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INFLATION AND GROWTH

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ABSTRACT

Models of inflation and growth in the sixties emphasized the portfolio substitution mechanism by which higher inflation made capital more attractive to hold relative to money, leading to higher capital intensity, and in the transition period to higher growth. The empirical evidence, however, is that growth and inflation are negatively correlated. Reasons for this negative correlation are investigated, and then embodied in a simple monetary maximizing model. Higher inflation is associated with lower growth because lower real balances reduce the efficiency of factors of production, and because there may be a link between government purchases and the use of the inflation tax. Comparative steady states and comparative dynamics is analyzed and the generally negative association between inflation and growth, both in steady states and in transition processes, is demonstrated.

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INFLATION AND GROWTH

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Much of Miguel Sidrauski's research focussed on the relationship between inflation and economic growth. Two of his contributions stand out. The first is the famous intertemporal optimizing model (1967a)--the Sidrauski model--in which the basic result is that changes in the growth rate of money have no effect on the capital stock in the long run. The second is his paper (1967b) developing Tobin's (1965) money and growth model, with constant savings rate, in which the equilibrium capital intensity increases with the inflation rate. In the short run in this model, though, an increase in the growth rate of money reduces the growth rate of output while increasing the inflation rate.

Sidrauski's approach to the money, inflation, and growth question emphasized portfolio substitution as the basic driving force in determining the relationship between inflation and growth. For instance, in his descriptive model (Sidrauski (1967b)), an increase in the growth rate of money leads to an increase in the expected rate of inflation, thereby reducing the demand for real balances. With the savings rate given, more of savings takes the form of physical capital and less is in (high-powered) money. Equilibrium capital intensity is thereby increased. Fiscal factors enter in the background, since in both these papers the money stock is changed through lump sum transfer payments.

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In his work with Duncan Foley (1970 and 1971), Sidrauski was able both to use a more sophisticated definition of monetary policy and to take a step away from the treatment of money growth as exogenous to the economy. There are three assets in the Foley-Sidrauski model, money, bonds, and capital. The three assets are imperfect portfolio substitutes. Monetary policy is now defined by changes in the ratio of money to bonds, caused for instance by an open market operation. Because money and bonds are imperfect substitutes, open market operations have real effects.[1] In this work the government is viewed as having desired paths for the inflation rate and other endogenous variables and as using its policy variables--including the composition of the debt, and the growth rate of the nominal stock of debt--to achieve those paths.

In the Foley-Sidrauski model, changes in the actual and expected inflation rates typically increase the steady state capital stock.[2] The change in the capital stock is again the consequence of the portfolio shift away from money towards capital that accompanies a reduction in the expected real return on money, combined with the specification of the consumption function. If the change in the inflation rate occurs as a result of a change in the rate of growth of the nominal stock of debt, with the composition of the debt being adjusted

[1] The importance of assumptions about asset substitutability was emphasized by Tobin (1961). It is clear that the Foley-Sidrauski model owes part of its structure to the Yale tradition in which Foley was trained. This is evident not only in the asset menu but also in the key role played in the model by the relative price of capital, which in this two-sector model is equivalent to Tobin's "q" in a one-sector model.

[2] The analysis is presented in Chapter 12 of Foley-Sidrauski (1971). It is interesting to note that in this chapter, the authors take great care that jumps in the price-level are prevented through appropriate monetary and fiscal policy when the inflation rate changes. This is so that the government does not "dissappoint private expectations" (p.190). However, in the exercises presented in Chapter 12, a change in the expected rate of inflation may change the relative price of capital unanticipatedly, so that the analysis cannot be taken as assuming perfect foresight.

through monetary policy to prevent a jump in the price level when the inflation rate changes, there is no effect on the capital stock of a change in the inflation rate.

