RARE DISASTERS AND EXCHANGE RATES

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ABSTRACT

We propose a new model of exchange rates, which yields a theory of the forward premium puzzle. Our explanation combines two ingredients: the possibility of rare economic disasters, and an asset view of the exchange rate. Our model is frictionless, has complete markets, and works for an arbitrary number of countries. In the model, rare worldwide disasters can occur and affect each country's productivity. Each country's exposure to disaster risk varies over time according to a mean-reverting process. Risky countries command high risk premia: they feature a depreciated exchange rate and a high interest rate. As their risk premium mean reverts, their exchange rate appreciates. Therefore, currencies of high interest rate countries appreciate on average. To make the notion of disaster risk more implementable, we show how options prices might in principle uncover latent disaster risk, and help forecast exchange rate movements. We then extend the framework to incorporate two factors: a disaster risk factor, and a business cycle factor. We calibrate the model and obtain quantitatively realistic values for the volatility of the exchange rate, the forward premium puzzle regression coefficients, and near-random walk exchange rate dynamics. Finally, we solve a model of stock markets across countries, which yields a series of predictions about the joint behavior of exchange rates, bonds, options and stocks across countries. The evidence from the options market appears to be supportive of the model.

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1   Introduction

We propose a new model of exchange rates, which yields a theory of exchange rate "excess volatility" and of the forward premium puzzle. We build on the idea proposed by Rietz (1988), Barro (2006) and Weitzman (2007) that the possibility of rare but extreme events is a major determinant of risk premia in asset markets, and develop a tractable framework to study its consequences for international asset prices.

Our model allows us to propose a solution for a major puzzle in international macroeconomics, the failure of uncovered interest rate parity (UIP). According to the UIP equation, the expected depreciation of a currency should be equal to the interest rate differential between that country and the reference region. A regression of exchange rate changes on interest rate differentials should yield a coefficient of 1. However, empirical studies starting with Hansen and Hodrick (1980) and Fama (1984), and recently surveyed by Froot and Thaler (1990), Lewis (1995) and Engel (1996), consistently produce a regression coefficient that is less than 1, and often negative. This invalidation of UIP has been termed the *forward premium puzzle*: currencies with high interest rates tend to appreciate. In other words, currencies with high interest rates feature positive predictable excess returns. There are four possible explanations: time-varying risk premia, Peso problems, expectations errors, and illiquid markets[1].

Our paper provides a theory of international time-varying risk premia in a complete markets, frictionless and rational framework. In our model, the exchange rate is both a relative price of non-traded and traded goods, and an asset price: it is the net present value of the export sector's productivity. For this, we develop a "stock view" of the exchange rate. We postulate (to the best of our knowledge, in a novel way), a linear technology by which an initial investment of non-tradable goods yields a stream of future export goods, with stochastic productivity. As a result, the exchange rate is the present value of the country's future export productivities, discounted using the international pricing kernel. This "stock view" of the exchange rate could be used in other contexts, e.g. with habit formation. In our model, risky countries are those whose productivity will fall by a lot during disasters.

Risky countries command high risk premia: they feature a depreciated exchange rate and a high interest rate. As their risk premium reverts to the mean, their exchange rate appreciates. Therefore, the currencies of high interest rate countries appreciate on average. This provides an explanation for the forward premium puzzle. The model is consistent with a forward premium puzzle, both in sample with and without disasters. Therefore it does not suffer from a Peso problem. The driving force of our result is that the risk premium covaries positively with interest rate. In other words, our theory does not rely on mismeasurement of expectations. We calibrate a version of the model and obtain quantitatively realistic values for the volatility of the exchange rate, the forward premium

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[1]Maynard and Phillips (2001) argue that the forward premium puzzle is the result of misspecification issues. The Fama (1984) regression assumes short-memory stationarity of the data, but evidence contradicts this. Misspecification arises from long memory in the forward premium (the independent variable), and this may induce a bias away from 1.

puzzle regression coefficients, and near-random walk exchange rate dynamics.

The model is very tractable, and expressions for the exchange rate, interest rate, risk premia, and forward premium puzzle coefficients are obtained in closed forms[2]. For this, it uses the modelling of environments with stochastic intensity of disasters proposed in a closed economy in Gabaix (2007b) (Rietz (1988) and Barro (2006) have constant intensity of disasters), and the "linearity-generating" processes developed in Gabaix (2007a). Our framework is also very flexible. We show that it is remarkably easy to extend the basic model to incorporate several factors corresponding to productivity, disaster risk and inflation.

Currency option prices potentially contain rich information both on the probability and severity of disasters and on currency risk premia. The model generates a rich pattern of implicit volatility curves for a given pair of currencies, with both a time varying "smile" – higher implicit volatilities for out of the money options – and a time varying "skew" – higher implicit volatilities for out of the money puts than out of the money calls. Indeed, the presence of extreme events generates a smile and the possibility for a country to be riskier than the other generates a skew. We show that in the model, the price of out of the money risk-reversals – and indicator of the premium of out of the money puts on out of the money calls – can be directly linked to currency risk premia. According to our theory, when out of the money puts are relatively more expensive, the corresponding currency should be depreciated and expected to appreciate. Campa, Chang and Reider (1998) and Carr and Wu (2007) provide evidence that, as predicted by the model, when out of the money put prices increase relative to out of the money call prices, the corresponding currency simultaneously depreciates. Farhi, Gabaix, Ranciere and Verdelhan (2008) confirm this result on a larger sample of currencies for a longer time period; in addition, this paper shows that high risk-reversal prices predict currency appreciations and tests a number of other joint predictions of the model for currency option prices and other international and domestic asset prices.

Finally, we provide a calibration of the model. We show that under some simple conditions, we match the volatility of exchange rates, interest rates, risk premia, and their half-life. Those conditions involves taking Barro's (2006) numbers, which imply that rare disasters matter 10 times as much as they would if agents were risk neutral. As tail events have their importance multiplied by 10, changes in beliefs about disasters translate into meaningful volatility. This is why the model yields a sizable volatility, which is hard to obtain with more traditional models (e.g., Obstfeld Rogoff 1995).

The model also offers a number of additional predictions. First, there should be a clear link between equity and currency risk premia through interest rates. High domestic interest rates imply high currency risk premia – an expected appreciation of the domestic currency – and low equity risk premia. Fama and Schwert (1977) and Campbell and Yogo (2006) provide evidence of the link between equity excess returns and nominal interest rates. Hau and Rey (2004) find that for Japan,

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[2]Pavlova and Rigobon (2007) also provide an elegant and tractable framework for analyzing the joint behavior of bonds, stocks and exchange rates, which is successful at accounting for comovements among international assets. However, the model is based on a tradition consumption CAPM and therefore generates small risk premia and small departures from UIP.

France, Germany and Switzerland, a negative shock to the foreign stock market – relative to the US– lead to a foreign currency appreciation.

Second, the model has rich implications for the relation between the relative shape of the yield curves between two countries and the expected change in the bilateral exchange rate. Boudoukh, Richardson and Whitelaw (2006) regress the exchange rate movement on the T-period forward rate from T periods ago, and find that the regression coefficient increases towards 1 with the horizon T. Indeed, our theory is consistent with this empirical finding in a context where risk-premia are rapidly mean-reverting, and productivity is slowly mean-reverting.

The pricing kernels we derive are flexible and attractive reduced-form candidates for models with time varying risk premia associated with large currency movements. This is, we believe, a prevalent feature of foreign exchange markets: the carry trade is often referred to "picking dimes in front a of steam roller." Although we take the extra-step of linking these risk-premia to aggregate consumption risk, this step is not needed to derive the asset pricing implications of our framework: our model does not live or die on this particular hypothesis.

Moreover, time varying disasters are inherently difficult to assess. As such, they might be especially vulnerable to expectations errors. Although our model embodies rational expectations, it can also be interpreted along behavioral lines as a consistent framework to analyze the impact of investor sentiment on international asset prices[3].

Relation to the literature. This paper adds to a large body of theoretical work on the UIP condition. On the empirical side, Frankel and Engel (1984) show that a simple CAPM has difficulty explaining deviations from UIP. Most papers test the UIP condition on nominal variables. Two recent studies cast the puzzle in terms of real variables. Hollifield and Yaron (2003) decompose the currency risk premium into conditional inflation risk, real risk, and the interaction between inflation and real risk. They find evidence that real factors, not nominal ones, drive virtually all of the predictable variation in currency risk premia. Lustig and Verdelhan (2007a) find that real aggregate consumption growth risk is priced on currency markets. This provides support for a model which – like ours – focuses on real risk, abstracting from money and inflation. However, Burnside, Eichenbaum, Kleschelski and Rebelo (2006) document that forward premium strategies yield very high Sharpe ratios, but argue that the payoffs of such strategies are not correlated with traditional risk factors. This disagreement spurred a debate on whether or not consumption growth risk explains excess returns on currency speculation (Burnside 2007, Lustig and Verdelhan 2007b).

Our paper also contributes to the large literature on Peso problems in international finance. See Lewis (2008) for a recent survey. Of most interest to us is Kaminsky (1993) and Evans and Lewis (1995). Kaminsky (1993), extending the work of Engel and Hamilton (1990), considers the possibility for rare events to explain investors expectations about the exchange rate. Rare events in her model are infrequent switches from contractionary to expansionary monetary policy. She provides evidence that investors' expectations are consistent with the model. However, she does

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[3]See Frankel and Froot (1989) for an empirical investigation of the irrationality of investors' expectations, and more recently Frankel (2007) for an insightful discussion.

not examine the forward premium puzzle, and only considers one exchange rate (dollar-sterling) and a short time period. Interestingly, Evans and Lewis (1995) show that a reasonably calibrated regime switching model induces important biases in Fama regressions in small samples. These biases disappear, however, in large samples.

To the best of our knowledge, we are the first to adapt the Rietz-Barro paradigm to exchange rates. Guo (2007), subsequently, also adopts this paradigm, in the context of a monetary model, whereas the essence of our model is real.

On the theory side, numerous studies have attempted to explain the UIP puzzle in rational expectations settings. Few models, however, are able to reproduce the negative UIP slope coefficient.[4] Here we concentrate on some of the most successful studies. We start by reviewing arguments that rely on counter-cyclical risk premia. We then discuss the literature that departs from rational expectations and introduces behavioral biases.

Frachot (1996) shows that a two-country Cox, Ingersoll, and Ross (1985) framework can account for the UIP puzzle but it does not provide an economic interpretation of the currency risk premium. Backus, Foresi and Telmer (2001) pursue a similar line of research and show that affine models of the term structure can only rationalize the forward premium puzzle if either the state variables have asymmetric effects on state prices in different currencies or nominal interest rates take on negative value with positive probability. Our model is outside of the scope of their criticism: it is entirely real and does not belong to the affine class.

Alvarez, Atkeson, and Kehoe (2002) rely on a model with endogenously segmented markets to generate qualitatively the forward premium anomaly. In their model, higher money growth leads to higher inflation. This induces more agents to enter the asset market because the cost of non-participation is higher. This, in turn, decreases risk premia. They later extend this type of mechanism in Alvarez, Atkeson and Kehoe (2006).

Most recently, Verdelhan (2007) generates counter-cyclical risk premia via the varying habit formation models pioneered by Abel (1990) and Campbell and Cochrane (1999). In his model, the domestic investor expects to receive a positive foreign currency excess return in bad times when he is more risk-averse than his foreign counterpart. Times of high risk-aversion correspond to low interest rates at home. Thus domestic investors expect positive currency excess returns when domestic interest rates are low and foreign interest rates are high.

Finally, Colacito (2006) and Colacito and Croce (2006) apply Bansal and Yaron (2004)'s model with Epstein-Zin-Weil preferences to international economics. Bansal and Shaliastovich (2007) have two-country setting, rely on a perfect cross-country correlation among shocks to the long run components of consumption growth rates to reproduce the UIP puzzle.

Turning to explanations based on behavioral biases, Bacchetta and van Wincoop (2006) develop a model where information is costly to acquire and to process. Because of these costs, many investors optimally choose to assess available information and revise their portfolios infrequently. This rational inattention mechanism produces a negative UIP coefficient along the lines suggested

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[4]For the difficulty of generating a negative coefficient, see Hollifield and Uppal (1997).

by Froot and Thaler (1990) and Lyons (2001): if investors are slow to respond to news of higher domestic interest rates, there will be a continued reallocation of portfolios towards domestic bonds and a appreciation of the currency subsequent to the shock.

Finally, another strand of the literature departs from the assumption of frictionless markets. Using microstructure frictions, Burnside, Eichenbaum and Rebelo (2007) rely on asymmetric information and behavioral biases to explain the forward premium puzzle.

Outline. The rest of the paper is organized as follows. In Section 2, we set up the basic model and derive its implications for the forward premium puzzle. In Section 3, we extend the model to incorporate a business cycle factor in addition to the disaster risk factor. In Section 4, we present a calibration of the model. In Section 5, we derive the implications of our model for the pricing of currency options and explain how to extract information about currency risk premia from the volatility smile. The rest further extends the model. In Section 6, we look at the joint properties of currency risk premia and bond risk premia by examining the information contained in the yield curve. We also develop a nominal version of the model which allows for nominal risk premia. In Section 7, we analyze the relationship between currency risk premia and equity premia. Section 8 concludes. Most proofs are in Appendix B.

2 Model setup

2.1 Macroeconomic environment: The stock view of the exchange rate

We consider a stochastic infinite horizon open economy model. There are N countries indexed by i. Each country i is endowed with two goods, a traded good, called T, and a non-traded good, called NT_i. The traded good is common to all countries, the non-traded good is country-specific.

Preferences. In country i, agents value consumption streams $\left(C_{it}^T, C_{it}^{NT_i}\right)_{t \geq 0}$ according to

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \exp(-\delta t) \frac{\left(C_{it}^T\right)^{1-\gamma} + \left(C_{it}^{NT_i}\right)^{1-\gamma}}{1 - \gamma} \right] \tag{1}$$

Note that the two goods enter separably in the utility function. Together with the assumption of complete markets, this will allow us to derive a simple expression for the pricing kernel.

Numéraires. Our choice of numéraires follows the Harberger convention: we choose the traded good T to be the international numéraire, and the non-traded good NT_i to be the numéraire in country i. We will sometimes call the traded good the "international good" or the "world currency".

We call e_{it} the exchange rate of country i in terms of the international good, with the convention that a high e_{it} means a "high value" domestic currency (when e_{it} increases, the domestic currency appreciates).[5] Hence, if a good has a price p_{it} in the currency of country i, it has price $p_t^* = e_{it}p_{it}$ in terms of the world currency. Stars (*) denote values in terms of the international good.

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[5]This choice of numéraire, although it does not follow the tradition which is to define the numéraire as a basket of goods in the country, brings tractability to the analysis.

As the non-traded good NT_i is the numéraire in country i, its price in country i is $p_{it}^{NT_i} = 1$. Hence, its price in terms of the traded good is $p_t^{NT_i*} = e_{it} p_{it}^{NT_i}$, so that

$$e_{it} = p_t^{NT_i*} \tag{2}$$

The exchange e_{it} rate of country i, in terms of the international currency (i.e., in terms of the traded good), is simply the price of the non-traded good of country i in terms of the traded good.

So, the exchange rate between country i and country j is the ratio of the e's of the two countries, e_{it}/e_{jt}.

Markets. Markets are complete: there is perfect risk sharing across countries in the consumption of international goods. Let C_t^{T*} be the world consumption of the traded good. The pricing kernel in terms of the traded good can therefore be expressed as

$$M_t^* = \exp(-\delta t) \left(C_t^{T*} \right)^{-\gamma}.$$

The pricing kernel means that an asset producing a stochastic stream $(D_{t+s})_{s \geq 0}$ of the traded good, has a price: $\mathbb{E}_t \left[\sum_{s=0}^{\infty} M_{t+s}^* D_{t+s} \right] / M_t^*$.

Technology. There is a linear technology to convert the non-traded good of country i into the traded good. By investing one unit of the non-traded good at time t, one obtains $\exp(-\lambda s)\omega_{i,t+s}$ units of the international good, at all periods $s \geq t$. The interpretation is that ω_{it} is the productivity of the export technology, and the initial investment depreciates at a rate λ.

Hence, the non-traded good is a capital good that produces dividends $D_{t+s} = \exp(-\lambda s)\omega_{i,t+s}$. So, in terms of the traded good, the price of the non-traded good NT_i of country i is:

$$p_t^{NT_i*} = \mathbb{E}_t \left[\sum_{s=0}^{\infty} M_{t+s}^* \exp(-\lambda s)\omega_{i,t+s} \right] / M_t^*$$

Given that $e_{it} = p_t^{NT_i*}$ (Eq. 2), the following obtains.

