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DEBT, DEFICITS AND FINITE HORIZONS

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Debt, Deficits and Finite Horizons

ABSTRACT

Many issues in macroeconomics, such as the level of the steady state interest rate, or the dynamic effects of government deficit finance, depend crucially on the horizon of economic agents. This paper develops a simple analytical model in which such issues can be examined and in which the horizon of agents is a parameter which can be chosen arbitrarily.

The first three sections of the paper characterize the dynamics and steady state of the economy in the absence of a government. The focus is on the effects of the horizon index on the economy. The paper clarifies in particular the separate roles of finite horizons and declining labor income through life in the determination of steady state interest rates.

The next three sections study the effects and the role of fiscal policy. The focus is on the effects of deficit finance both in closed and open economies. The paper clarifies the respective roles of government spending, deficits and debt in the determination of interest rates.

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This paper characterizes the dynamic behavior of an economy where agents have finite horizons. It then analyzes the effects and the role of government debt and deficits, of intertemporal reallocations of taxes.

There is in general no simple aggregate consumption function in an economy composed of finitely lived agents. This is because agents differ in two respects. Being of different ages, they have different levels and compositions of wealth. Having different horizons, they have different propensities to consume out of wealth. This systematic relation between wealth level, wealth composition and propensity to consume makes exact or approximate aggregation impossible (Modigliani [1966]).

In view of this problem, the solution chosen by Diamond [1965] was to choose a very simple population and age structure, avoiding altogether the need for aggregation. The solution chosen in this paper is, instead, to make assumptions which allow aggregation. The central assumption is that agents face, throughout their life, a constant instantaneous probability of death  $p$ . Thus their expected life is  $p^{-1}$ ; furthermore, it is constant throughout their life. Agents are of different ages, have different levels of wealth but have the same horizon, the same propensity to consume. This allows one to solve the aggregation problem.

The main advantage of this approach is its flexibility. If we think of  $p^{-1}$  as the horizon index, we can choose it anywhere between zero and infinity, and study the effects of the horizon of agents on the behavior of the economy. In particular, by letting  $p$  go to zero, we obtain the infinite horizon case as a limiting case.

The main drawback of this approach is that it captures the finite horizon aspect of life, but not the change in behavior over life, the "life cycle" aspect of life. In that respect it is closer to the initial formulation of permanent income by Friedman [1957] than to that of life cycle by Modigliani [1966]. It is

well adapted to issues where the finite horizon aspect is important, such as the issue of government debt. It is poorly adapted to issues where differences in propensities to consume across agents are potentially important.

The first three sections characterize the behavior of the economy in the absence of a government. Section I derives the behavior of both individual and aggregate consumption functions. The specification of aggregate consumption resembles some existing specifications: aggregate consumption is a function of aggregate financial and human wealth. The rate used to discount labor income is higher than the interest rate. Section II studies the dynamics and steady state of an economy composed of agents with finite horizons, both in closed and open economies. Section III considers two extensions. The first focuses on the effects of the elasticity of substitution of consumption. The second allows for declining labor income throughout life, to capture the effects of the "saving for retirement" motive on capital accumulation.

The next three sections characterize the effects and role of debt and deficits. The government is assumed to have lump-sum taxes at its disposal and the focus is on intertemporal reallocations of taxes, rather than on changes in spending. Section IV introduces the government budget constraint and shows how finite horizons imply a role for debt policy. It studies whether and how an index of fiscal policy can be defined and how it evolves over time for particular reallocations of taxes. Section V shows the steady state effects of fiscal policy in the closed and open economies; Section VI characterizes its dynamic effects, by considering two examples. The first, inspired by the current fiscal policy in the U.S. is that of a reallocation which creates high deficits followed later by surpluses. The second, which leads to the study of optimal debt policy, studies the role of debt policy in smoothing aggregate consumption in the face of regular fluctuations in output.

## Section I. The Aggregate Consumption Function

The derivation of aggregate consumption is based on two major assumptions.

The first specifies the probability of death and the structure of population: Time is continuous. Each agent, throughout his life faces a constant instantaneous probability of death  $p^{1,2}$ . At any instant of time, a large cohort, whose size is normalized to be  $p$ , is born.

If the probability of death is constant, the expected remaining life for an agent of any age is given by  $\int_0^{\infty} tpe^{-pt} dt = p^{-1}$ . We shall refer to  $p^{-1}$  as the horizon index. As we let  $p$  go to zero,  $p^{-1}$  goes to infinity: agents have infinite horizons.

The assumption that cohorts are large implies that, although each agent is uncertain about time of death, the size of a cohort declines non-stochastically through time. A cohort born at time zero has a size, as of time  $t$ , of  $pe^{-pt}$ , and the size of the population at anytime  $t$  is  $\int_0^t pe^{-p(t-s)} ds = 1$ .

In the absence of insurance, uncertainty about death implies that agents may leave unanticipated bequests although they have no bequest motive. They may also be constrained to maintain a positive wealth position if they are prohibited from leaving debt to their heirs. Under our assumptions, private markets may however provide insurance risklessly and it is reasonable and convenient to assume that they do so. This motivates the second assumption:<sup>3</sup>

There exist life insurance companies. Agents may contract to make (or receive) a payment contingent upon their death.

Because of the large number of identical agents, such contracts may be offered risklessly by life insurance companies. Given free entry and a zero

profit condition, and given a probability of death  $p$ , agents will pay (receive) a rate  $p$  to receive (pay) one good contingent on their death.

In the absence of a bequest motive, and if negative bequests are prohibited, agents will contract to have all of their wealth (positive or negative) return to the life insurance company contingent on their death. Thus, if their wealth is  $w$ , they will receive  $pw$  if they do not die, pay  $w$  if they die.

These two assumptions are sufficient to characterize aggregate consumption. For simplicity, we shall however make two further assumptions. The first, which implies a simple individual consumption function, is that utility is logarithmic. The second, which implies a simple form for aggregate human wealth is that labor income is distributed equally across agents. We shall relax these two assumptions in Section III.

#### Individual consumption

Denote by  $c(s,t)$ ,  $y(s,t)$ ,  $w(s,t)$ ,  $h(s,t)$  consumption, non-interest income, non-human and human wealth of an agent born at time  $s$ , as of time  $t$ . Let  $r(t)$  be the interest rate at time  $t$ . Under the assumption that instantaneous utility is logarithmic, the agent maximizes:

$$E_t \left( \int_t^\infty \log c(s,v) e^{\theta(t-v)} dv \right), \quad \theta \geq 0.$$

Given the constant probability of death, and if the only source of uncertainty is about the time of death, maximizing the above is equivalent to maximizing:

$$\int_t^\infty \log c(s,v) e^{(\theta+p)(t-v)} dv \tag{1}$$

The effective discount rate is therefore  $(\theta+p)$ . Even if  $\theta$  is equal to zero, agents will discount the future if  $p$  is positive.

If an agent has wealth  $w(s,t)$  at time  $t$ , he receives  $r(t)w(s,t)$  in interest and  $pw(s,t)$  from the insurance company. Thus its dynamic budget constraint is:

$$dw(s,t)/dt = (r(t)+p) w(s,t) + y(s,t) - c(s,t) \quad (2)$$

An additional transversality condition is needed to prevent agents from going infinitely into debt and protecting themselves by buying life insurance. We impose a condition which is the extension of that used in the deterministic case.<sup>5</sup> The solution must be such that if the agent is still alive at time  $v$  :

$$\lim_{v \rightarrow \infty} e^{-\int_t^v (r(\mu)+p)d\mu} w(s,v) = 0 .$$

If this is the case, the budget constraint can be integrated to give:

$$\int_t^{\infty} c(s,v) e^{-\int_t^v (r(\mu)+p)d\mu} dv = w(s,t) + h(s,t) \quad (3)$$

$$\text{where } h(s,t) = \int_t^{\infty} y(s,v) e^{-\int_t^v (r(\mu)+p)d\mu} dv$$

The agent maximizes (1) subject to (3). This problem is very similar to the deterministic case, except for the presence of  $(\theta+p)$  and  $(r+p)$  instead of  $\theta$  and  $r$ . The solution is simply :

$$c(s,t) = (p+\theta) (w(s,t) + h(s,t)) . \quad (4)$$

Individual consumption depends on total individual wealth, with propensity  $(\theta+p)$ . The discount rate used to discount labor income is  $(r+p)$ , the same as the rate at which non-human wealth accumulates.

Aggregate consumption.

Denote aggregate consumption, non-interest income, non-human wealth and human wealth at time  $t$  by  $C(t)$ ,  $Y(t)$ ,  $W(t)$  and  $H(t)$ . Then

$$C(t) = \int_{-\infty}^t c(s,t) p e^{p(s-t)} ds$$

$$Y(t) = \int_{-\infty}^t y(s,t) p e^{p(s-t)} ds$$

$$W(t) = \int_{-\infty}^t w(s,t) p e^{p(s-t)} ds$$

$$H(t) = \int_{-\infty}^t h(s,t) p e^{p(s-t)} ds = \int_{-\infty}^t \left( \int_t^{\infty} y(s,v) e^{-\int_t^v (r(\mu)+p)d\mu} dv \right) p e^{p(s-t)} ds$$

From equation (4), aggregate consumption is given by:

$$C(t) = (p+\theta) (H(t) + W(t)).$$

The next step is to derive the dynamic behavior of  $H(t)$  and  $W(t)$ .

The dynamic behavior of  $H(t)$  depends on the distribution of labor income.

We shall assume for the moment that labor income is equally distributed across agents, i.e. that all agents work and have the same productivity:  $y(s,v) = Y(v) V_s$ . This implies that all agents alive have the same human wealth.

