h) = G(x,t) \times \Pr\{\text{no lower cost arrives in } (t,t+h)\}.$$

We have

$$\begin{aligned}
\Pr\{\text{no lower cost arrives in } (t,t+h)\} &= \Pr\{\text{no ideas arrive in } (t,t+h)\} \\
+\Pr\{\text{one idea} &> x \text{ arrives from } G \ \text{ in } (t,t+h)\} \\
+\Pr\{\text{one idea} &> x \text{ arrives from } H \ \text{ in } (t,t+h)\} \\
+\Pr\{\text{more than one idea} &> x \text{ arrives in } (t,t+h)\}
\end{aligned}$$

$$= 1 - \alpha h - \beta h + \alpha h G(x,t) + \beta h H(x,t) + o(h)$$

(where the function $o(h)$ satsfies $\lim_{h\to\infty} o(h)/h = 0$). Then

$$G(x,t+h) = G(x,t)\left[1 - \alpha h - \beta h + \alpha h G(x,t) + \beta h H(x,t) + o(h)\right].$$

Rearranging, dividing through by h, and letting $h \to 0$ gives

$$\frac{\partial \log(G(x,t))}{\partial t} = -\alpha[1 - G(x,t)] - \beta[1 - H(x,t)]. \tag{1.1}$$

Deterministic Idea Arrivals

Now consider a very similar economy in which the internal and external source distributions G and H have exactly the above interpretations. Again, we fix x and motivate a differential equation for $G(x,t)$ by

$$G(x, t+h) = G(x,t) \times \Pr\{\text{no lower cost arrives in } (t, t+h)\}.$$

In this case, we assume that in an interval $(t, t+h)$ a producer gets *exactly* αh independent draws from the internal source G and exactly βh draws from the external source H.

$$\begin{aligned}
\Pr\{\text{no lower cost arrives in } (t, t+h)\} &= \Pr\{\text{all } \alpha h \text{ ideas from } G \text{ exceed } x\} \\
&\quad \times \Pr\{\text{all } \beta h \text{ ideas from } H \text{ exceed } x\} \\
&= G(x,t)^{\alpha h} H(x,t)^{\beta h}
\end{aligned}$$

Then

$$G(x, t+h) = G(x,t) G(x,t)^{\alpha h} H(x,t)^{\beta h}$$

and it follows that

$$\frac{G(x,t+h) - G(x,t)}{G(x,t)h} = \frac{G(x,t)^{\alpha h} H(x,t)^{\beta h} - 1}{h}.$$

Taking the limits as $h \to 0$, we have

$$\frac{\partial \log(G(x,t))}{\partial t} = \alpha \log(G(x,t)) + \beta \log(H(x,t)). \tag{1.2}$$

The two equations (1.1) and (1.2) represent two different physical realities: They are not mathematically equivalent, nor is there any reason to expect them to be intimately related. In the rest of this note we develop some of the properties of both equations. In doing so, we keep in mind the convenience of exponential (or Frechet) distributions in Eaton-Kortum type trade theory. Section 2 treats equation (1.1) with an external source only. Section 3 deals with (1.1) with internal sources, and in general. Section 4 goes over the same ground for equation (1.2). In all cases, we characterize balanced path solutions completely, and provide conditions on initial distributions that ensure convergence to balanced path solutions.

2. Poisson Arrivals with an External Idea Source

With $\alpha = 0$, (1.1) is reduced to

$$\frac{\partial \log(G(x,t))}{\partial t} = -\beta[1 - H(x,t)], \tag{2.1}$$

where $H(x,t)$ is a given source function. We integrate (2.1) to get

$$G(x,t) = G(x,0)\exp\{-\beta \int_0^t [1 - H(x,s)]ds\}. \tag{2.2}$$

The following examples illustrate some possibilities.

Example 2.1. Rectangular source distribution.