Despite the ambiguities of the relationship between inflation and capital intensity in Sidrauski's work, the overall results suggest that higher inflation will be associated with higher capital intensity. This has no implications for the inflation-growth relationship since in steady state the the rate of growth is determined by population and productivity increase. But the steady state capital intensity-inflation relationship implies a relationship between inflation and growth in the adjustment process. If a higher inflation rate is associated with higher capital intensity, then in the transition from one steady state to another the growth rate of output has on average to be above the growth rate of population and productivity. Thus if higher inflation is associated with greater capital intensity, we should expect that on average higher inflaton rates will be associated with higher growth rates of output. Further, even in the Sidrauski optimizing model, increases in the inflation rate that do not affect the steady state capital stock may increase the growth rate of output in the short run. (Fischer, 1979).[3]

Sidrauski wrote his papers on inflation and growth in the second half of the 1960's. After the stagflationary decade of the 1970's, we would be surprised to find any systematic relationship between the growth rate of output and the inflation rate. Equations (R1) and (R2) present the results of cross-section

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[3] In Sidrauski (1967b) an increase in the growth rate of money in the short run is associated with a lower growth rate of output, despite the positive relationship between steady state inflation and capital intensity. This result depends on the assumption of adaptive expectations; it disappears if expectations are formed rationally and the price level is allowed to jump when the growth rate of money changes.

time-series regressions for the periods 1961-1973 and 1973-1981 respectively for a sample of 53 countries, chosen purely by the availability of data on the International Financial Statistics tapes.[4]

(R1) $gy_{i,t} = \alpha_i + \alpha_t - .049gy_{i,t-1} - .192gp_{i,t} + .060gp_{i,t-1}$

$\qquad\qquad\qquad\qquad (.044) \qquad\quad (.022) \qquad\quad (.023)$

$\bar{R}^2 = .508$

(R2) $gy_{i.t} = \alpha_i + \alpha_t + .341gy_{i,t-1} - .868gp_{i,t} + .288gp_{i,t-1}$

$\qquad\qquad\qquad\qquad (.047) \qquad\quad (.021) \qquad\quad (.045)$

$\bar{R}^2 = .803$

In these regressions $gy_{i,t}$ is the growth rate of output in country i in period t, and $gp_{i,t}$ is the inflation rate. The time period and country specific coefficients, α_t and α_i are not presented.

Both regressions show the same pattern: a negative contemporaneous relationship between growth and the inflation rate, and a positive lagged relationship. The coefficients of the inflation variables are strongly significant in all cases. Taking averages across the 54 countries for the two time periods, we obtain the data presented in Table 1. As we would have expected, there is no sign of a positive relationship between inflation and growth between the two periods. And even within each sub-period, the regressions show a predominately negative rather than positive relationship between inflation and growth.

[4] The sample is not random, since it is the higher income countries that typically maintain more complete data. The countries are listed in appendix 1.

Table 1: Inflation and Growth, Annual Averages Across Countries and Years (% per annum)

Years	1961-1973	1973-1981
Inflation rate	6.83	14.31
Growth rate	5.10	4.33

The facts thus disagree with the implication that inflation and growth are positively correlated: there is a significant negative contemporaneous correlation. In the next section I review hypotheses about the links among inflation, output, and growth. Then in the remainder of the paper I present a simple Sidrauski type optimizing model based on some of these links that suggests why we might observe a negative relationship between inflation and growth.

1. The Mechanisms Linking Inflation, Output, and Growth.

There are several economic mechanisms linking inflation, output, and growth. (i) The first is the portfolio link emphasized by Sidrauski. An increase in the expected rate of inflation causes a shift from money to capital in the portfolio. In a two sector model this increases the relative price of installed capital, leading to higher rates of investment and growth, and ultimately to higher capital intensity in production. However, savings behavior may sever the link. For instance, if savings drives the real interest rate to equality with the rate of time preference (adjusted for growth), changes in the inflation rate may leave the steady state capital intensity unaffected, as in

Sidrauski (1967a).[5]

(ii) Non-adjustment of the tax system for inflation may make the after-tax real return to capital a decreasing function of the inflation rate, even if the pre-tax rate of return is independent of the inflation rte. This is the mechanism emphasized by Feldstein (1976). This mechanism tends to reduce steady state capital intensity as the inflation rate rises.