$H(t)$  is then given by:

$$H(t) = \int_t^{\infty} Y(v) e^{-\int_t^v (r(\mu)+p)d\mu} dv$$

or in a differential form:

$$dH(t)/dt = (r(t)+p)H(t) - Y(t); \quad \lim_{v \rightarrow \infty} H(v) e^{-\int_t^v (r(\mu)+p)d\mu} = 0.$$

Differentiating now  $W(t)$  with respect to time gives:

$$dW(t)/dt = w(t,t) - pW(t) + \int_{-\infty}^t \frac{dw(s,t)}{dt} pe^{p(s-t)} ds$$

The first term on the right is the financial wealth of newly born agents, which is equal to zero. The second term is the wealth of those who die. The third is the change in the wealth of those alive. Using equation (2) gives:

$$dW(t)/dt = r(t)W(t) + Y(t) - C(t)$$

Collecting equations, dropping the time index  $t$ , denoting  $d./dt$  by a dot, gives a first characterization of aggregate consumption:

$$C = (p+\theta)(H + W) \quad (5)$$

$$\dot{H} = (r+p)H - Y; \quad \lim_{V \rightarrow \infty} H_V e^{-\int_t^V (r_\mu + p) d\mu} = 0. \quad (6)$$

$$\dot{W} = rW + Y - C \quad (7)$$

Whereas individual wealth accumulates, for those alive, at rate  $r+p$ , aggregate wealth accumulates at rate  $r$ . This is because the amount  $pW$  is a transfer, through life insurance companies of those who die to those who remain alive; it is not therefore an addition to aggregate wealth.

Thus aggregate consumption is a linear function of total aggregate wealth. If agents have finite horizons, if  $p > 0$ , the discount rate used to discount non-interest income is higher than the interest rate.<sup>6</sup> (Human wealth can be either thought of as the present value of non-interest income discounted at rate  $(r+p)$ , or as the present value of non-interest income accruing in the future to agents alive today, discounted at rate  $r$ ).

There is an alternative characterization of the behavior of aggregate consumption which will be useful. Differentiating (5) and eliminating  $\dot{H}$  and  $\dot{W}$  gives:

$$\dot{C} = (r-\theta)C - p(p+\theta)W \quad (8)$$

$$\dot{W} = rW + Y - C \quad (9)$$

If agents have infinite horizons,  $p = 0$  and equation (8) reduces to the standard equation (Hall [1978] for example). If  $p > 0$ , the rate of change of  $C$  depends also on non-human wealth. Note that even if  $p$  is positive, individual consumption follows  $\dot{c} = (r - \theta)c$ . Thus if  $r = \theta$ , individual consumption will be constant but aggregate consumption will in general vary.



## Section II. Dynamics and Steady State

We consider in turn the cases of a closed and of an open economy.

### The Closed Economy

There are two factors of production, capital  $K$ , and labor; from above the size of the labor force is equal to the size of population and equal to unity. Let  $\tilde{F}(K,1)$  be the constant returns to scale production function, and  $\delta$  be the depreciation rate. Define

$$F(K) \equiv \tilde{F}(K,1) - \delta K$$

Non human wealth  $W$  is equal to  $K$ .  $Y$  is labor income, which we shall denote now by  $\omega(K)$  and  $r$  is the net marginal product of capital,  $r(K)$ , which may be positive or negative. Using equations (8) and (9), equilibrium is characterized by:

$$\dot{C} = (r(K) - \theta)C - p(p + \theta)K \quad (10)$$

$$\dot{K} = F(K) - C \quad (11)$$

Figure 1 characterizes the phase diagram associated with the system. Let's define two values of  $K$ ,  $K^*$  such that  $r(K^*) = \theta$ , and  $K^{**}$  such that  $r(K^{**}) = \theta + p$ . The locus  $\dot{C} = 0$  is upward sloping, going through the origin and reaching asymptotically  $K^*$ . The locus  $\dot{K} = 0$  traces the net production function.

The equilibrium is unique with a saddle point structure. The stable arm  $SS$  is upward sloping. Any other trajectory can be shown to imply a negative level of  $C$  or  $K$  in finite time and thus the stable arm is the only acceptable trajectory: given  $K$ ,  $C$  is uniquely determined.

The steady state level of capital  $\bar{K}$  is such that  $r(\bar{K})$  is between  $\theta$  and  $\theta + p$ .  $r(\bar{K}) > \theta$  follows from inspection of the phase diagram.  $r(\bar{K}) < \theta + p$  is shown

as follows: Suppose  $r(\bar{K}) \geq \theta + p$ , so that:

$$r(\bar{K}) - \theta \geq p \Rightarrow (r(\bar{K}) - \theta)\bar{C} \geq p\bar{C}$$

$$\dot{C} = 0 \text{ and equation (10)} \Rightarrow p(p+\theta)\bar{K} \geq p\bar{C},$$

$$\dot{K} = 0 \text{ and equation (11)} \Rightarrow p(p+\theta)\bar{K} \geq pF(\bar{K}) \Rightarrow (p+\theta)\bar{K} \geq F(\bar{K})$$

$$\text{As } r(\bar{K}) \geq \theta + p \Rightarrow r(\bar{K})\bar{K} \geq F(\bar{K}) \text{ which is impossible.}$$

If  $p = 0$ , so that agents have infinite horizons,  $r(\bar{K}) = \theta$ . The modified golden rule obtains. When  $p$  is positive,  $r(\bar{K})$  is larger than  $\theta$ . The reason is clear from the individual consumption equation  $\dot{c} = (r - \theta)c$ : In steady state, labor income is constant throughout life. In order to generate positive aggregate capital, agents must be saving initially, consumption must be increasing. Thus  $r$  must be larger than  $\theta$ .

Furthermore,  $r$  is an increasing function of  $p$ . The shorter the horizon, the lower the level of capital. We shall see however in the next section that declining labor income throughout life has the opposite effect.

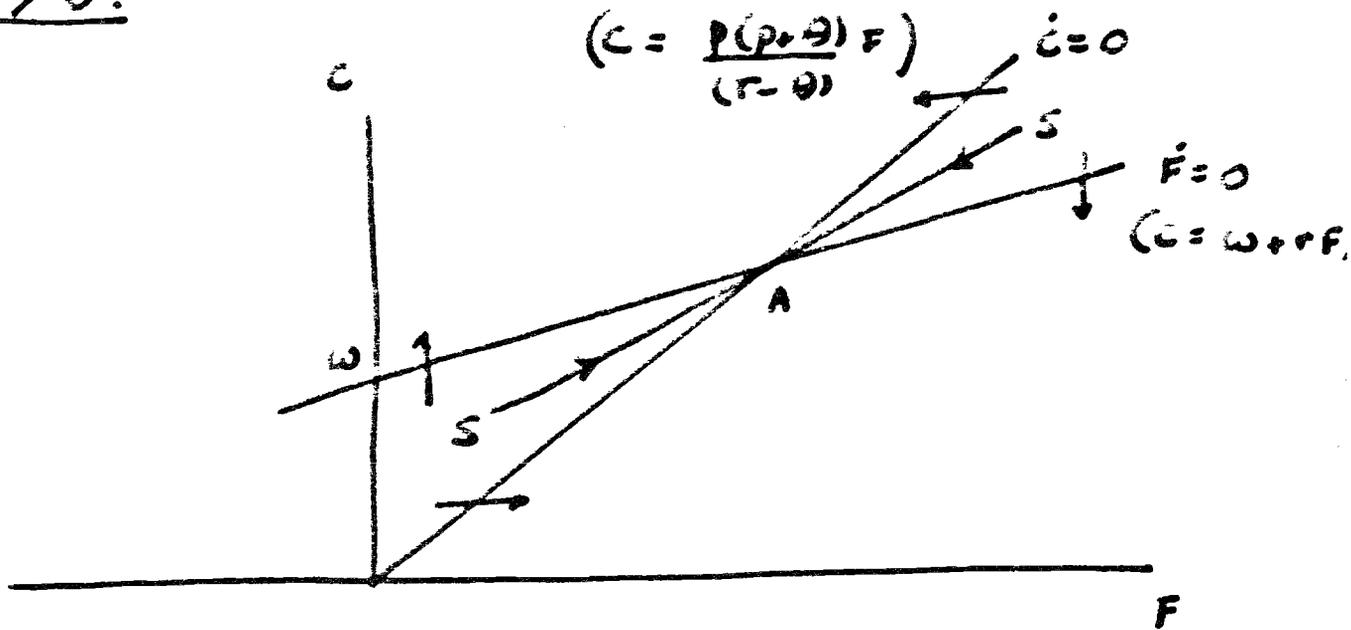
Does the decentralized economy generate too much or too little capital accumulation? If we use the subjective discount rate  $\theta$  as the relevant rate,  $r(\bar{K}) > \theta$  and the economy has too small a capital stock. If, however, we use the effective subjective discount rate  $\theta + p$ ,  $r(\bar{K}) < \theta + p$  and the economy has too large a capital stock.