Let $1 - H(x,s) = x/m$, $0 \leq x \leq m$, for some $m > 0$. Let $G(x,0)$ be a truncated exponential with parameter λ :

$$\begin{aligned} G(x,0) &= e^{-\lambda x} \quad \text{if } \; x < m \\ &= 0 \qquad \text{if } \; x \geq m \end{aligned}$$

(That is, the distribution has mass $e^{-\lambda m}$ at $x = m$.) Then

$$G(x,t) = 0 \;\; \text{for all } x \geq m,\, t \geq 0$$

and for all t, $x < m$,

$$\begin{aligned} G(x,t) &= e^{-\lambda x}\exp\{-\beta \int_0^t \frac{x}{m} ds\} \\ &= e^{-\lambda x}\exp\{-\beta \frac{xt}{m}\} \\ &= \exp\{-(\lambda + \frac{\beta t}{m})x\} \end{aligned}$$

The truncated exponential is preserved, with parameter

$$\lambda(t) = \lambda + \frac{\alpha t}{m}.$$

Note that mass is piling up near 0; truncation becomes irrelevant. But also note that

$$\frac{1}{\lambda}\frac{d\lambda}{dt} \to 0.$$

A constant source distribution doesn't work as engine of growth. To get growth, we need to build it in. We can do this through the arrival rate $\alpha(t)$ or through the source distribution H.

Example 2.2. A shifting, rectangular source distribution.

In Example 2.1, the technology frontier is always improving, but at a decreasing rate. We can get sustained growth by introducing an ever-improving source distribution, as follows. Let

$$1 - H(x,t) = x/m(t) \quad \text{for} \quad 0 \le x \le m(t)$$

and 0 otherwise, where

$$m(t) = m_0 e^{-\nu t}.$$

In this case

$$\begin{aligned} \int_0^t [1 - H(x,s)]ds &= \frac{x}{m_0}\int_0^t e^{\nu s} ds \\ &= \frac{x}{m_0}\frac{1}{\nu}\left(e^{\nu t} - 1\right) \end{aligned}$$

and

$$G(x,t) = G(x,0)\exp\{-\frac{\alpha}{m_0 \nu}\left(e^{\nu t} - 1\right)x\}.$$

Let $G(x,0)$ be a truncated exponential with parameter λ_0 :

$$\begin{aligned} G(x,0) &= e^{-\lambda_0 x} \quad \text{if} \quad x < m_0 \\ &= 0 \qquad \text{if} \quad x > m_0 \end{aligned}$$

Then for $x \leq m(t)$,

$$\begin{aligned} G(x,t) &= e^{-\lambda_0 x} \exp\{-\frac{\alpha}{m_0 \nu}\left(e^{\nu t}-1\right)x\} \\ &= e^{-\lambda(t)x} \end{aligned}$$

where

$$\lambda(t) = \lambda_0 + \frac{\alpha}{m_0 \nu}\left(e^{\nu t}-1\right).$$

For $m(t) \leq x \leq m_0$, the mass at x declines at the rate α from the time at which $m(t) = x$. We have

$$\frac{d\lambda(t)}{dt} = \frac{\alpha}{m_0} e^{\nu t}$$

$$\begin{aligned} \frac{1}{\lambda}\frac{d\lambda(t)}{dt} &= \left[\lambda_0 - \frac{\alpha}{m_0 \nu} + \frac{\alpha}{m_0 \nu} e^{\nu t}\right]^{-1} \frac{\alpha}{m_0} e^{\nu t} \\ &= \left[1 + e^{-\nu t}\left(\frac{\lambda_0 \alpha \nu}{m_0} - 1\right)\right]^{-1} \nu \\ &\rightarrow \nu \quad \text{as} \quad t \rightarrow \infty \end{aligned}$$

Example 2.3. Growth in the arrival rate of ideas.