(iii) If the growth rate of the money stock is given, an increase in the growth rate of output tends to reduce the inflation rate because the demand for real balances increases more rapidly. Equivalently, an adverse supply shock will lead to an increase in the inflation rate at the same time as the level of output or, in the short run, the growth rate of output, declines.

(iv) If government spending is financed in part through the printing of money, then a reduction in the growth rate of output, reducing the demand for real balances, may require the government to increase the growth rate of money. This is a special case of "passive money", the approach emphasized by Olivera (1970).

(v) Increases in the inflation rate lead to lower real balances and the diversion of real resources to the making of transactions. This reduces output. More generally, it appears that higher rates of inflation--anticipated as well as unanticipated--reduce the efficiency of the price system, and reduce factor productivity.

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[5] Dornbusch and Frenkel (1973) show that the relationship between inflation and equilibrium capital intensity depends on the reduced form effect of an increase in the inflation rate on consumption.

(iv) The short-run Phillips curve implies that increases in output and the inflation rate (perhaps the unanticipated inflation rate) are positively associated.

In this lecture I use a variant of the Sidrauski optimizing model to study the relationships among inflation, output, and growth under the assumptions that money enters the production function, and that the printing of money is used to finance government spending. I thus embody in the model two of the potential links between inflation and growth that were outlined above: reductions in real balances reduce the productivity of factors of production (point (v)); and that changes in the inflation rate may be a result of changes in government spending (point (iv)). I shall also examine the effects of supply shocks, thus allowing also for the impact of autonomous changes in ouput on the inflation rate (point (iii)). The general results to be expected are that inflation and growth will be negatively related.

2. An Optimizing Model with Money in the Production Function.

I start with a simple model in which the growth rate of money is exogeneous but money enters the production function rather than the utility function. As noted by Sidrauski in his thesis (1967c, p.53), the inclusion of money in the production function removes the superneutrality of money that is the best known result of his monetary optimizing model. Money is included in the production function to represent the notion that higher levels of real balances free labor and other resources for productive use. The practice can be rigorously justified. (Fischer, 1974).

The representative family maximizes the additively separable utility functional

(1) $V_o = \int_o^\infty u(c_t)e^{-\delta t}dt$

subject to the constraints

(2) $\dot{a}_t = r_t k_t + w_t + i_t m_t - \pi_t m_t + v_t - c_t - na_t$

(3) $a_t = k_t + m_t$

Most letters represent the standard variables and all relevant variables are per capita: a is wealth, r the real interest rate, w the wage rate, i the nominal interest rate, π the expected inflation rate, v is the real per capita value of the transfers through which money balances are introduced, n is the growth rate of the family. The family has amount a_t of wealth at time t that it allocates between capital and the holding of money.

The money stock is growing at a constant write θ, so that the real value, per capita, of transfer payments v_t is equal to $\frac{\dot{M}}{PN} = \frac{\dot{M}}{M}\frac{M}{PN} = \theta m$ where P is the price level and N is population.

Physical capital and real balances are rented to firms at rental rates of r and i respectively. Firms maximize profits, producing output by the neoclassical production function

(4) $y_t = f(k_t, m_t)$

where again all variables are per capita.

First order conditions for an interior maximum for the household are

(5) $u' = \lambda_1$

(6) $i_t - \pi_t = r_t$

(7) $\frac{\dot{\lambda}_1}{\lambda_1} = \frac{u''\dot{c}}{u'} = \delta + n - r_t$

Here λ_1 is the multiplier on the flow budget constraint (2). Note that for an interior maximum the real returns on capital and real balances have to be equal.