### The open economy.

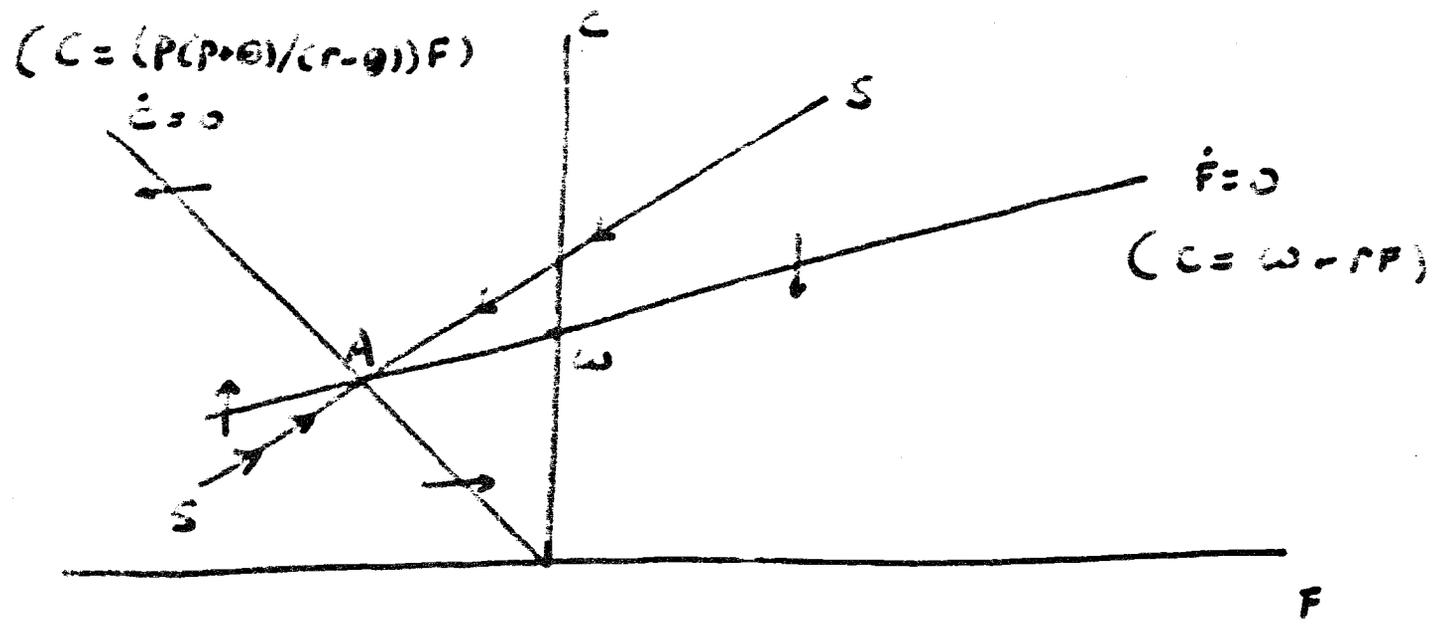
In the open economy, the interest rate is the world interest rate,  $r$ , which is fixed and at which consumers can freely borrow or lend. For simplicity, there is no capital and the only assets are therefore the net holdings of foreign assets, denoted  $F$ . Non-interest income is exogenous and denoted as  $\omega$ . Using equations (8) and (9), equilibrium is characterized in this case by:

# Figure 2. Open Economy : Dynamics.

2a :  $r > \theta$ .



2b :  $r < \theta$



$$\dot{C} = (r-\theta)C - p(p+\theta)F \quad (12)$$

$$\dot{F} = rF + \omega - C \quad (13)$$

The first question is whether there exists a (saddle point) stable steady state. The dynamic system is linear in  $C$  and  $F$ . It is saddle point stable if  $r$  is between  $-p$  and  $\theta+p$ . If  $r$  was larger than  $\theta+p$ , individual consumption would increase at a rate larger than  $p$ , that is, larger than the rate of death; aggregate consumption would therefore increase forever. If we exclude this case, we have a well-defined steady state, with associated values for  $C$  and  $F$ . This result differs sharply from the infinite horizon case where a steady state value of  $C$  exists only if  $r = \theta$  and where in this case the value of  $F$  is indeterminate (or more precisely, depends on the path of adjustment).

We construct now the phase diagram. The slope of  $\dot{C} = 0$  is positive if  $r > \theta$ , negative if  $r < \theta$ . The slope of  $\dot{F} = 0$  is positive. Both cases are represented in Figures 2a and b. In both cases, the stable arm is upward sloping.

What are the characteristics of this steady state? If  $r = \theta$ , the value of  $F$  is zero. Agents have flat labor income and consumption profiles; they do not save or dissave. If  $r$  is greater than  $\theta$ , individual consumption is increasing, agents are accumulating over their life and the level of foreign assets is positive. If  $r$  is smaller than  $\theta$ , agents are decumulating and, as a result, the level of foreign assets is negative. The country is a net debtor in steady state.

An increase in  $r$  increases the level of foreign assets. An increase in  $p$  pushes the level of foreign assets towards zero, reducing it if positive, increasing it if negative: shorter horizons imply smaller accumulation or decumulation.

An alternative way of looking at aggregate behavior is to return to the specification giving aggregate consumption as a function of wealth and to derive an aggregate savings function:

$$C = (p+\theta)\left(\frac{\omega}{r+p} + F\right) \text{ whereas income is } \omega + rF, \text{ so that:}$$

$$S \equiv Y-C = \left(\frac{r-\theta}{r+p}\right)\omega + (r-p-\theta)F$$

As  $r < \theta+p$ , savings is a decreasing function of wealth  $F$ ; the effect of non-interest income  $\omega$  is ambiguous and depends on  $(r-\theta)$ . In steady state  $S = 0$ . If  $r = \theta$ , the equilibrium level of foreign assets is zero. It is positive if  $r > \theta$ , negative if  $r < \theta$ .

### Section III. Two extensions

We consider now two extensions to the original model. The first focuses on the effects of the elasticity of substitution of consumption, the second on the effects of declining labor income through life.

#### The role of the elasticity of substitution

We now consider the class of isoelastic utility functions:

$$\begin{aligned} u(c) &= c^{1-\sigma}/(1-\sigma) && , \sigma \neq 1 \\ &= \log c && , \sigma = 1. \end{aligned}$$

The elasticity of substitution is given by  $\sigma^{-1}$ . All other assumptions are unchanged.

Following the same steps as in Section I, the first order condition to the individual maximization problem is:

$$dc(s,t)/dt = \sigma^{-1}(r(t)-\theta)c(s,t) \quad (14)$$

Solving for  $c(s,v)$ ,  $v \geq t$  as a function of  $c(s,t)$ , replacing in the budget constraint given by equation (3), and solving for  $c(s,t)$  gives:

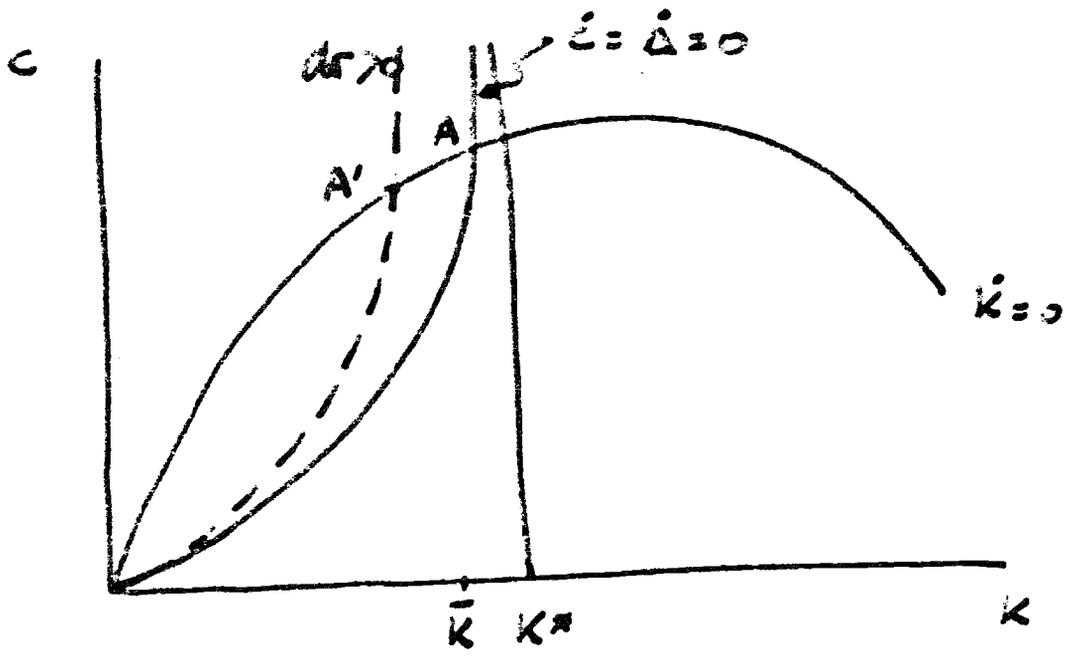
$$\begin{aligned} c(s,t) &= (\Delta(t))^{-1}(w(s,t) + h(s,t)) \\ \Delta(t) &= \int_t^\infty e^{\sigma^{-1} \int_t^v [(1-\sigma)(r(u)+p) - (\theta+p)] du} dv \end{aligned}$$

As before, consumption is a linear function of total wealth. The propensity to consume is now a function of the sequence of expected interest rates; it is however not a function of age and is therefore the same for all agents. The behavior of  $\Delta(t)$  can equivalently be characterized by a differential equation (and a transversality condition):