Alternatively, we can get sustained growth by assuming that the rate of arrival of ideas $\alpha\left(t\right)$ grows at a constant rate λ. Consider general stationary external source distribution of new ideas $H\left(x\right)$. Let

$$\alpha\left(t\right) = \beta \lambda e^{\lambda t}.$$

In this case,

$$\begin{aligned} G\left(x,t\right) &= G\left(x,0\right) \exp\left\{-\beta \int_0^t \lambda e^{\lambda s}\left[1 - H\left(x\right)\right] ds\right\} \\ &= G\left(x,0\right) \exp\left\{-\beta\left(e^{\lambda t}-1\right)\left[1 - H\left(x\right)\right]\right\} \end{aligned}$$

In a balanced growth path the stationary normalized technology frontier $xe^{-\lambda t}$

solves:

$$\begin{aligned}\lim_{t\to\infty} G\left(e^{-\lambda t}x, t\right) &= \lim_{t\to\infty} G(x,0)\exp\left\{-\beta\left(e^{\lambda t}-1\right)\left[1-H\left(e^{-\lambda t}x\right)\right]\right\} \\ &= \exp\left\{-\beta \lim_{t\to\infty}\frac{1-H\left(e^{-\lambda t}x\right)}{e^{-\lambda t}}\right\}\end{aligned}$$

Using L'Hospital's rule

$$\begin{aligned}\lim_{t\to\infty} G\left(e^{-\lambda t}x, t\right) &= -\beta \lim_{t\to\infty}\frac{\lambda H'\left(e^{-\lambda t}x\right)e^{-\lambda t}x}{-\lambda e^{-\lambda t}} \\ &= \exp\left\{\beta H'(0)\,x\right\}\end{aligned}$$

Independently of other features of the external source distribution, the asymptotic distribution of $G(x,t)$ is exponential with parameter $-\beta H'(0)$.[1]

3. Poisson Arrivals with an Internal Idea Source

Here we set $\beta = 0$ so that (1.1) is reduced to an autonomous equation in $G(x,\cdot)$:

$$\frac{\partial \log(G(x,t))}{\partial t} = -\alpha[1-G(x,t)]. \tag{3.1}$$

We fix x and let $y(t) = G(x,t)$ and $y(0) = G(x,0) = y_0$. In this case. (3.1) becomes

$$\frac{dy}{dt} = -\alpha y(1-y), \quad y(0) = y_0.$$

The unique solution $y : \mathbf{R}_+ \to [0,1]$ is

$$y(t) = \frac{y_0}{y_0 + e^{\alpha t}(1-y_0)}. \tag{3.2}$$

In the original notation, the solution is then

$$G(x,t) = \frac{G(x,0)}{G(x,0) + e^{\alpha t}(1-G(x,0))}. \tag{3.3}$$

We ask whether (3.3) has a "balanced growth path," which is to say whether there is a function φ and a number $\nu > 0$ such that

$$G(x,t) = \varphi(e^{\nu t}x)$$

[1] This example is closely related to Proposition 3.2 in Kortum (1997).

solves (3.3). If so, then

$$\varphi(e^{\nu t}x) = \frac{\varphi(x)}{\varphi(x) + e^{\alpha t}(1-\varphi(x))}.$$

Differentiate both sides with respect to t :

$$\varphi'(e^{\nu t}x)\nu e^{vt}x = -\frac{\varphi(x)}{[\varphi(x) + e^{\alpha t}(1-\varphi(x))]^2}\alpha e^{\alpha t}(1-\varphi(x))$$

If we choose $\nu = \alpha$, evaluate at $t = 0$, and cancel we get

$$\varphi'(x) = -\frac{1}{x}\varphi(x)(1-\varphi(x)).$$

The solution $\varphi : \mathbf{R}_+ \to [0,1]$ is

$$\varphi(x) = \frac{1}{1+\phi x} \tag{3.4}$$

where ϕ is a parameter to be determined. Note that for any value of ϕ, (3.4) defines a cdf $1-\varphi(x)$ on $\mathbf{R}_+$.We have a one parameter family of balanced growth paths, depending on the initial condition, analogous to the balanced paths of an Ak growth model.