Proposition 1 *(Stock view of the exchange rate) In terms of the "international currency," the exchange rate e_{it} of country i is the discounted present value of its future export productivity:*

$$e_{it} = \mathbb{E}_t \left[\sum_{s=0}^{\infty} M_{t+s}^* \exp(-\lambda s)\omega_{i,t+s} \right] / M_t^* \tag{3}$$

with the convention that an increase in e_{it} means an appreciation of country i's currency.

In Eq. 3, $\omega_{i,t+s}$ is the productivity of country i's export sector at time $t + s$. M_{t+s}^* is the international pricing kernel, and is independent of country i.

To our knowledge, the above formulation is novel, complete-market microfoundation for the "asset view" of the exchange rate (Engel and West 2005 survey earlier "asset view" models, that feature incomplete markets). The exchange rate is the relative price of two goods, the traded and

the non-traded good.[6] At the same time, Eq. 3 gives us a stock view of the exchange rate: the exchange is a present value of future levels of productivity in the country. The above formulation could be used for many other models of the exchange rate. For instance, the stochastic discount factor M_{t+s}^* could come from a model with habit formation (Abel 1990, Bekaert 1996, Campbell Cochrane 1999), long run risk (Bansal and Yaron 2004), or first order risk aversion (Bekaert, Hodrick and Marshall 1997). We choose to study disasters, in part because they have been less studied.

Potential variants. To keep the model simple, we made a series of modelling choices that could be modified.

Let us first discuss our specification of technology. We have made the arguably strong assumption that the investment technology transforms non traded goods in a flow of traded goods. This assumption, however, is not crucial. We could have assumed, for example, that investment goods are a composite of traded and non traded goods. Let $f(e_t)$ be the relative price of investment goods – corresponding to the technology for producing investment goods – in terms of traded goods. We would then have the following formula for the exchange rate: $f(e_{it}) = \mathbb{E}_t \left[\sum_{s=0}^{\infty} M_{t+s}^* \exp(-\lambda s) \omega_{i,t+s} \right] / M_t^*$. Similarly, we could have let the output of the investment technology be a basket of traded and non-traded goods. The stochastic process for the exchange rate would then have had to be solved as the fixed point of a functional equation. The economics of the model would not be changed materially, but the analysis would become much more complex and closed form solutions would be lost.

Likewise, we could have used a different price index than the "Harberger" index puts a weight of 1 on non-tradables. This more sophisticated index would put a small positive weight on non-tradables. This would not importantly change our results, but would make the analysis more complex.

Last but not least, the utility function (1) could be changed to:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \exp\left(-\delta t\right) \frac{\left(C_t^T\right)^{1-\gamma}}{1-\gamma} \right] + V(\{C_t^{NT_i}\}_{t \geq 0}), \tag{4}$$

where V is any utility function over non-traded goods consumption processes $\{C_t^{NT_i}\}_{t \geq 0}$. With this formulation, our formulas for the exchange rate (e.g., Eq. 10-11) would still hold. The only thing that matters here is the marginal utility from one unit of tradable. Were we to follow this route, our model would generate a very imperfect correlation between total consumption and real exchange rates, which Backus and Smith (1993) have demonstrated holds in the data. For instance, V could incorporate habit formation or adjustment costs.

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[6] In our model, the real exchange rate is the relative price of non-tradables in terms of tradables. The merit of this identity has been investigated in the literature. The debate has focused on whether the relative price of non-tradables is as volatile as the real exchange rate. An important obstacle that any study in this area has to overcome is the identification in the data of pure tradable goods and pure non-tradable goods. Engel (1999) used different measures of tradable goods to conclude that variations in their relative price accounted for a small fraction of real exchange rate movements. Burstein, Eichenbaum and Rebelo (2005, 2006) emphasize that that retail prices of tradable goods contain a substantial fraction of non-tradable inputs. Using prices of traded goods at the dock, they find that the price of nontradable goods relative to tradable goods accounts for a substantial fraction of the movements in the real exchange rate.

2.2 Macroeconomic environment: Disaster risk

World consumption of the traded good. We will study equilibria where the world consumption of the traded good C^{T*} follows the following stochastic process. As Rietz (1988) and Barro (2006), we assume that in each period $t+1$, a disaster may happen, with a probability p_t. If a disaster does not happen, $C_{t+1}^{T*}/C_t^{T*} = \exp(g)$, where g is the normal-times growth rate of the economy. If a disaster happens, then $C_{t+1}^{T*}/C_t^{T*} = \exp(g)B_{t+1}$, with $B_{t+1} > 0$.[7] For instance, if $B_{t+1} = 0.7$, consumption falls by 30%. To sum up:

$$\frac{C_{t+1}^{T*}}{C_t^{T*}} = \begin{cases} \exp(g) & \text{if there is no disaster at } t+1 \\ \exp(g)B_{t+1} & \text{if there is a disaster at } t+1 \end{cases} \tag{5}$$

Hence the pricing kernel is given by

$$\frac{M_{t+1}^*}{M_t^*} = \begin{cases} \exp(-R) & \text{if there is no disaster at } t+1 \\ \exp(-R)B_{t+1}^{-\gamma} & \text{if there is a disaster at } t+1 \end{cases} \tag{6}$$

where

$$R = \delta + \gamma g_c$$

is the risk-free rate in an economy that would have a zero probability of disasters. For future reference, we refer to it as the Ramsey interest rate.

Process (5) can be rationalized as the general equilibrium outcome in a model with a finite number of countries, provided the endowments of those countries satisfy some conditions spelled out in Lemma 1 of Appendix B.

Productivity. We assume that productivity of country i follows:

$$\frac{\omega_{i,t+1}}{\omega_{i,t}} = \begin{cases} \exp(g_{\omega_i}) & \text{if there is no disaster at } t+1 \\ \exp(g_{\omega_i})F_{i,t+1} & \text{if there is a disaster at } t+1 \end{cases}$$

i.e. during disaster, the relative productivity of the traded good is multiplied by $F_{i,t+1}$. For instance, if productivity falls by 20%, then $F_{i,t+1} = 0.8$. We define the "resilience" of country i as:

$$H_{it} = p_{it}\left(\mathbb{E}_t\left[B_{t+1}^{-\gamma}F_{it+1} \mid \text{Disaster at } t+1\right] - 1\right) = H_{i*} + \widehat{H}_{it}. \tag{7}$$

where H_{i*} and $\widehat{H}_{it}$ are respectively the constant and variable part of the resilience. This is a measure of how well productivity is insulated from world disaster.[8] In (7), the probability p_t and world intensity of disasters B_{t+1} are common to all countries, but the recovery rate $F_{i,t+1}$ is country-specific. Of course, the recovery rates could be correlated across countries. In order to facilitate taking the continuous time limit, it is useful to define $h_{i*} = \ln(1 + H_{i*})$.

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[7] Typically, extra i.i.d. noise is added, but given that it never materially affects the asset prices, it is omitted here.

[8] This model addresses the concern of Brandt, Cochrane and Santa-Clara (2006), who note that discount factors must be highly correlated across countries. They are in this model, because the crisis affect all countries.

For tractability, we postulate a Linearity-Generating process (Appendix A) for $M_t^* e^{-\lambda t} (1, \omega_{it})$. The law of motion for $\widehat{H}_{it}$ is:

$$\widehat{H}_{it+1} = \frac{1 + H_{i*}}{1 + H_{it}} \exp(-\phi_{H_i}) \widehat{H}_{it} + \varepsilon_{i,t+1}^H, \tag{8}$$

where $\mathbb{E}_t \left[\varepsilon_{i,t+1}^H \right] = \mathbb{E}_t \left[\varepsilon_{i,t+1}^H \mid \text{Disaster at } t+1 \right] = 0$.

Eq. 8 means that $\widehat{H}_t$ mean-reverts to 0, but as a "twisted" autoregressive process. As H_{it} hovers around H_{i*}, $\frac{1 + H_{i*}}{1 + H_{it}}$ is close to 1, so that the process behaves much like a regular AR(1): $\widehat{H}_{it+1} \simeq \exp(-\phi_{H_i}) \widehat{H}_{it} + \varepsilon_{i,t+1}^H$, an equation that holds up to second order terms. The $\frac{1 + H_*}{1 + H_t}$ term is a "twist" term that makes the process very tractable. It is best thought as economically innocuous, and simply an analytical convenience. Gabaix (2007a, Technical Appendix) shows that the process, physically, behaves indeed like an AR(1).[9]

Its continuous time analogue is:

$$\widehat{H}_{it} = - \left(\phi_{H_i} + \widehat{H}_{it} \right) \widehat{H}_{it} dt + dN_{it}^H, \tag{9}$$

where N_t^H is a martingale, $E_t \left[dN_t^H \right] = \mathbb{E}_t \left[dN_t^H \mid \text{Disaster at } t+1 \right] = 0$.

This assumption allows us to derive the equilibrium exchange rate in closed form.

Proposition 2 *(Level of the exchange rate) In terms of the "international currency," the exchange rate of country i is:*

$$e_{it} = \frac{\omega_{it}}{1 - \exp(-r_{ei})} \left(1 + \frac{\exp(-r_{ei} - h_{i*})}{1 - \exp(-r_{ei} - \phi_H)} \widehat{H}_{it} \right) \tag{10}$$

where ω_t is the current productivity of the country. In the limit of small time intervals, the exchange rate is:

$$e_{it} = \frac{\omega_{it}}{r_{ei}} \left(1 + \frac{\widehat{H}_{it}}{r_{ei} + \phi_{H_i}} \right) \tag{11}$$

with

$$r_{ei} \equiv R + \lambda - g_{\omega_i} - h_{i*}. \tag{12}$$

Formula (11) is a modified version of Gordon's formula. It can be verified that e_{it} is decreasing in r_{ei}: the exchange rate is decreasing in the Ramsey interest rate R, decreasing in the depreciation rate of capital λ, increasing in the growth of productivity g_ω. Formula (11) implicitly exhibits a Balassa-Samuelson effect: more productive countries – countries with a higher ω_t – have a higher real exchange rate. Countries with a high expected productivity growth also have high exchange rates.

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[9]Here, changes in resiliences are changes in the technological exposure of the country. One might endogenize them more, as optimal use of growth options by the countries, along the lines of Berk, Green and Naik (1999)'s model of use of growth options by firms.

Importantly, e_t is increasing in h_* and $\widehat{H}_t$: Risky countries have a low exchange rate. Finally, at this stage, the volatility of the exchange rate comes from the volatility of its resilience $\widehat{H}_t$. Later, we generalize the setup and introduce other factors.

In Section (5), we explain how to infer a country's resilience from currency options data and provide evidence that riskier countries have depreciated real exchange rates.

2.3 The forward premium puzzle

Consider a one period domestic bond in country i, that yields 1 unit of the currency of country i at time $t + 1$. It will be worth $e_{i,t+1}$ of the international currency. Hence the domestic price of that bond is given by:[10]

$$\frac{1}{1 + r_{it}} = \mathbb{E}_t \left[\frac{M_{t+1}^* e_{i,t+1}}{M_t^* e_{i,t}} \right] \tag{13}$$

where r_t is the domestic interest rate – the nominal interest rate in domestic currency.

Proposition 3 *(Level of the domestic short term interest rate, when there is no inflation on the home good). The value of the domestic short term rate in country i is*

$$r_{it} = \exp\left(r_{ei} - \lambda\right) \left[1 - \frac{\left(1 - \exp\left(-r_{ei}\right)\right) \exp\left(-h_{i*}\right) \widehat{H}_{it}}{1 - \exp\left(-r_{ei} - \phi_H\right) + \exp\left(-h_{i*}\right) \widehat{H}_{it}} \right] - 1 \tag{14}$$

In the limit of small time intervals, the interest rate is:

$$r_{it} = r_{ei} - \lambda - \frac{r_{ei} \widehat{H}_{it}}{r_{ei} + \phi_H + \widehat{H}_{it}} \tag{15}$$

When a country is very "risky", ($\widehat{H}_{it}$ low), its interest rate is high (15), because its currency has a high risk of depreciating in bad states of the world. Note that this risk is a risk of depreciation, not a default risk. [11]

Hence, countries with high interest rates will see their exchange rate appreciate – that's the "forward exchange rate premium puzzle" or "uncovered interest rate parity puzzle" highlighted by Hansen and Hodrick (1980) and Fama (1984), and replicated for various countries and time periods many times since (Engel 1996, Lewis 1995 provide surveys).

We analyze the predictions of our model for Fama regressions in two different types of samples: with disaster and with no disaster. We consider countries with identical constant parameters, but possibly different $\widehat{H}_t$ and ω_t.

We start with the regression in a sample without disasters. In the continuous time limit, the expected growth rate of the exchange rate, conditional on no disasters is, dropping the index i for

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[10]The derivation is standard. In the international currency, the payoff of the bond is e_{t+1}, so its price is $\mathbb{E}_t \left[\frac{M_{t+1}^* e_{t+1}}{M_t^*} \right]$, and its domestic price is (13).

[11]Safe countries can borrow at a lower interest rate, which may explain while historically the dollar or Swiss Franc interest rates were low (Gourinchas and Rey 2007).

country i,

$$\mathbb{E}_t \left[\frac{1}{e_t} \frac{de_t}{dt} \right] = g_\omega + \frac{\mathbb{E}_t \left[\frac{d\widehat{H}_t}{dt} \right]}{r_e + \phi + \widehat{H}_t} = g_\omega - \frac{\left(\phi + \widehat{H}_t \right) \widehat{H}_t}{r_e + \phi + \widehat{H}_t}.$$

In a first order approximation in $\widehat{H}_t$:

$$\mathbb{E}_t \left[\frac{1}{e_t} \frac{de_t}{dt} \right] = g_\omega - \frac{\phi}{r_e + \phi} \widehat{H}_t$$

When the country is very risky, $\widehat{H}_t$ is high, and its exchange rate is low (11); as the exchange rate mean-reverts, its exchange rate will appreciate, so that $\mathbb{E}_t \left[\frac{de_t}{dt} \right] / e_t > 0$.

Similarly, in a first order approximation in $\widehat{H}_t$:

$$r_t = r_e - \lambda - \frac{r_e \widehat{H}_t}{r_e + \phi}$$

Hence

$$\mathbb{E}_t \left[\frac{1}{e_t} \frac{de_t}{dt} \right] = \frac{\phi}{r_e} r_t + g_\omega - \frac{\phi (r_e - \lambda)}{r_e}$$

Consider the Fama (1984) regression of the changes in the exchange rate between countries A and B regressed on the difference in interest rates, in a sample with no disasters:

$$\text{Fama regression: } \mathbb{E}_t \left[\frac{e_{t+1}^A - e_t^A}{e_t^A} - \frac{e_{t+1}^B - e_t^B}{e_t^B} \right] = \alpha - \beta (r_t^A - r_t^B) \tag{16}$$

The expectation hypothesis predicts $\beta = 1$. The present model however predicts a negative coefficient. For simplicity, we consider the case where the two countries, A and B, have the same r_e.

Proposition 4 *(Coefficient in the Fama regression, conditionally on no disasters). In the Fama regression (16), in a sample with no disasters, the coefficient is:*

$$\beta = -\frac{\phi}{r_e}. \tag{17}$$

In the Fama regression (16), in an unconditional sample, in the case where $B_t \equiv B$, the coefficient is:

$$\beta^{Full} = 1 - \frac{\phi}{r_e} (1 - B^\gamma) \tag{18}$$

With the calibrated numbers $\phi = 20\%/\text{year}$ and $r_e = 10\%/\text{year}$, the coefficient in a yearly regression should be $\beta = -2$. This is in the order of magnitude of the estimates in the literature. We conclude that even quantitatively, the UIP puzzle seems to be accounted for by the framework.

The Fama coefficient in a sample without disasters does not depend on B. Even when disasters are not associated with risk premia – $B = 1$– the Fama regression in a small sample with no disaster would indicate a violation of UIP. Time varying risk premia are crucial to explain the

forward premium puzzle in a sample with disasters. In particular, for $B = 1$, there is no disaster risk (consumption doesn't fall during disasters), so that $\beta^{Full} = 1$. Hence, the Fama regression yields a negative coefficient only if disaster risk is high enough. We note that the negative β^{Full} does not come from a Peso problem explanation in the sense that, in the model, even in a sample that includes disasters, there can be a negative coefficient in the Fama regression.