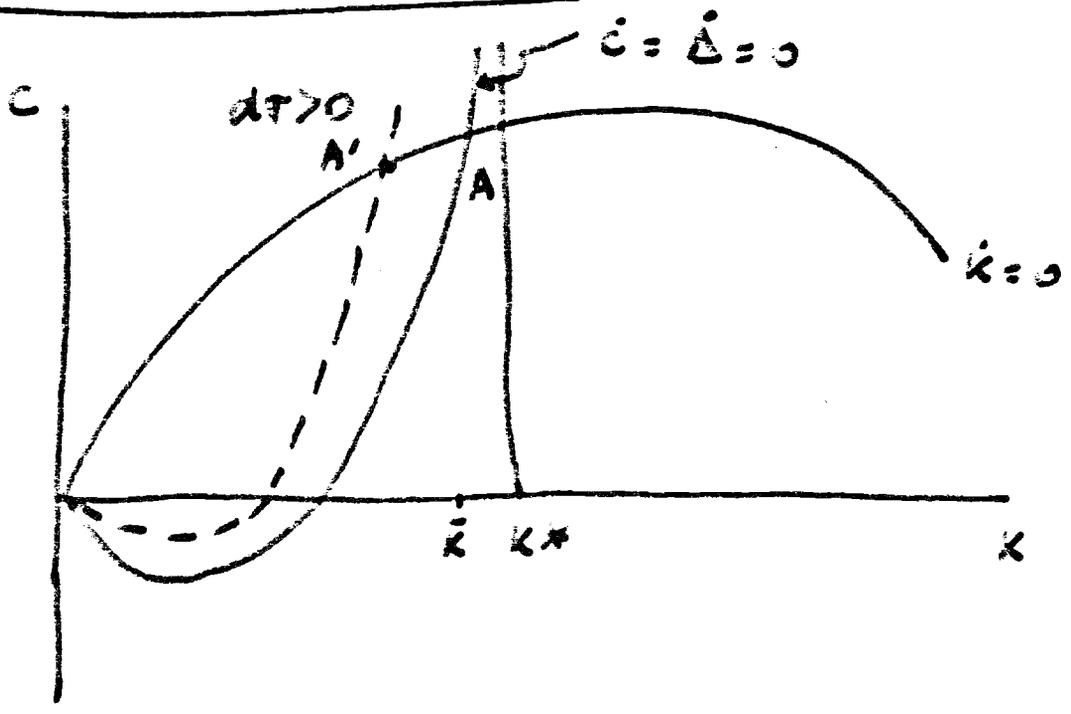
$$d\Delta(t)/dt = -1 - \sigma^{-1} [(1-\sigma)(r(t) + p) - (\theta+p)] \Delta(t)$$

Figure 3 . Effect of the elasticity of substitution

3a.  $\sigma > 1$ . low elasticity



3b.  $\sigma < 1$ . high elasticity



Following the same steps as in Section I gives the dynamic behavior of aggregate consumption and aggregate wealth:

$$C = \Delta^{-1}(H + W)$$

$$\dot{H} = (r + p)H - Y$$

$$\dot{W} = rW + Y - C$$

$$\dot{\Delta} = -1 - \sigma^{-1} [(1-\sigma)(r+p) - (\theta+p)]\Delta$$

Some manipulation gives an alternative characterization of aggregate consumption, corresponding to equation (8) in Section I:

$$\dot{C} = \sigma^{-1}(r-\theta)C - p\Delta^{-1}W$$

We shall limit our analysis to the effects of  $\sigma$  on steady state capital in a closed economy. The steady state is characterized by:

$$\dot{C} = \sigma^{-1} (r(K) - \theta)C - p\Delta^{-1} K = 0 \quad (15)$$

$$\dot{K} = F(K) - C = 0 \quad (16)$$

$$\dot{\Delta} = -1 - \sigma^{-1} [(1-\sigma)(r(K)+p) - (\theta+p)]\Delta = 0 \quad (17)$$

Replacing  $\Delta$  in (15) by its value for  $\dot{\Delta} = 0$  in (17), gives the  $\dot{C} = 0$  locus:

$$C = p (\theta + p + (\sigma-1)(r(K) + p))/(r(K) - \theta)K. \text{ This locus is drawn in Figure}$$

3. For  $\sigma > 1$ , and  $C > 0$ , this locus starts from the origin, is upward sloping and approaches  $K^*$  asymptotically, where  $K^*$  is such that  $r(K^*) = \theta$ . For  $\sigma < 1$ , the locus is initially downward sloping, then upward sloping. It also approaches  $K^*$  asymptotically. These two cases are drawn in Figures 3a and 3b.

The  $\dot{K} = 0$  locus traces the net production function.

The steady state capital stock is smaller than  $K^*$ , the net marginal product  $r^*$  is greater than  $\theta$ . An increase in  $\sigma$  shifts the  $\dot{C} = 0$  to the left, decreasing steady state capital. Thus, the lower the elasticity of substitution, the lower steady state capital. Intuition for this result is obtained by examining equation (14), giving the behavior of individual consumption. As in Section II,

a positive aggregate capital requires savings initially in life. As labor income is flat, this requires low and increasing consumption. The lower  $\sigma^{-1}$ , the larger the interest rate needed to twist the consumption path. Equivalently, the lower  $\sigma^{-1}$ , the lower initial individual savings given the interest rate.

### The effects of declining labor income

What we want to capture here are the effects of "saving for retirement" on aggregate capital accumulation. Introducing retirement, that is zero labor income after some given length of life is not analytically convenient. We assume instead that labor income declines through life, at rate  $\alpha$ . More precisely, we assume:

$$y(s,v) = aY(v)e^{\alpha(s-v)} \quad (18)$$

$a$  is a constant to be determined. The share of labor income received by an agent is an exponentially decreasing function of age. The value of  $a$  is determined by the condition that:

$$\begin{aligned} Y(v) &= \int_{\infty}^v y(s,v)pe^{p(s-v)}ds \\ &= aY(v) \int_{\infty}^v pe^{(p+\alpha)(s-v)}ds = (pa/(p+\alpha))Y(v) \end{aligned}$$

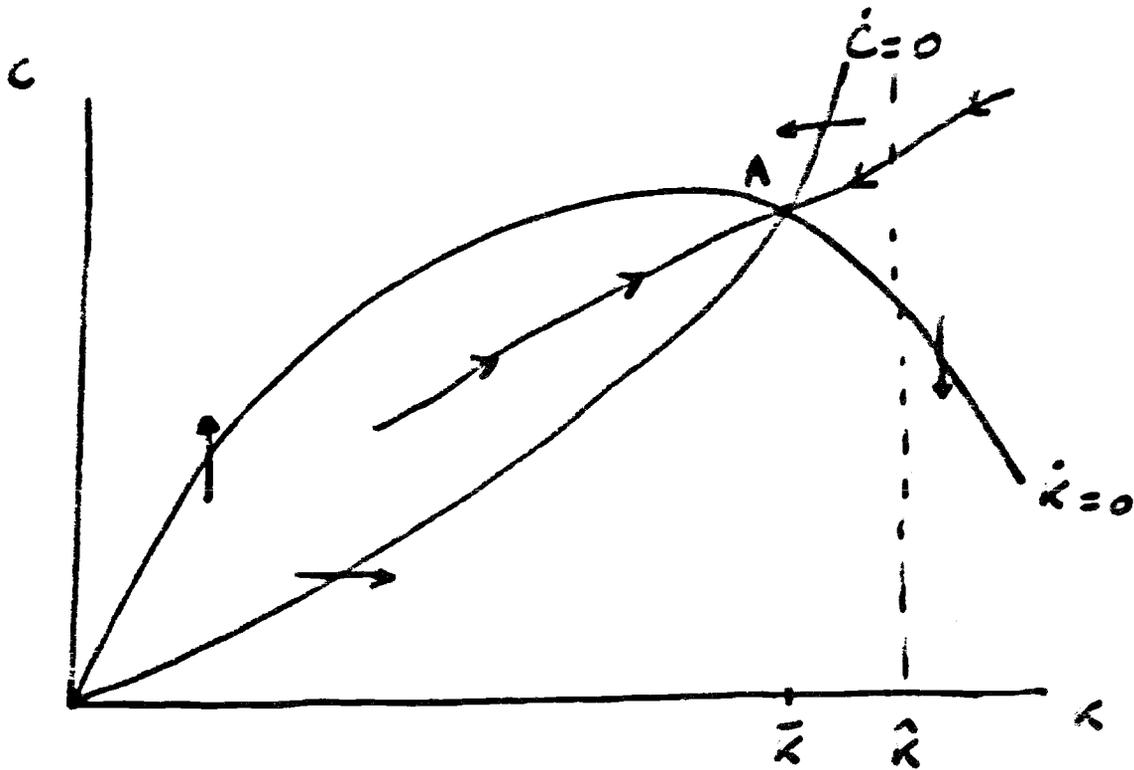
$$\Rightarrow a = (p + \alpha)/p.$$

Note that the case where  $\alpha$  is positive is well defined only if  $p$  is strictly positive. If  $p=0$ , agents are infinitely long lived. Individual labor income is the same, up to a constant, as aggregate labor income; it cannot be a decreasing function of age.

The derivation of individual consumption is identical to that of Section I. The derivation of aggregate consumption is also the same, except for aggregate human wealth. Given (18), individual human wealth is given by:

$$h(s,t) = e^{\alpha(s-t)} \left[ \left( \frac{p+\alpha}{p} \right) \int_t^{\infty} Y(v) e^{-\int_t^v (\alpha+r(\mu)+p)d\mu} dv \right]$$

Figure 4. Declining labor income



$$\hat{k} | r(\hat{k}) = \theta - \alpha$$

Note that the term in brackets is the same for all agents. Thus aggregate human wealth is:

$$\begin{aligned} H(t) &\equiv \int_{-\infty}^t h(s,t) p e^{p(s-t)} ds \\ &= \int_t^{\infty} Y(v) e^{-\int_t^v (\alpha + r(\mu) + p) d\mu} dv \end{aligned}$$

The effect of declining labor income is to increase the discount rate on future aggregate labor income. This is because agents currently alive will receive, even if still alive in the future, a smaller and smaller share of total income.