First order conditions for the figure are:

(8) $f_k(k,m) = r_t$

(9) $f_m(k,m) = i_t = r_t + \pi_t$

For later purposes it is useful to solve (8) and (9) for π as a function of k and m:[6]

(10) $\pi = \pi(k,m)$ $\qquad \pi_1 = f_{mk} - f_{kk} > 0$

$\pi_2 = f_{mm} - f_{km} < 0$

From the definition of the per capita money stock $m = \frac{M}{PN}$ we obtain

(11) $\dot{m} = \lfloor \theta - (\pi(k,m) + n) \rfloor m$

The Steady State: In steady state k, m, c, and π are constant. From (11) we obtain that in steady state:

(12) $\pi^* = \theta - n = \pi(k,m)$

where π^* is the steady state rate of inflation. From (7) we obtain:

(13) $r^* = \delta + n = f_k(k,m)$

implying that the real interest rate is invariant to the steady state inflation rate.

However, money is not superneutral in this model. Using (12) and (13) and making the assumption that $f_{km} > 0$ we show in Figure 1 the loci in (k,m) space on which $\dot{c}$ and $\dot{m}$ are zero: these are equations (13) and (12) respectively. It can be shown that the $(\dot{c} = 0)$ locus is steeper than the $(\dot{m} = 0)$ schedule.

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[6] It is more intuitive to think of (8) and (9) as giving demand functions for the factors of production as functions of r and π. Equation (10) then merely inverts the relationship. For simplicity I henceforth assume $f_{mk} > 0$.

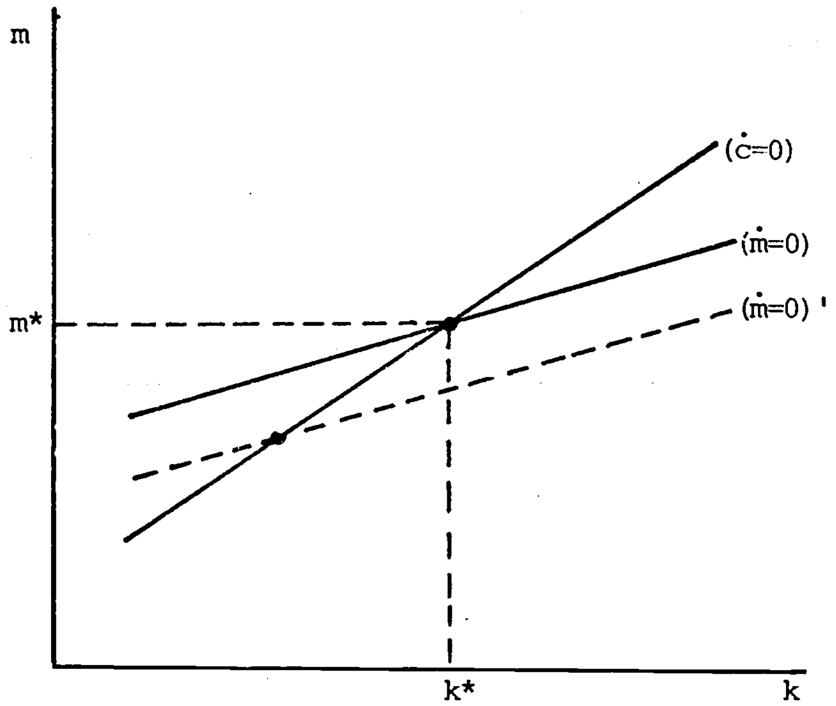

Figure 1: Determination of Steady-State Real Balances and Capital.