Other Interpretations The model can be interpreted alternatively in two different ways. First, while we have derived our different pricing kernels from a CCAPM framework adapted for disasters, there are other possibilities. The pricing kernels we derive are flexible and attractive reduced-form candidates for models with time varying risk premia associated with large currency movements. This is, we believe, a prevalent feature of foreign exchange markets: the carry trade is often referred to "picking dimes in front a of steam roller." Although we take the extra-step of linking these risk-premia to aggregate consumption risk, this step is not needed to derive the asset pricing implications of our framework: our model does not live or die on this particular hypothesis. In fact, this particular dimension of the model is hard to test since the problem of estimating time varying correlations with consumption is compounded in our case by the fact that rare disasters that might not occur in short samples are crucial to computing the relevant correlations. Using option prices in Farhi, Gabaix, Ranciere and Verdelhan (2008), we test a higher level implication of the model – that there are time-varying risk premia associated with large movements in exchange rate.

Second, while the model is presented as rational, it can also be viewed as a tractable way to capture time-varying perception of risk, or investor sentiment. Disasters are rare by nature and their probability of occurrence as well as their severity make them prone to expectations errors, herds and other behavioral biases. In the model, the varying beliefs about the probability and intensity of crashes could be rational, or behavioral. Under this interpretation, this paper offers a way to model time-varying "perception of risk", "risk appetite" or "sentiment" – people's estimate of how their asset would perform in a crisis. Under that interpretation, one doesn't need to use the "macroeconomic consumption drop" interpretation. In fact, one can interpret more loosely disasters as small probability bad events as "financial crashes." Our international pricing kernel (6) does not refer to consumption. The agents basically follow an expected value maximization, except that $B^{-\gamma}$ term increases the effective weight put on low probability events, consistent with Prospect theory. The model then offers a coherent way to think about the joint behavior of sentiment and prices.

3 A setup with a risk factor and a business cycle factor

The above setup gave the essence of the disaster mechanism, but it has only one factor, so that, controlling for current productivity, exchange rate and risk premia are perfectly correlated, which in a variety of contexts is not a desirable feature. Accordingly, we extend the framework to a two-factor model, a risk factor, and a business cycle factor.

3.1 Setup with a risk factor and a business cycle factor

In the baseline model, the real rate varies only because of the risk premium. We can easily extend the model to business cycle movements in the interest rates. For ease of notations, we typically drop the index i for country i. We say that the country's productivity is $\omega_t = \overline{\omega}_t (1 + y_t)$, where $\overline{\omega}_t$ is the "permanent" component of productivity, and y_t is a "business cycle" fluctuation or "deviation of productivity from trend". We model:

$$\frac{\overline{\omega}_{t+1}}{\overline{\omega}_t} = \begin{cases} \exp(g_\omega) & \text{in normal times} \\ \exp(g_\omega) F_{t+1} & \text{if disaster} \end{cases}$$

and LG-twisted process for y_t:

$$\mathbb{E}_t [y_{t+1}] = \frac{1 + H_*}{1 + H_t} \exp(-\phi_y) y_t$$

with innovation uncorrelated to the ones of ω_t and M_t. As before, that means that, to a first order, y_t follows an AR(1), $\mathbb{E}_t [y_{t+1}] = \exp(-\phi_y) y_t + O(y_t^2)$. The "twist" term, $\frac{1+H_*}{1+H_t}$, is close to 1 in practice, and makes the process tractable without affecting importantly the dynamics of y_t. This allows to calculate the exchange rate.

Proposition 5 *(Exchange rate with a business cycle factor) The exchange rate is*

$$e_t = \frac{\overline{\omega}_t}{1 - \exp(-r_e)} \left(1 + \frac{\exp(-r_e - h_*)}{1 - \exp(-r_e - \phi_H)} \widehat{H}_t + \frac{1 - \exp(-r_e)}{1 - \exp(-r_e - \phi_y)} y_t \right) \tag{19}$$

and in the continuous time limit, it is:

$$e_t = \frac{\overline{\omega}_t}{r_e} \left(1 + \frac{\widehat{H}_t}{r_e + \phi_H} + \frac{r_e y_t}{r_e + \phi_y} \right) \tag{20}$$

while the interest rate is:

$$r_t = r_e - \lambda + \frac{-\frac{r_e}{r_e + \phi_H} \widehat{H}_t + \frac{r_e \phi_y}{r_e + \phi_y} y_t}{1 + \frac{\widehat{H}_t}{r_e + \phi_H} + \frac{r_e y_t}{r_e + \phi_y}} \tag{21}$$

In this setup, the resilience $\widehat{H}_t$ has the same effect as before. But there is an additional factor, the deviation of productivity from trend y_t, which is not associated with any risk premium. As would be expected, when productivity is high, the exchange rate is high. Also, it is expected to depreciate, so that the interest rate rate is high.

3.2 Fama regression with two factors

Let us revisit the Fama regression (16):

$$\mathbb{E}_t\left[\frac{e^A_{t+1} - e^A_t}{e^A_t} - \frac{e^B_{t+1} - e^B_t}{e^B_t}\right] = \alpha' - \beta'(r^A_t - r^B_t).$$

The next Proposition relates the coefficient β' in a sample with no disaster, and the coefficient β'^{Full} in full sample, to their corresponding values β and β^{Full} previously derived for the one-factor model.

Proposition 6 *(Value of the β coefficient in the Fama regression, with two factors). Up to second order terms, in the Fama regression, the coefficients are:*

$$\beta' = \nu\beta + 1 - \nu \tag{22}$$
$$\beta'^{Full} = \nu\beta^{Full} + 1 - \nu \tag{23}$$

where β and β^{Full} are given in Eqs. 17 and 18, and ν is the share of variance in the interest rate due to $\widehat{H}_t$,

$$\nu = \frac{\left(\frac{r_e}{r_e+\phi_H}\right)^2 \text{Var}\left(\widehat{H}_t\right)}{\left(\frac{r_e}{r_e+\phi_H}\right)^2 \text{Var}\left(\widehat{H}_t\right) + \left(\frac{r_e\phi_y}{r_e+\phi_y}\right)^2 \text{Var}\left(y_t\right)}. \tag{24}$$

In Eq. 22, β' is the weighted average of two Fama coefficients. One, β, comes from the variations in the risk premium. The second, 1, comes from the cyclical variations in productivity, and is the value predicted by the expectation hypothesis. The weight ν is the relative share of the two factors in the variance of the interest rate.

4 Calibration

4.1 Choice of Parameter Values

We use yearly units.

Disaster and risk aversion. To keep the calibration parsimonious, the probability and intensity of disasters are constant. The probability of disaster is $p = 1.7\%$, as estimated by Barro (2006). The key parameter in the calibration is $p\mathbb{E}[B^{-\gamma}]$, the risk-adjusted probability of disasters: disasters are overweighted compared to their physical probability by a factor $\mathbb{E}[B^{-\gamma}]$. This factor is very sensitive to the severity of disasters and to the coefficient of relative risk aversion. Increasing γ from 2 to 4 for example increases the risk-adjusted probability of disasters by a factor of 2.8. Going from $\gamma = 4$ to $\gamma = 6$ increases it by another factor of 5.

We present three main scenarios which differ by the implied risk-adjustment factor $\mathbb{E}[B^{-\gamma}]$. These scenarios can be thought alternatively as varying the coefficient of relative risk aversion or

the actual severity of disasters:[12] these two parameter enter our results only through the summary statistics $\mathbb{E}\left[B^{-\gamma}\right]$. Under our preferred calibration (Calibration 1), the implied risk-adjustment factor is $\mathbb{E}\left[B^{-\gamma}\right] = 10.9$. This number is in line the numbers reported by Barro (2006) and Barro and Ursua (2008) for a coefficient of relative risk aversion $\gamma = 4$. We also present a Calibration 2 with more benign disasters (higher B or lower γ) where $\mathbb{E}\left[B^{-\gamma}\right] = 2.44$. Finally, we present a Calibration 3 with more extreme disasters (lower B or higher γ) where $\mathbb{E}\left[B^{-\gamma}\right] = 24.4$.

Growth rates. We choose the growth rates so that in normal times, consumption of nontradables grows at rate $g_c = 3\%$. We set $g_\omega = g_c$, but values are not really sensitive to that parameter. We make sure that the riskless domestic short term rate is on average around 1%, which pins down the rate of time preference, δ.[13]

Exchange rate. An initial investment depreciates at a usual rate $\lambda = 8\%$. To specify the volatility of the recovery rate F_t, we specify that it has a baseline value $F_* = 0.8$, and its range is $F_t \in [F_{\min}, F_{\max}] = [0.2, 1.2]$. That means that the technology of transforming domestic goods into international goods could improve. This is because ω_t is really the ratio between two productivities – to produce domestic or international goods, so that relative ratio could increase or decrease. This possibility of a worst-case fall of productivity to 0.2 of its initial level may seem high. Perhaps it proxies for disruptions not directly linked to productivity, e.g. the introduction of taxes, regulation, or a loss of property rights (as in Barro 2006), though we do not model those interpretations here.

The speed of mean-reversion is $\phi_H = 0.2$, which gives a high-life of $\ln 2/\phi_H = 3.5$ years, and is in line with typical estimates from the exchange rate predictability literature (Rogoff 1996).

This translates into a range for $\widehat{H}_t = p\left(B^{-\gamma}F_t - 1\right)$, $\left[\widehat{H}_{\min}, \widehat{H}_{\max}\right]$. We parametrize the volatility according to Appendix C, with

$$\sigma^2\left(\widehat{H}\right) = 2v\phi_H \left|\widehat{H}_{\min}\right| \widehat{H}_{\max} \left(1 - \widehat{H}/\widehat{H}_{\min}\right)^2 \left(1 - \widehat{H}/\widehat{H}_{\max}\right)^2,$$

which guaranties that $\widehat{H}$ remains within $\left[\widehat{H}_{\min}, \widehat{H}_{\max}\right]$, as the volatility dies down fast enough at the boundaries. The parameter v controls the volatility $\widehat{H}$ and F. For instance, a country with volatile riskiness will have a high v.

To calibrate the exchange rate fluctuations, we start from (11), and take the benchmark of a constant productivity ω_t during the "normal times" period under study. Then, the only changes in the real exchange rates are due to expectation about the "resilience" of a country if a disaster happens. Differentiation of (11) gives a bilateral exchange rate volatility between two uncorrelated

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[12] Here we take the Barro and Ursua numbers at face value: each observation is a realization of B. Another possibility would have been to estimate a distribution of B for world consumption. This should decrease the importance of disasters, via diversification. However, many disasters are correlated (as they often occur during wars or global economic or financial crises). There are several difficulties in implementing this approach, from pure data issues to conceptual issues – for instance, distinguishing between tradable and non-tradable consumption. The data is not available yet, but Barro and Ursua are in the process of computing these numbers.

[13] Note that for Calibration 2, this forces us to adopt a negative value for δ. Alternatively, for the purpose of this particular calibration, we could have adjusted the consumption growth rate g_c; we could also have introduced Epstein Zin Weill preferences with a higher intertemporal elasticity of substitution while maintaining a coefficient of relative risk aversion of 4. We chose to vary as few parameters as possible.

exchange rates[14] $\sigma_{e_{ij}} \simeq \sqrt{2}\sigma_H/(r_e + \phi)$. If two countries are perfectly correlated, then $\sigma_{ei} = 0$, while if they have a correlation of -1, then $\sigma_{e_{ij}} \simeq 2\sigma_H/(r_e + \phi)$. We report the values for the uncorrelated case.

Default risk. To keep the model parsimonious, we assume no default risk on debt. This is the cleanest assumption for developed countries. Of course, in many cases (e.g. to price sovereign debt), default can be added without changing anything to the exchange rate.

4.2 Implications for levels, volatilities and correlations

Table 1 presents the main results. In Calibration 1, with $v = 0.2$, the volatility of the bilateral exchange rate is 11%, and with $v = 0.1$, 8%. In our low risk calibration (Calibration 2), the corresponding volatilities of the exchange rate are almost trivial: 1.8% and 2.5%. In our high risk calibration (Calibration 3) these volatilities are magnified and become 25.7% and 18.7%, respectively. Hence, the model can reasonably generate a high volatility of the exchange rate. The reason is that disasters have a high importance: their importance is magnified by $\mathbb{E}[B^{-\gamma}]$, which is 10.9 (Calibration 1), and 24.4 (Calibration 3). By contrast, in Calibration 2, disasters are milder, risk premia are lower and less volatile, and as a consequence, exchange rates themselves are much smoother.

Table 1: Calibrations of the Model with Medium, Low and High Disaster Risk

	Calibration 1 Medium risk	Calibration2 Low risk	Calibration 3 High risk
Postulated importance of disasters $\mathbb{E}[B^{-\gamma}]$	10.9	2.44	24.4
Implied values			
Range of domestic riskless short rate (in %)	$-0.9; 1.5; 5.1$	$1.0; 1.5; 2.3$	$-3.8; 1.5; 9.58$
Volatility of domestic riskless short rate (in %)	0.7	0.17	1.7
Range for FX $e_t/(\omega_t/r_e)$	$0.62; 1; 1.25$	$0.91; 1; 1.06$	$0.16; 1; 1.56$
Baseline Fama regression coefficient, $-\phi/r_e$	-2.1	-2.1	-2.1
Bilateral FX volatility σ_e (in %) when $v = 0.1; 0.2$	$9.0; 12.7$	$1.8; 2.5$	$17.9; 25.3$

Explanation: Each calibration starts with a postulated importance of disasters, $\mathbb{E}[B^{-\gamma}]$. For instance if $\mathbb{E}[B^{-\gamma}] = 10.9$, rare disasters matter 10.9 times as much as they would if agents were risk neutral. We then fit the rate of time preference δ, to get a typical value of the interest rate close to 1.5%. We report the minimum, typical (corresponding to $\widehat{H}_t = 0$) and maximum range for the domestic short term interest rate; the minimum, typical and maximum value for the exchange rate over "steady state fundamentals" (ω_t/r_e). Finally, we report for volatility of the bilateral exchange rate for currencies with two uncorrelated fundamentals. Perfectly correlated currencies have 0 bilateral FX volatility, perfectly anticorrelated currencies, a volatility equal to the one reported in the table, times $\sqrt{2}$. The time unit is the year.

In all three calibrations, the volatility of F_t (defined as $\text{Var}^{1/2}(F_{t+1} - F_t)$) is 9% per year when

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[14]This is because $\frac{de_t}{e_t} = \frac{\widehat{H}_t}{r+\phi+\widehat{H}_t} \simeq \frac{\widehat{H}_t}{r+\phi}$, and the bilateral exchange rate $e_{ij} = e_i/e_j$ has twice the variance of any of the exchange rates, if the $\widehat{H}_t$ shocks are uncorrelated.

$\nu = 0.1$, and 12.7% when $\nu = 0.2$: expectations about recovery rate vary pretty rapidly from year to year. As $\widehat{H}_t$ is quite volatile, the exchange rate is hard to forecast (the same way stocks are hard to forecast). At short horizons, it behaves like a random walk (qualitatively consistent with Meese and Rogoff 1983).

In the remaining of this section, we focus on our preferred calibration (Calibration 1) and dig deeper into its implications. We turn to the model's implications for the correlation between the changes in the exchange rate and the changes in the interest rate. More precisely we are interested in comparing $\text{Corr}\left(\frac{\Delta\left(e_t^A/e_t^B\right)}{e_t^A/e_t^B}, \Delta\left(r_t^A - r_t^B\right) \right)$ in the data and in the model. In the data, this correlation is close to 0. In the simple version of Calibration 1 we have only one real factor. Hence the correlation is -1 almost by definition. However, a two-factor version of our model similar to that developed in Section 3 performs much better. The reason for this is that although the disaster factor pushes towards a negative correlation, the business cycle factor by contrast, pushes towards a positive correlation. For the purpose of the calibration, it is more intuitive to model business cycle shocks as shocks to productivity trend growth rather than shocks to the level of productivity.[15]

We choose the average standard deviation of the innovation in $\widehat{g}_{\omega,t}$ to be $\sigma_{\widehat{g}_{\omega,t}} = 1.4\%$ and a mean-reversion coefficient $\phi_g = 10\%$. With these numbers, we have

$$\text{Corr}\left(\frac{\Delta\left(e_t^A/e_t^B\right)}{e_t^A/e_t^B}, \Delta\left(r_t^A - r_t^B\right) \right) = -0.027.$$

The volatility of the bilateral exchange is 11.1%, the volatility of the domestic interest rate is 1.0%. The Fama coefficient is still negative but smaller in absolute value: $\beta' = -0.04$. This of course, is to be anticipated because we have introduced a business-cycle factor with significant variance. To some extent, there is tension in the assumed volatility of the business-cycle factor $\sigma_{\widehat{g}_{\omega,t}}$, between matching small correlation $\text{Corr}\left(\frac{\Delta\left(e_t^A/e_t^B\right)}{e_t^A/e_t^B}, \Delta\left(r_t^A - r_t^B\right) \right)$ and generating a large negative Fama regression coefficient.