Collecting equations gives the following:

$$C = (p + \theta)(H + W) \quad (19)$$

$$\dot{H} = (r + p + \alpha)H - Y \quad (20)$$

$$\dot{W} = rW + Y - C \quad (21)$$

The equation for human wealth is given in differential form; the only effect of  $\alpha$  is to further increase the discount rate on future labor income above the interest rate.

The alternative characterization of the dynamics, obtained by differentiating (14) and eliminating  $\dot{H}$  and  $\dot{W}$  using (20) and (21) is:

$$\dot{C} = (r + \alpha - \theta)C - (p + \alpha)(p + \theta)W$$

$$\dot{W} = rW + Y - C$$

Let's now turn to the dynamics and steady state of the closed economy. The dynamics are characterized by:

$$\dot{C} = (r(K) + \alpha - \theta)C - (p + \alpha)(p + \theta)K \quad (22)$$

$$\dot{K} = F(K) - C \quad (23)$$

The phase diagram associated with (22) and (23) is drawn in Figure 4. The  $\dot{C} = 0$  locus goes through the origin, is convex and reaches asymptotically  $\hat{K}$ , where  $r(\hat{K}) = \theta - \alpha$ . Note that, as  $\theta - \alpha$  can be positive or negative,  $\hat{K}$  can be on

either side of the golden rule capital stock.<sup>7</sup> The  $\dot{K}$  locus traces the net production function.

The equilibrium is saddle point stable. The stable arm SS is upward sloping.

What are the characteristics of the steady state capital stock? As in Section II, the capital stock  $\bar{K}$  is such that  $r(\bar{K}) < \theta + p$ . Suppose it was not. Then:

$$r(\bar{K}) \geq \theta + p \Rightarrow$$

$$(r(\bar{K}) - \theta + \alpha)\bar{C} \geq (p + \alpha)\bar{C}.$$

$$\text{Equation (22) and } \dot{C} = 0 \Rightarrow (p + \alpha)(p + \theta)\bar{K} \geq (p + \alpha)\bar{C}$$

$$\text{Equation (23) and } \dot{K} = 0 \Rightarrow (p + \theta)\bar{K} \geq F(\bar{K})$$

$$\text{As } r(\bar{K}) \geq \theta + p \Rightarrow r(\bar{K})\bar{K} \geq F(\bar{K}) \text{ which is impossible.}$$

There is however no guarantee that  $r(\bar{K}) > \theta$ . Indeed, for  $\alpha$  large enough,  $r(\bar{K})$  may be negative. Using a standard terminology, the equilibrium may be, as in Diamond, inefficient. An increase in  $\alpha$  increases steady state capital. To see this, consider

$$\frac{dC}{d\alpha} \Big|_{\dot{C} = 0, K = \bar{K}} = \frac{(p + \theta)(r(\bar{K}) - \theta - p)}{(r(\bar{K}) - \theta - \alpha)}$$

$$\text{As } r(\bar{K}) < \theta + p, \text{ and } r(\bar{K}) > \theta - \alpha,$$

$$\frac{dC}{d\alpha} \Big|_{\dot{C} = 0, K = \bar{K}} < 0.$$

In the neighborhood of steady state, an increase in  $\alpha$  shifts the  $\dot{C} = 0$  locus to the right, increasing steady state capital.

To summarize, the effect of finite horizons per se is to decrease capital accumulation. The effect of declining labor income, of saving for retirement is however to increase it. The net effect is ambiguous and the steady state can be inefficient.

We now turn to the effects of fiscal policy.

Section IV. Effects of taxes on aggregate demand.

The government budget constraint.

We now introduce a government which spends on goods which do not affect the marginal utility of private consumption and finances spending either by lump-sum taxes <sup>8</sup>, or by debt. Its dynamic budget constraint is:

$$\dot{D} = rD + G - T$$

D is debt, G is spending, T taxes. We shall refer to  $T - G$  as the surplus, or deficit as the case may be, and to  $\dot{D}$  as the change in debt. This is only a semantic convention. The government is also required to satisfy the transversality condition:

$$\lim_{t \rightarrow \infty} D_t e^{-\int_s^t r_v dv} = 0$$

This condition, together with the dynamic budget constraint is equivalent to the statement that the level of debt is equal to the present discounted value of future surpluses:

$$D_t + \int_t^{\infty} G_s e^{-\int_t^s r_v dv} ds = \int_t^{\infty} T_s e^{-\int_t^s r_v dv} ds \quad (24)$$

The presence of taxes modifies slightly the aggregate consumption function of Section I which becomes:

$$C_t = (p+\theta)(H_t + W_t); W_t = D_t + K_t \quad (25)$$

$$H_t = \int_t^{\infty} Y_s e^{-\int_t^s (r_v+p)dv} ds - \int_t^{\infty} T_s e^{-\int_t^s (r_v+p)dv} ds \quad (26)$$

$$\dot{W}_t = r W_t + Y_t - C_t - T_t \quad (27)$$

Financial wealth,  $W$ , now includes government debt  $D$  and other assets,  $K$ .

Human wealth is the present discounted value of non-interest income minus taxes, discounted at rate  $(r+p)$ .

#### Effects of a reallocation of taxes.

Consider a decrease in taxes at time  $t$  associated with an increase at time  $t + \tau$ . Given the government budget constraint (24), and given the level of debt  $D_t$  and the path of  $G$  being unchanged, these changes must satisfy:

$$dT_{t+\tau} = -e^{-\int_t^{t+\tau} r_v dv} dT_t$$

The effect on the consumer at time  $t$  is given by the effect on human wealth.

Equation (26) implies an effect of:

$$-dT_t - dT_{t+\tau} e^{-\int_t^{t+\tau} (r_v+p)dv}$$

or using the government budget constraint, an effect of  $-dT_t(1 - e^{-p\tau})$ . Thus, unless  $p = 0$ , a decrease in taxes today increases human wealth and consumption.

The longer taxes are deferred, the larger the effect. This effect of reallocation of taxes comes from the different discount rates in the government budget constraint and in the definition of human wealth. This in turn reflects the fact that taxes are partly shifted to future generations.

An index of fiscal stance

Fiscal policy, that is, the sequence of current and anticipated taxes, spending and debt, affects aggregate demand in three ways. Debt is part of wealth and affects consumption; the sequence of taxes affects human wealth and thus consumption. The level of government spending affects aggregate demand directly. It is useful, both conceptually and technically, to summarize these effects by an index of fiscal policy. Let  $g$  denote this index, so that, collecting all the terms in aggregate demand which depend directly on fiscal policy:

$$g_t = (p+\theta) \left( D_t - \int_t^\infty T_s e^{-\int_t^s (r_v+p)dv} ds \right) + G_t$$

This index can be rewritten as:

$$g_t = G_t - (p+\theta) \int_t^\infty G_s e^{-\int_t^s (r_v+p)dv} ds + (p+\theta) \left( D_t + \int_t^\infty (G_s - T_s) e^{-\int_t^s (r_v+p)dv} ds \right) \quad (28)$$

The first line gives the effects of the sequence of spending on aggregate demand if it is financed exclusively by contemporaneous taxes. Changes in spending are not the subject of this paper. Note however that if  $r = \theta$ , a permanent change in spending has no effect on aggregate demand. The second line gives the effects of financing. Note first that if  $p = 0$ , the government budget constraint (24) implies that this line is identically equal to zero; once again financing is irrelevant. If  $p$  is positive, this is not the case, and if  $D_t$  is positive, this term is positive.

How does this index evolve over time for a given fiscal policy? Consider a policy characterized by large deficits initially, followed later, as debt accumulates, by surpluses so as to satisfy the intertemporal government budget

constraint. Does the initial increase in  $g$  disappear as surpluses appear, or does  $g$  remain positive as debt accumulates? An example, inspired by the current U.S. fiscal policy, will suggest the answer. The specific fiscal policy we consider is characterized by:

$$\dot{D}_t = rD_t - T_t; T_t = \beta D_t - \sigma; D_0 = 0 \quad (29)$$

Government spending is equal to zero for simplicity. Taxes are an increasing function of the level of debt. A necessary and sufficient condition for the government transversality condition to be satisfied is that  $\beta \geq r$ , so that an increase in debt reduces  $\dot{D}$ . The interest rate  $r$  is assumed constant.

Consider the effect of an increase of  $\sigma$  from zero, which implies initially a sequence of deficits. As debt accumulates, taxes increase, surpluses appear and debt converges to a new steady state value,  $D_\infty$ , where  $D_\infty = \sigma/(\beta-r)$ . The smaller  $(\beta-r)$ , the larger the sequence of deficits, the higher the steady state level of debt.