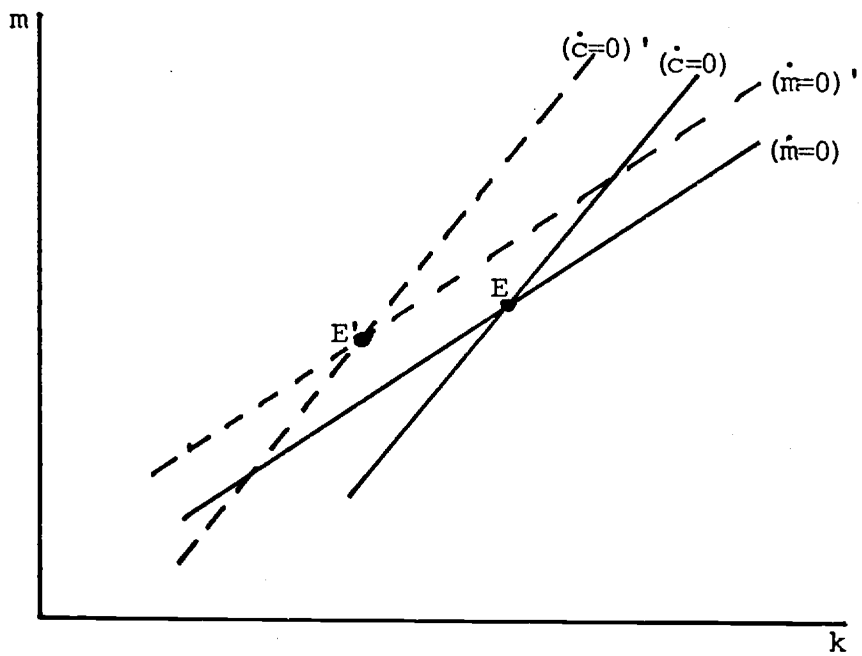

Figure 2: The Effects of an Increase in the Growth Rate of Population.

Now, using (12), an increase in the growth rate of money (which increases the steady state inflation rate) shifts the $(\dot{m} = 0)$ locus down to $(\dot{m} = 0)'$. Both k and m fall, and equilibrium output per capita therefore falls. Thus, as is to be expected in this case, higher inflation reduces output. In the transition between steady states, higher inflation is accompanied by lower output growth.

Similarly, in Figure 2 we analyze the effects of an increase in the rate of population growth, n. Both the $(\dot{m} = 0)$ and $(\dot{c} = 0)$ loci shift back as shown, to $(\dot{m} = 0)'$ and $(\dot{c} = 0)'$ respectively. It can be shown that the shifts are such as to reduce both k^* and m^*. The economy thus moves from a position like E to one like E'. Per capita steady state output is thus reduced. Since an increase in n reduces the steady state inflation rate we once again have a negative relationship between inflation and output, this time across steady states. By the same token, we should expect a negative relationship in the transition between steady states.

<u>The Steady State with Productivity Growth</u>. Assume next that there is labor augmenting technical progress at rate μ, and that the utility function is of the form:

$$(14) \quad U(c) = \frac{c^{1-R}}{1-R} \qquad\qquad R > 0$$

where R is the coefficient of relative risk aversion and also the elasticity of marginal utility.

In this case there exists a steady state in which ratios of variables to effective labor units are constant. The steady state real interest rate is given by

$$(15) \quad r^* = \delta + n + R\mu$$

and the steady state inflation rate is

$$(16)\quad \pi^* = \theta - (n + \mu)$$

The growth rate and the inflation rate are inversely related in this case too. Further, it appears likely, though not certain, that the short run relationship between output growth and inflation is also negative. The ambiguity arises because there may be a phase of capital decumulation and perhaps lower growth immediately after the increase in the rate of productivity growth.[7]

<u>The Adjustment Path</u>. I assume there is perfect foresight so that actual and expected inflation rates are the same. In that case the dynamic equations of the model are (17), (11) and (7):

$$(17)\quad \dot{k}_t = f(k_t,m_t) - c_t - nk_t$$

$$(7)\quad \dot{c}_t = \frac{u'}{u''}\lfloor\delta + n - f_k(k,m)\rfloor$$

$$(11)\quad \dot{m} = \lfloor\theta - (\pi(m,k) + n)\rfloor m$$

Equation (17) is not an explicit constraint on the behavior of any individual agent in the model, but has to be satisfied in perfect foresight equilbrium.