5 Option prices and exchange rate risk premia

5.1 Theory

Option prices incorporate direct information about the probability and severity of disasters. In particular, consider the implied volatility smile of a pair of currencies: a risky currency and a safe currency. The smile will be much steeper on the risky currency side. A high "smile-skew" should

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[15]More precisely, we assume that productivity growth has deviations from trend, $\widehat{g}_{\omega,t}$

$$\frac{\omega_{t+1}}{\omega_t} = (1 + \widehat{g}_{\omega,t})\exp(g_\omega) \times \begin{cases} 1 & \text{in normal times} \\ F_{t+1} & \text{if disaster} \end{cases}$$

which follows a twisted AR(1) process, $\mathbb{E}_t\left[\widehat{g}_{\omega,t+1}\right] = \frac{1+H_*}{1+H_t}\frac{\exp(-\phi_g)\widehat{g}_{\omega,t}}{1+\widehat{g}_{\omega,t}}$. Then, Eq. 20 and 21 hold to a first order, replacing $r_e y_t$ by $\widehat{g}_{\omega,t}$.

predict currency appreciation, high interest rate differential and high bond returns.

Consider two countries A and B. The currency B price at date 0 of a call that gives the option to buy at date 1 one unit of currency B for $K\frac{e_0^B}{e_0^A}$ units of currency A is $\frac{1}{e_0^B}\mathbb{E}_0\left[\frac{M_T^*}{M_0^*}\left(e_1^B - K\frac{e_0^B}{e_0^A}e_1^A\right)^+\right]$, i.e.

$$V^{Call}(K) = \mathbb{E}_0\left[\frac{M_1^*}{M_0^*}\left(\frac{e_1^B}{e_0^B} - K\frac{e_1^A}{e_0^A}\right)^+\right]. \tag{25}$$

Likewise, the currency B price at date 0 of a put that gives the option to sell at date 1 one unit of currency B for $K\frac{e_0^B}{e_0^A}$ units of currency A is $V^{Put}(K) = \mathbb{E}_0\left[\frac{M_1^*}{M_0^*}\left(K\frac{e_1^A}{e_0^A} - \frac{e_1^B}{e_0^B}\right)^+\right]$.

In order to gain in tractability, we make two simplifying assumptions, as in Gabaix (2007b). First we assume that if a disaster occurs in period $t+1$, ε_{t+1}^H is equal to zero. Second, we assume that the distribution of e_{t+1} conditional on date t information and no disaster occurring in period $t+1$ is lognormal with drift $\mu_{i,t}$ and volatility $\sigma_{i,t}$ where $i \in \{A, B\}$ indexes countries: $e_1^{i,ND}/e_0^i = \exp\left(\mu_i + \varepsilon_i - \sigma_i^2/2\right)$ with $\varepsilon_i \sim N(0, \sigma_i^2)$.[16]

Proposition 7 *The price of a call with strike K is:*

$$
\begin{aligned}
V^{Call}(K) =\ & \exp\left(-R + \mu_B\right)(1 - p_0)C^{BS}\left(K\exp\left(\mu_A - \mu_B\right), \sigma_{A|B}\right) + \\
& + \exp\left(-R + \mu_B\right)p_0\mathbb{E}_0^D\left[B_1^{-\gamma}\left(F_{1,B} - K\exp\left(\mu_A - \mu_B\right)F_{1,A}\right)^+\right]
\end{aligned} \tag{26}
$$

where $C^{BS}(K, \sigma)$ is the Black-Scholes call value when the strike is K, the volatility σ, the interest rate 0, the maturity 1, the spot price 1 and

$$\sigma_{A|B} \equiv \left(\sigma_A^2 + \sigma_B^2 - 2\rho_{A,B}\sigma_A\sigma_B\right)^{1/2} = \left(Var^{ND}\ln\left(\frac{e_1^B}{e_0^B}\bigg/\frac{e_1^A}{e_0^A}\right)\right)^{1/2}$$

is the standard deviation of the bilateral log exchange rate if there is no disaster. The price of a put is given by the Put-Call parity equation:

$$V^{Put}(K) = V^{Call}(K) + \frac{K}{1 + r_A} - \frac{1}{1 + r_B} \tag{27}$$

The option price (26) is the sum of two terms. The first one is a familiar Black-Scholes term. The second is a purely disaster term. For strikes close to the spot, the Black-Scholes term dominates. But far out of the money, if the disaster term is not 0, it will typically dominate.

5.2 Risk Reversals

Option prices reveal the risk-neutral distribution – that is, the physical probabilities adjusted for risk– of future exchange rates. Disasters are associated with large movements in exchange rates.

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[16]This can be insured as in Gabaix (2007b). We assume that, if there is no disaster, then $\omega_{t+1}/\omega_t = e^g\left(1 + \varepsilon_{t+1}^\omega\right)$, with $E_t\left[\varepsilon_{t+1}^\omega\right] = E_t\left[\varepsilon_{t+1}^\omega\varepsilon_{t+1}^H\right] = 0$. This does not change any of the formulas for the exchange rate and the interest rate. The disturbance term ε_{t+1}^ω can be designed to insure that $e_1^{i,ND}/e_0^i$ has the lognormal noise announced above.

Information about disaster risk-premia can logically be extracted from out of the money option prices. Indeed, consider the price of a "risk reversal":

$$RR\left(k\right) \equiv \frac{1}{T}\left[V^{Put}\left(\frac{1+r_A}{1+r_B}k\right) - V^{Call}\left(\frac{1+r_A}{1+r_B}k^{-1}\right)\right] \qquad (28)$$

with $k > 0$, and T is the duration of a "period" in years. This is the price of a put at a fraction $k \leq 1$ of the forward rate, minus the price of a call at a fraction $1/k \geq 1$ of the forward rate (indeed, $\frac{1+r_A}{1+r_B}$ is the forward rate of the currency, as a multiple of the current spot price). If there is a high crash risk for currency B (i.e., currency B will do worse than currency A during disasters), then RR should be positive. If there is higher crash risk for currency A, RR should be negative. If there is no disaster risk, then RR is very close to 0. Risk-reversals are used routinely by traders engaged in currency speculation and in the carry trade in particular as it allows them to bound the gains and losses on their positions.

Calculations show that it is approximately, for short time periods T, and for $|k - 1|$ large compared to $\sigma_{AB}\sqrt{T}$ (so that the disaster terms mostly influence the value of RR):

$$RR \simeq \Psi\left(\widehat{H}_B - \widehat{H}_A, (1-k)H_*/T\right) \qquad (29)$$

where $\Psi\left(x,y\right) = \left(x-y\right)^+ - \left(-x-y\right)^+$, a relation that has a shape illustrated in the left panel of Figure 2. The risk reversal position is such that the Black-Scholes component of the put and call have the same price in the limit of short maturities. This allows us to extract the disaster intensities from option prices. When $\left|\widehat{H}_B - \widehat{H}_A\right|$ is large compared to $|1-k|H_*/T$,

$$RR \simeq \widehat{H}_B - \widehat{H}_A. \qquad (30)$$

Hence, the risk reversal is approximately the difference in resilience between the two countries.

Conditionally on no disasters, the excess return on a position long on currency B, and short in currency A, is:

$$\mathbb{E}_t^{ND}\left[\frac{e_{t+T}^B - e_t^B}{e_t^B} - \frac{e_{t+T}^A - e_t^A}{e_t^A}\right]/T + r_t^A - r_t^B = H_t^B - H_t^A \qquad (31)$$

which is close to RR. That is, a currency with a high put price should have a low price, and should subsequently appreciate. This is because it has a high risk premium, that affects both the put value, and a low value of the exchange rate. Equation 31 expresses quantitatively the magnitude of the effect.

5.3   Risk Reversals and Currency Returns

We illustrate the potential power of risk reversals to forecast currency returns. We consider two countries, A and B, with same permanent resilience ($H_{A*} = H_{B*} = H_*$), but perhaps different transient resiliences, $\widehat{H}_{At}, \widehat{H}_{Bt}$. We take currency A's at its steady state, $\widehat{H}_{At} = 0$. We take a period
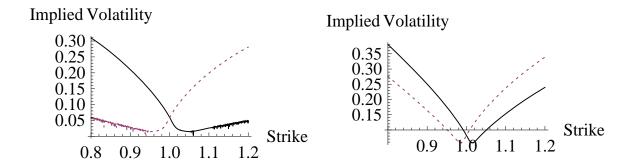
Figure 1: **Implied volatility vs Strike.** Units are annualized. Country B is riskier than country A. So, the implied volatility of out of the money puts is high. The dotted line is the mirror image of the solid curve, replacing the strike k by its symmetrical value around the forward exchange rate, $1/k$. It illustrates that for deep puts, the implied volatility of an out of the money put on currency B is higher than the implied volatility of an out of the money symmetrical call. The values chosen are: $\widehat{H}_A = 0, \widehat{H}_B = -5\%$ in annualized values. Finally, the left panel corresponds to no noise, and the right panel corresponds to a noise $\nu = 0.5$.

length of T equal to 3 months.

At this stage, we need to specify a detail that did not matter before, the volatility of the realization of F_A in (26): for an option, the uncertainty about the realization of the risk matters. We choose to say that if $\overline{F}_A$ is the expected recovery of country A, its actual realization is $F_A = \overline{F}_A (1 \pm \nu)$, where the noise ν will be chosen either equal to 0 (no uncertainty) or 0.5 (high uncertainty), no correlations across countries. Given we have no strong prior on the value of that uncertainty, we show the plots for two values. In Figures 1-3, the left panel corresponds to no noise ($\nu = 0$), and the right panel corresponds to a noise $\nu = 0.5$.

Implied volatility and risk reversals. Figure 1 shows the implied volatility of a put as a function of the strike. We take the case where country B is riskier than country A. We take $\widehat{H}_B = -5\%$ in annualized values. So, the implied volatility of out of the money puts is high. The dotted line is the mirror image of the solid curve, replacing the strike k by $1/k$. It illustrates that for deep puts, implied volatilities of the crash of currency B are higher than implied volatility of a boom. We see that more noise increase implied volatility, as it should, and also decreases the difference between the put implied volatility (evaluated at $k < 1$) and the symmetrical call volatility (evaluated at $1/k > 1$).

Resilience and risk reversals. Figure 2 analyzes the impact of country B's resilience ($\widehat{H}_B$) on the value of the risk-reversal, assuming a relative strike $k = 0.8$. The Figure illustrates that when country B is riskier, its risk-reversal is more expensive. When there is enough noise, the relation is approximately linear.

Currency returns and risk reversals. Figure 3 plots the expected returns on a position of one unit long of a currency B bill, and one unit short of currency A short term bill, vs the risk reversal. We again assume a relative strike $k = 0.8$. When country B is riskier, its risk-reversal is higher
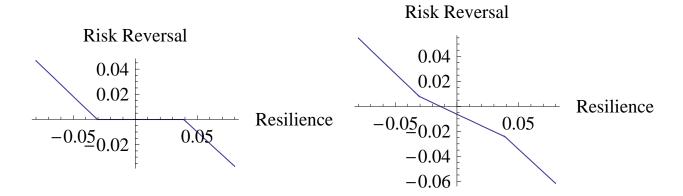
Figure 2: **Risk-Reversal vs Resilience.** Units are annualized. The figure plots the Risk-Reversal (28) vs the resilience $\widehat{H}_B$, assuming the country A is at the steady state, $\widehat{H}_A = 0$. The Figure illustrates that when country B is riskier, its risk-reversal is higher. The left panel corresponds to no noise, and the right panel corresponds to a noise $\nu = 0.5$.
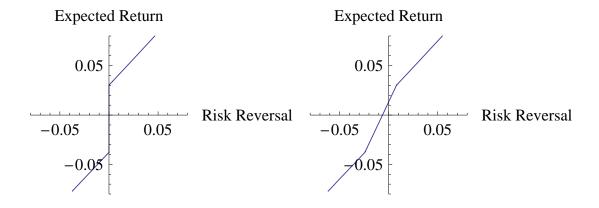

Figure 3: **Risk-Reversal vs Expected Returns.** Units are annualized. The figure plots the expected returns on a position of 1 unit long of a currency B bill, and 1 unit short of currency A short term bill, vs the the risk reversal. We assume that country A is at the steady state, $\widehat{H}_A = 0$. The Figure illustrates that when country B is riskier, its risk-reversal is higher and its expected returns are higher. The left panel corresponds to no noise, and the right panel corresponds to a noise $\nu = 0.5$.

and its expected returns are higher. We see that when the noise is higher, the relation becomes approximately linear as in (30)-(31). Hence, one can hope to use risk-reversals to forecast currency returns. We explore these issues in Farhi, Gabaix, Ranciere and Verdelhan (2008).

5.4 Some Evidence: Contemporaneous Correlations between Currency Movements and Risk Reversals

Carr and Wu (2007) compute the risk-reversals for two pairs of countries: UK and Japan versus the US. They find a high correlation between changes in the price of risk-reversal options and changes in nominal exchange rates: currencies that become riskier – for which puts become relatively more expensive than calls – experience a simultaneous depreciation. Farhi, Gabaix, Ranciere and Verdelhan (2008) extends their analysis to a sample of 25 countries. Their analysis confirms the finding of Carr and Wu (2007) and shows that it also holds for real exchange rates, providing direct evidence in favor of our model.[17]

We next turn to two extensions of our model, to nominal rates, and to stocks.

6 Yield curve, forward rates, and exchange rates, real and nominal

6.1 Exchange rates and long term real rates

To study the forward premium puzzle for long term rates, we first derive the price of long term bonds. The price of a bond yield one unit of the currency at time $t+T$ is: $Z_t(T) = \mathbb{E}_t \left[\frac{M^*_{t+T} e_{t+T}}{M^*_t e_t} \right]$.

The yield at maturity T, $Y_t(T)$, and the forward rates $f_t(T)$ are defined by

$$Z_t(T) = \exp\left(-Y_t(T)T\right) = \exp\left(-\sum_{T'=1}^{T} f_t(T')\right).$$

Proposition 8 *(Price of a domestic bond, when there is no inflation on the home goods) The domestic forward rate is, in the continuous time limit, up to second order terms in $\widehat{H}_t$ and y_t,*

$$f_t(T) = r_e - \lambda - \frac{r_e}{r_e + \phi_H} \exp\left(-\phi_H T\right) \widehat{H}_t + \frac{r_e \phi_y}{r_e + \phi_y} \exp\left(-\phi_y T\right) y_t \tag{32}$$

The proof to this Proposition also calculates the expressions for bonds, yields, forward rates, in discrete and continuous time.

The proof is in Appendix B.

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[17]In a contemporaneous paper, Brunnermeier, Nagel and Pedersen (2008) provide an interesting analysis of the link between expected returns, crash risk (measured by realized skewness) and risk reversals.

To illustrate the economics, suppose that the country has a very high $\widehat{H}_t$, i.e. is very safe. Future $\widehat{H}_t$ will, on average, mean-revert to 0. Hence, by (11), the exchange rate (which is high now) will depreciate. The short terms rates are low (Eq. 15), which is the forward premium puzzle. Eq. 32 says that long term rates are low (the bond price is high because $\widehat{H}_t$.is high). Hence, perhaps paradoxically at first, investors expect the exchange rate to depreciate in the long term, and also, long term rates are low. In the model, this is because investors perceive the country as very "safe", and require a small risk premium on it.

6.1.1 Fama regression with forward rates

Boudoukh, Richardson and Whitelaw (BRW, 2006) propose to regress the exchange rate movement on the $T-$period forward rate from T periods ago:

$$\text{BRW regression: } \mathbb{E}_t \left[\frac{e_{t+1}^A - e_t^A}{e_t^A} - \frac{e_{t+1}^B - e_t^B}{e_t^B} \right] = \alpha^{Fwd}(T) - \beta^{Fwd}(T)\left(f_{t-T}^A(T+1) - f_{t-T}^B(T+1)\right) \tag{33}$$

Our model's prediction is in the next Proposition.