Solving (29) for the path of  $D$  and  $T$ , and replacing in (28) gives:

$$g_t = (p+\theta)\sigma \frac{p}{\beta-r} \left( \frac{1}{r+p} - \frac{1}{\beta+p} e^{-(\beta-r)t} \right), \text{ so that:} \quad (30)$$

$$g_0 = (p+\theta)\sigma \frac{p}{(r+p)(\beta+p)}$$

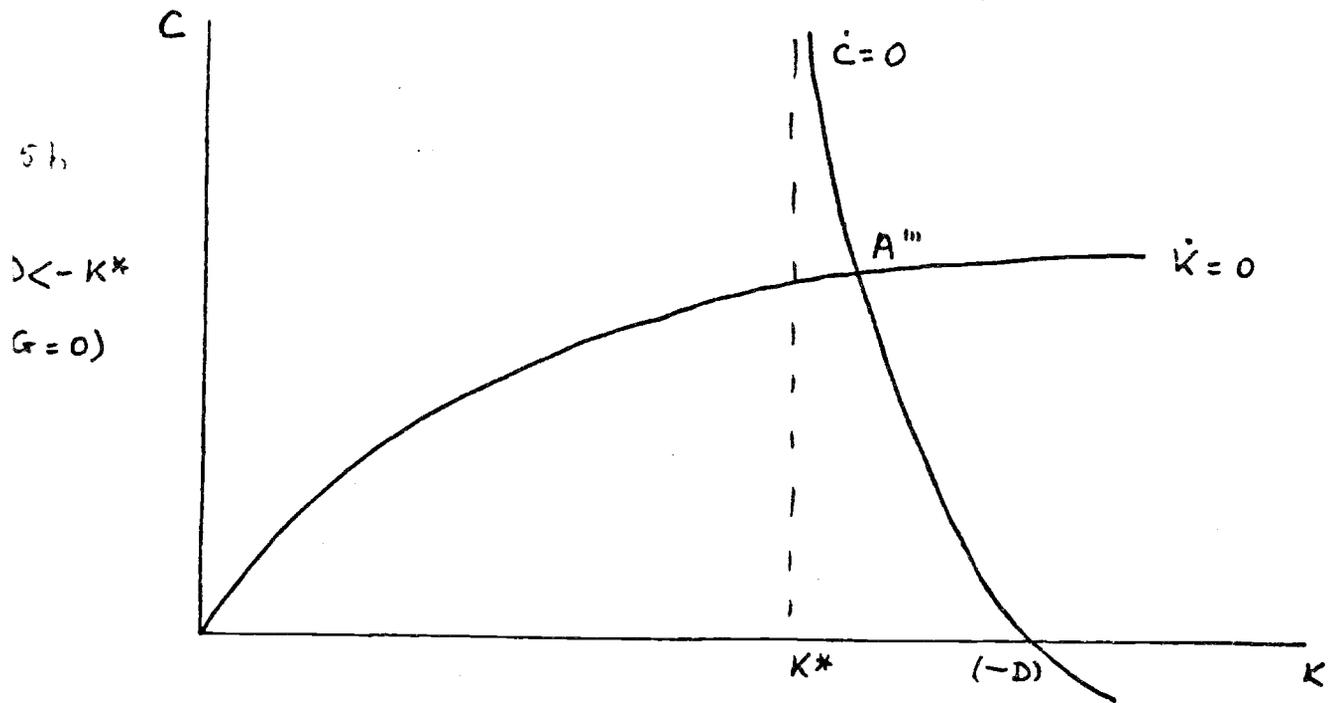
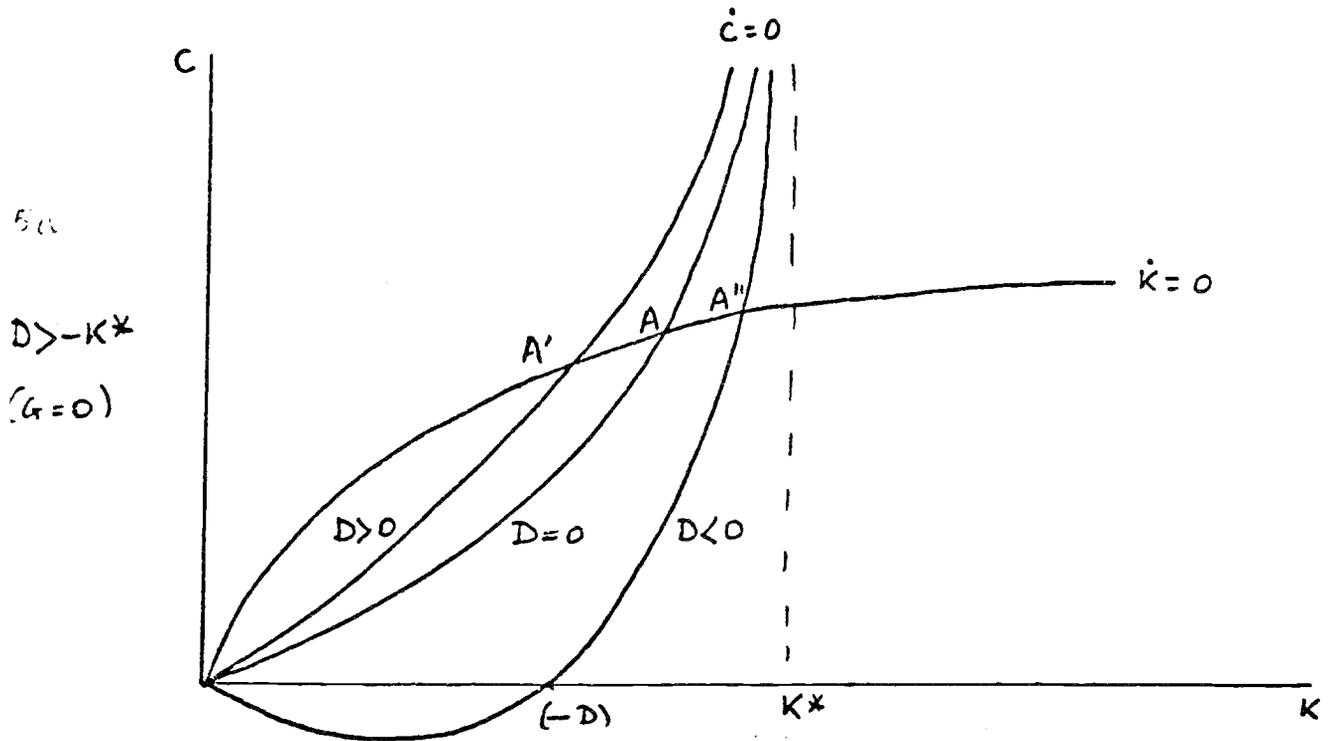
$$g_\infty = (p+\theta)\sigma \frac{p}{(r+p)(\beta-r)} = (p+\theta) \frac{p}{r+p} D_\infty > g_0$$

The index of fiscal policy therefore increases over time. At time zero, it depends on the sequence of anticipated deficits, and thus on  $\beta$ .

It increases over time to reach  $g_\infty$ . The value of  $g_\infty$  is easy to understand. In steady state, taxes required to pay interest on the debt are equal to  $r D_\infty$ . Their present value in human wealth is  $(r/(r+p))D_\infty$  and thus the net effect of debt is  $(p+\theta)(D_\infty - (r/(r+p))D_\infty)$ , which is equal to  $g_\infty$ .

We now turn to the general equilibrium effects of fiscal policy.

Figure 5 Closed economy. Steady state effects of debt.



Section V. Steady state effects of fiscal policy.

We return to the closed and open economies of Section II, and characterize the effects of fiscal policy on steady state levels of assets and consumption.

The closed economy

In the presence of fiscal policy, the equations of motion become:

$$\dot{C} = (r(K) - \theta)C - p(p+\theta)(D + K)$$

$$\dot{K} = F(K) - C - G$$

$$\dot{D} = r(K)D + G - T$$

This does not completely characterize the dynamics. A characterization of movements in  $T$  and  $G$  which satisfies the government transversality condition is also needed. The dynamics of the system are therefore, in general, difficult to characterize. However we can study the dynamics of the  $(C, K)$  system for constant values of  $D$  and  $G$ . If we consider constant values of  $D$ ,  $\dot{D}$  is obviously equal to zero and thus implicitly, taxes vary to satisfy  $T = r(K)D + G$ .

We must distinguish between two cases depending on whether the level of debt is less or greater than  $(-K^*)$  where, as before,  $K^*$  is such that  $r(K^*) = \theta$ . These two cases are drawn in Figures 5a and b. Figure 5a draws the  $\dot{C} = 0$  locus for three different levels of debt, zero as in Section II, positive or negative. Figure 5b draws the  $\dot{C} = 0$  locus when  $D < -K^*$ .

If  $G = 0$ , there exists, for any level of debt, a unique equilibrium. This equilibrium is saddle point stable and the stable arm is upward sloping. All other trajectories imply negative consumption or capital in finite time. If  $G$  is positive however (which, on the phase diagram, shifts the  $\dot{K} = 0$  locus vertically downwards by an amount  $G$ ), there may be, if debt is larger than  $(-K^*)$ , zero or

two equilibria. If there are two equilibria, one is saddle point stable and the other strictly unstable. The case of zero or two equilibria is not considered further.

The levels of capital and of consumption are decreasing functions of the level of debt and the government can choose any level of capital by an appropriate choice of debt. If the government uses  $(\theta+p)$  as the social discount rate, it may achieve the desired level of capital  $K^{**}$ , such that  $r(K^{**}) = \theta+p$ , by issuing a positive amount of debt. If instead it uses  $\theta$  as the social discount rate, it must issue a negative amount of debt, in this case precisely in amount  $(-K^*)$ . The mechanism is clear: an increase in debt implies, as debt is partly perceived as net wealth, an increase in consumption and a decrease in capital accumulation until a lower steady state is reached.