It is shown in Appendix 2 that there is a unique stable root for the dynamic system (17), (7) and (11), linearized around the steady state. The solution for the optimal path of the capital stock can be written as

$$(18)\quad k_t - k^* = (k_0-k^*)e^{zt}$$

where z is the unique stable root of the system, k^* is the steady state capital stock, and k_0 is the initial capital stock.

The behavior of the inflation rate in the adjustment process can be derived from (10), given the dynamics of m and k. In particular, we obtain

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[7] Because k and m are now ratios of capital and real balances respectively to <u>effective</u> and not <u>actual</u> labor, it is difficult to compare the pre- and post-productivity increase situations.

$$(19) \quad \dot{\pi} = \frac{\pi_1 z}{z + \pi_2 m} \dot{k}$$

Both numerator and denominator of the ratio in (19) are negative, so that the ratio itself is positive. This means that the rate of inflation is increasing so long as capital is being accumulated. This result may appear paradoxical until we realize that on the adjustment path the growth rate of the capital stock is falling as capital is accumulated. Thus we find a negative relationship between the rate of inflation and the rate of growth on the adjustment path.

A Supply Shock: Suppose that the economy is in a steady state, and that there is a Hicks-neutral supply shock. We represent this by rewriting the production function (4) as

$$(4)' \quad y_t = Af(k_t, m_t)$$

The supply shock is represented by a fall in A from 1 to a lower level.

We can analyze the consequences of a fall in A by considering its effects on the equilibrium capital stock. Using equations (12) and (13), it can be shown that an adverse Hicks-neutral supply shock reduces both the equilibrium capital stock and the equilibrium stock of real balances. Dynamic adjustment thus takes the form of a gradual reduction of the capital stock, accompanied by a falling inflation rate.

Since the steady state inflation rate after the permanent supply shock is back to the level at which it started, the transition period is marked by higher than steady state inflation, along with lower than steady state growth as capital is being decumulated. Again, there is a negative relationship between the rate of growth of ouput and the inflation rate in the adjustment process.

In the model of this section in which the rate of growth of the money stock is exogenous, the predominant relationship between the inflation rate on one hand and the growth rate or the level of ouput on the other hand is negative. In every case we have examined, we have found such a negative relationship. The primary difference between the structure of the present model and that of Sidrauski (1967a) is that money is treated in this section as a factor of production, rather than an argument of the utility function. This--as noted by Sidrauski--makes for non-neutrality of the inflation rate.

Beyond this, the theory also suggests that the inflation rate and the growth rate of output should in general be negatively correlated. Such a result also obtains in Sidrauski's model for certain of the exercises we have examined. For instance, given the growth rate of the money stock, an increase in the rate of productivity growth will reduce the steady state inflation rate in both this model and Sidrauski's. Thus there is no implication even in the original Sidrauski model that there can be only a one-directional relationship between inflation and growth.

3. Money Financing of Government Spending.

The final possible link we examine between inflation and output arises from the use of the inflation tax to finance government spending.[8] The set-up changes very little. In equation (2), the consumer's flow budget constraint, the term v_t disappears since the government no longer makes transfer payments. First order conditions for both households and firms are unchanged, as is the inflation equation (10).

[8] See Dornbusch (1977) for an earlier analysis of endogenous money.

The changes come in the description of the dynamic system, equations (7), (11) and (17). Assume that government purchases each period are equal to γc. The ratio of government purchases to consumption is now constant.[9] The dynamic equations are:

$$(20)\quad \dot{k} = f(k,m) - c(1 + \gamma) - nk$$

$$(21)\quad \dot{c} = \frac{u'}{u''}(\delta + n - f_k(k,m))$$

$$(22)\quad \dot{m} = \gamma c - [\pi(m,k) + n]m$$

The following results can be demonstrated, for small values of γ. First, with constant γ there is a unique perfect foresight path for the economy, with the same dynamic behavior as that in Section 2. Second, there are the expected comparative steady state results. An increase in γ--an increase in government spending financed through the inflation tax--reduces steady state real balances and capital stock, and increases the steady state inflation rate.