Proposition 9 *(Value of the β coefficient in the Fama regression, with two factors, with forward rates). Up to second order terms, in the BRW (33) regression with forward rates, the coefficients are:*

$$\beta^{Fwd}(T) = \nu(T)\beta + 1 - \nu(T) \tag{34}$$

and

$$\beta^{Fwd,Full}(T) = \nu(T)\beta^{Full} + 1 - \nu(T) \tag{35}$$

where β and β^{Full} are given in Eqs. 17 and 18, and

$$\nu(T) = \frac{\left(\frac{r_e}{r_e+\phi_H}\right)^2 \text{Var}\left(\widehat{H}_t\right) \exp\left(-2\phi_H T\right)}{\left(\frac{r_e}{r_e+\phi_H}\right)^2 \text{Var}\left(\widehat{H}_t\right) \exp\left(-2\phi_H T\right) + \left(\frac{r_e\phi_y}{r_e+\phi_y}\right)^2 \text{Var}\left(y_t\right) \exp\left(-2\phi_y T\right)} \tag{36}$$

is the share of variance in the forward rate due to $\widehat{H}_t$. In particular, when $\phi_H > \phi_y$, the long horizon regression have coefficient going to 1: $\lim_{T\to\infty} \beta^{Fwd}(T) = \lim_{T\to\infty} \beta^{Fwd,Full}(T) = 1$.

BRW (2006) find that $\beta^{fwd}(T)$ increases toward 1 with the horizon. Our theory is consistent with this empirical finding. Indeed, to interpret Proposition 9, consider the case where risk-premia are fast mean-reverting, and productivity is slowly mean reverting, $\phi_H > \phi_y$. Then, large T, $\nu(T)$ tends to 0, which means that, at long horizons, the forward rate is mostly determined by the level of y_t, not of the risk premium. Hence, at long horizon the model behaves like a model without risk premia, hence generates a coefficient β close to 1.

6.2 A simple model of exchange rates and nominal yield curves

Until recently, forward real interest rates were not available. Only their nominal counterparts were the support of actively traded securities. Even today, most bonds are nominal bonds.

To model nominal bonds, we build on the real two factor model developed above. Let $Q_t = Q_0 \prod_{s=1}^{t} (1 - i_s)$ be the value of money (the inverse of the price level). The nominal interest rate $\widetilde{r}_t$ satisfies $\frac{1}{1+\widetilde{r}_t} = \mathbb{E}_t \left[\frac{M^*_{t+1} e_{t+1}}{M^*_t e_t} (1 - i_t) \right]$, so that, in the continuous time limit,

$$\widetilde{r}_t = r_t + i_t, \tag{37}$$

the nominal interest rate is the real interest rate, plus inflation. The Fisher neutrality applies: there is no burst of inflation during disasters. With a burst of inflation, even short term bonds would command a risk premium.

Inflation hovers around i_*, according to the LG process:

$$i_{t+1} = i_* + \frac{1 - i_*}{1 - i_t} \exp\left(-\phi_i\right) (i_t - i_*) + \varepsilon^i_{t+1} \tag{38}$$

where ε^i_{t+1} has mean 0, and is uncorrelated with innovations in M_{t+1}, in particular with disasters. One could correlate this, but the analysis is a bit more complicated (the analysis is available upon request). The expected value of 1 unit of currency T period later is:

$$\mathbb{E}_t \left[\frac{Q_{t+T}}{Q_t} \right] = (1 - i_*)^T \left(1 - \frac{1 - \exp\left(-\phi_i T\right)}{1 - \exp\left(-\phi_i\right)} \frac{i_t - i_*}{1 - i_*} \right) \tag{39}$$

or $\mathbb{E}_t \left[\frac{Q_{t+T}}{Q_t} \right] = \exp\left(-i_* T\right) \left(1 - \frac{1 - \exp(-\phi_i T)}{\phi_i} (i_t - i_*) \right)$ in the continuous time limit.

To fix notations, we denote nominal variables with a tilde. The price of long term nominal bonds yielding one unit of the currency at time $t + T$ is $\widetilde{Z}_t(T) = \mathbb{E}_t \left[\frac{M^*_{t+T} e_{t+T} Q_{t+T}}{M^*_t e_t Q_t} \right]$. Because we assume that shocks to inflation are uncorrelated with disasters, the value present value of one nominal unit of the currency is

$$\widetilde{Z}_t(T) = \mathbb{E}_t \left[\frac{M^*_{t+T} e_{t+T} Q_{t+T}}{M^*_t e_t Q_t} \right] = \mathbb{E}_t \left[\frac{M^*_{t+T} e_{t+T}}{M^*_t e_t} \right] \mathbb{E}_t \left[\frac{Q_{t+T}}{Q_t} \right]$$

Hence, the value of the zero coupon bond is:

Proposition 10 *(Price of a nominal domestic bond, with no inflation risk premia) The domestic nominal forward rate is, in the continuous time limit, up to second order terms in $\widehat{H}_t$ and y_t, $i_t - i_*$:*

$$f_t(T) = r_e - \lambda - \frac{r_e}{r_e + \phi_H} \exp\left(-\phi_H T\right) \widehat{H}_t + \frac{r_e \phi_y}{r_e + \phi_y} \exp\left(-\phi_y T\right) y_t + i_* + \exp\left(-\phi_i T\right) (i_t - i_*) \tag{40}$$

The proof to this Proposition also calculates the expressions for bonds, yields, forward rates, in

discrete and continuous time.

The nominal forward rate in (40) depends on real and nominal factors. The real factors are the resilience of the economy (the $\widehat{H}_t$) term, the expected growth rate of productivity $(-\phi_y y_t)$. The nominal factor is inflation i_t.

Each of the three terms is multiplied by a term of the type $\exp(-\phi_H T)$. For small speeds of mean reversion ϕ's, it means that the forward curve is fairly flat.

With Q_t the value of money, the nominal exchange rate is: $\widetilde{e}_t = e_t Q_t$. The expected depreciation of the nominal exchange rate is, up to second order terms, and conditionally on no disasters:

$$\mathbb{E}_t\left[\frac{d\widetilde{e}_t}{\widetilde{e}_t}\right]/dt = g_\omega - \frac{\phi_H \widehat{H}_t}{r_e + \phi_H} - \frac{r_e \phi_y y_t}{r_e + \phi_y} - i_t \tag{41}$$

We can derive the implications of our model for a Fama regression in nominal terms:

$$\mathbb{E}_t\left[\frac{\widetilde{e}_{t+1}^A - \widetilde{e}_t^A}{\widetilde{e}_t^A} - \frac{\widetilde{e}_{t+1}^B - \widetilde{e}_t^B}{\widetilde{e}_t^B}\right] = \widetilde{\alpha}^{nom} - \widetilde{\beta}^{nom}(\widetilde{r}_t^A - \widetilde{r}_t^B) \tag{42}$$

where $\widetilde{r}_t^A$ and $\widetilde{r}_t^B$ are now, with some a slight abuse of notational, the nominal interest rates in countries A and B. Our model's prediction is in the next Proposition.

Proposition 11 *(Value of the β coefficient in the Fama regression in nominal terms). Up to second order terms, in the nominal Fama regression (42) regression with forward rates, the coefficients are:*

$$\widetilde{\beta}^{nom} = \nu^{nom}\beta + 1 - \nu^{nom} \text{ and } \widetilde{\beta}^{nom,Full} = \nu^{nom}\beta^{Full} + 1 - \nu^{nom} \tag{43}$$

where β and β^{Full} are the coefficients in the Fama regression defined in propositions (4) and (??), and

$$\nu^{nom} = \frac{\left(\frac{r_e}{r_e+\phi_H}\right)^2 \text{Var}\left(\widehat{H}_t\right)}{\left(\frac{r_e}{r_e+\phi_H}\right)^2 \text{Var}\left(\widehat{H}_t\right) + \left(\frac{r_e\phi_y}{r_e+\phi_y}\right)^2 \text{Var}\left(y_t\right) + \text{Var}(i_t)} \tag{44}$$

is the share of variance in the forward rate due to $\widehat{H}_t$.

In this simple model with no inflation risk premia, the higher the variance of inflation , the closer to 1 is β^{nom}. Hence, countries with very variable inflation (typically, those are also countries with high average inflation) satisfy approximately the uncovered interest rate parity conditions. When disaster risks are very variable —and the real exchange rate is very variable – then β^{nom} is more negative.

6.3   A richer model with nominal risk premia

We now develop a richer model with an inflation-specific risk premium. We extend the framework developed in the previous section by incorporating inflation risk along the lines of Gabaix (2007a).

The variable part of inflation now follows the process:

$$\widehat{i}_{t+1} = \frac{1 - i_*}{1 - i_t} \cdot \left(\exp\left(-\phi_i\right) \widehat{i}_t + 1_{\{\text{Disaster at } t+1\}} \left(j_* + \widehat{j}_t\right) \right) + \varepsilon^i_{t+1} \tag{45}$$

In case of a disaster, inflation jumps by an amount $j_t = j_* + \widehat{j}_t$. This jump in inflation makes long term bonds particularly risky. j_* is the baseline jump in inflation, $\widehat{j}_t$ is the mean-reverting deviation from baseline. It follows a twisted auto-regressive process, and, for simplicity, does not jump during crises:

$$\widehat{j}_{t+1} = \frac{1 - i_*}{1 - i_t} \cdot \exp\left(\phi_\pi\right) \widehat{j}_t + \varepsilon^j_{t+1} \tag{46}$$

We define $\pi^i_t \equiv \frac{pB^{-\gamma}F}{1+H}\widehat{j}_t$, which is the mean-reverting part of the "risk adjusted" expected increase in inflation if there is a disaster. We parametrize the typical jump in inflation j_* in terms of a number $\kappa \leq (1 - \rho_i)/2$:

$$\frac{pB^{-\gamma}Fj_*}{1 + H} = (1 - i_*)^2 \kappa \left(1 - \rho_i - \kappa\right).$$

κ represents a risk premium for the risk that inflation increases during disasters. Also, we define $i_{**} \equiv i_* + \kappa$ and $\psi_\pi \equiv \phi_\pi - \kappa$. They represent the "risk adjusted" trend and mean-reversion parameter in the inflation process.

We denote nominal variables with a tilde. The price of a long term nominal bond yielding one unit of the currency at time $t + T$ is $\widetilde{Z}_t\left(T\right) = \mathbb{E}_t \left[\frac{M^*_{t+T} e_{t+T} Q_{t+T}}{M^*_t e_t Q_t} \right]$.

The yield at maturity T, $\widetilde{Y}_t\left(T\right)$, and the forward rates $\widetilde{f}_t\left(T\right)$ are defined by

$$\widetilde{Z}_t\left(T\right) = \exp\left(-\widetilde{Y}_t\left(T\right)T\right) = \exp\left(-\sum_{T'=1}^{T} \widetilde{f}_t\left(T'\right)\right).$$

The forward rates can be derived in closed form.

Proposition 12 *(Price of a domestic nominal bond, with inflation risk premia) In the continuous time limit, in up to second order terms in $\left(\widehat{H}_t, y_t, i_t, \pi^i_t\right)$:*

$$\begin{aligned}
\widetilde{f}_t\left(T\right) &= r_e - \lambda - \frac{r_e}{r_e + \phi_H}\exp\left(-\phi_H T\right)\widehat{H}_t + \frac{r_e \phi_y}{r_e + \phi_y}\exp\left(-\phi_y T\right)y_t + \tag{47} \\
&\quad + i_{**}\left(1 - \exp\left(-\phi_i T\right)\right) + \exp\left(-\phi_i T\right)i_t + \frac{\exp\left(-\phi_i T\right) - \exp\left(-\psi_\pi T\right)}{\psi_\pi - \phi_i}\pi^i_t
\end{aligned}$$

The nominal forward rate in (47) depends on real and nominal factors. The real factors are the resilience of the economy (the $\widehat{H}_t$) term, the expected growth rate of productivity $\left(-\phi_y y_t\right)$. The nominal factors are inflation i_t, and the variable component of the the risk premium for inflation jump risk, π^i_t.

When a disaster occurs, inflation increases (on average). As very short term bills are essentially

immune to inflation risk, while long term bonds lose value when inflation is higher, long term bonds are riskier, hence they get a higher risk premium. Hence, the yield curve slope up on average – as implied by the term $i_{**}(1 - \exp(-\phi_i T)) \sim i_{**}\phi_i T$.

Each of the three terms is multiplied by a term of the type $\exp(-\phi_H T)$. For small speeds of mean reversion ϕ's, it means that the forward curve is fairly flat. The last term, however, is close to T for small maturities $(\frac{\exp(-\phi_i T) - \exp(-\psi_\pi T)}{\psi_\pi - \phi_i} \sim T)$. It creates a variable slope in the forward curve. Hence, we obtain a rich, potentially realistic, forward curve.

Nominal yield curves contain a lot of potentially information useful to predict exchange rates. We now explain how to best extract the relevant information to compute exchange rate risk premia. As above, the expected depreciation of the nominal exchange rate is, up to second order terms, and conditionally on no disasters:

$$\mathbb{E}_t\left[\frac{d\widetilde{e}_t}{\widetilde{e}_t}\right]/dt = g_\omega - \frac{\phi_H \widehat{H}_t}{r_e + \phi_H} - \frac{r_e \phi_y y_t}{r_e + \phi_y} - i_t \tag{48}$$

It involves three factors that are also reflected in the nominal forward curve. Note however, that it does not involve the inflation risk premium π_t^i. So, an optimal combination of forward rates should predict expected currency returns with more accuracy than the simple Fama regression. It can be shown that when $\phi_y \neq \phi_H$, the currency risk premium is a linear combination of the traditional yield curve factors: level r_t, slope $\partial_T \widetilde{f}_t(0)$, and curvature $\partial_T^2 \widetilde{f}_t(0)$.

7 Equity premia and exchange-rate risk premia

Our model allows us to think in a tractable way about the joint determination of exchange rate and equity values.

7.1 Local market price of risk and local maximal Sharpe ratios

A clean way of getting at this question is to characterize the maximal Sharpe ratio and the market price of risk in local currency. The stochastic discount factor in local currency is $m_{t+1} = \frac{M_{t+1} e_{t+1}}{M_t e_t}$. The maximal Sharpe ratio is given by: $\mathcal{S}_t \equiv \frac{\text{Var}_t^{1/2}(m_{t+1})}{\mathbb{E}_t(m_{t+1})}$. It is given by the formula[18]

$$\mathcal{S}_t = \sqrt{\sigma_{e,t}^2 + (1 + \sigma_{e,t}^2)\frac{1 - p_t}{p_t}\frac{H_t^2}{(1 + H_t)^2}}$$

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[18]$m_{t+1} = \mathbb{E}_t\left[\frac{e_{t+1}}{e_t} \mid \text{No disaster}\right]e^{-R}(1 + \varepsilon_{t+1})(1 + B_t^{-\gamma}F_t J_{t+1})$, where $var(\varepsilon_{t+1}) = \sigma_e^2$, and $J_{t+1} = 0$ with probability $1 - p_t$, and 1 with probability p_t.

where $\sigma_{e,t} = \text{Var}_t^{1/2}(e_{t+1} \,|\text{No disaster})/\mathbb{E}_t(e_{t+1} \,|\text{No disaster})$ is the standard deviation of the log exchange rate in normal times. In the continuous time, limit, we can derive a very simple expression

$$\mathcal{S}_t = \sqrt{\sigma_{e,t}^2 + \frac{H_t^2}{p_t}} \quad \text{with } \sigma_{e,t}^2 = \text{Var}_t\left(\frac{de_t}{e_t} \,|\, \text{No disaster}\right)$$

The only source of time variation in $\sigma_{e,t}^2$ comes from time variations in the variance and covariance of the structural shocks to H_t and y_t: ε_t^H and ε_t^y.

The maximum Sharpe ratio $\mathcal{S}_t$ is high when resiliency H_t is high. Therefore, countries that demand low currency risk premia will feature high local Sharpe ratio and high local equity premia.

7.2 Explicit stock values

Another way to proceed is to take a stand on what fraction of present and future endowments is capitalized in each stock market. A commonly taken route in Lucas-tree economies is to equate the market to a claim on the entirety of the present and future national endowments of goods. However, listed stocks only account for a very small and potentially non-representative fraction of future GDP. Hence, we model stocks without taking a specific stand on the link between the aggregate dividend of listed stocks and GDP.

7.2.1 Firm producing the international good

Consider first the case of a of domestic firm, that produces the international good. More precisely, the dividend D_t follows the following process

$$\frac{D_{t+1}}{D_t} = \begin{cases} e^{g_D}(1 + \varepsilon_{D,t+1}) & \text{in normal times} \\ e^{g_D}(1 + \varepsilon_{D,t+1})F_{D,t} & \text{if crisis} \end{cases}$$

where an idiosyncratic shock uncorrelated with the stochastic discount factors.