### The Open Economy

In the presence of fiscal policy, the equations of motion become:

$$\dot{C} = (r-\theta)C - p(p+\theta)(D + F)$$

$$\dot{F} = rF + \omega - C - G$$

$$\dot{D} = rD + G - T$$

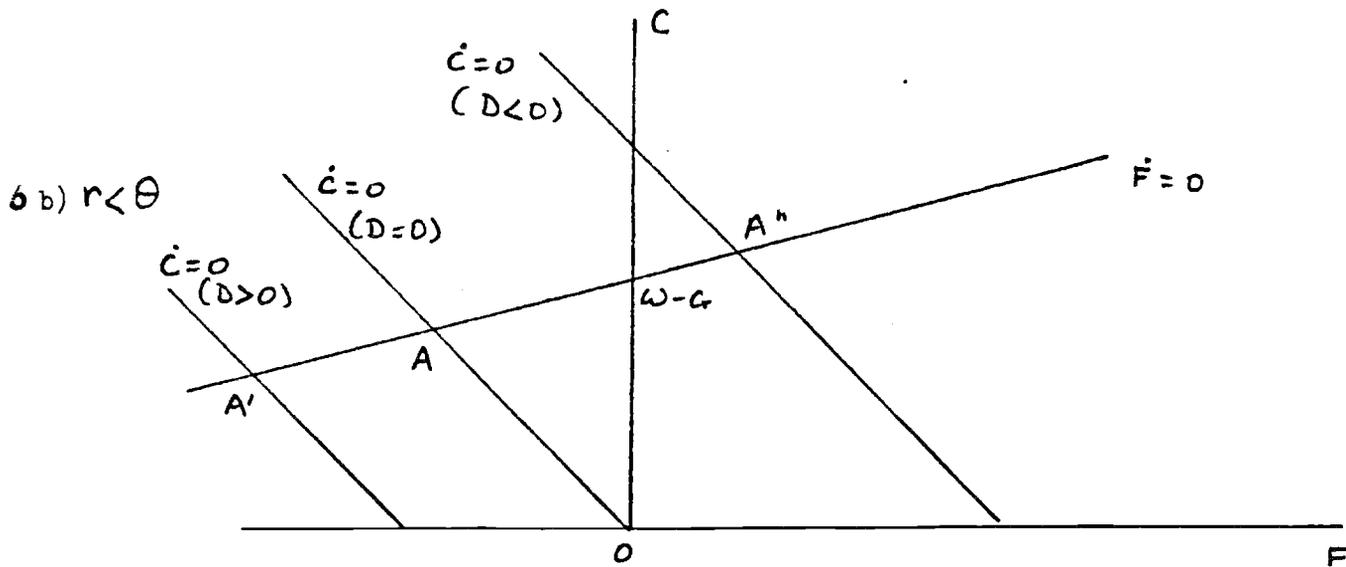
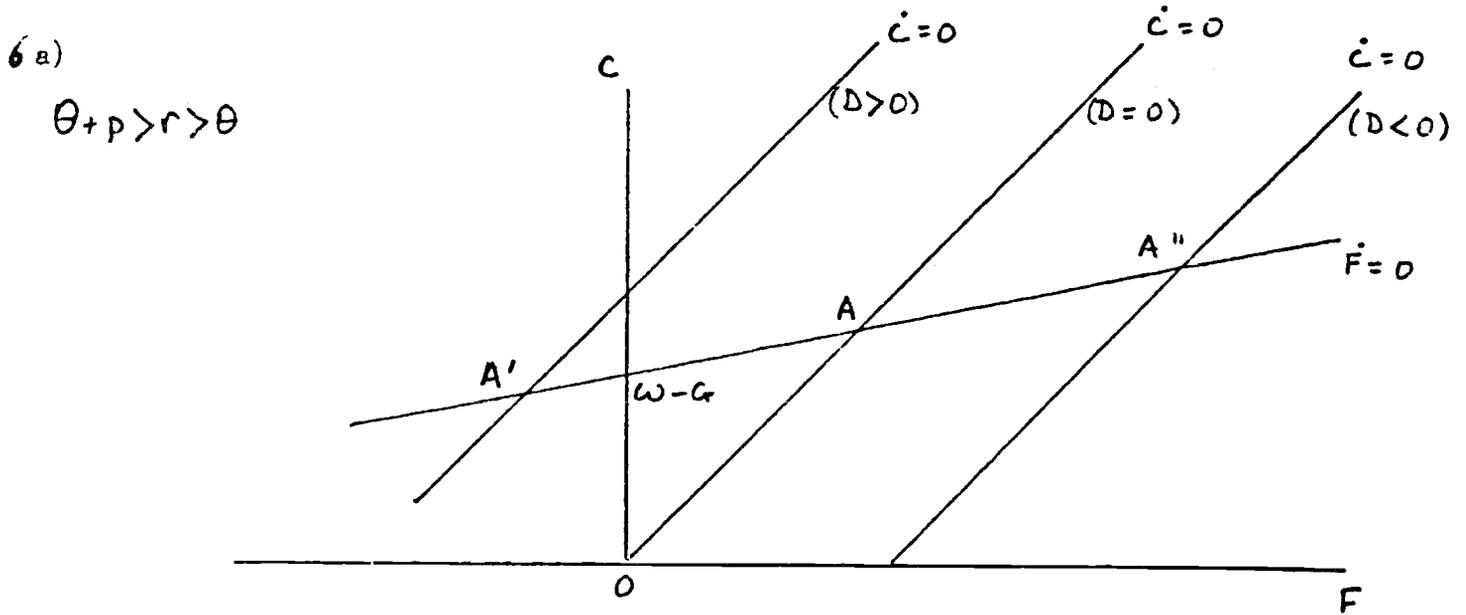
We may again describe the dynamics of  $C$  and  $F$  given constant values of  $D$  and  $G$ , that is, assuming  $\dot{D} = 0$  in the last equation and thus  $T = G + rD$ .

The phase diagram is shown in Figures 6a and b, which consider the cases  $r > \theta$  and  $r < \theta$ . The system is saddle point stable for any value of  $D$ . The effect of an increase in  $D$  is always to decrease the steady state level of  $F$  and of  $C$ . This effect is easier to understand if we return to the savings function. Consumption and income are now given respectively by

$$(p+\theta) \left( \frac{\omega-T}{r+p} + D + F \right) \quad \text{and} \quad \omega - T + r(D+F).$$

This gives the savings function:

Figure 6. Open economy. Steady state effects of debt.



$$S = ((r-\theta)/(r+p))(\omega-G) + (r-p-\theta)F - ((p+\theta)/(r+p))pD$$

Under the stability condition that  $r$  is between 0 and  $\theta+p$ , the propensity to save is negative out of government debt and other financial assets, ambiguous out of labor income. In particular if  $r = \theta$ :

$$S = -pF - pD$$

This implies, as steady state savings equal zero,

$$\frac{dF}{dD} < 0 \text{ and } \frac{dF}{dD} \geq -1 \text{ as } r \geq \theta$$

Thus government debt displaces foreign assets in the agents' wealth. The displacement is one for one if  $r = \theta$ , but may be much larger if  $r$  is greater than  $\theta$ . These results again differ sharply from the infinite horizon where the level of debt has no effect on the steady state level of  $F$ . By its debt policy, the government can choose any level of private consumption.

We now turn to the dynamic effects of specific fiscal policies.

Section VI. Dynamic Effects of Fiscal Policy.

If a fiscal policy implies low debt and high deficits initially, and high debt and surpluses in steady state, we may suspect that its dynamic effects are quite different from its steady state effects. If the economy is closed, we want to know the effects of the fiscal policy on the sequence of interest rates and asset accumulation. If the economy is open and interest rates given, the focus is on the current account and asset accumulation. We shall limit ourselves to the analytically simpler case of an open economy and consider two examples.

Government and Current Account Deficits

We characterize the effects of a fiscal policy which resembles that of the US. The fiscal policy is the same as the policy studied in Section IV, that is of an initial increase at time  $t = 0$  in deficits, followed later on by surpluses. This policy is assumed to be unanticipated as of  $t < 0$ . The economy is the open economy studied in Section V. The equations of motion are given by:

$$C = (p+\theta)(F + H + D)$$

$$\dot{H} = (r+p)H - (\omega-T); \quad T = \alpha D - \sigma$$

$$\dot{F} = rF + \omega - C$$

$$\dot{D} = (r-\alpha)D - \alpha; \quad G = 0; \quad D_0 = 0$$

The alternative characterization of consumption, as a linear function of wealth is here more convenient. In this case, it is also very convenient to use the index of fiscal policy  $g$ , defined in Section III and derived for this particular fiscal policy, also in Section IV. The system becomes:

$$C = (p+\theta)(F + \omega/(r+p)) + g$$

$$\dot{F} = rF + \omega - C$$

$$\dot{g} = -(\beta-r)g + (p+\theta)\sigma p/(r+p); \quad g_0 = (p+\theta)\sigma p/(r+p)(\alpha+p)$$

The effect of fiscal policy on the dynamics of  $C$  and  $F$  is entirely captured by the single index  $g$ . The behavior of  $g$ , for the specific fiscal policy we consider was derived in equation (30) and is written here equivalently as a differential equation.