[Where has the distribution (3.4) been seen before? The *standard logistic distribution* has the cdf

$$F(x) = \frac{e^x}{1+e^x}$$

If $y = \log(x)$ has the distribution φ in (3.4), the corresponding cdf for y is

$$K(y) = \frac{\phi e^y}{1+\phi e^y}$$

So $y = \log(x^\phi)$ is a standard logistic random variable. Is this a useful fact?]

We next ask the stability question: Given an initial distribution $G(x,0)$, when does

$$\lim_{t\to\infty} G(e^{-\alpha t}x, t) = \frac{1}{1+\phi x}$$

for some $\phi > 0$? From (3.3),

$$\begin{aligned}\lim_{t\to\infty} G(e^{-\alpha t}x, t) &= \lim_{t\to\infty} \frac{G(e^{-\alpha t}x, 0)}{G(e^{-\alpha t}x, 0) + e^{\alpha t}(1 - G(e^{-\alpha t}x, 0))} \\ &= \lim_{t\to\infty} \frac{1}{1 + \left[1/G(e^{-\alpha t}x, 0) - 1\right]/e^{-\alpha t}}\end{aligned}$$

Apply L'Hospital's rule to get

$$\begin{aligned}\lim_{t\to\infty} \frac{1/G(e^{-\alpha t}x, 0) - 1}{e^{-\alpha t}} &= \lim_{t\to\infty} \frac{1/\left[G(e^{-\alpha t}x, 0)\right]^2 G_x(e^{-\alpha t}x, 0)\alpha e^{-\alpha t}x}{-\alpha e^{-\alpha t}} \\ &= -G_x(0, 0)x\end{aligned}$$

since $G(0, 0) = 1$. The term $-G_x(0, 0)$ is the density of $G(x, 0)$ at $x = 0$. We have proved

Theorem. Suppose the distribution defined by $G(x, 0)$ has a density $-G_x(x, 0)$ with $\phi = -G_x(0, 0) < \infty$. Then for all $x \geq 0$,

$$\lim_{t\to\infty} G(e^{-\alpha t}x, t) = \frac{1}{1 + \phi x}.$$

The asymptotic behavior of $G(x, t)$ is thus determined entirely by the value of the initial density at 0. The density function corresponding to (3.4) is

$$\frac{\phi}{(1 + \phi x)^2}.$$

It does not have a mean. Its mode, ϕ, is attained at $x = 0$.

For completeness, we provide the solution to (1.1) in a system with both internal and external sources. The basic DE in this case is

$$\frac{\partial \log(G(x, t))}{\partial t} = -\alpha[1 - G(x, t)] - \beta[1 - H(x, t)].$$

Fix x and let $y(t) = G(x,t)$, $y(0) = G(x,0) = y_0$, and $1 - H(x,t) = u(t)$. This variable y satisfies

$$\frac{dy}{dt} = -\alpha y(1-y) - \beta y u(t), \qquad y(0) = y_0. \tag{3.5}$$

As before, we use the change of variable $z = y^{-1}$, so that (3.5) is equivalent to

$$\begin{aligned}
\frac{dz}{dt} &= -y^{-2}\frac{dy}{dt} \\
&= y^{-2}\left[\alpha y(1-y) + \beta y u\right] \\
&= \alpha y^{-1}(1-y) + \beta y^{-1} u \\
&= (\alpha + \beta u)z - \alpha
\end{aligned}$$

For $z(0) = z_0$, the unique solution $z : \mathbf{R}_+ \to [0,1]$ is

$$z(t) = e^{\int_0^t (\alpha+\beta u(s))ds}\left(z_0 - \alpha \int_0^t e^{-\int_0^s (\alpha+\beta u(\tau))d\tau} ds\right) \tag{3.6}$$

4. Deterministic Arrivals

We turn to the deterministic arrival case described in (1.2). With an external idea source only, $\alpha = 0$, (1.2) is reduced to

$$\frac{\partial \log(G(x,t))}{\partial t} = \beta \log(H(x,t)). \tag{4.1}$$

where $H(x,t)$ is a given source function. We integrate (4.1) to get

$$\log(G(x,t)) = \log(G(x,0)) + \beta \int_0^t \log(H(x,s))ds. \tag{4.2}$$

The following examples illustrate some possibilities.