Define the resilience $H_{D,t}$ of the stock as

$$H_{D,t} = p_t \left(\mathbb{E}_t \left[B_{t+1}^{-\gamma} F_{D,t+1} \right] - 1 \right) = H_{D*} + \widehat{H}_{D,t}.$$

It is convenient to define $H_{D*} = e^{h_{D*}} - 1$. The law of motion for $\widehat{H}_{D,t}$ is:

$$\widehat{H}_{D,t+1} = \frac{1 + H_{D*}}{1 + H_{D,t}} \exp\left(-\phi_{H_D}\right) \widehat{H}_{Dt} + \varepsilon_{t+1}^{H_D},$$

where ϕ_{H_D} is the speed of mean reversion of the resilience of the stock.

Proposition 13 *(Domestic stocks producing international goods). The domestic price of the stock $P_{D,t}$ is*

$$P_{D,t} = \frac{D_t}{e_t} \frac{1 + \frac{\exp(-r_D - h_{D*})}{1 - \exp\left(-r_D - \phi_{H_D}\right)} \widehat{H}_{D,t}}{1 - \exp\left(-r_D\right)}$$

In the continuous time limit

$$P_{D,t} = \frac{D_t}{e_t} \frac{1 + \frac{\widehat{H}_{D,t}}{r_D + \phi_{H_D}}}{r_D} \tag{49}$$

A more resilient stock (high $\widehat{H}_{D,t}$) has a higher price-dividend and lower future returns. Controlling for this resilience, if the currency is strong (because the country as a whole is safe), then the stock price in domestic currency is low.[19] As e_t is expected to depreciate, the expected return of the stock in local currency is high. In this sense, currency risk premia and local currency equity premia are negatively correlated. Hence, the theory provides an explanation for Hau and Rey (2006)'s evidence that the home-currency stock price is decreasing in the exchange rate.

7.2.2 Firm producing the domestic good

We now turn to a domestic producer producing D_s quantities of the domestic good. Its stock price, in the international currency, is $P_t^* = E_t \left[\sum_{s \geq t} M_s e_s D_s \right]$, so that its domestic price is $P_t = P_t^*/e_t$, hence:

$$P_t/D_t = \frac{\mathbb{E}_t \left[\sum_{s \geq t} M_s e_s D_s \right]}{e_t D_t} \tag{50}$$

We postulate the following process for D_t

$$\frac{D_{t+1}}{D_t} = \begin{cases} \exp(g) & \text{if there is no disaster at } t+1 \\ \exp(g) F_t^i & \text{if there is a disaster at } t+1 \end{cases}$$

F_t^i is the recovery rate in the dividend of that firm. We postulate the F_t^i also follows a LG process, hovering around F_*^i, and mean-reverts at a rate ϕ_F, with a twist spelled out in Eq. (70).

Proposition 14 *(Price of a domestic stock producing non-traded goods). To a first order approximation, the price of stock producing domestic goods is, in terms of the international currency:*

$$P_t^* = P_t e_t = \omega_t D_t \left[\frac{1}{r_D} + \frac{(H_* + p_*)\widehat{F}_{it}}{r_D(r_D + \phi_F)} + \left(\frac{F_*^i}{r_D} + \frac{1}{r_e + \phi_H} \right) \frac{\widehat{H}_t}{r_D + \phi_H} \right] \tag{51}$$

and in the domestic currency,

$$P_t = \frac{r_e}{r_D} D_t \left[1 + \frac{(H_* + p_*)\widehat{F}_{it}}{r_D + \phi_F} + \left(F_*^i - \frac{\phi_H}{r_e + \phi_H} \right) \frac{\widehat{H}_t}{r_D + \phi_H} \right] \tag{52}$$

where $r_D = R - g_D - g_\omega - (H_* + p_*) F_*^i$.

To analyze the above expression, we take the polar case where $\widehat{F}_{it}$ (the resiliency of the firm's technology) is uncorrelated with $\widehat{H}_t$ (the country's resilience). The international price of the stock

---

[19] Of course, if the resilience of the stocks has a strong covariance with the currency's resilience, the relationship is inverted: good news about resilience increases both e_t and $\widehat{H}_{D,t}$, and increases (49).

(51) increases with $\widehat{H}_t$, hence with the exchange rate.

The domestic price (52) of the stock can decrease or decrease with the exchange rate, depending on the sign of $F_*^i - \frac{\phi_H}{r_e + \phi_H}$. The price of resilient stock increases with the exchange rate, while the price of non-resilient stocks decreases with the exchange rate. The reason for this ambiguous result can be see in Eq. 50, where an increase in e_t increases both the numerator and the denominator. Take a resilient stock, with F_*^i close to 1. A increase in the country's resilience, $\widehat{H}_t$, increases the present value of future dividends (the numerator of Eq. 50), because future resiliences are high, and the discount rate is lower. Hence, the numerator in (50) increases a lot. The denominator increases also, but not as much. The net effect is that the domestic stock price increases: The cash flows that the firm produces are more valuable, and less risky. However, take a stock with $F_*^i = 0$, i.e. a stock that will be bankrupt after a disaster. Then, there is no "discount rate effect" in the numerator of (50), as cash-flows always have maximal riskiness (they disappear in a disaster). So, the effect due to the rise in the denominator is stronger. Hence, the stock falls, when the exchange rate increases.

All in all, we see that the price of domestic stocks producing nontradables increases with the exchange rate, when it is expressed in international currency, but, expressed in domestic currency, it increases only for the most resilient stocks. Again, one might hope to test that prediction.

8   Conclusion

This paper proposes a simple, tractable model of exchange rates and interest rates, and offers a theory of the forward premium puzzle. Its main modelling contributions are, first, to develop an "exchange rate as a stock" view of the exchange rate, in a complete market setting (Proposition 1). Second, to work out the exchange rate in a stochastic disaster framework, and to obtain closed forms for the value of the exchange rate, and the forward premium puzzle coefficients.

The paper suggests several questions for future research. First, it would be good to examine new predictions that the model might generate, including the relationships between bonds, options and exchange rate premia and predictability. Second, it would be interesting to extend the model to stocks, so as to study the link between exchange rates and stock markets. Third, given that the model is very simple to state, and to solve (thanks to the modeling "tricks" allowed by linearity-generating processes), it can serve as a simple framework for various questions. This gives hope that a solution to more puzzles in international economics (Obstfeld and Rogoff 2001) may be within reach.

Appendix A. Results for Linearity-Generating processes

The paper repeatedly uses the Linearity-Generating (LG) processes identified and analyzed in Gabaix (2007a). This Appendix gathers the main results. LG processes are given by $M_t D_t$, a pricing kernel M_t times a dividend D_t, and X_t, a n-dimensional vector of factors (that can be thought as stationary). For instance, for bonds, the dividend is $D_t = 1$.

Discrete time By definition, a process $M_t D_t (1, X_t)$ is a LG process with generator $\Omega = \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix}$ if and only if it follows, for all t's:

$$\mathbb{E}_t \left[\frac{M_{t+1} D_{t+1}}{M_t D_t} \right] = \alpha + \delta' X_t \tag{53}$$

$$\mathbb{E}_t \left[\frac{M_{t+1} D_{t+1}}{M_t D_t} X_{t+1} \right] = \gamma + \Gamma X_t \tag{54}$$

Higher moments need not be specified.

For instance, the functional form of the noise does not matter, which makes LG processes parsimonious. Stocks and bonds have simple closed-form expressions.

The price of a stock, $P_t = \mathbb{E}_t \left[\sum_{s \geq t} M_s D_s \right] / M_t$, is:

$$P_t = D_t \frac{1 + \delta' (I_n - \Gamma)^{-1} X_t}{1 - \alpha - \delta' (I_n - \Gamma)^{-1} \gamma} \tag{55}$$

The price-dividend ratio of a "bond", $Z_t (T) = \mathbb{E}_t [M_{t+T} D_{t+T}] / (M_t D_t)$, is:[20]

$$\begin{aligned} Z_t (T) &= \begin{pmatrix} 1 & 0_n \end{pmatrix} \Omega^T \begin{pmatrix} 1 \\ X_t \end{pmatrix} \\ &= \alpha^T + \delta' \frac{\alpha^T I_n - \Gamma^T}{\alpha I_n - \Gamma} X_t \text{ when } \gamma = 0 \end{aligned}$$

Continuous time In continuous time, $M_t D_t (1, X_t)$ is a LG process with generator $\omega = \begin{pmatrix} a & \beta \\ b & \Phi \end{pmatrix}$ if and if only it follows:

$$\mathbb{E}_t \left[\frac{d (M_t D_t)}{M_t D_t} \right] = -(a + \beta' X_t) \, dt \tag{56}$$

$$\mathbb{E}_t \left[\frac{d (M_t D_t X_t)}{M_t D_t} \right] = -(b + \Phi X_t) \, dt \tag{57}$$

---

[20] Here 0_n denotes a $n-$dimensional row of zeros.

The price of a stock, $P_t/D_t = \mathbb{E}_t\left[\int_t^\infty M_s D_s ds\right]/(M_t D_t)$, is:

$$P_t/D_t = \frac{1 - \beta'\Phi^{-1}X_t}{a - \beta'\Phi^{-1}b}$$

and the price-dividend ratio of a "bond" is: $Z_t(T) = \mathbb{E}_t\left[M_{t+T}D_{t+T}\right]/(M_t D_t)$

$$
\begin{aligned}
Z_t(T) &= \begin{pmatrix} 1 & 0_n \end{pmatrix} \cdot \exp\left[-\begin{pmatrix} a & \beta' \\ b & \Phi \end{pmatrix} T\right] \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} \qquad (58) \\
&= \exp(-aT) + \beta' \frac{\exp(-\Phi T) - \exp(-aT)I_n}{\Phi - aI_n} X_t \text{ when } b = 0
\end{aligned}
$$

To ensure that the process is well-behaved (hence prevent prices from being negative), the volatility of the process has to go to zero near some boundary. Gabaix (2007a) details these conditions.

9 Appendix B. Proofs

For simplicity, we drop the country index i in most proofs.

Proof of Proposition 2 By Proposition 1, we have

$$\frac{e_t}{\exp(-\lambda t)\omega_t} = \mathbb{E}_t\left[\sum_{s=0}^\infty M_{t+s}^* \frac{\exp(-\lambda(t+s))\omega_{t+s}}{\exp(-\lambda t)\omega_t}\right]/M_t^* \qquad (59)$$

Let $D_t = \exp(-\lambda t)\omega_t$ and $X_t = \widehat{H}_t$. With this notation,

$$
\begin{aligned}
\mathbb{E}_t\left[\frac{M_{t+1}^* D_{t+1}}{M_t^* D_t}\right] &= \exp(-R - \lambda + g_\omega)\left\{(1 - p_t) + p_t \mathbb{E}_t\left[B_{t+t}^{-\gamma}F_{t+1}\right]\right\} \\
&= \exp(-R - \lambda + g_\omega)(1 + H_t) = \exp(-R - \lambda + g_\omega)(1 + H_*) + \exp(-R - \lambda + g_\omega)\widehat{H}_t \\
&= \exp(-R - \lambda + g_\omega)(1 + H_*) + \exp(-R - \lambda + g_\omega)X_t = \exp(-r_e) + \exp(-r_e - h_*)X_t,
\end{aligned}
$$

using $r_e = R + \lambda - g_\omega - h_*$. Also:

$$
\begin{aligned}
\mathbb{E}_t\left[\frac{M_{t+1}^* D_{t+1}}{M_t^* D_t} X_{t+1}\right] &= \mathbb{E}_t\left[\frac{M_{t+1}^* D_{t+1}}{M_t^* D_t}\right]\mathbb{E}_t[X_{t+1}] \\
&= e^{-R - \lambda + g_\omega}(1 + H_t)\frac{1 + H_*}{1 + H_t}e^{-\phi_H}\widehat{H}_t \\
&= e^{-R - \lambda + g_\omega - \phi_H}(1 + H_*)\widehat{H}_t = e^{-r_e - \phi_H}X_t
\end{aligned}
$$

There are two ways to conclude. *The first way* uses the notations of Appendix A: The above

two moment calculations show that $Y_t = M_t^* D_t (1, X_t)$ is a LG process, with generator Ω:

$$\Omega = \begin{pmatrix} \exp(-r_e) & \exp(-r_e - h_*) \\ 0 & \exp(-r_e - \phi_H) \end{pmatrix}$$

Using equation 55, we find

$$e_t = \frac{\omega_t}{1 - \exp(-r_e)} \left(1 + \frac{\exp(-r_e - h_*)}{1 - \exp(-r_e - \phi_H)} \widehat{H}_t \right) \tag{60}$$

which proves the proposition.

The second way (which is less rigorous, but does not require to know the results on LG processes), is to look for a solution of the type $e_t = \omega_t \left(a + b\widehat{H}_t \right)$, for some constants a and b, which satisfies: $e_t = \omega_t + \mathbb{E}_t \left[M_{t+1}^* \exp(-\lambda) \omega_{t+1} / M_t^* \right]$. Dividing by ω_t, this is:

$$
\begin{aligned}
a + b\widehat{H}_t &= 1 + \mathbb{E}_t \left[\frac{M_{t+1}^* D_{t+1}}{M_t^* D_t} \left(a + b\widehat{H}_{t+1} \right) \right] = 1 + a\mathbb{E}_t \left[\frac{M_{t+1}^* D_{t+1}}{M_t^* D_t} \right] + b\mathbb{E}_t \left[\frac{M_{t+1}^* D_{t+1}}{M_t^* D_t} \widehat{H}_t \right] \\
&= 1 + a \left[e^{-r_e} + e^{-r_e - h_*} \widehat{H}_t \right] + b e^{-r_e - \phi_H} \widehat{H}_t,
\end{aligned}
$$

which should hold for all $\widehat{H}_t$. Solving for a and b, we get $a = 1 + e^{-r_i} a$, $b = e^{-r_e - h_*} a + b e^{-r_i - \phi_H}$, and (60).

The lower bound for $\widehat{H}_t$ is: $\exp(-r_e)\widehat{H}_t > \exp(-\phi_H) - 1$, i.e., in the continuous time limit, $\widehat{H}_t > -\phi_H$.

A Lemma on the existence of the equilibrium

Lemma 1 *(Existence of the Equilibrium) There are (infinitely many) endowment processes that generate the equilibrium described in the paper.*

Proof. Call $\eta_{i,t}^a$ and $\eta_{i,t}^b$ country i's endowment of the international good, and domestic good, respectively. We work out under which conditions they generate the announced equilibrium.

Say that the equilibrium is described by a social planner's maximization of $\sum_i \lambda_i^\gamma U_i$, where $U_i = \mathbb{E}_0 \left[\sum_{t=0}^\infty \exp(-\delta t) \frac{\left(C_t^T \right)^{1-\gamma} + \left(C_t^{NT_i} \right)^{1-\gamma}}{1-\gamma} \right]$ is country i's utility, and λ_i^γ the Negishi weight on country i. We normalize $\sum \lambda_i = 1$. Calling q_t the Arrow-Debreu price of 1 unit of the international good at date t, and Y_{ta} the world production of the international good. Among other things, the planner optimizes the consumptions of the domestic good, so solves:

$$\max_{C_{it}^a} \sum_i \lambda_i^\gamma \sum_{t=0}^\infty \exp(-\delta t) \frac{\left(C_{it}^T \right)^{1-\gamma} + \left(C_{it}^{NT_i} \right)^{1-\gamma}}{1-\gamma} + \sum_t q_t \left(Y_t^T - \sum_i C_{it}^T \right)$$

where so that $\exp(-\delta t) \lambda_i^\gamma \left(C_{it}^T \right)^{-\gamma} - q_t = 0$, and $C_{it}^T = \lambda_i q_t^{-1/\gamma} \exp(\delta t / \gamma)$. Using $Y_t^T = \sum_i C_{it}^T$, we get: $C_{it}^T = \lambda_i Y_t^T$.