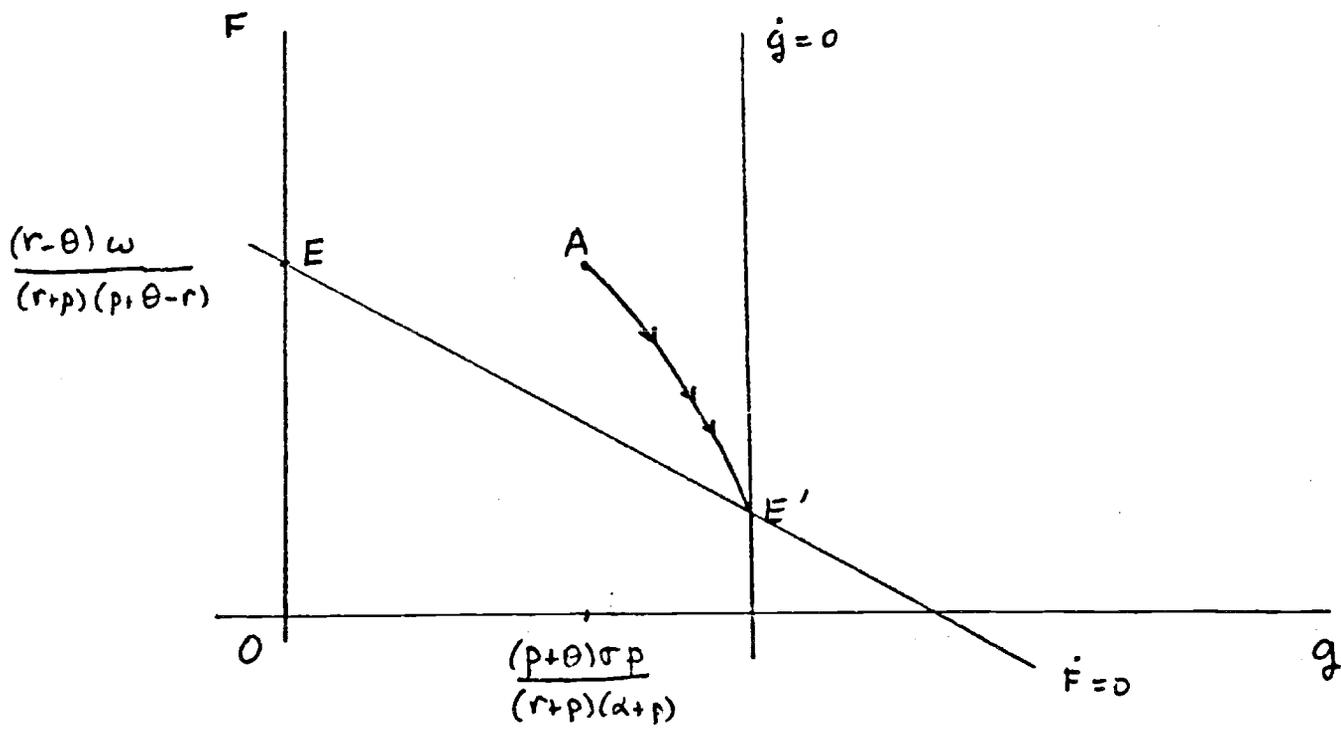
The dynamics of  $F$  and  $g$  are characterized in Figure 7. The  $\dot{F} = 0$  locus is downward sloping, with slope  $(p+\theta-r)^{-1}$  (we maintain the stability condition  $r < p + \theta$ ). The initial steady state is at point  $E$ . At  $t = 0$ , the locus  $\dot{g} = 0$  shifts to the right,  $g$  jumps from zero to a positive value,  $\sigma(p+\theta)p/(r+p)(\alpha+p)$ . Over time the economy converges from  $A$  to  $E'$ .

The effect of fiscal policy is thus of a decumulation of foreign assets as government debt accumulates. The rates of foreign asset accumulation and government debt accumulation are not however related in any simple way: the rate of foreign asset decumulation depends not only on the current deficit, but on the entire sequence of deficits (surpluses). For example, if  $r = \theta$ , we have seen in the previous section that in the long run, the decrease in foreign assets is equal to the increase in government debt. At  $t = 0$  however, the effect of the rate of change of government debt on the rate of change in foreign assets is given by:

$$(d\dot{F}/d\dot{D})|(t=0) = -g_0/\sigma = p/(\alpha+p)$$

Thus the effect is less than one for one. It tends to one as  $p$  tends to infinity, as the horizon of agents shortens. It tends to  $p/(r+p)$  as  $\alpha$  reaches its lower bound (for stability) of  $r$ .

Figure 7.



We learn from this example that we may think of the effects of fiscal policy through its effects on the index  $g$ , and that although fiscal deficits will be associated with current account deficits, there is no simple relation between the two.

### Output Cycles and the Role of Debt Policy.

This second example examines the effects of regular fluctuations in output on the decentralized economy, and the role of debt in such a case.

The economy is open, and for simplicity we assume  $r = 0$ . If we define total financial wealth  $W$  as the sum of foreign assets and government debt, the equations of motion of the previous section can be rewritten as

$$\dot{C} = -p(p+\theta)W$$

$$\dot{W} = rW + \omega - C - T$$

$$(\dot{D} = rD + G - T).$$

Suppose now that  $\omega_t$  follows  $\omega_t = \sigma + \sin t$ .

What will, in the absence of fiscal policy ( $T = D = G = 0$ ), the behavior of  $C$  and  $W$  be?

Some algebra yields:

$$\begin{aligned}
C_t &= \alpha \sin t + \beta \cos t + \alpha \\
W_t &= \delta \sin t + \gamma \cos t + \sigma && \text{where} \\
\Delta &\equiv (\theta^2 + (1 + p(p+\theta))^2)^{-1} && \text{and} \\
\alpha &= \Delta p(p+\theta)(1 + p(p+\theta)) && \doteq p(p+\theta) \\
\beta &= -\Delta p(p+\theta)\theta && \doteq p(p+\theta)\theta \\
\gamma &= -\Delta (1 + p(p+\theta)) && \doteq -1 \\
\delta &= \Delta \theta && \doteq \theta
\end{aligned}$$

The approximations to  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$  hold if  $p$  and  $\theta$  are not far from zero.

If  $p = 0$ , agents completely smooth out consumption. Aggregate consumption is constant. Agents accumulate foreign assets when  $\omega$  is high, decumulate when  $\omega$  is low.

If  $p > 0$ , aggregate consumption is cyclical. As  $r = \theta$ , each agent still has a flat consumption path. Newly born do not however have the same level of consumption as those who die. If  $\theta = 0$ , aggregate consumption moves in phase with income, but by less.

These movements in aggregate consumption suggest a role for fiscal policy. As  $r = \theta$ , the consumption of each individual is constant throughout life. Different cohorts however have different levels of consumption. Thus, if the social welfare function is concave in individual utilities, it is desirable to smooth consumption across cohorts. Fiscal policy can achieve constant aggregate consumption over time. The equations of motion above show how this can be done:  $T$  must simply equal  $\sin t$ . If all deviations of  $\omega$  from its mean  $\sigma$  are taxed, consumption and wealth will be constant. Government debt and foreign assets follow symmetric but opposite paths. As government debt increases for example, it displaces one for one foreign assets in the agents' portfolio.

This change in portfolio composition has no effect, in an open economy, on either  $w$  or  $r$ . This would not be the case if the same debt policy was pursued in a closed economy: variations in the capital stock would affect both  $w$  and  $r$ . A characterization of optimal policy would be substantially more difficult.

Conclusion

The purpose of this paper was to characterize rigorously the effects of intertemporal reallocations of taxes when agents have finite horizons. To that end, many assumptions, such as the existence of lump-sum taxes, or a constant employment level, were made which need to be removed to obtain a more realistic characterization of the effects of debt and deficits. The aggregate consumption function developed here seems well adapted to the task. The index of fiscal policy should also prove useful both conceptually and empirically.

Footnotes.

<sup>1</sup>An alternative interpretation, suggested by Robert Barro, is to think not of agents, but of families.  $p$  is then the probability that either the family ends, that is that members of the family die without children, or that the current members of the family have no bequest motive. The assumption of a constant  $p$  is more acceptable under this interpretation.

<sup>2</sup>How unrealistic is the assumption of a constant  $p$ ? Evidence on mortality rates suggests low and approximately constant probabilities from age 20 to 40. After this, mortality rates are well summarized by "Gompertz's Law" (see Wetterstrand [1983]):  $h_y = 1 - e^{-\mu_y}$ ;  $\mu_y = BC^y$

where  $y$  is age,  $g$  is the mortality rate,  $B$  and  $C$  are positive constants.

Estimates are for example:  $g_{50} = 1\%$ ;  $g_{60} = 3\%$ ;  $g_{80} = 16\%$ ;  $g_{100} = 67\%$ .

<sup>3</sup>The role of insurance when there is uncertainty about time of death was studied by Yaari [1965] An equivalent assumption is that there exists actuarial bonds. Lenders lend to intermediaries. These claims are cancelled by the death of the lenders. Borrowers borrow from intermediaries; these claims are cancelled by death of the borrowers. Intermediation can again be done risklessly.

<sup>4</sup>The assumption of a constant probability of death implies that the objective function (equation (1)) does not change through time. There is therefore no issue of time consistency.

<sup>5</sup>This condition is the extension to infinite time of the condition proposed by Yaari [1965].

<sup>6</sup>Such a specification, allowing for a higher discount rate for human wealth, has been estimated by Hayashi [1982]. His estimated coefficients,  $\alpha$ ,  $\mu$ ,  $\rho$  are related to  $p$ ,  $\theta$ ,  $r$  by:  $p = \mu - \rho$ ;  $\theta = \alpha - \mu + \rho$ ;  $r = \rho$ . His estimates (Table 1,

$\lambda = 0$ ) imply, at annual rates:

$$p = .10; \quad r = .03; \quad \theta = -.03$$

<sup>7</sup>As there is no population growth in this model, the golden rule is that the net marginal product of capital be equal to zero,  $r(K) = 0$ .

<sup>8</sup>The assumption that the government can use lump-sum taxation is made to focus on the effects of finite horizons. As is well understood, if lump-sum taxation is not available, there is a role for debt policy even if agents have infinite horizons.

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