It was to be expected, given that real balances enter the production function, that we would find here as in the previous model a negative steady-state relationship between inflation and output. However, the result is not dependent on the inclusion of real balances in the production function, for even if money were a consumer good as in the original Sidrauski maximizing model, we would find a similar relationship.

4. Concluding Comments.

We have shown, using essentially the model introduced by Sidrauski, that several economic factors can each account for the observed negative relationship between inflation and growth or between inflation and output--even overlooking

[9] In a more complete analysis with a maximizing government, this ratio could be endogenous.

the possibly significant tax effects. It is only the Phillips curve that suggests we might find an opposite relationship between inflation and growth, at least in the short run. But the present model, in which there is full information and full flexibility of prices, is not well-suited to the analysis of the Phillips curve.

Of course, the contrast between the results in this paper and those in Sidrauski (1967a) suggest a warning that is conveyed also by considering the history of the Phillips curve: there is unlikely to be a unique correlation between two endogenous variables that is independent of the disturbances--policy induced or otherwise--that cause them to change. Typically someone examining a relationship like that between inflation and growth has a policy question in mind: for instance, does money financing of a deficit result in a higher capital stock than debt financing? It is preferable to address the policy question directly than to ask about the general relationship between inflation and growth.

Many of the results presented in this lecture are derived from comparative steady state analysis. But it is noteworthy that it is also possible to use the models to examine impact effects of policy changes and to study the adjustment process. The only innovation here is the use of perfect foresight, for in his thesis Sidrauski examined adjustment processes in his maximizing model, using adaptive expectations. Sidrauski of course did not use rational expectations assumptions in his work. But in his building of a complete macro model with maximizing agents, he was thoroughly in the spirit that is dominant in much of macroeconomics today.

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Appendix 1

Countries Included in the Data Sample.

Argentina	Korea
Australia	Luxembourg
Austria	Malawi
Belgium	Nepal
Burma	Netherlands
Canada	New Zealand
Columbia	Nigeria
Costa Rica	Norway
Cyprus	Pakistan
Denmark	Paraguay
Ecuador	Peru
El Salvador	Phillipines
Finland	Singapore
France	South Africa
Germany	Spain
Greece	Sweden
Guyana	Switzerland
Haiti	Thailand
Honduras	Tunisia
Iceland	Turkey
India	United Kingdom
Indonesia	United States
Ireland	Venezuela
Israel	Yugoslavia
Italy	Zaire
Jamaica	Zambia
Japan	

Appendix 2: Stability Analysis

1. The model discussed in Section 2 is described by equations (17), (7), and (11). Linearizing those equations around the steady state we obtain:

$$\text{(A1)} \quad \begin{bmatrix} \dot{k} \\ \dot{c} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \delta & -1 & f_m \\ -\frac{u'}{u''} f_{kk} & 0 & -\frac{u'}{u''} f_{km} \\ -\pi_1 m & 0 & -\pi_2 m \end{bmatrix} \begin{bmatrix} dk \\ dc \\ dm \end{bmatrix}$$

2. The determinant of the above system is

$$\text{(A2)} \quad \Delta = \frac{m \cdot u'}{u''} [f_{kk} f_{mm} - f_{mk}^2] < 0$$

Since the determinant is the product of the roots, there are either one or three negative roots.

3. The trace of the above matrix is

$$\text{(A3)} \quad Tr = \delta - \pi_2 m > 0$$

The trace is the sum of the roots. Since the sum of the roots is positive, not all roots can be negative. Accordingly there is only one negative root.