Let us now study country i's consumption and investment decisions. Country i at time t, solves $\max_{C_{it}^T, C_{it}^{NTi}} \frac{\left(C_{it}^T\right)^{1-\gamma} + \left(C_{it}^{NTi}\right)^{1-\gamma}}{1-\gamma}$ s.t. $C_{it}^T + e_{it} C_{it}^{NTi} =$ expenditure at time t, so $\left(C_{it}^{NTi}\right)^{-\gamma} = e_{it}\left(C_{it}^T\right)^{-\gamma}$, hence $C_{it}^{NTi} = e_{it}^{-1/\gamma} \lambda_i Y_t^T$. The investment in the capital good is $\eta_{it}^{NTi} - C_{it}^{NTi} = \eta_{it}^{NTi} - e_{it}^{-1/\gamma} \lambda_i Y_t^T$, so that the accumulated quantity of the capital good is $K_{it} = \sum_{s=0}^{\infty} e^{-\lambda s}\left(\eta_{i,t-s}^{NTi} - e_{i,t-s}^{-1/\gamma} \lambda_i Y_{t-s}^T\right)$. As country i produces $K_{it}\omega_{it}$ of the world good, and also has an endowment η_{it}^T of it, the total available consumption of the world good at time t is:

$$Y_t^T = \sum_i \eta_{it}^T + \sum_i \omega_{it} \sum_{s=0}^{\infty} e^{-\lambda s}\left(\eta_{i,t-s}^{NTi} - e_{i,t-s}^{-1/\gamma} \lambda_i Y_{t-s}^T\right). \tag{61}$$

The first term is the endowment of the world good, and second is the production of it.

The equilibrium is described as in the paper, if the endowment processes $\eta_{i,t}^T$ and $\eta_{i,t}^{NTi}$ satisfy (61), with $Y_t^T = C_t^{T*}$. By inspection there is an infinity of such endowment processes.

Proof of Proposition 3 In this proof, it is useful to define $x_t = e^{-h_*}\widehat{H}_t$. Then, $\mathbb{E}_t\left[\frac{M_{t+1}^* \omega_{t+1}}{M_t^* \omega_t}\right] = \exp\left(-R + g_\omega\right)(1 + H_t) = \exp\left(-r_e + \lambda\right)(1 + x_t)$.

Also, $\mathbb{E}_t\left[x_{t+1}\right] = \exp\left(-\phi\right)\frac{x_t}{1+x_t}$, and $e_t = \omega_t A\left(1 + B x_t\right)$, with $A = 1/\left(1 - \exp\left(-r_e\right)\right)$, $B = \frac{\exp(-r_e)}{1-\exp(-r_e-\phi_H)}$,

$$
\begin{aligned}
1 + r_t &= \frac{M_t^* e_t}{\mathbb{E}_t\left[M_{t+1}^* e_{t+1}\right]} = \frac{A\left(1 + B x_t\right)}{\mathbb{E}_t\left[\frac{M_{t+1}^* \omega_{t+1}}{M_t^* \omega_t} A\left(1 + B x_{t+1}\right)\right]} = \frac{1 + B x_t}{\mathbb{E}_t\left[\frac{M_{t+1}^* \omega_{t+1}}{M_t^* \omega_t}\right] \mathbb{E}_t\left[1 + B x_{t+1}\right]} \\
&= \frac{1 + B x_t}{\exp\left(-r_e + \lambda\right)(1 + x_t)\left(1 + B\frac{\exp(-\phi_H)x_t}{1+x_t}\right)} = \exp\left(r_e - \lambda\right)\frac{1 + B x_t}{1 + x_t\left(1 + B\exp\left(-\phi_H\right)\right)} \\
&= \exp\left(r_e - \lambda\right)\frac{1 + \frac{\exp(-r_e)}{1-\exp(-r_e-\phi_H)} x_t}{1 + \frac{1}{1-\exp(-r_e-\phi_H)} x_t} \\
&= \exp\left(r_e - \lambda\right)\left[1 - \frac{\left(1 - \exp\left(-r_e\right)\right)\exp\left(-h_*\right)\widehat{H}_t}{1 - \exp\left(-r_e - \phi_H\right) + \exp\left(-h_*\right)\widehat{H}_t}\right].
\end{aligned}
$$

Proof of Proposition 4 Eq. (17) was derived in the text leading to the Proposition For the unconditional regression, the reasoning is thus.

Unconditional Fama regressions. We next turn to the unconditional Fama regression. Using Eq. 13, we have

$$\frac{1 + r_t^B}{1 + r_t^A} = \frac{\mathbb{E}_t\left[\frac{M_{t+1}^* e_{t+1}^A}{M_t^* e_t^A}\right]}{\mathbb{E}_t\left[\frac{M_{t+1}^* e_{t+1}^B}{M_t^* e_t^B}\right]}$$

which in the continuous time limit can be expressed as

$$r_t^B - r_t^A = \mathbb{E}_t\left[\frac{e_{t+1}^A - e_t^A}{e_t^A} - \frac{e_{t+1}^B - e_t^B}{e_t^B}\right] + \text{Cov}_t\left(\frac{M_{t+1}^*}{M_t^*}, \frac{e_{t+1}^A - e_t^A}{e_t^A} - \frac{e_{t+1}^B - e_t^B}{e_t^B}\right)$$

i.e.

$$\mathbb{E}_t\left[\frac{e_{t+1}^A - e_t^A}{e_t^A} - \frac{e_{t+1}^B - e_t^B}{e_t^B}\right] = r_t^B - r_t^A - \text{Cov}_t\left(\frac{M_{t+1}^*}{M_t^*}, \frac{e_{t+1}^A - e_t^A}{e_t^A} - \frac{e_{t+1}^B - e_t^B}{e_t^B}\right)$$

This expression highlights the role of the risk premium $\pi_t^{A,B}$:

$$\pi_t^{A,B} = -\text{Cov}_t\left(\frac{M_{t+1}^*}{M_t^*}, \frac{e_{t+1}^A - e_t^A}{e_t^A} - \frac{e_{t+1}^B - e_t^B}{e_t^B}\right)$$

Consider now the Fama (1984) regression of the changes in the exchange rate between countries A and B regressed on the difference in interest rates in a full sample:

$$\text{Fama regression: } \mathbb{E}_t\left[\frac{e_{t+1}^A - e_t^A}{e_t^A} - \frac{e_{t+1}^B - e_t^B}{e_t^B}\right] = \alpha^{Full} - \beta^{Full}(r_t^A - r_t^B) \tag{62}$$

The coefficient β^{Full} is now given by

$$\beta^{Full} = 1 - \frac{\text{Cov}(\pi_t^{A,B}, r_t^A - r_t^B)}{\text{Var}(r_t^A - r_t^B)}$$

Therefore, we can have $\beta^{Full} < 0$ if and only if the risk premium covaries positively enough with the interest rate differential. It is easy to compute

$$\pi_t^{A,B} = (1-\beta)(r_t^A - r_t^B) + p_t\mathbb{E}_t\left[F_{t+1}^A - F_{t+1}^B\right]$$

which leads to

$$\beta^{Full} = \beta - \frac{\text{Cov}\left(p_t\mathbb{E}_t\left[F_{t+1}^A - F_{t+1}^B\right], r_t^A - r_t^B\right)}{\text{Var}(r_t^A - r_t^B)}$$

$$\beta^{Full} = \beta + (1-\beta)\frac{\text{Cov}\left(p_t\mathbb{E}_t\left[F_{t+1}^A - F_{t+1}^B\right], \widehat{H}_t^A - \widehat{H}_t^B\right)}{\text{Var}\left(\widehat{H}_t^A - \widehat{H}_t^B\right)} \tag{63}$$

In the case where B_{t+1} is constant and equal to B, and $p_t\mathbb{E}_t\left[F_{t+1}^A - F_{t+1}^B\right] = \left(\widehat{H}_t^A - \widehat{H}_t^B\right)B^\gamma$, so:

$$\beta^{Full} = \beta + (1-\beta)B^\gamma = -\frac{\phi}{r_e} + \left(1 + \frac{\phi}{r_e}\right)B^\gamma$$

Proof of Proposition 5 *Derivation of the exchange rate.* Call $m_t = M_t^*\exp(-\lambda t)\overline{\omega}_t$. We show that $m_t\left(1, \widehat{H}_t, y_t\right)$ is a LG process. As in the Proof of Proposition 2:

$$\mathbb{E}_t\left[\frac{m_{t+1}}{m_t}\right] = \exp(-r_e)\left(1 + \exp(-h_*)\widehat{H}_t\right) = \exp(-r_e - h_*)(1 + H_t)$$

$$\mathbb{E}_t\left[\frac{m_{t+1}}{m_t}\widehat{H}_t\right] = \exp(-r_e - \phi_H)\widehat{H}_t$$

The new moment is:

$$
\mathbb{E}_t \left[\frac{m_{t+1}}{m_t} y_{t+1} \right] = \mathbb{E}_t \left[\frac{m_{t+1}}{m_t} \right] \mathbb{E}_t \left[y_{t+1} \right]
$$

$$
= \exp\left(-r_e - h_*\right) \left(1 + H_t\right) \frac{1 + H_*}{1 + H_t} \exp\left(-\phi_y\right) y_t = \exp\left(-r_e - \phi_y\right) y_t
$$

So $m_t \left(1, \widehat{H}_t, y_t \right)$ is a LG process, with generator:

$$
\Omega = \exp\left(-r_e\right) \begin{pmatrix} 1 & \exp\left(-h_*\right) & 0 \\ 0 & \exp\left(-\phi_H\right) & 0 \\ 0 & 0 & \exp\left(-\phi_y\right) \end{pmatrix}. \tag{64}
$$

The exchange rate follows:

$$
\frac{e_t}{\overline{\omega}_t} = \mathbb{E}_t \left[\sum_{s=0}^{\infty} \frac{M^*_{t+s}}{M^*_t} \exp\left(-\lambda s\right) \overline{\omega}_{t+s} \left(1 + g_t\right) \right] = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}' \cdot \left(I_3 - \Omega\right)^{-1} \cdot \begin{pmatrix} 1 \\ \widehat{H}_t \\ y_t \end{pmatrix}
$$

$$
= \frac{1}{1 - \exp\left(-r_e\right)} \left(1 + \frac{\exp\left(-r_e - h_*\right)}{1 - \exp\left(-r_e - \phi_H\right)} \widehat{H}_t \right) + \frac{1}{1 - \exp\left(-r_e - \phi_y\right)} y_t
$$

The last equation comes from the fact that $I_3 - \Omega$ is bloc-diagonal. This yields the announced expression.

Derivation of the interest rate. In the continuous time limit,

$$
\mathbb{E}_t \left[d\widehat{H}_t \right] = -\left(\phi_H + \widehat{H}_t \right) \widehat{H}_t dt \tag{65}
$$

$$
\mathbb{E}_t \left[dy_t \right] = -\left(\phi_y + \widehat{H}_t \right) y_t dt \tag{66}
$$

so the interest rate satisfies:

$$-r_t = \mathbb{E}_t\left[\frac{d\left(M_t^* e_t\right)}{M_t^* e_t}\right]/dt = \mathbb{E}_t\left[\frac{dM_t^*}{M_t^*}\,\middle|\, \text{no disaster}\right] + \mathbb{E}_t\left[\frac{de_t}{e_t}\,\middle|\, \text{no disaster}\right]$$

$$+p_t\left(\mathbb{E}_t\left[\frac{M_{t+}e_{t+}}{M_t e_t} - 1\,\middle|\, \text{disaster}\right]\right)$$

$$= -R + g_\omega + \frac{\frac{\mathbb{E}_t[d\widehat{H}_t]/dt}{r_e+\phi_H} + \frac{r_e\mathbb{E}_t[dy_t]/dt}{r_e+\phi_y}}{1 + \frac{\widehat{H}_t}{r_e+\phi_H} + \frac{r_e y_t}{r_e+\phi_y}} + p_t\left(B_t^{-\gamma}F_t - 1\right)$$

$$= -R + g_\omega + \frac{\frac{-\left(\phi_H+\widehat{H}_t\right)\widehat{H}_t}{r_e+\phi_H} + \frac{-r_e\left(\phi_y+\widehat{H}_t\right)y_t}{r_e+\phi_y}}{1 + \frac{\widehat{H}_t}{r_e+\phi_H} + \frac{r_e y_t}{r_e+\phi_y}} + H_* + \widehat{H}_t$$

$$= -r_e + \lambda + \frac{\frac{r_e}{r_e+\phi_H}\widehat{H}_t - \frac{r_e\phi_y}{r_e+\phi_y}y_t}{1 + \frac{\widehat{H}_t}{r_e+\phi_H} + \frac{r_e y_t}{r_e+\phi_y}}.$$

Proof of Proposition 6 We start by the case of the regression in a sample that does not contain disasters. As in the proof of Proposition 5,

$$\mathbb{E}_t\left[\frac{de_t}{e_t}\right]/dt = g_\omega + \frac{\frac{-\left(\phi_H+\widehat{H}_t\right)\widehat{H}_t}{r_e+\phi_H} + \frac{-r_e\left(\phi_y+\widehat{H}_t\right)y_t}{r_e+\phi_y}}{1 + \frac{\widehat{H}_t}{r_e+\phi_H} + \frac{r_e y_t}{r_e+\phi_y}}$$

So, up to second order terms in $\widehat{H}_t$ and y_t,

$$\mathbb{E}_t\left[\frac{de_t}{e_t}\right]/dt = g_\omega + \frac{-\phi_H\widehat{H}_t}{r_e+\phi_H} + \frac{-r_e\phi_y y_t}{r_e+\phi_y} \equiv a\widehat{H}_t + by_t + c$$

$$r_t = r_e - \lambda - \frac{r_e}{r_e+\phi_H}\widehat{H}_t + \frac{r_e\phi_y}{r_e+\phi_y}y_t \equiv A\widehat{H}_t + By_t + C$$

so

$$\beta' = -\frac{\text{Cov}\left(\mathbb{E}_t\left[\frac{de_t}{e_t}\right]/dt, r_t\right)}{\text{Var}\left(r_t\right)} = -\frac{aA\,\text{Var}\left(\widehat{H}_t\right) + bB\,\text{Var}\left(y_t\right)}{A^2\,\text{Var}\left(\widehat{H}_t\right) + B^2\,\text{Var}\left(y_t\right)}$$

$$= -\nu\frac{a}{A} - (1-\nu)\frac{b}{B} = \nu\beta + 1 - \nu$$

where $\nu = \frac{A^2\,\text{Var}\left(\widehat{H}_t\right)}{A^2\,\text{Var}\left(\widehat{H}_t\right)+B^2\,\text{Var}(y_t)}$.

The case of the full sample regression is proved similarly.

Proof of Proposition 7 We start with the call price.

$$V^{Call}(K) = \mathbb{E}_0\left[\frac{M_1^*}{M_0^*}\left(\frac{e_1^B}{e_0^B} - K\frac{e_1^A}{e_0^A}\right)^+\right]$$

$$= (1-p_0)\mathbb{E}_0^{ND}\left[\frac{M_1^*}{M_0^*}\left(\frac{e_1^B}{e_0^B} - K\frac{e_1^A}{e_0^A}\right)^+\right] + p_0\mathbb{E}_0^D\left[\frac{M_1^*}{M_0^*}\left(\frac{e_1^B}{e_0^B} - K\frac{e_1^A}{e_0^A}\right)^+\right]$$

$$= (1-p_0)e^{-R}\mathbb{E}_0^{ND}\left[\left(\frac{e_1^B}{e_0^B} - K\frac{e_1^A}{e_0^A}\right)^+\right] + p_0 e^{-R}\mathbb{E}_0^D\left[B_1^{-\gamma}\left(\frac{e_1^B}{e_0^B} - K\frac{e_1^A}{e_0^A}\right)^+\right]$$

where ND and D superscript denote expectation conditional, respectively, on no disaster and disaster. The next calculation uses the following Lemma, which is standard.[21]

Lemma 2 *(Discrete time Girsanov) Suppose that (x,y) are jointly Gaussian distributed under P. Consider the measure $dQ/dP = \exp(x - \mathbb{E}[x] - \text{Var}(x)/2)$. Then, under Q, y is Gaussian, with distribution*

$$y \sim^Q \mathcal{N}(\mathbb{E}[y] + \text{Cov}(x,y), \text{Var}(y)) \tag{67}$$

where $\mathbb{E}[y], \text{Cov}(x,y), \text{Var}(y)$ are calculated under P.