Example 4.1. Suppose $G(x,0) = \exp(-\lambda(0)x)$ and $H(x,t) = \exp(-\mu(t)x)$. Then (4.2) implies

$$\log(G(x,t)) = -\lambda(0)x - \beta x \int_0^t \mu(s)ds,$$

from which it follows that $G(x,t) = \exp(\lambda(t))$ where

$$\lambda(t) = \lambda(0) + \beta \int_0^t \mu(s)ds.$$

With an internal idea source only, $\beta = 0$, (1.2) becomes

$$\frac{\partial \log(G(x,t))}{\partial t} = \alpha \log(G(x,t)). \tag{4.3}$$

The solution is

$$\log(G(x,t)) = \log(G(x,0))e^{\alpha t}.$$

If $G(x,0) = \exp(-\lambda x)$ then $\log(G(x,0)) = -\lambda x$, $\log(G(x,t)) = -\lambda(t)x$, and

$$\lambda(t) = \lambda e^{\alpha t}. \tag{4.4}$$

Thus exponential distributions are preserved exactly with either external or internal sources. All exponential distributions with $\lambda(t)$ given by (4.4) satisfy the balanced path condition

$$G(x,t) = \varphi(e^{\nu t}x)$$

with $\nu = \alpha$.

We next investigate stability. Under what conditions on the initial distribution $G(x,0)$ will it be the case that

$$\lim_{t\to\infty} \log\left(G(e^{-\alpha t}x,t)\right) = -\lambda x$$

for some $\lambda > 0$? From the solution to (4.3) we have

$$\log(G(e^{-\alpha t}x,t)) = \log(G(e^{-\alpha t}x,0))e^{\alpha t}.$$

Then

$$\begin{aligned}\lim_{t\to\infty} \log\left(G(e^{-\alpha t}x,t)\right) &= \lim_{t\to\infty} \log(G(e^{-\alpha t}x,0))e^{\alpha t} \\ &= \lim_{t\to\infty} \frac{\log(G(e^{-\alpha t}x,0))}{e^{-\alpha t}}\end{aligned}$$

and applying L'Hospital's rule gives

$$\begin{aligned}\lim_{t\to\infty} \log\left(G(e^{-\alpha t}x,t)\right) &= \lim_{t\to\infty} \left[-e^{-\alpha t}\right]^{-1} \left[\frac{-\alpha G_x(e^{-\alpha t}x,0)e^{-\alpha t}x}{G(e^{-\alpha t}x,0)}\right] \\ &= \lim_{t\to\infty} \left[\frac{G_x(e^{-\alpha t}x,0)x}{G(e^{-\alpha t}x,0)}\right] \\ &= G_x(x,0)x\end{aligned}$$

As in Section 3, we have the

Theorem.[2] Suppose the distribution defined by $G(x,0)$ has a density $-G_x(x,0)$ with $\lambda = -G_x(0,0) > 0$. Then for all $x \geq 0$,

$$\lim_{t\to\infty} G(e^{-\alpha t}x, t) = -\lambda x.$$

Again, the asymptotic behavior of $G(x,t)$ is determined entirely by the value of the initial density at 0.

REFERENCES

[1] Eaton, Jonathon, and Samuel S. Kortum. 1999. "International Technology Diffusion: Theory and Measurement." *International Economic Review*, 40: 537-570.

[2] Falk, Michael and Frank Marohn. 1993. "Von Mises Conditions Revisited." *Annals of Probability*, 21: 1310-1328.

[3] Kortum, Samuel S. 1997. "Research, Patenting, and Technological Change." *Econometrica*, 65: 1389-1419.

[2] This theorem is a version of a class of well-known results known as Von Mises conditions. See Falk and Marohn (1993) and the references cited therein.