To do the calculation, write $\frac{e_1^{B,ND}}{e_0^B} = \exp(\mu_B + \varepsilon_B - \sigma_B^2/2)$, and the analogue for A. We call $\eta = \varepsilon_B - \varepsilon_A$, and calculate:

$$V_1 = \mathbb{E}_0^{ND}\left[\left(\frac{e_1^{B,ND}}{e_0^B} - K\frac{e_1^{A,ND}}{e_0^A}\right)^+\right] = \mathbb{E}_0^{ND}\left[\left(\exp(\mu_B + \varepsilon_B - \sigma_B^2/2 - K\exp(\mu_A + \varepsilon_A - \sigma_A^2/2))\right)^+\right]$$

$$= \exp(\mu_A)\mathbb{E}_0^{ND}\left[\exp(\varepsilon_A - \sigma_A^2/2)\left(\exp(\mu_B - \mu_A - \sigma_B^2/2 + \sigma_A^2/2 + \eta) - K\right)^+\right]$$

We define $dQ/dP = \exp(\varepsilon_A - \sigma_A^2/2)$, and use Lemma 2. Under Q, $y = \mu_B - \mu_A - \sigma_B^2/2 + \sigma_A^2/2 + \eta$ is a Gaussian variable with variance σ_η^2 and mean:

$$\mathbb{E}^Q[y] = \mu_B - \mu_A - \sigma_B^2/2 + \sigma_A^2/2 + \text{Cov}(\varepsilon_B - \varepsilon_A, \varepsilon_A)$$

$$= \mu_B - \mu_A - \sigma_B^2/2 - \sigma_A^2/2 + \sigma_{A,B} = \mu_B - \mu_A - \text{Var}(\eta)/2$$

Hence,

$$V_1 = \exp(\mu_A)\mathbb{E}^Q\left[(e^y - K)^+\right] = \exp(\mu_A)\mathbb{E}^Q\left[\left(\exp(\mu_B - \mu_A - \text{Var}(\eta)/2 + \eta) - K\right)^+\right]$$

$$= \exp(\mu_B)C^{BS}(K, \mu_B - \mu_A, \sigma_{A/B})$$

---

[21] To verify it, we calculate that the characteristic function of y is the characteristic function of distribution (67):

$$E^Q\left[e^{ky}\right] = E\left[e^{x-E[x]-\sigma_x^2/2}e^{ky}\right] = \exp\left(kE[y] + \frac{k^2\sigma_y^2}{2} + k\,cov(x,y)\right) = \exp\left[k(E[y] + cov(x,y)) + \frac{k^2\sigma_y^2}{2}\right].$$

where $\sigma_{A|B} = (\text{Var}\,(\varepsilon_B - \varepsilon_A))^{1/2}$, and $C^{BS}\,(K \exp{(-r)}, \sigma) = E\left[(\exp{(\sigma u - \sigma^2/2)} - \exp{(-r)}\,K)^+\right]$ (with u a standard Gaussian) is the Black-Scholes calls value when the interest rate is 0, the maturity 1, the strike K, the spot price 1, and the volatility σ.

Next, we observe that:

$$\mathbb{E}_0^D\left[B_1^{-\gamma}\left(\frac{e_1^B}{e_0^B} - K\frac{e_1^A}{e_0^A}\right)^+\right] = \mathbb{E}_0^D\left[B_1^{-\gamma}\left(\exp{(\mu_B)}\,F_B - K\exp{(\mu_A)}\,F_A\right)^+\right]$$

We conclude that the value of the call is (26).

Put price. We use put-call parity. Using the identity $x^+ = x + (-x)^+$, and the fact that $\mathbb{E}_0\left[\frac{M_1^* e_1^A}{M_0^* e_0^A}\right] = \exp{(-r_A)}$,

$$
\begin{aligned}
V^{Put}\,(K) &= \mathbb{E}_0\left[\frac{M_1^*}{M_0^*}\left(K\frac{e_1^A}{e_0^A} - \frac{e_1^B}{e_0^B}\right)^+\right] = \mathbb{E}_0\left[\frac{M_1^*}{M_0^*}\left(K\frac{e_1^A}{e_0^A} - \frac{e_1^B}{e_0^B}\right)\right] + \mathbb{E}_0\left[\frac{M_1^*}{M_0^*}\left(\frac{e_1^B}{e_0^B} - K\frac{e_1^A}{e_0^A}\right)^+\right] \\
&= \frac{K}{1 + r_A} - \frac{1}{1 + r_B} + V^{Call}\,(K)\,.
\end{aligned}
$$

The following analogue of (26) also holds:

$$
\begin{aligned}
V^{Put}\,(K) &= \exp{(-R + \mu_B)}\,(1 - p_0)\,C_{Put}^{BS}\left(K\exp{(\mu_A - \mu_B)}, \sigma_{A|B}\right) + \\
&\quad + \exp{(-R + \mu_B)}\,p_0\mathbb{E}_0^D\left[B_1^{-\gamma}\left(-F_{1,B} + K\exp{(\mu_A - \mu_B)}\,F_{1,A}\right)^+\right]
\end{aligned}
$$

Proof of Proposition 8 The proof of Proposition 5 showed that $M_t^* \exp{(-\lambda t)}\,\overline{\omega}_t\left(1, \widehat{H}_t, y_t\right)$ is a LG process, with generator given by (64). Writing $e_t = \overline{\omega}_t\left(a + b\widehat{H}_t + cy_t\right)$, we have

$$
\begin{aligned}
Z_t &= \mathbb{E}_t\left[\frac{M_{t+T}^* e_{t+T}}{M_t^* e_t}\right] = \frac{\exp{(\lambda T)}}{a + b\widehat{H}_t + cy_t}\mathbb{E}_t\left[\frac{M_{t+T}^* \exp{(-\lambda\,(t + T))}\,\overline{\omega}_{t+T}\left(a + b\widehat{H}_t + cy_t\right)}{M_t^* \exp{(-\lambda t)}\,\overline{\omega}_t}\right] \\
&= \frac{\exp{(\lambda T)}}{a + b\widehat{H}_t + cy_t}\begin{pmatrix}a \\ b \\ c\end{pmatrix}' \Omega^T \begin{pmatrix}1 \\ \widehat{H}_t \\ y_t\end{pmatrix} \quad \text{by the rules on LG processes} \\
&= \frac{\exp{(\lambda T)}}{a + b\widehat{H}_t + cy_t}\begin{pmatrix}a \\ b \\ c\end{pmatrix}' \exp{(-r_e T)}\begin{pmatrix}1 & \exp{(-h_*)}\frac{1 - \exp{(-\phi_H T)}}{1 - \exp{(-\phi_H)}} & 0 \\ 0 & \exp{(-\phi_H T)} & 0 \\ 0 & 0 & \exp{(-\phi_y T)}\end{pmatrix}\begin{pmatrix}1 \\ \widehat{H}_t \\ y_t\end{pmatrix} \\
&= \exp{(-(r_e - \lambda)T)}\frac{a + \left(a\exp{(-h_*)}\frac{1 - \exp{(-\phi_H T)}}{1 - \exp{(-\phi_H)}} + b\right)\widehat{H}_t + c\exp{(-\phi_y T)}\,y_t}{a + b\widehat{H}_t + cy_t} \\
&= \exp{(-(r_e - \lambda)T)}\frac{1 + \left(a\exp{(-h_*)}\frac{1 - \exp{(-\phi_H T)}}{1 - \exp{(-\phi_H)}} + b\right)\widehat{H}_t + c\exp{(-\phi_y T)}\,y_t}{1 + \frac{\exp{(-r_e - h_*)}}{1 - \exp{(-r_e - \phi_H)}}\widehat{H}_t + \frac{1 - \exp{(-r_e)}}{1 - \exp{(-r_e - \phi_y)}}y_t}\,,
\end{aligned}
$$

So the zero-coupon price is:

$$Z_t\left(T\right)=e^{-(r_e-\lambda)T}\frac{1+\frac{1-\exp(-r_e-\phi_H)-(1-\exp(-r_e))\exp(-\phi_H T)}{\left(1-e^{-\phi_H}\right)\left(1-e^{-r_e-\phi_H}\right)}\exp\left(-h_*\right)\widehat{H}_t+\exp\left(-\phi_y T\right)\frac{1-\exp(-r_e)}{1-\exp\left(-r_e-\phi_y\right)}y_t}{1+\frac{\exp(-r_e-h_*)}{1-\exp(-r_e-\phi_H)}\widehat{H}_t+\frac{1-\exp(-r_e)}{1-\exp\left(-r_e-\phi_y\right)}y_t}$$

Taking Taylor expansions,

$$Z_t\left(T\right)=e^{-(r_e-\lambda)T}\left[1+\frac{\left(1-e^{-r_e}\right)\left(1-e^{-\phi_H T}\right)}{\left(1-e^{-\phi_H}\right)\left(1-e^{-r_e-\phi_H}\right)}e^{-h_*}\widehat{H}_t-\frac{\left(1-e^{-r_e}\right)\left(1-e^{-\phi_y T}\right)}{1-e^{-r_e-\phi_y}}y_t\right]+o\left(\widehat{H}_t,y_t\right)$$

$$Y_t\left(T\right)=r_e-\lambda-\frac{\left(1-e^{-r_e}\right)\left(1-e^{-\phi_H T}\right)}{T\left(1-e^{-\phi_H}\right)\left(1-e^{-r_e-\phi_H}\right)}e^{-h_*}\widehat{H}_t+\frac{\left(1-e^{-r_e}\right)\left(1-e^{-\phi_y T}\right)}{\left(1-e^{-r_e-\phi_y}\right)T}y_t+o\left(\widehat{H}_t,y_t\right) \quad (68)$$

$$f_t\left(T\right)=r_e-\lambda-\frac{\left(1-e^{-r_e}\right)e^{-\phi_H(T-1)}}{1-e^{-r_e-\phi_H}}e^{-h_*}\widehat{H}_t+\frac{\left(1-e^{-r_e}\right)\left(1-e^{-\phi_y}\right)e^{-\phi_y(T-1)}}{\left(1-e^{-r_e-\phi_y}\right)}y_t+o\left(\widehat{H}_t,y_t\right) \quad (69)$$

and in the continuous time limit,

$$f_t\left(T\right)=r_e-\lambda-\frac{r_e}{r_e+\phi_H}e^{-\phi_H T}\widehat{H}_t+\frac{r_e\phi_y}{r_e+\phi_y}e^{-\phi_y T}y_t+o\left(\widehat{H}_t,y_t\right)$$

Proof of Proposition 12 The real part of the forward rate was calculated in Eq. 32. The nominal part is calculated in Gabaix (2007b). The two expressions add up, because we do a Taylor expansion.

Proof of Proposition 14 $P_t=E_t\left[\sum_{s\geq t}M_s\omega_s D_s\left(1+\frac{\widehat{H}_s}{R_e+\phi}\right)\right]/\left(e_t D_t\right)$. We calculate the corresponding LG moments. We start with:

$$\frac{M_{t+1}\omega_{t+1}D_{t+1}}{M_t\omega_t D_t}=\exp\left(-R+g_\omega+g_D\right)\times\begin{cases}1 & \text{if there is no disaster at }t+1\\ F_t F_t^i & \text{if there is a disaster at }t+1\end{cases}$$

where as before F_t is the reduction in the country productivity in producing the international good. We postulate that the process for F_t^i allows the decomposition:

$$p_t B_t^{-\gamma}F_t F_t^i-p_t=\left(H_*+p_*\right)F_*^i-p_*+F_*^i\widehat{H}_t+\left(H_*+p_*\right)\widehat{F}_t^i$$

This decomposition is the natural one, as the central value of $p_t B_t^{-\gamma}F_t$ is H_*+p_*, and the central value of F_t^i is called F_*^i. The process for F_{it} is a LG-twisted autoregressive process:

$$E_t\left[d\widehat{F}_{it}\right]/dt=-\left(\phi_{F_i}+F_*^i\widehat{H}_t+\left(H_*+p_*\right)\widehat{F}_t^i\right)\widehat{F}_{it} \quad (70)$$

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We define $r_D = R - g_D - g_\omega - (H_* + p_*) F_*^i + p_*$ The LG moments are (normalizing $g_D = g_\omega = 0$ in the derivations):

$$E_t \left[\frac{d(M\omega D)_t}{(M\omega D)_t} \right] / dt = -R + p_t \left(B_t^{-\gamma} F_t F_t^i - 1 \right) = -r_D + F_*^i \widehat{H}_t + (H_* + p_*) \widehat{F}_t^i$$

$$E_t \left[\frac{d\left(M\omega D \cdot \widehat{F}_{it} \right)_t}{(M\omega D)_t} \right] / dt = -R\widehat{F}_{it} - \left(\phi_{F_i} + F_*^i \widehat{H}_t + H_* \widehat{F}_t^i \right) \widehat{F}_{it} + p \left(B^{-\gamma} F_t F_t^i \cdot F_t^i - \widehat{F}_{it} \right)$$

$$= -(r_D + \phi_F) \widehat{F}_{it}$$

$$E_t \left[\frac{d\left(M\omega D \widehat{H}_t \right)_t}{(M\omega D)_t} \right] / dt = \left(-r_D + F_*^i \widehat{H}_t + (H_* + p_*) \widehat{F}_t^i \right) \widehat{H}_t - \left(\phi + \widehat{H}_t \right) \widehat{H}_t = -(r_D + \phi) \widehat{H}_t + h.o.t.$$

Hence the last expression involves a linearization. So, to a first order, $M_t \omega_t D_t \left(1, \widehat{H}_t, \widehat{F}_{it} \right)$ is a

LG process, with generator $\omega = \begin{pmatrix} r_D & -F_*^i & -(H_* + p_*) \\ 0 & r_D + \phi_H & 0 \\ 0 & 0 & r_D + \phi_F \end{pmatrix}$. So (50) gives, in virtue of the rule

on LG processes (Gabaix 2007a, Theorem 4 and Proposition 4):

$$P_t e_t = \begin{pmatrix} 1 \\ 1/(r_e + \phi_H) \\ 0 \end{pmatrix}' \omega^{-1} \begin{pmatrix} 1 \\ \widehat{H}_t \\ \widehat{F}_{it} \end{pmatrix} D_t,$$

which yields (51). Eq. 52 comes from a Taylor expansion.

Appendix C. Variance processes

Suppose an LG process centered at 0, $dX_t = -(\phi + X_t) X_t dt + \sigma(X_t) dW_t$, where W_t is a standard Brownian motion. Because of economic considerations, the support of the X_t needs to be some $(X_{\min}, X_{\max})$, with $-\phi < X_{\min} < 0 < X_{\max}$. The following variance process makes that possible:

$$\sigma^2(X) = 2K \left(1 - X/X_{\min} \right)^2 \left(1 - X/X_{\max} \right)^2 \tag{71}$$

with $K > 0$. K is in units of $[\text{Time}]^{-3}$. The average variance of X is $\overline{\sigma}_X^2 = \mathbb{E}\left[\sigma^2(X_t) \right] = \int_{X_{\min}}^{X_{\max}} \sigma(X)^2 p(X) dX$, where $p(X)$ is the steady state distribution of X_t. It can be calculated via the Forward Kolmogorov equation, which yields

$$d \ln p(X) / dX = 2X(\phi + X) / \sigma^2(X) - d \ln \sigma^2(X) / dX.$$

Numerical simulations shows that the process volatility is fairly well-approximated by: $\overline{\sigma}_X \simeq K^{1/2}\xi$, with $\xi = 1.3$. Also, the standard deviation of X's steady state distribution is well-approximated by $(K/\phi)^{1/2}$.

Asset prices often require to analyze the standard deviation of expressions like $\ln(1 + aX_t)$. Numerical analysis shows that the Taylor expansion approximation is a good one: Average volatility of: $\ln(1 + aX_t) \simeq aK^{1/2}\xi$, which numerical simulations prove to be a good approximation too.

For the steady-state distribution to have a "nice" shape (e.g., be unimodal), the following restrictions appear to be useful: $K \leq 0.2 \cdot \phi \left|X_{\min}\right| X_{\max}$.

When the process is not centered at 0, one simply centers the values. For instance, in our calibration, the recovery rate of the country productivity, F_t, has support $[F_{\min}, F_{\max}]$, centered around F_*. The probability and intensity of disasters (p and B) are constant. Define $H_t = p\left(B^{-\gamma}F_t - 1\right)$, and the associated $H_{\min}, H_{\max}, H_*$. The associated centered process is $X_t = \widehat{H}_t = H_t - H_*$. We take the volatility parameter to be: $K = v \cdot \phi_H \left|X_{\min}\right| X_{\max}$, with the volatility parameter $v \in [0, 0.2]$. This yields a volatility of $\widehat{H}_t$ equal to $\overline{\sigma}_{\widehat{H}_t} = \xi\left(v \cdot \phi_H \left|\widehat{H}_{\min}\right| \widehat{H}_{\max}\right)^{0.5}$, a volatility of F_t equal to $\overline{\sigma}_F = \overline{\sigma}_{\widehat{H}_t}/\left(pB^{-\gamma}\right)$, and a volatility of the bilateral exchange rate (between two uncorrelated countries) equal to $\sqrt{2}\overline{\sigma}_{\widehat{H}_t}/\left(r_e + \phi_H\right)$.

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